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On the analogy between Arithmetic Geometry and foliated spaces

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Abstract: *Christopher Deninger has developed an infinite dimensional cohomological formalism which allows to prove the expected properties of the arithmetical Zeta functions (including the Riemann Zeta function). These cohomologies are (in general) not yet constructed. Deninger has argued that these cohomologies might be constructed as leafwise cohomologies of suitable foliated spaces. We shall review some recent results which support this hope.*

1 – Introduction

Christopher Deninger's approach to the study of arithmetic zeta functions proceeds in two steps.

In the **first step**, he postulates the existence of infinite dimensional cohomology groups satisfying some "natural properties". From these data, he has elaborated a formalism allowing him to prove the expected properties for the arithmetic zeta functions: functional equation, conjectures of Artin, Beilinson, Riemann ... etc. There it is crucial to interpret the so called explicit formulae for the arithmetic zeta function as a Lefschetz trace formula.

The **second step** consists in constructing these cohomologies. Deninger has given some hope that these cohomologies might be constructed as leafwise cohomologies of suitable foliated spaces. Very little is known in this direction

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at the moment, but this second step seems to be a good motivation to develop interesting mathematics even if they are far from the ultimate goal.

In Section 2 we recall Deninger's cohomological formalism in the case of the Riemann zeta function. We point out a dissymmetry in the explicit formula (1) between the coefficients of $\delta_{k \log p}$ and $\delta_{-k \log p}$, see Comment 1.

In Section 3 is devoted to the description of the Lesfchetz trace formula for a flow acting on a codimension one foliated space. In Section 3.1 we recall the Guillemin-Sternberg trace formula which is indeed an important computational tool for this goal. By comparison with (1), it suggests that there should exist a flow $(\phi^t)_{t\in\mathbb{R}}$ acting on a certain space $S_{\mathbb{Q}}$ with the following property. To each prime number p [resp. the archimedean place of $\mathbb{Q}$] there should correspond a closed orbit with length $\log p$ [resp. a stationary point] of the flow ϕ^t.

In Section 3.2 we recall the theorem of Alvarez-Lopez and Kordyukov.

They consider a flow $(\phi^t)_{t\in\mathbb{R}}$ acting on $(X, \mathcal{F})$ where the compact three dimensonal manifold X is foliated by Riemann surfaces. They assume that $(\phi^t)_{t\in\mathbb{R}}$ preserves globally the foliation and is transverse to the foliation. Then Alvarez-Lopez and Kordyukov define a suitable leafwise Hodge cohomology on which ϕ^t acts and they prove a Lefschetz trace (à la Atiyah-Bott) formula which has some similarities with (1) for t real positive. But the dissymmetry mentioned above for (1) does not hold.

In Section 3.2 we consider the case of an elliptic curve E_0 over a finite field $\mathbb{F}_q$. The explicit formula (7) for its zeta function ζ_{E_0} exhibits a dissymmetry between the coefficients of $\delta_{k \log Nw}$ and $\delta_{-k \log Nw}$, where w is a closed point of E_0. It is quite analogous to the one mentioned above for (1). We review briefly our result which, using the work of Deninger and results from Alvarez-Lopez and Kordyukov, allows to interpret (7) as an Atiyah-Bott Lefschetz trace formula and to provide a dynamical interpretation of this dissymmetry.

In Section 4, we first recall the statement of Lichtenbaum's conjecture for a number field K. Then we explain briefly how Deninger proved an analogue of this conjecture in the case of a foliation $(X, \mathcal{F}, \phi^t)$ with the following properties. X is a smooth compact 3-dimensional manifold endowed with a codimension 1 foliation $\mathcal{F}$ and the flow ϕ^t preserves globally the foliation and is transverse to it. We shall explain how the reduced leafwise cohomology enters as a crucial ingredient of the proof.

In Section 5, we make a synthesis of various results of Deninger. We state several axioms for a laminated foliated space $(S_{\mathbb{Q}}, \mathcal{F}, g, \phi^t)$ which (if satisfied!) allow to construct the required cohomology groups for the Riemann zeta function. We compare carefully the contribution of the archimedean place of $\mathbb{Q}$ in (1) with the contribution of a stationary point in the Guillemin-Sternberg formula.

2 – Deninger's Cohomological formalism in the case of the Riemann zeta function

The (completed) Riemann zeta function is given by:

$$\widehat{\zeta}(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2)\prod_{p\in\mathcal{P}}\frac{1}{1-p^{-s}}$$

where $\mathcal{P} = \{2,3,5,\dots\}$ denotes the usual set of prime numbers. The following well known explicit formulas express a connection between $\mathcal{P}\cup\{\infty\}$ and the zeroes of $\widehat{\zeta}$. Let $\alpha \in C^{\infty}_{\text{compact}}(\mathbb{R},\mathbb{R})$ and for real s, set $\Phi(s) = \int_{\mathbb{R}} e^{st}\alpha(t)dt$; Φ belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$. Then one can prove the following formula:

$$\begin{aligned}&\Phi(0) - \sum_{\rho\in\widehat{\zeta}^{-1}\{0\},\,\Re\rho\geq 0}\Phi(\rho) + \Phi(1) = \\ &= \sum_{p\in\mathcal{P}}\log p\left(\sum_{k\geq 1}\alpha(k\log p) + \sum_{k\leq -1}p^k\alpha(k\log p)\right) + W_\infty(\alpha),\end{aligned}\tag{1}$$

where

$$W_\infty(\alpha) = \alpha(0)\log\pi + \int_0^{+\infty}\left(\frac{\alpha(t)+e^{-t}\alpha(-t)}{1-e^{-2t}} - \alpha(0)\frac{e^{-2t}}{t}\right)dt.$$

Now recall the standard Lefschetz trace formula for a smooth map with non degenerate fixed points $\phi : V \to V$ where V is an oriented compact Riemann surface:

$$\sum_{j=0}^{2}(-1)^j\,\mathrm{TR}(\phi^* : H^j(V,\mathbb{R})) = \sum_{\phi(v)=v}(-1)\mathrm{sign}\det(\mathrm{Id}-D\phi)(v).$$

Deninger's philosophy is motivated by the fact that the left hand side of (1)

$$\Phi(0) - \sum_{\rho\in\widehat{\zeta}^{-1}\{0\},\,\Re\rho\geq 0}\Phi(\rho) + \Phi(1)$$

is reminiscent of a Lefschetz trace formula of the form

$$\mathrm{TR}\int_{\mathbb{R}}\alpha(t)e^{t\,\Theta_0}\,dt - \mathrm{TR}\int_{\mathbb{R}}\alpha(t)e^{t\,\Theta_1}\,dt + \mathrm{TR}\int_{\mathbb{R}}\alpha(t)e^{t\,\Theta_2}\,dt,$$

where the following two assumptions should be satisfied.

- $\Theta_0 = 0$ acts on $H^0 = \mathbb{R}$, $\Theta_2 = \mathrm{Id}$ acts on $H^2 = \mathbb{R}$.
- The unbounded operator, Θ_1 acts on an infinite dimensional real vector (pre-Hilbert) space H^1, for any $\alpha \in C^\infty_{\text{compact}}(\mathbb{R}, \mathbb{R})$ the operator $\int_{\mathbb{R}} \alpha(t) e^{t\,\Theta_1}\, dt$ is trace class. The eigenvalues of $\Theta_1 \otimes Id_{\mathbb{C}}$ acting on $H^1 \otimes_{\mathbb{R}} \mathbb{C}$ coincide with the non trivial zeroes of $\widehat{\zeta}$.

Moreover in Deninger's approach one first assumes the existence of a Poincare duality pairing:

$$H^1 \times H^1 \to H^2$$

$$(\alpha, \beta) \to \alpha \cup \beta$$

satisfying

$$\forall \alpha, \beta \in H^1,\ e^{t\,\Theta_1}\alpha \cup e^{t\,\Theta_1}\beta = e^t(\alpha \cup \beta), \tag{2}$$

where the e^t is dictated by the fact that $\Phi_2 = Id$.

Second, one assumes the existence of a Hodge star $\star$ on H^1 such that $e^{t\,\Theta_1}\star = \star e^{t\,\Theta_1}$ and $< \alpha; \beta >= \alpha \cup \star\beta$ defines a real scalar product on the real vector space H^1.

Then with these data, Deninger's formalism implies the following:

$$\forall \alpha \in H^1,\ \langle e^{t\,\Theta_1}\alpha; e^{t\,\Theta_1}\alpha\rangle = e^t\langle \alpha; \alpha\rangle. \tag{3}$$

Therefore,

$$\frac{d}{dt}\langle e^{t\,\Theta_1}\alpha; e^{t\,\Theta_1}\alpha\rangle_{t=0} = \langle \Theta_1(\alpha); \alpha\rangle + \langle \alpha; \Theta_1(\alpha) >=< \alpha; \alpha\rangle,$$

and

$$\langle (\Theta_1 - 1/2)(\alpha); \alpha\rangle + \langle \alpha; (\Theta_1 - 1/2)(\alpha)\rangle = 0.$$

Thus one gets that $\Theta_1 - \frac{1}{2}$ is antisymmetric on the real vector space H^1. Therefore, the eigenvalues s of Θ_1 (which coincide by (1) to the non trivial zeroes of $\widehat{\zeta}$) satisfy $s - 1/2 \in i\mathbb{R}$ or equivalently: $\Re s = \frac{1}{2}$. Therefore Deninger's formalism should imply the Riemann hypothesis!!! This argument comes from an idea of Serre [Se60] and has been formalized in the foliation case in [De-Si02]. Of course, we have described only a very small part of Deninger's formalism which deals also with L-functions of motives, Artin conjecture, Beilinson conjectures ... etc.

Comment 1. There is a dissymmetry in (1) between the coefficients of $\alpha(k \log p)$ and $\alpha(-k \log p)$ for $k \in \mathbb{N}^*$. In the framework of Deninger's formalism the explanation is the following. Equation (2) allows to prove (3) which in turn implies that the transpose of $e^{t\Theta_1}$ is $e^t e^{-t\Theta_1}$. Therefore, if we have a Lefschetz cohomological interpretation of (1) in Deninger's formalism for a test function α with support in $]0, +\infty[$ then we have also a cohomological proof of (1) for α

with support in $]-\infty, 0[$. In this formalism, (2) (and the above dissymmetry) is quite connected to the Riemann hypothesis.

Notice that Ralf Meyer [Meyer03] has provided a nice spectral interpretation of the zeroes of $\widehat{\zeta}$ and an original proof of (1). Unfortunately, he cannot prove the Riemann Hypothesis because he is obliged to work with Frechet spaces rather than Hilbert spaces. To our opinion, the geometry underlying his constructions is not sufficient. Recall that also Alain Connes [Co99] has reduced the validity of the Riemann hypothesis (for $L-$function of the Hecke characters) to a trace formula.

The idea of the proof of (1) is the following: apply the residue theorem to

$$\left(\int_0^{+\infty} \sqrt{t}\,\alpha(\log t) t^s \frac{dt}{t}\right) \frac{\widehat{\zeta}'}{\widehat{\zeta}}(s)$$

on the interior of the rectangle of $\mathbb{C}$ defined by the four points:

$$1+\epsilon+iT,\ -\epsilon+iT,\ -\epsilon-iT,\ 1+\epsilon-iT,$$

then use the functional equation $\widehat{\zeta}(s)=\widehat{\zeta}(1-s)$ and the formula:

$$\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+\frac{1}{4}\right)=\int_0^{+\infty}\left(\frac{e^{-u}}{u}-\frac{e^{-u(\frac{s}{2}+\frac{1}{4})}}{1-e^{-u}}\right)du,$$

lastly let T goes to $+\infty$.

3 – Analogy with the foliation case

3.1 – The Guillemin-Sternberg trace formula

Consider a smooth compact manifold X with a smooth action:

$$\phi : X \times \mathbb{R} \to X,\ (x,t) \to \phi^t(x),$$

so that $\phi^{t+t'} = \phi^t \circ \phi^{t'}$ for any $t, t' \in \mathbb{R}$. Let $D_y\phi^t$ denote (for fixed $t \in \mathbb{R}$) the differential of the map $y \in X \to \phi^t(y)$. One has: $D_y\phi^t(\partial_s\phi^s_{|s=0}(x)) = \partial_s\phi^s_{|s=0}(x)$. In other words, the vector field associated with the flow ϕ^t belongs to $\ker(D_y\phi^t - \mathrm{Id})$.

Consider also a smooth vector bundle $E \to X$. Assume that E is endowed with a smooth family of maps

$$\psi^t : (\phi^t)^*E \to E,\ t \in \mathbb{R},$$

satisfying the following cocycle condition:

$$\forall u \in C^\infty(X;E),\ \forall t,t' \in \mathbb{R},\ \psi^{t'}(\psi^t(u\circ\phi^t)\circ\phi^{t'}) = \psi^{t+t'}(u\circ\phi^{t+t'}).$$

So we require that the maps $K^t : u \to \psi^t(u\circ\phi^t) = K^t(u)$ define an action of the additive group $\mathbb{R}$ on $C^\infty(X;E)$. Notice that in the case of $E = \wedge^* T^* X$ and $\psi^s = {}^t D\phi^s$ (the transpose of the differential of ϕ^s) this condition is satisfied.

We shall assume that the graph of ϕ meets transversally the "diagonal" $\{(x,x,t),\ x \in X,\ t \in \mathbb{R}\setminus\{0\}\}$. Guillemin-Sternberg have checked ([G-S77]) that the trace $\mathrm{Tr}(K^t|C^\infty(X;E))$ is defined as a distribution of $t \in \mathbb{R}\setminus\{0\}$ by the formula:

$$\mathrm{Tr}(K^t|C^\infty(X;E)) = \int_X K^t(x,x)$$

where $K^t(x,y)$ denote Schwartz (density) kernel of K^t. We warn the reader that, in general, for $\alpha \in C^\infty_{\mathrm{compact}}(\mathbb{R})\setminus\{0\}$, $\int_{\mathbb{R}} \alpha(t) K^t dt$ is not trace class.

Now, we give the name $T_x^0 = \partial_t \phi^t(x)_{t=0}\,\mathbb{R}$ to the real line generated by the vector field $\partial_t\phi^t(x)_{t=0}$ of ϕ^t at a point x where $\partial_t\phi^t(x)_{t=0} \neq 0$.

Proposition 1 (Guillemin-Sternberg [G-S77]). *The following formula holds in $\mathcal{D}'(\mathbb{R}\setminus\{0\})$.*

$$\mathrm{Tr}(K^t;C^\infty(X;E)) = \sum_\gamma l(\gamma) \sum_{k\in\mathbb{Z}^*} \frac{\mathrm{Tr}(\psi^{kl(\gamma)}_{x_\gamma};E_{x_\gamma})}{|\det(1 - D_y\phi^{kl(\gamma)}(x_\gamma)\,;\,T_{x_\gamma}X/T^0_{x_\gamma})|}\delta_{kl(\gamma)} + \\ + \sum_x \frac{\mathrm{Tr}(\psi^t_x;E_x)}{|\det(1 - D_y\phi^t(x);T_xX)|}.$$

In the first sum, γ runs over the periodic primitive orbits of ϕ^t, x_γ denotes any point of γ, $l(\gamma)$ is the length of γ, $\phi^{l(\gamma)}(x_\gamma) = x_\gamma$. In the second sum, x runs over the fixed points of the flow: $\phi^t(x) = x$ for any $t \in \mathbb{R}$.

Comment 2. Recall that $D_y\phi^t$ denotes, for fixed t, the differential of the map $y(\in X) \to \phi^t(y)$. The non vanishing of the two determinants in Proposition 1 is equivalent to the fact that the graph of ϕ meets transversally the "diagonal" $\{(x,x,t),\ x\in X,\ t\in\mathbb{R}\setminus\{0\}\}$.

Note that the following elementary observation is the main ingredient of the proof the Proposition 1. It is important with respect to Subsection 3.3. Let $A \in GL_n(\mathbb{R})$ and $\delta_0(\cdot)$ denote the Dirac mass at $0 \in \mathbb{R}^n$. Then one computes the distribution $\delta_0(A\cdot)$ in the following way. For any $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, one has:

$$\langle \delta_0(A\cdot); f(\cdot)\rangle = \int_{\mathbb{R}^n} \delta_0(Ax)f(x)dx = \int_{\mathbb{R}^n} \delta_0(y) f(A^{-1}y)\frac{1}{\mathrm{Jac}(A)}dy = \frac{1}{\mathrm{Jac}(A)} f(0)$$

where dy denotes the Lebesgue measure. Therefore: $\delta_0(A\cdot) = \frac{1}{\mathrm{Jac}(A)}\delta_0(\cdot)$.

3.2 – The Lesfchetz trace formula of Alvarez-Lopez and Kordyukov

Now we shall assume that X is a (still compact) three dimensional manifold and endowed with a codimension one foliation $(X, \mathcal{F})$. We shall also assume that the flow ϕ^t preserves the foliation $(X, \mathcal{F})$, is transverse to it and thus has no fixed point. Therefore $(X, \mathcal{F})$ is a Riemannian foliation. We shall apply later Proposition 1 with $E = \wedge^* T^* \mathcal{F} \to X$.

Comment 3. A typical example is $X = \frac{L \times \mathbb{R}^{+*}}{\Lambda}$, where Λ is a subgroup of $(\mathbb{R}^{+*}, \times)$ and $\phi^t(l, x) = (l, xe^{-t})$. See Section 4.

Now, we get a so called bundle like metric g_X on $(X, \mathcal{F})$ in the following way. We require that $g_X(\partial_t \phi^t(z)) = 1$, $\partial_t \phi^t(z) \perp T\mathcal{F}$ for any $(t, z) \in \mathbb{R} \times X$, and that $(g_X)_{|T\mathcal{F}}$ is a given leafwise metric. By construction, with respect to g_X, the foliation $(X, \mathcal{F})$ is defined locally by riemannian submersions.

In this setting Alvarez-Lopez and Kordyukov [A-K01] have proved the following Hodge decomposition theorem $(0 \leq j \leq 2)$:

$$C^\infty(X, \wedge^j T^* \mathcal{F}) = \ker \Delta^j_\tau \oplus \overline{\operatorname{Im} \Delta^j_\tau} \tag{4}$$

where Δ^j_τ denotes the leafwise Laplacian. Since we have $\frac{\ker d_\mathcal{F}}{\operatorname{Im} d_\mathcal{F}} = \ker \Delta^j_\tau$, we call the vector space $H^j_\tau = \ker \Delta^j_\tau$ a reduced leafwise cohomology group.

Let π^j_τ denote the projection of the vector space of leafwise differential forms $C^\infty(X, \wedge^j T^* \mathcal{F})$ onto $H^j_\tau = \ker \Delta^j_\tau$ according to (4) with $0 \leq j \leq 2$. Then Alvarez-Lopez and Kordyukov [A-K00] have proved the following Lefschetz trace formula.

Theorem 1 ([A − K00]). *Let* $\alpha \in C^\infty_{\text{compact}}(\mathbb{R})$ *Then the operators*

$$\int_\mathbb{R} \alpha(s)\, \pi^j_\tau \circ (\phi^s)^* \circ \pi^j_\tau \, ds$$

are trace class for $0 \leq j \leq 2$. *Let* χ_Λ *denote the leafwise measured Connes Euler characteristic of* $(X, \mathcal{F})$ ([Co94]). *Then one has*:

$$\begin{aligned} &\sum_{j=0}^{2} (-1)^j \mathrm{TR} \int_\mathbb{R} \alpha(s)\, \pi^j_\tau \circ (\phi^s)^* \circ \pi^j_\tau \, ds = \\ &= \chi_\Lambda \alpha(0) + \sum_\gamma \sum_{k \geq 1} l(\gamma) \left(\epsilon_{-k\gamma} \alpha(-kl(\gamma)) + \epsilon_{k\gamma} \alpha(kl(\gamma)) \right) \end{aligned} \tag{5}$$

where γ *runs over the primitive closed orbits of* ϕ^t, $l(\gamma)$ *is the length of* γ, $x_\gamma \in \gamma$ *and* $\epsilon_{\pm k\gamma} = \operatorname{sign} \det(\mathrm{id} - D\phi^{\pm kl(\gamma)}_{|\mathrm{T}_{x_\gamma}\mathcal{F}})$.

PROOF. (Sketch of the idea). The case where the support of α is included in a suitably small interval $[-\epsilon, +\epsilon]$ is treated separately. When the (compact) support of α is included in $\mathbb{R} \setminus \{0\}$, the authors show by highly non trivial arguments based on (4), that

$$\sum_{j=0}^{2}(-1)^j \mathrm{TR} \int_{\mathbb{R}} \alpha(s)\, \pi_\tau^j \circ (\phi^s)^* \circ \pi_\tau^j \, ds \; = \sum_{j=0}^{2}(-1)^j \mathrm{Tr} \int_{\mathbb{R}} \alpha(s) (\phi^s)^* \, ds.$$

But Proposition 1 (with $E = \wedge^j T^* \mathcal{F}$) shows that the right handside is equal to:

$$\sum_{\gamma} l(\gamma) \sum_{k \in \mathbb{Z}^*} \sum_{j=0}^{2} (-1)^j \, \frac{\mathrm{Tr}\big((D_y \phi^{\pm kl(\gamma)}(x_\gamma))^* : \wedge^j \mathrm{T}^*_{x_\gamma} \mathcal{F} \mapsto \wedge^j \mathrm{T}^*_{x_\gamma} \mathcal{F}\big)}{|\det(\mathrm{id} - D_y \phi^{\pm kl(\gamma)}_{|\mathrm{T}_{x_\gamma} \mathcal{F}})|} \alpha(kl(\gamma)).$$

One then gets immediately the result.

Comment 4. Notice that here (unlike in (1)) there is no dissymmetry for the coefficients of $\alpha(-kl(\gamma))$ and $\alpha(kl(\gamma)$. The reason for this absence of dissymmetry is explained by the Guillemin-Sternberg formula as we have seen in the proof. Here is another way to rephrase this explanation when X is orientable. If we had a leafwise metric g satisfying $(\phi^t)^* g = e^t g$ then we should get in (5) the same dissymmetry as the one already mentioned in (1). Assume that for a fix real β one has $\forall t \in \mathbb{R}$, $(\phi^t)^* g = e^{\beta t} g$. Consider the bundlelike metric g_X on X as above. Its volume form ω_X is such that $(\phi^t)^* \omega_X = e^{\beta t} \omega_X$, $\forall t \in \mathbb{R}$. But we know that the degree $\frac{\int_X (\phi^t)^* \omega_X}{\int_X \omega_X}$ has to be an integer for any $t \in \mathbb{R}$. Therefore $\beta = 0$.

The Ruelle zeta function is defined by

$$\zeta_R(s) = \prod_{\gamma \,\text{primitive orbit}} \frac{1}{(1 - e^{-sl(\gamma)})^{\epsilon_\gamma}}, \quad \Re s \gg 1.$$

The induced action of $(\phi^s)^*$ on H^j_τ is of the form $e^{s\theta_j}$. Deninger's results (e.g. [De98], [De07a]) suggest to conjecture that for $0 \leq j \leq 2$, $s \to \det_\infty(s \,\mathrm{Id} - \theta_j : H^j_\tau)$ defines an entire holomorphic function and that

$$\zeta_R(s) = \prod_{j=0}^{2} \det\nolimits_\infty(s \,\mathrm{Id} - \theta_j : H^j_\tau))^{(-1)^{j+1}}$$

where $\det_\infty$ denotes an infinite regularized determinant (see [De94] for definitions). If this last equality is true then (5) should constitute the explicit formula for ζ_R. Notice moreover that in Theorem 1 there is no term similar to

$W_\infty(\alpha)$ in (1) because the flow $(\phi^t)_{t\in\mathbb{R}}$ is assumed to have no fixed (e.g. stationary) point. Assume morever that there exists a leafwise metric g such that $\forall t \in \mathbb{R}$, $(\phi^t)^* g = g$. Then, by considering the associated bundlelike metric g_X one defines easily a scalar product $\langle ; \rangle$ on H^1_τ such that $e^{t\theta_1}$ becomes a unitary operator on the Hilbert completion of H^1_τ. Therefore, all the zeroes of ζ_R are on the real line $\Re z = 0$.

Recall that it is not always possible to find a leafwise metric g such that $\forall t \in \mathbb{R}$, $(\phi^t)^* g = g$. Here is an example communicated to me by Alvarez-Lopez. Let h be a diffeomorphism of S^2 fixing the two poles. Assume that for the corresponding $\mathbb{Z}$-action the north pole is attractive and the south pole is repulsive. Set $X = \frac{S^2\times\mathbb{R}}{\mathbb{Z}}$ where the action of $m \in \mathbb{Z}$ is defined by $m\cdot(l,x) = (h^m(l), m+x)$. The sets $S^2 \times \{x\}$ induce a foliation $\mathcal{F}$. Consider the flow ϕ^t defined by $\phi^t(l,x) = (l, x+t)$. Then there is no leafwise metric g on $(X,\mathcal{F})$ such that $\forall t \in \mathbb{R}$, $(\phi^t)^* g = g$.

Comment 5. Alvarez-Lopez and Kordyukov are working on a proof of a Lefschetz trace formula when the flow ϕ^t is allowed to have stationary points. They work with a notion of "adiabatic cohomology" and their programme is promising.

3.3 – Foliated spaces with a p–adic transversal

We shall now describe an example of foliated space where on can prove a Lefschetz trace formula exhibiting a dissymmetry quite similar to the one mentioned in Comment 1.

Let E_0 be an elliptic curve over a finite field $\mathbb{F}_q$ where $q = p^f$ and the prime number p is the characteristic of $\mathbb{F}_q$. Recall that the zeta function $\zeta_{E_0}(s)$ of E_0 is given by:

$$\zeta_{E_0}(s) = \prod_{w\in|E_0|} \frac{1}{1-(Nw)^{-s}} = \frac{(1-\xi q^{-s})(1-\overline{\xi} q^{-s})}{(1-q^{-s})(1-q^{1-s})} \tag{6}$$

where $|E_0|$ denotes the set of closed points of E_0 and ξ is a complex number which by Hasse's theorem satisfies $|\xi| = \sqrt{q}$. The explicit formula for $\zeta_{E_0}(s)$ takes the following form. Let $\alpha \in C_c^\infty(\mathbb{R},\mathbb{R})$ and set for any real s, $\Phi(s) = \int_{\mathbb{R}} e^{st}\alpha(t)\,dt$. Then, one has:

$$\begin{aligned} &\sum_{\nu\in\mathbb{Z}} \Phi\left(\frac{2\pi\nu i}{\log q}\right) - \sum_{\rho\in\zeta_{E_0}^{-1}\{0\}} \Phi(\rho) + \sum_{\nu\in\mathbb{Z}} \Phi\left(1+\frac{2\pi\nu i}{\log q}\right) = \\ &= \sum_{w\in|E_0|} \log Nw \left(\sum_{k\geq 1} \alpha(k\log Nw) + \sum_{k\leq -1} (Nw)^k \alpha(k\log Nw)\right). \end{aligned} \tag{7}$$

The idea of the proof is to apply the residue theorem to

$$s \to \left(\int_0^{+\infty} \sqrt{t}\, \alpha(\log t) t^s \frac{d\,t}{t} \right) \frac{\zeta'_{E_0}}{\zeta_{E_0}}(s)$$

and to use the functional equation $\zeta_{E_0}(s) = \zeta_{E_0}(1-s)$.

Let $\phi_0 : E_0 \to E_0$ be the $q-$th power Frobenius endomorphism of E_0 over $\mathbb{F}_q$. Deninger has used (see [De02]) the following result due to Oort [Oor73]:

Lemma 1. *There exists*:

1) *a complete local integral domain R with field of fractions L a finite extension of $\mathbb{Q}_p$ $(q = p^f)$ such that $R/\mathcal{M} = \mathbb{F}_q$ where $\mathcal{M}$ is the maximal ideal of R.*
2) *an elliptic curve $\mathcal{E}$ over spec R together with an endomorphism $\phi : \mathcal{E} \to \mathcal{E}$ such that*:

$$(\mathcal{E}, \phi) \otimes \mathbb{F}_q = (E_0, \phi_0).$$

So $(\mathcal{E}, \phi)$ is a lift of (E_0, ϕ_0) in characteristic zero.

Remark 1.

1) If the elliptic curve E_0 is ordinary, then one may take for R the ring of Witt vectors of $\mathbb{F}_q$, $W(\mathbb{F}_q)$, and then there is a canonical choice of the lifting $(\mathcal{E}, \phi)$. On the contrary, if E_0 is supersingular [Si92, page 137], then there is no canonical choice of $(\mathcal{E}, \phi)$.
2) It is possible to lift a curve of genus ≥ 2 (over $\mathbb{F}_q$) in characteristic zero, but Hurwitz's formula [Si92, page 41] shows that one cannot lift its Frobenius morphism.

Now (still following [De02]), we denote by $E = \mathcal{E} \otimes_R L$ the generic fibre. Then $\mathrm{End}_L(E) \otimes \mathbb{Q} = K$ is a field K which is either $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$. We fix an embedding $L \subset \mathbb{C}$ and consider the complex analytic elliptic curve $E(\mathbb{C})$. Let ω be a non zero holomorphic one form on $E(\mathbb{C})$ and let Γ be its period lattice. Then the Abel-Jacobi map:

$$E(\mathbb{C}) \to \mathbb{C}/\Gamma,\ p \to \int_0^p \omega \mod \Gamma$$

induces an isomorphism. Next we choose the embedding $K \subset \mathbb{C}$ such that for any $\alpha \in K$, $\Theta(\alpha)$ induces the multiplication by α on the Lie algebra $\mathbb{C}$ of $\mathbb{C}/\Gamma$ where Θ is the natural homomorphism:

$$\Theta :\ K = \mathrm{End}_L(E) \otimes \mathbb{Q} \to \mathrm{End}(\mathbb{C}/\Gamma) \otimes \mathbb{Q}.$$

Next we consider the unique element $\xi \in \Theta^{-1}(\mathrm{End}_L(E)) \subset K$ such that $\Theta(\xi) = \phi \otimes L$. By construction one has $\xi\Gamma \subset \Gamma$ and the complex elliptic curve $\mathbb{C}/\Gamma$ endowed with the multiplication by ξ represents a lift of (E_0, ϕ_0). Now, we set

$$V = \cup_{n\in\mathbb{N}} \xi^{-n}\Gamma, \ \ \mathrm{T}\Gamma = \lim_{+\infty \leftarrow n} \frac{\Gamma}{\xi^n\Gamma}, \ \text{ and } \ V_\xi\Gamma = \mathrm{T}\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The set $\mathrm{T}\Gamma$ is a Tate module defined by a projective limit and $V_\xi\Gamma$ is a $\mathbb{Q}_p$-vector space of dimension 1 or 2. Any element v of V acts on $\mathbb{C} \times V_\xi\Gamma$ by $v.(z, \hat{v}) = (z + v, \hat{v} - v)$, we denote by $\frac{\mathbb{C}\times V_\xi\Gamma}{V}$ the quotient space.

Lemma 2. *The natural homomorphism*:

$$\frac{\mathbb{C} \times \mathrm{T}\Gamma}{\Gamma} \to \frac{\mathbb{C} \times V_\xi\Gamma}{V}$$

defines a $\{\xi^l,\ l \in \mathbb{Z}\}$-equivariant isomorphism where the action of ξ is induced by the diagonal action on $\mathbb{C} \times \mathrm{T}\Gamma$ and $\mathbb{C} \times V_\xi\Gamma$ respectively.

Now, any element $q^\nu \in q^{\mathbb{Z}}$ acts on $\frac{\mathbb{C}\times V_\xi\Gamma}{V} \times \mathbb{R}^{+*}$ by

$$q^\nu.([z, \hat{v}], x) = ([\xi^\nu z, \xi^\nu \hat{v}], xq^\nu).$$

In [De02], Deninger has introduced the (compact) laminated Riemannian foliated space $(S(E_0), \mathcal{F})$ where

$$S(E_0) \ = \ \frac{\mathbb{C} \times V_\xi\Gamma}{V} \times_{q^{\mathbb{Z}}} \mathbb{R}^{+*},$$

and the leaves of $\mathcal{F}$ are the images of the sets $\mathbb{C} \times \{\hat{v}\} \times \{x\}$ by the natural map $\pi : \mathbb{C} \times V_\xi\Gamma \times \mathbb{R}^{+*} \to S(E_0)$. Observe that the domain of a typical foliation chart is locally isomorphic to $D \times \Omega \times]1, 2[$ where D is an open disk of $\mathbb{C}$, Ω is an open subset of $\mathrm{T}\Gamma$ so that the leaves are given by $D \times \{\omega\} \times \{x\}$ for $(\omega, x) \in \Omega \times]1, 2[$; the term "laminated" refers to the fact that the local transversal to the foliation $\mathcal{F}$ is the disconnected space $\Omega \times]1, 2[$.

Remark 2. Using the fact that V (resp. $q^{\mathbb{Z}}$) acts freely on $V_\xi\Gamma$ (resp. $\mathbb{R}^{+*}$), the reader will check that $(S(E_0), \mathcal{F})$ has trivial holonomy.

One defines a flow ϕ^t acting on $(S(E_0), \mathcal{F})$ and sending each leaf into another leaf by: $\phi^t(z, \hat{v}, x) = (z, \hat{v}, xe^{-t})$. Let μ_ξ denote a Haar measure on the group $V_\xi\Gamma$ then, one has the following

Lemma 3 ([De02]).

1) *The measure*

$$dx_1dx_2 \otimes \mu_\xi \otimes \frac{dx}{x}$$

on $\mathbb{C} \times V_\xi\Gamma \times \mathbb{R}^{+*}$ induces a measure μ on $S(E_0)$.

2) *The measure μ is invariant under the action of ϕ^t.*

PROOF.

1) We just have to check that for any $\nu \in \mathbb{N}^*$ and any borel subset A of $V_\xi\Gamma$, one has
$$\mu_\xi(\xi^\nu A) = |\xi|^{-2\nu}\mu_\xi(A) = q^{-\nu}\mu_\xi(A).$$
Since $(\xi^\nu)_*\mu_\xi$ is also a Haar measure on $V_\xi\Gamma$ it suffices to check this equality for $A = \mathrm{T}\Gamma$. But this is an immediate consequence of the fact that
$$\mathrm{T}\Gamma/(\xi^\nu \mathrm{T}\Gamma) \simeq \Gamma/(\xi^\nu\Gamma)$$
has $|\xi|^{2\nu} = q^\nu$ elements.
2) This is obvious.

Using the fact that $|\xi| = \sqrt{q}$, one checks that the Riemannian metric on the bundle $\mathrm{T}\mathbb{C} \times V_\xi\Gamma \times \mathbb{R}^{+*}$ given by:
$$g_{z,\hat{v},x}(\eta_1, \eta_2) = x^{-1}\mathrm{Re}\,(\eta_1\overline{\eta_2})$$
induces a Riemannian metric g along the leaves of $(S(E_0), \mathcal{F})$ so that the following property is satisfied:

$$\forall \eta \in \mathrm{T}_{[z,\hat{v},x]}\mathcal{F},\ g(D_{[z,\hat{v},x]}\phi^t(\eta), D_{[z,\hat{v},x]}\phi^t(\eta)) = e^t g(\eta, \eta). \tag{8}$$

Compare with Comment 4.

THEOREM 2 (DENINGER [De02]). *There is a natural bijection between the set of valuations w of the function field $K(E_0)$ of E_0 and the set of primitive compact $\mathbb{R}-$orbits of ϕ^t on $S(E_0)$. It has the following property. If w corresponds to $\gamma = \gamma_w$, then*
$$l(\gamma_w) = \log N(w).$$

Deninger has also provided a nice spectral cohomological interpretation of the left hand side of (7).

Now we are going to recall briefly the definition of the leafwise Hodge cohomology that has allowed us to give in [Lei07] an Atiyah-Bott-Lefschetz proof à la Alvarez-Lopez Kordyukov of the explicit formula (7).

First we introduce carefully a natural transverse measure on $(S(E_0), \mathcal{F})$ and point out its important role. For more on these notions, the reader may have a look at: [Co94], [Sul93] and [Ghys99].

Set $\mathcal{L}_{E_0} = \frac{\mathbb{C}\times \mathrm{T}\Gamma}{\Gamma}$, this is a compact laminated space which is foliated by its path-connected components. Any element $q^\nu \in q^{\mathbb{Z}}$ acts on $[z, \widehat{v}] \in \mathcal{L}_{E_0}$ by $q^\nu \cdot [z, \widehat{v}] = [\xi^\nu z, \xi^\nu \widehat{v}]$. The Haar measure μ_ξ of $\mathrm{T}\Gamma$ induces a transverse measure, still denoted μ_ξ, of $\mathcal{L}_{E_0}$. For any Borel transversal T of $\mathcal{L}_{E_0}$ one has $\mu_\xi(q \cdot T) = q^{-1}\mu_\xi(T)$.

Moreover, the metric $\widetilde{g} = (dx_1)^2 + (dx_2)^2$ (where $z = x_1 + ix_2$) defines a leafwise metric on $\mathcal{L}_{E_0}$, let $\lambda_{\widetilde{g}}$ be the associated leafwise volume form. Then $\lambda_{\widetilde{g}}\mu_\xi$ defines a $q^{\mathbb{Z}}-$invariant measure of $\mathcal{L}_{E_0}$.

The leafwise metric g in (8) of $(S(E_0), \mathcal{F})$ is defined by $g = x^{-1}\widetilde{g}$ and its associated leafwise volume form is given by $\lambda_g = x^{-1}dx_1 \wedge dx_2$.

Definition 1.

1) Let $\mathcal{A}^j_{\mathcal{F}}(S(E_0))$ denote the vector space of leafwise differential forms which, in the local coordinates $(z, \widehat{v}, x)$, are of the form
$$u(z, \widehat{v}, x)d^a \Re z d^b \operatorname{Im} z$$
where $a + b = j \in \{0, 1, 2\}$ and $(z, \widehat{v}, x) \to D^\beta_{\Re z, \operatorname{Im} z, x} u(z, \widehat{v}, x)$ is continuous for any multiindex of differentiation $\beta \in \mathbb{N}^3$.
2) One defines the Sobolev space $H_{+\infty}(S(E_0)\,;\wedge^j \mathrm{T}^*\mathcal{F})$ in the same way but we simply require that the functions $(z, \widehat{v}, x) \to D^\beta_{\Re z, \operatorname{Im} z, x} u(z, \widehat{v}, x)$ are locally L^2.

Now it is clear that $\frac{\mu}{\lambda} = \mu_\xi dx$ defines a transverse measure on $(S(E_0), \mathcal{F})$ with associated Ruelle-Sullivan current $C(\frac{\mu}{\lambda})$. We can pair sections of $\mathcal{A}^2_{\mathcal{F}}(S(E_0))$ with $C(\frac{\mu}{\lambda})$, for instance the measure μ may be recovered by the formula:
$$\forall f \in C^0(S(E_0)),\ \left(f\lambda; C\left(\frac{\mu}{\lambda}\right)\right) = \int_{S(E_0)} f\, d\,\mu.$$
One defines a scalar product by the following formula:
$$\forall \omega, \omega' \in \mathcal{A}^j_{\mathcal{F}}(S(E_0)),\ \langle \omega; \omega' \rangle = \left(\omega \cup *\omega'; C\left(\frac{\mu}{\lambda}\right)\right)$$
where $*$ denotes the leafwise Hodge star operator associated to g.

Theorem 3 ([Lei07]).

1) *One has a Hodge decomposition* (*for* $0 \le j \le 2$)*:*
$$H_{+\infty}(S(E_0), \wedge^j \mathrm{T}^*\mathcal{F}) = H^j_\tau \oplus^\perp \overline{\operatorname{Im} \Delta_\tau}.$$
Let π^j_τ *denote the associated projection onto the vector space of leafwise harmonic forms* H^j_τ.
2) *Let* $\alpha \in C^\infty_{\text{compact}}(\mathbb{R}; \mathbb{R})$*. Then* $\int_{\mathbb{R}} \alpha(t)(\phi^t)^*\pi^j_\tau dt$ *is trace class and*
$$\sum_{j=0}^{2} (-1)^j TR \int_{\mathbb{R}} \alpha(t)(\phi^t)^*\pi^j_\tau dt =$$
$$= \sum_\gamma \sum_{k \ge 1} l(\gamma)\left(e^{-kl(\gamma)}\alpha(-kl(\gamma)) + \alpha(kl(\gamma))\right)$$

where γ *runs over the set of primitive closed orbits of* $(S(E_0), \mathcal{F})$*. According to Theorem* 2*, we obtain in this way an Atiyah-Bott-Lefschetz proof* (*along the lines of* [A-K00]) *of the explicit formula* (7)*.*

COMMENT 6. Actually, in [Lei07] we have defined in an abstract way (being motivated by the work of Deninger) a class of laminated foliated spaces for which the previous Theorem still holds true.

As noticed by Deninger, the dissymmetry in (7) of the coefficients of $\alpha(kl(\gamma))$ for $k \leq -1$ and $k \geq 1$ is due to property (8) (see the remark following Corollary 1 of [Lie03]). We are going to propose a dynamical explanation, à la Guillemin-Sternberg, of this dissymmetry. Consider a point $(z_0, \widehat{v}_0, 1) \in S(E_0)$, with $\widehat{v}_0 \in \mathrm{T}\Gamma$, such that $\phi^{-\log q}[z_0, \widehat{v}_0, 1] = [z_0, \widehat{v}_0, q] = [z_0, \widehat{v}_0, 1]$. Recall that by definition $(\xi^{-1} z_0, \xi^{-1}\widehat{v}_0, q^{-1} q) \sim (z_0, \widehat{v}_0, q)$. So $[\xi^{-1} z_0, \xi^{-1}\widehat{v}_0, 1] = [z_0, \widehat{v}_0, 1]$ and there exists $\gamma \in \xi^{-1}\Gamma$ such that

$$\xi^{-1} z_0 = z_0 + \gamma, \quad \xi^{-1}\widehat{v}_0 = \widehat{v}_0 - \gamma. \tag{9}$$

The operator $(\phi^t)^*$ acting on $\mathcal{A}^j_{\mathcal{F}}(S(E_0))$ admits a Schwartz kernel defined by the formula:

$$\forall \omega \in \mathcal{A}^j_{\mathcal{F}}(S(E_0)),\ (\phi^t)^*(\omega)(y) = \int_{S(E_0)} (D\phi^t)^* \delta_{\phi^t(y)=y'}\, \omega(y')\, d\mu(y').$$

Consider a point $y = [z, \widehat{v}, x]$ belonging to a small neighborhood of $\{\phi^t[z_0, \widehat{v}_0, 1], -\log q \leq t \leq 0\}$. Then, with the previous notations, one has:

$$\phi^t(y) = (\xi^{-1} z - \gamma, \xi^{-1}\widehat{v} + \gamma, q^{-1} x e^{-t}). \tag{10}$$

The following lemma shows basically that the graph of the flow $(\phi^t)_{t \in \mathbb{R}\setminus\{0\}}$ is transverse to the diagonal and computes $\delta_{\phi^t(y)=y}$.

LEMMA 4.

1) *$z \in \mathbb{C} \to \xi^{-1} z - \gamma - z$ and $\widehat{v} \in V_\xi\Gamma \to \xi^{-1}\widehat{v} + \gamma - \widehat{v}$ are invertible and their jacobians are respectively given by:*

$$\mathrm{Jac}(\xi^{-1} z - \gamma - z) = |\xi^{-1} - 1|^2,\ \mathrm{Jac}(\xi^{-1}\widehat{v} + \gamma - \widehat{v}) = q.$$

2) *Let V be an open neighborhood of $(z_0, \widehat{v}_0)$, set:*

$$U = \{(z, \widehat{v}, e^{-s}) /\ s \in]-\log q, 0],\ (s, \widehat{v}) \in V\}.$$

Consider $\epsilon > 0$ and V small enough so that $t \in [-\log q, 0] \to (z_0, \widehat{v}_0, e^{-t})$ is the only closed orbit of ϕ^t contained in U with length in $]-\epsilon-\log q, \epsilon-\log q[$. Then one has the following equality as a distribution on $U\times]-\epsilon-\log q, \epsilon-\log q[$:

$$\delta_{\phi^t(y)=y} = \frac{1}{|\xi^{-1} - 1|^2} \delta_{z - z_0} \otimes \frac{1}{q} \delta_{\widehat{v} - \widehat{v}_0} \otimes \delta_{t + \log q}.$$

PROOF.

1) We prove only the second equality. Recall that $\mathrm{T}\Gamma$ is an open compact subset of $V_\xi \Gamma$. Then, since $\widehat{v} \to \widehat{v} - \xi \widehat{v}$ defines an automorphism of $\mathrm{T}\Gamma$ whose inverse is $\widehat{v} \to \sum_{n \in \mathbb{N}} \xi^n \widehat{v}$, one has $\mathrm{Jac}\,(\widehat{v} - \xi \widehat{v}) = 1$. Now recall that the proof of Lemma 3 shows that $\mu_\xi(\xi \mathrm{T}\Gamma) = \frac{1}{q} \mu_\xi(\mathrm{T}\Gamma)$ so that $\mathrm{Jac}\,(\xi \widehat{v}) = \frac{1}{q}$. By combining the last two equalities for Jac, one gets:

$$\mathrm{Jac}(\xi^{-1}\widehat{v} + \gamma - \widehat{v}) = q.$$

2) Using the change of variable formula for $\int d\mu_\xi$ and the equality $\xi^{-1}\widehat{v}_0 + \gamma - \widehat{v}_0 = 0$, one sees that for $\widehat{v}$ close to $\widehat{v}_0$ one has

$$\delta_{\xi^{-1}\widehat{v} + \gamma - \widehat{v}} = \frac{1}{\mathrm{Jac}(\xi^{-1}\widehat{v} + \gamma - \widehat{v})} \delta_{\widehat{v} - \widehat{v}_0}.$$

Then a computation using (9) and (10) shows (see also [Co99, Section IV]), that for $y = [z, \widehat{v}, x] \in U$ and $t \in] - \epsilon - \log q, \epsilon - \log q[$ one has:

$$\delta_{\phi^t(y) = y} = \frac{1}{\mathrm{Jac}(\xi^{-1}z - \gamma - z)} \delta_{z - z_0} \otimes \frac{1}{\mathrm{Jac}(\xi^{-1}\widehat{v} + \gamma - \widehat{v})} \delta_{\widehat{v} - \widehat{v}_0} \otimes \delta_{t + \log q}.$$

By combining 1) with this equality one gets the result.

Recall now that $d\mu(y) = dx_1 dx_2 \otimes \mu_\xi \otimes \frac{dx}{x}$. The formula $| \int_0^{-\log q} \frac{d\, e^{-s}}{e^{-s}} | = \log q$ and Lemma 4. 2) show that for t close to $-\log q$ the distributional trace

$$\int_{S(E_0)} \mathrm{Tr}\,(D\phi^t)^* \delta_{\phi^t(y) = y}\, d\mu(y)$$

is well defined (near $-\log q$) and is equal to:

$$\log q \sum_{\gamma_w,\, l(\gamma_w) = \log q} \frac{1}{q} \delta_{-l(\gamma_w)}$$

where γ_w runs over the set of closed orbits of ϕ^t of length $l(\gamma_w) = \log q$.

Since $\mathrm{Jac}(\xi \widehat{v} + \gamma - \widehat{v}) = 1$ a similar argument shows that for t close to $\log q$ the distributional trace

$$\int_{S(E_0)} \mathrm{Tr}\,(D\phi^t)^* \delta_{\phi^t(y) = y}\, d\mu(y)$$

is well defined (near $\log q$) and equal to:

$$\log q \sum_{\gamma_w,\, l(\gamma_w) = \log q} \delta_{l(\gamma_w)}.$$

Therefore, we have given a dynamical explanation of the dissymmetry occuring in (7).

Now we come to another analogy. In [De07b], Deninger has suggested that, in the case of the Riemann zeta function, the distribution

$$\sum_{\rho\in\widehat{\zeta}^{-1}\{0\},\ \mathrm{Im}\,\rho>0} e^{\rho z}$$

might be interpreted as a trace involving a transversal wave operator. We refer to [De07b, Section 5] for a list of interesting open problems in this direction. In this Section, we simply check that, in the case of ζ_{E_0}, Deninger's intuition is right (see also the end of the next Section). Recall that local coordinates on $S(E_0)=\frac{\mathcal{L}_{E_0}\times\mathbb{R}^{+*}}{q^{\mathbb{Z}}}$ (where $\mathcal{L}_{E_0}=\frac{\mathbb{C}\times\mathrm{T}\Gamma}{\Gamma}$) are given by $(z,\widehat{v},x)$. We endow $S(E_0)$ with the bundle like metric:

$$\frac{dzd\overline{z}}{x}\oplus\frac{dx^2}{x^2}.$$

We have a notion of transverse exterior derivative d_T:

$$d_T:\Gamma(S(E_0)\,;\wedge^1\mathrm{T}^*\mathcal{F})\to\Gamma(S(E_0)\,;\wedge^1\mathrm{T}^*\mathcal{F}\otimes\wedge^1\mathrm{T}^*\mathcal{F}^{\perp}).$$

Let δ_T be its adjoint. Set $\Delta_T=\delta_T d_T$, it acts on $\Gamma(S(E_0)\,;\wedge^1\mathrm{T}^*\mathcal{F})$. We write, locally, an element of $\Gamma(S(E_0)\,;\wedge^{1,0}\mathrm{T}^*\mathcal{F})$ as $a(z,\widehat{v},x)dz$.

Lemma 5. *One has*:

$$\Delta_T(adz)=(-\partial_x x)(x\partial_x)(a)\,d\,z.$$

Consider the following transverse operator defined by:

$$\widetilde{\Delta}_T(adz)=\left(-\partial_x x+\frac{1}{2}\right)\left(x\partial_x+\frac{1}{2}\right)(a)\,d\,z.$$

Lemma 6. *Assume for simplicity that the zeta function $\zeta_{E_0}(s)$ of the elliptic curve E_0 does not vanish on $\mathbb{R}$. The following equality holds, between distributions of the variable $t\in\mathbb{R}$:*

$$\mathrm{Tr}\,\pi^1_\tau e^{it\sqrt{\widetilde{\Delta}_T}}=2\sum_{z\in\zeta_{E_0}^{-1}(0),\ \mathrm{Im}\,z>0} e^{it\,\mathrm{Im}\,z}.$$

4 – Foliations provide a simple analogue of Lichtenbaum conjecture for zeta functions

4.1 – Lichtenbaum's conjecture

Let K be a number field. Stephen Lichtenbaum has conjectured ([Licht]) the existence of certain Weil étale cohomology groups with and without compact support $H_c^j(K\,;\mathbb{Z})$, $H_c^j(K\,;\mathbb{R})$, and $H^j(K\,;\mathbb{Z})$, $H^j(K\,;\mathbb{R})$ (for $j\in\mathbb{N}$). These groups are additive (abelian) and should be related to the zeta function ζ_K of K as follows.

Conjecture 1 (Lichtenbaum). The groups $H_c^j(K\,;\mathbb{Z})$ are finitely generated and vanish for $j\geq 4$. Giving $\mathbb{R}$ the usual topology one has:

$$H_c^j(K\,;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}=H_c^j(K\,;\mathbb{R}).$$

Moreover, there exists a canonical element $\psi\in H^1(K,\mathbb{R})$ which is functorial with respect to K and such that the following three properties hold.

1) The complex
$$\dots\xrightarrow{D}H_c^j(K,\mathbb{R})\xrightarrow{D}H_c^{j+1}(K,\mathbb{R})\to\dots$$
where $Dh=\psi\cup h$ is acyclic. Notice that $D^2=0$ because $\deg\psi=1$.

2) One has near $s=0$,
$$\zeta_K(s)=s^{\left(\sum_{j=0}^{3}(-1)^j j\,\mathrm{rank}\,H_c^j(K,\mathbb{Z})\right)}\,\zeta_K^*(s).$$

3)
$$\zeta_K^*(0)=\frac{\displaystyle\prod_{0\leq j\leq 3}|H_c^j(K,\mathbb{Z})_{\mathrm{torsion}}|^{(-1)^j}}{det(H_c^\bullet(K,\mathbb{R}),D,f^\bullet)}$$
where for each $j\in\{0,1,2,3\}$, f^j is a basis of $\frac{H_c^j(K,\mathbb{Z})}{\mathrm{torsion}}$.

We explain the meaning of 3). Since the complex in 1) is acyclic, we have a canonical isomorphism:

$$\otimes_{0\leq j\leq 3}(\det H_c^j(K,\mathbb{R}))^{(-1)^j}\simeq\mathbb{R}.$$

When j is odd we take the dual of this real line. Then each f^j induces a basis $\widehat{f^j}$ of $\det H_c^j(K,\mathbb{R})$ and $\otimes_{0\leq j\leq 3}(\widehat{f^j})^{(-1)^j}$ defines a real number denoted $\det(H_c^\bullet(K,\mathbb{R}),D,f^\bullet)$ which does not depend on the choice of $f^\bullet$.

Lichtenbaum [Licht05] has proven, in the function field case, the analogue of his conjecture.

4.2 – A dynamical foliation analogue

We follow Deninger [De07a]. Recall that in Section 3.2 we have defined the Ruelle zeta function ζ_R of $(X, \mathcal{F}, g, \phi^t)$. We assume now that all the hypothesis, conjectures and notations stated after Comment 4 at the end of Section 3.2. are satisfied by $(X, \mathcal{F}, g, \phi^t)$ and ζ_R. In particular the leafwise metric g is ϕ^t-invariant and the flow ϕ^t is transverse to the foliation $\mathcal{F}$ with no fixed point. Moreover for simplicity we assume that all the $\epsilon_\gamma = 1$.

Comment 7. By a structure theorem we could assume that $(X, \mathcal{F}, \phi^t)$ is of the form: $X = \frac{L\times\mathbb{R}}{\Gamma}$, where Γ is a subgroup of $(\mathbb{R}, +)$, L is a fixed (noncompact) Riemann surface, leaf = image of $L \times \{pt\}$, $\phi^s(l, t) = (l, t + s)$.

Set $Y_\phi = \frac{d\phi^t}{dt} = \partial_t$. Define $\omega \in C^\infty(X, \mathrm{T}^*X)$ by $\omega(Y_\phi) = 1$ and $\omega_{|\mathrm{T}\mathcal{F}} = 0$. Then ω is closed (one has $\omega = dt$ in the previous comment).

Theorem 4 (Deninger [De07a]).

1) *The complex*

$$\ldots \xrightarrow{D} H^j(X, \mathbb{R}) \xrightarrow{D} H^{j+1}(X, \mathbb{R}) \to \ldots$$

where $Dh = [\omega] \cup h$ *is acyclic and* $H^j(X, \mathbb{R})$ *denotes the standard singular cohomology.*

2) *One has near* $s = 0$,

$$\zeta_R(s) = s^{\left(\sum_{j=0}^{3}(-1)^j j \operatorname{rank} H^j(X,\mathbb{Z})\right)}\, \zeta_R^*(s).$$

3)

$$\zeta_R^*(0) = \frac{\prod_{0\le j\le 3} |H^j(X, \mathbb{Z})_{\text{torsion}}|^{(-1)^j}}{\det(H^\bullet(X, \mathbb{R}), D, f^\bullet)}$$

where for each $j \in \{0, 1, 2, 3\}$, f^j *is a basis of* $\frac{H^j(X,\mathbb{Z})}{\text{torsion}}$.

– **Analogy with Lichtenbaum conjecture**

In this situation, the role of Lichtenbaum's Weil étale cohomology is played by the ordinary singular cohomology with $\mathbb{Z}$ or $\mathbb{R}-$coefficients. Since X is compact, we do not have to worry about compact supports. The Ruelle zeta function ζ_R is expressed in terms of Hodge leafwise cohomology $\ker \Delta_\tau$ (cf. (4)), which is related to $H^\bullet(X, \mathbb{R})$ via the decompositions (11) below. Recall that here ζ_R has no Gamma factors (since ϕ^t has no fixed point) and that the zeroes of ζ_R are located on $\Re s = 0$.

Proof. Sketch.

Define a metric g_X on X by: $g_X = g \oplus^\perp g_0$ on $TX = T\mathcal{F} \oplus \mathbb{R}Y_\phi$ where $g_0(Y_\phi) = 1$.

One has the bigrading:

$$\wedge^n T^* X = \oplus_{p+q=n} \wedge^p T^* \mathcal{F} \otimes \wedge^q (\mathrm{I\!R} Y_\phi)^*.$$

We have $\Delta = \Delta_\tau \oplus -\theta^2$ where θ denotes the infinitesimal generator of $(\phi^s)^*(= e^{s\,\theta})$ acting on $C^\infty(X\,;\,\wedge^* \mathrm{T}^* \mathcal{F})$ and Δ_τ denotes the leafwise Laplacian. Moreover one has

$$\ker \Delta^n = \omega \wedge (\ker \Delta_\tau^{n-1})^{\theta=0} \oplus (\ker \Delta_\tau^n)^{\theta=0}. \tag{11}$$

Then, using techniques from the heat equation proof of the index theorem, Deninger proves that

$$\zeta_R^*(0) = \exp\left(- \sum_{0\le j\le 3} (-1)^j \frac{j}{2} \zeta'_{\Delta^j}(0) \right) = T(X, g_X)^{-1}$$

where $T(X, g_X)$ denotes the Ray-Singer analytic torsion. The metric g_X induces a scalar product on $\ker \Delta^j$ and on $H^j(X, \mathrm{I\!R})$ via the Hodge isomorphism. Consider an orthonormal basis h^j on $H^j(X, \mathrm{I\!R})$, denote by h_j the dual basis on $H_j(X, \mathrm{I\!R})$. Consider also the basis f_j of $\frac{H_j(X,\mathbb{Z})}{\text{torsion}}$ which is dual to f^j $(0 \le j \le 3)$. Now recall that the Reidemeister torsion is defined by:

$$\tau(X) = \prod_{j=0}^{3} |H_j(X, \mathbb{Z})_{\text{torsion}}|^{(-1)^j}.$$

Using the Poincare duality isomorphism $H^j(X, \mathbb{Z}) \simeq H_{3-j}(X, \mathbb{Z})$ one then gets:

$$\tau(X) = \prod_{j=0}^{3} |H^j(X, \mathbb{Z})_{\text{torsion}}|^{(-1)^{j+1}}.$$

By the Cheeger-Mueller theorem one has:

$$T(X, g_X) = \tau(X) \prod_{j=0}^{3} |\det_{f_j} h_j|^{(-1)^j}.$$

Now, using the decompositions (11) Deninger shows that

$$1 = |\det(H^\bullet(X, \mathrm{I\!R}), D, h^\bullet)|.$$

By construction, one has:

$$|\det(H^\bullet(X, \mathrm{I\!R}), D, h^\bullet)| = |\det(H^\bullet(X, \mathrm{I\!R}), D, f^\bullet)| \prod_{j=0}^{3} |\det_{f_j} h_j|^{(-1)^{j+1}}.$$

By combining the last six identities one then gets the theorem.

In the proof of Theorem 4, we have seen that the operator $|\theta|$ appears as a transverse square root Laplacian. Assume that the Ruelle zeta function does not vanish at 0. Then one checks that the following holds as distributions of the variable $t \in \mathbb{R}$.

$$2 \sum_{\rho \in \zeta_R^{-1}(0),\, \operatorname{Im} \rho > 0} e^{t\rho} \;=\; \operatorname{Tr} \pi_\tau^1 e^{it|\theta|}.$$

See the end of Section 3.3.

5 – Remarks about a conjectural dynamical foliated space $(S_{\mathbb{Q}}, \mathcal{F}, g, \phi^t)$ associated to the Riemann zeta function

The following Section is speculative in nature. It should be viewed as a working programme or a motivation for developing interesting mathematics.

5.1 – Structural Assumptions and their consequences

We assume, following Deninger (e.g. [De01b], [De01]), that to $\operatorname{Spec} \mathbb{Z} \cup \{\infty\}$, one can associate a Riemannian (laminated) foliated space $(S_{\mathbb{Q}}, \mathcal{F}, g, \phi^t)$ satisfying the following assumptions.

1. The leaves are Riemann surfaces and the path connected components of $S_{\mathbb{Q}}$ are three dimensional. Moreover, g denotes a leafwise riemannian metric, $(\phi^t)_{t \in \mathbb{R}}$ is a flow acting on $(S_{\mathbb{Q}}, \mathcal{F})$ and permuting the leaves.

2. To each prime $p \in \mathcal{P}$ there corresponds a unique primitive closed orbit γ_p of ϕ^t of length $\log p$. To the archimedean absolute value of $\mathbb{Q}$ there corresponds a unique fixed point $x_\infty = \phi^t(x_\infty), \forall t \in \mathbb{R}$, of the flow. The flow is transverse to all the leaves different from the one containing x_∞.

3. We assume that:

$$\forall t \in \mathbb{R},\; e^{-t/2} D_y \phi^t(x_\infty)_{|T_{x_\infty}\mathcal{F}} \in SO_2(T_{x_\infty}\mathcal{F}). \tag{12}$$

4. We have reduced real leafwise cohomology groups $\overline{H}^j_{\mathcal{F}}$ $(0 \leq j \leq 2)$ on which $(\phi^t)_{t \in \mathbb{R}}$ acts naturally such that $\overline{H}^0_{\mathcal{F}} \simeq \mathbb{R}$, $\overline{H}^2_{\mathcal{F}} \simeq \mathbb{R}$ and $\overline{H}^1_{\mathcal{F}}$ is infinite dimensional. Let $[\lambda_g]$ denote the class in $\overline{H}^2_{\mathcal{F}}$ of the leafwise kaehler metric λ_g associated to g. Then we assume that

$$\forall t \in \mathbb{R},\; (\phi^t)^*([\lambda_g]) = e^t[\lambda_g]. \tag{13}$$

5. The action of ϕ^t on $\overline{H}^1_{\mathcal{F}}$ commutes with the Hodge star $\star$ induced by g. Moreover there exists a transverse measure μ on $(S_{\mathbb{Q}}, \mathcal{F})$ such that $\int_{S_{\mathbb{Q}}} (\alpha \wedge \star\beta)\mu$ defines a scalar product on $\overline{H}^1_{\mathcal{F}}$.

6. For any $\alpha \in C^{\infty}_{\text{compact}}(\mathbb{R};\mathbb{R})$, $\int_{\mathbb{R}} \alpha(t)(\phi^t)^* dt$ acting on $\overline{H}^1_{\mathcal{F}}$ is trace class. The explicit formula (1) is interpreted as a Lefschetz trace formula for the riemannian foliated space $(S_{\mathbb{Q}}, \mathcal{F}, g, \phi^t)$ with respect to the leafwise cohomology groups $\overline{H}^j_{\mathcal{F}}$ $(0 \leq j \leq 2)$.

7. The fixed point $x_\infty \in S_{\mathbb{Q}}$ should be a limit point of a trajectory γ_∞ : $\lim_{t\to+\infty} \phi^t(y) = x_\infty$ for any $y \in \gamma_\infty$. Moreover, γ_∞ should have the following orbifold structure. Define an orbifold structure on $\mathbb{R}^{\geq 0}$ by requiring the following map to be an orbifold isomorphism:

$$Sq : \frac{\mathbb{R}}{\{1,-1\}} \to \mathbb{R}^{\geq 0},\ Sq(z) = z^2.$$

Notice that Sq transforms the flow $\phi^t_{\frac{\mathbb{R}}{\{1,-1\}}}(z) = ze^{-t}$ into the flow $\phi^t_{\mathbb{R}^{\geq 0}}(v) = ve^{-2t}$. Then we require that there exists an embedding $\Psi : \mathbb{R}^{\geq 0} \to \gamma_\infty$ such that $\Psi(0) = x_\infty$ and

$$\forall (t,v) \in \mathbb{R}\times\mathbb{R}^{\geq 0},\ \Psi(\phi^t_{\mathbb{R}^{\geq 0}}(v) = ve^{-2t}) = \phi^t(\Psi(v)). \tag{14}$$

Lastly we require that γ_∞ is transverse at x_∞ to $T_{x_\infty}\mathcal{F}$.

Comment 8. The stronger assumption $\forall t \in \mathbb{R}$, $(\phi^t)^*(g) = e^t g$ implies (12) (because $\phi^0 = \mathrm{Id}$), (13) and the fact that ϕ^t commutes with the Hodge star not only on $\overline{H}^1_{\mathcal{F}}$ but also on the vector space of leafwise differential 1−forms. Deninger told us privately that this assumption $(\phi^t)^*(g) = e^t g$ might be too strong. Assumption 5 and (13) implies equation (3) in Deninger's formalism. Therefore, the first six Assumptions imply the Riemann hypothesis as explained in Section 2. Assumption 7 is stated here as a hint about a possible way to prove Assumption 6. See the next subsection.

Comment 9. The results that we have described in Section 3.3 (e.g. Lemma 4) suggest that the disymmetry mentionned in Comment 1 might be explained in the following way. For each prime $p \in \mathcal{P}$, $(S_{\mathbb{Q}}, \mathcal{F})$ should exhibit a transversal of the type $]0,1[\times\mathbb{Z}_p$ and possibly the ring of finite Adeles might enter into the picture.

5.2 – Remarks about the contribution of the archimedean place in (1)

Now we apply formally the Guillemin-Sternberg trace formula for

$$\sum_{j=0}^{2}(-1)^j \operatorname{Tr}((\phi^t)^* ; \Gamma(S_{\mathbb{Q}} ; \wedge^j T^*\mathcal{F}))$$

where $\Gamma(S_{\mathbb{Q}}; \wedge^j T^*\mathcal{F})$ denotes the set of "smooth" sections.

Lemma 7 (Deninger [De01]).

1) *The contribution of the fixed point* x_∞ *in the previous Guillemin-Sternberg trace formula is*:

$$\frac{1}{|\det(1 - D_y\phi^t(x_\infty)\,;\, T_{x_\infty}S_{\mathbb{Q}}/T_{x_\infty}\mathcal{F})|}.$$

2)

$$\forall t \in \mathbb{R} \setminus \{0\},\ \frac{1}{|\det(1 - D_y\phi^t(x_\infty)\,;\, T_{x_\infty}S_{\mathbb{Q}}/T_{x_\infty}\mathcal{F})|} = \frac{1}{|1 - e^{-2t}|}.$$

Proof.

1) Using Proposition 1, one sees that the contribution of the fixed point x_∞ is equal to:

$$\frac{\displaystyle\sum_{j=0}^{2}(-1)^j \operatorname{Tr}((D_y\phi^t)^*(x_\infty); \wedge^j T^*_{x_\infty}\mathcal{F})}{|\det(1 - D_y\phi^t(x_\infty); T_{x_\infty}S_{\mathbb{Q}})|} =$$

$$= \frac{\det(1 - D_y\phi^t(x_\infty); T_{x_\infty}\mathcal{F})}{|\det(1 - D_y\phi^t(x_\infty); T_{x_\infty}\mathcal{F})|} \frac{1}{|\det(1 - D_y\phi^t(x_\infty); T_{x_\infty}S_{\mathbb{Q}}/T_{x_\infty}\mathcal{F})|}.$$

Using property (12) one checks easily that

$$\frac{\det(1 - D_y\phi^t(x_\infty); T_{x_\infty}\mathcal{F})}{|\det(1 - D_y\phi^t(x_\infty)\,;\, T_{x_\infty}\mathcal{F})|} = 1.$$

One then gets immediately 1).

2) Since $T_{x_\infty}S_{\mathbb{Q}}/T_{x_\infty}\mathcal{F}$ is a real line, there exists $\kappa \in \mathbb{R}$ such that:

$$\forall t \in \mathbb{R}, |\det(1 - D_y\phi^t(x_\infty)\,;\, T_{x_\infty}S_{\mathbb{Q}}/T_{x_\infty}\mathcal{F})| = |1 - e^{\kappa t}|.$$

By Assumption 7, γ_∞ is transverse at x_∞ to $T_{x_\infty}\mathcal{F}$ and (14) shows that $D_y\phi^t(x_\infty)$ acts as e^{-2t} on the real line $T_{x_\infty}S_{\mathbb{Q}}/T_{x_\infty}\mathcal{F}$. One then gets 2) immediately.

Recall that we wish to interpret (1) as a Lefschetz trace formula via the Guillemin-Sternberg formula.

PROPOSITION 2 (Deninger [De01]).

1) *The contribution of the real archimedean absolute value in* (1) *coincides for any real t positive with the contribution of the fixed point x_∞ in the Guillemin-Sternberg formula.*
2) *The contributions of the fixed point x_∞ for t real negative in the Guillemin-Sternberg formula and of the real archimedean absolute value in* (1) *do not coincide.*
3)

$$\forall t < 0,\ \frac{e^t}{1-e^{2t}} = \frac{\displaystyle\sum_{j=0}^{2}(-1)^j Tr((D_y\phi^t)^*(x_\infty)\,;\,\wedge^j T^*_{x_\infty}\mathcal{F})}{|\det(1-D_y\phi^{|t|}(x_\infty)\,;\,T_{x_\infty}S_{\mathbb{Q}})|} \tag{15}$$

where in the denominator we have written $\phi^{|t|}$.

PROOF.

1) This is part 2) of the previous lemma.
2) Indeed, the Guillemin-Sternberg formula gives

$$\frac{1}{|1-e^{-2t}|} = \frac{e^{2t}}{1-e^{2t}},$$

whereas (1) gives $\frac{e^t}{1-e^{2t}}$ for $t < 0$.
3) This follows from (12) and a simple computation.

COMMENT 10. It was in order to explain the factor -2 (instead of -1) in $\frac{1}{1-e^{-2t}}$ for $t > 0$ in (1) that Deninger has proposed in [De01b, Section 3] the Assumption 7.

– **Open Question**

Find a conceptual explanation of the equation (15) by a suitable generalization of Guillemin-Sternberg's trace formula to a "suitable singular setting".

5.3 – Remarks about the contribution of the archimedean place in the explicit formula for $\zeta_{\mathbb{Q}[i]}$

We recall the explicit formula for the zeta function $\zeta_{\mathbb{Q}[i]}$ of $\mathbb{Q}[i]$ as an equality between two distributions in $\mathcal{D}'(\mathbb{R}\setminus\{0\})$ (t being the real variable).

$$\begin{aligned} 1 - \sum_{\rho\in\zeta^{-1}_{\mathbb{Q}[i]}\{0\},\Re\rho\geq 0} e^{t\rho} + e^t = &\sum_{\mathcal{Q}} \log N\mathcal{Q} \sum_{k\geq 1}(\delta_{k\log N\mathcal{Q}} + (N\mathcal{Q})^{-k}\delta_{-k\log N\mathcal{Q}}) + \\ &+ \frac{1}{1-e^{-t}}\,1_{\{t>0\}} + \frac{e^t}{1-e^t}\,1_{\{t<0\}}, \end{aligned} \tag{16}$$

where $\mathcal{Q}$ runs over the set of non zero prime ideals of $\mathbb{Z}[i]$ and $N\mathcal{Q}$ denotes the norm of $\mathcal{Q}$.

Of course, one conjectures the existence of a Riemannian (laminated) foliated space $(S_{\mathbb{Q}[i]}, \mathcal{F}, g, \phi^t)$ satisfying a list of axioms quite similar to the ones stated in Section 5.1. We simply explain how Assumption 7 has to be modified for the pair of the two complex archimedean places $\{|\cdot|_{\mathbb{C}}, |\bar{\cdot}|_{\mathbb{C}}\}$ of $\zeta_{\mathbb{Q}[i]}$.

Substitute of 7 for $\mathbb{Q}[i]$ (cf. [De01b, Section 3]). There exists a stationary fixed point $z_\infty \in S_{\mathbb{Q}[i]}$ of ϕ^t and two trajectories $\gamma_\pm$ of the flow ϕ^t with end point z_∞. For any $z_\pm \in \gamma_\pm$, $\lim_{t\to+\infty} \phi^t(z_\pm) = z_\infty$. These two trajectories $\gamma_\pm$ are transverse to $\mathcal{F}$ at z_∞. Moreover there exists an embedding:

$$\Psi : \mathbb{R} \to \gamma_- \cup \gamma_+,$$

such that $\Psi(0) = z_\infty$, $\gamma_\pm \setminus \{0\} = \Psi(\mathbb{R}^\pm \setminus \{0\})$. Lastly, $\forall v, t \in \mathbb{R}$, $\Psi(ve^{-t}) = \phi^t(\Psi(v))$.

Therefore, the contribution of z_∞ in the Guillemin-Sternberg trace formula is:

$$\forall t \in \mathbb{R} \setminus \{0\},\ \frac{1}{|1 - e^{-t}|} = \frac{1}{|\det(1 - D_y\phi^t(z_\infty)\,;\, T_{z_\infty}S_{\mathbb{Q}[i]}/T_{z_\infty}\mathcal{F})|}.$$

Part 1) of the following proposition shows that the contribution of a *complex* archimedean place in the explicit formula is better understood than the one of a *real* archimedean place (cf. Proposition 2. 2)). Part 2) suggests that the previous open question should admit a conceptual answer.

Proposition 3 ([De01, Section 5]).

1) *The contribution of z_∞ in the Guillemin-Sternberg trace formula coincides, for $t \in \mathbb{R} \setminus \{0\}$, with the contribution of the two complex archimedean places in* (16).
2) *For t real negative, one has*:

$$\frac{e^t}{1 - e^t} = \frac{\sum_{j=0}^{2} (-1)^j Tr((D_y\phi^t)^*(z_\infty); \wedge^j T^*_{z_\infty}\mathcal{F})}{|\det(1 - D_y\phi^{|t|}(z_\infty)\,;\, T_{z_\infty}S_{\mathbb{Q}[i]})|}.$$

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