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Uniqueness of renormalized solutions for a class of parabolic equations with unbounded nonlinearities

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ABSTRACT: *We prove uniqueness and a comparison principle of renormalized solutions for a class of doubly nonlinear parabolic equations* $\frac{\partial b(x,u)}{\partial t} - \mathrm{div}(A(t,x)Du + \Phi(u)) = f$, *where the right side belongs to* $L^1((0,T)\times\Omega)$ *and where* $b(x,u)$ *is unbounded function of* u *and where* $A(t,x)$ *is a bounded symmetric and coercive matrix, and* Φ *is continuous function but without any growth assumption on* u.

1 – Introduction

In the present paper we establish the uniqueness and comparison principle for a renormalized solutions for a class of doubly nonlinear parabolic equations of the type

$$\frac{\partial b(x,u)}{\partial t} - \mathrm{div}\Big(A(t,x)Du + \Phi(u)\Big) = f \quad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$b(x,u)(t=0) = b(x,u_0) \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega\times(0,T). \tag{1.3}$$

In Problem (1.1)-(1.3) the framework is the following: Ω is a bounded domain of $\mathbb{R}^N$, $(N\geq 1)$, T is a positive real number while the data f and $b(x,u_0)$ in $L^1(\Omega\times(0,T))$ and $L^1(\Omega)$. And where b is a Carathéodory function such that,

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$b(x,s)$ is unbounded function of s. The matrix $A(t,x)$ is a bounded symmetric and coercive matrix. The function Φ is just assumed to be continuous on $\mathbb{R}$.

When Problem (1.1)-(1.3) is investigated one of the difficulties is due to the facts that the data f and $b(x,u_0)$ only belong to L^1 and the growths of $b(x,u)$ and $\Phi(u)$ are not controlled with respect to u (the function $\Phi(u)$ does not belong to $(L^1_{\text{loc}}((0,T)\times\Omega))^N$ in general), so that proving existence of a weak solution (i.e. in the distribution meaning) seems to be an arduous task. To overcome this difficulty we use the framework of renormalized solutions. The existence of a renormalized solutions of (1.1)-(1.3) is proved in H. Redwane [15].

The notion of renormalized soluion is introduced by Lions and Di Perna [14] for the study of Boltzmann equation (see also P.-L. Lions [10] for a few applications to fluid mechanics models). This notion was then adapted to elliptic version of (1.1)-(1.3) in Boccardo, J.-L. Diaz, D. Giachetti, F. Murat [8], in P.-L. Lions and F. Murat [11] and F. Murat [12], [13] (see also [2], [3], [4], [5], [6], [7]). At the same the equivalent notion of entropy solutions have been developed independently by Bénilan and al. [1] for the study of nonlinear elliptic problems.

The paper is organized as follows: Section 2 is devoted to specify the assumptions on b, Φ, f and u_0 needed in the present study and gives the definition and the existence (Theorem 2.0.3) of a renormalized solution of (1.1)-(1.3). In Section 3 we establish uniqueness and a comparison principle of such a solution (Theorem 3.0.4)

2 – Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open set on $\mathbb{R}^N$ $(N \geq 1)$, $T > 0$ is given and we set $Q = \Omega\times(0,T)$.

$$b : \Omega\times\mathbb{R}\to\mathbb{R} \text{ is a Carathéodory function such that;} \tag{2.1}$$

for every $x \in \Omega$: $b(x,s)$ is a strictly increasing C^1-function, with $b(x,0)=0$. For any $K > 0$, there exists $\lambda_K > 0$, a function A_K in $L^1(\Omega)$ and a function B_K in $L^2(\Omega)$ such that

$$\lambda_K \leq \frac{\partial b(x,s)}{\partial s} \leq A_K(x) \quad \text{and} \quad \left|\nabla_x\left(\frac{\partial b(x,s)}{\partial s}\right)\right| \leq B_K(x) \tag{2.2}$$

for almost every $x\in\Omega$, for every s such that $|s|\leq K$.

(2.3) $A(t,x)$ is a symmetric coercive matrix field with coefficients lying in $L^\infty(Q)$ i.e. $A(t,x) = (a_{ij}(t,x))_{1\le i,j\le N}$ with:

- $a_{ij}(t,x) \in L^\infty(Q)$ and $a_{ij}(t,x) = a_{ji}(t,x)$ a.e. in Q, $\forall\ i,\ j$
- $\exists\ \alpha > 0$ such that a.e. in Q, $\forall \xi \in \mathbb{R}^N \quad A(t,x)\xi\xi \ge \alpha \|\xi\|^2_{\mathbb{R}^N}$

$$\Phi\ :\ \mathbb{R} \to \mathbb{R}^N \text{ is a continuous function} \tag{2.4}$$

$$f \text{ is an element of } L^1(Q). \tag{2.5}$$

(2.6) u_0 is a measurable function defined on Ω such that $b(x,u_0) \in L^1(\Omega)$.

Remark 2.0.1. In (2.2), we denote by $\nabla_x\left(\frac{\partial b(x,s)}{\partial s}\right)$ the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions.

As already mentioned in the introduction Problem (1.1), (1.2), (1.3) does not admit a weak solution under assumptions (2.1)-(2.6), since the growths of $b(x,u)$ and $\Phi(u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs $L^2(0,T;W_0^{1,2}(\Omega))$).

Throughout this paper and for any non negative real number K we denote by $T_K(r) = \min(K, \max(r,-K))$ the truncation function at height K. The definition of a renormalized solution for Problem (1.1), (1.2), (1.3) can be stated as follows.

Definition 2.0.2. A measurable function u defined on Q is a renormalized solution of Problem (1.1), (1.2), (1.3) if

$$T_K(u) \in L^2(0,T;W_0^{1,2}(\Omega)) \text{ for any } K \ge 0 \text{ and } b(x,u) \in L^\infty(0,T;L^1(\Omega)); \tag{2.7}$$

$$\int_{\{(t,x)\in Q;\ n\le|u(x,t)|\le n+1\}} A(x,t)DuDu\,dx\,dt \ \longrightarrow 0 \ \text{ as } n \to +\infty; \tag{2.8}$$

and if, for every increasing function S in $W^{2,\infty}(\mathbb{R})$, which is piecewise C^1 and such that S' has a compact support, we have

$$\begin{aligned} \frac{\partial b_S(x,u)}{\partial t} &- \operatorname{div}(S'(u)A(t,x)Du) + S''(u)A(t,x)DuDu \\ &- \operatorname{div}(S'(u)\Phi(u)) + S''(u)\Phi(u)Du = fS'(u) \ \text{ in } D'(Q); \end{aligned} \tag{2.9}$$

$$b_S(x,u)(t=0) = b_S(x,u_0) \ \text{ in } \Omega; \tag{2.10}$$

where $b_S(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s} S'(s)\,ds$.

The existence theorem of renormalized solution of (1.1)-(1.3):

Theorem 2.0.3. *Under assumptions* (2.1)-(2.6) *there exists at least a renormalized solution u of Problem* (1.1)-(1.3).

Proof of Theorem 3.0.3. The existence theorem of renormalized solution of (1.1)-(1.3) is proved in H. Redwane [15]

3 – Comparison principle and uniqueness result

This section is concerned with a comparison principle (and an uniqueness result) for renormalized solutions. We establish the following theorem.

Theorem 3.0.4. *Assume that assumptions* (2.1), (2.2), (2.3), (2.4), (2.5) *and* (2.6) *hold true and moreover that.*

For any $K > 0$, *there exists a positive real number* $\beta_K > 0$, *such that*

$$\Big|\frac{\partial b(x,z_1)}{\partial s} - \frac{\partial b(x,z_2)}{\partial s}\Big| \leq \beta_K \,\Big|z_1 - z_2\Big| \tag{3.1}$$

for almost every x *in* Ω, *and for every* z_1 *and every* z_2 *such that* $|z_1| \leq K$ *and* $|z_2| \leq K$.

$$\Phi \text{ is a locally Lipschitz continuous function on } \mathbb{R}. \tag{3.2}$$

Let then u_1 *and* u_2 *be renormalized solutions corresponding to the data* (f_1, u_0^1) *and* (f_2, u_0^2). *If these data satisfying* $f_1 \leq f_2$ *and* $u_0^1 \leq u_0^2$ *almost every where, we have*

$$u_1 \leq u_2 \textit{ almost every where.}$$

Proof of Theorem 3.0.4. The proof is divided into two steps. In Step 1, we define a smooth approximation S_n of T_n, and we consider two renormalized solutions u_1 and u_2 of (1.1), (1.2), (1.3) for the data (f_1, u_0^1) and (f_2, u_0^2) respectively. We plug the test function $\frac{1}{\sigma}{T_\sigma}^+\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))\Big)$ in the difference of equations (2.9) for u_1 and u_2 in which we have taken $S = S_n$.

In Step 2, we investigate the behaviour of the different terms in the estimate obtained in Step 1 (estimates (3.5)) as σ tends to 0 and when n tends to $+\infty$.

Step 1. Remark that when Φ is locally Lipschitz continuous on $\mathbb{R}$ the following derivation is licit for any function S and u satisfying the conditions mentioned in Definition 2.0.2.

$$\operatorname{div}\Big(S'(u)\Phi(u)\Big) - S''(u)\Phi(u)Du = S'(u)\Phi'(u)Du = \operatorname{div}(\Phi_S(u)). \tag{3.3}$$

Where $\Phi_S = (\Phi_{S,1}, \Phi_{S,2}, \cdots, \Phi_{S,N})$ with

$$\Phi_{S,i}(r) = \int_0^r \Phi'_{S,i}(t) S'(t)\, dt.$$

Let us now introduce a specific choice of function S in (2.9). For all $n > 0$, let $S_n \in C^1(\mathbb{R})$ be the function defined by $S_n(0) = 0$; $S_n'(r) = 1$ for $|r| \leq n$; $S_n'(r) = n + 1 - |r|$ for $n \leq |r| \leq n + 1$ and $S_n'(r) = 0$ for $|r| \geq n + 1$.

It yields, taking $S = S_n$ in (2.9)

$$\frac{\partial b_{S_n}(x, u_i)}{\partial t} - \operatorname{div}\Big(S'(u_i)A(t, x)Du_i\Big) + S''(u_i)A(t, x)Du_iDu_i + \tag{3.4}$$
$$- \operatorname{div}\Big(\Phi_{S_n}(u_i)\Big) = f_iS_n'(u_i) \text{ in } D'(Q);$$

for $i = 1, 2$ and where $b_{S_n}(x, r) = \int_0^r \frac{\partial b(x,s)}{\partial s} S_n'(s)\, ds$.

We use $\frac{1}{\sigma}T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2))$ as a test function in the difference of equations (3.4) for u_1 and u_2.

$$\frac{1}{\sigma}\int_0^T\int_0^t\Big\langle\frac{\partial(b_{S_n}(x, u_1) - b_{S_n}(x, u_2))}{\partial t}; T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2))\Big\rangle ds\, dt + A_n^\sigma = \tag{3.5}$$
$$= B_n^\sigma + C_n^\sigma + D_n^\sigma,$$

for any $\sigma > 0$, $n > 0$, and where

$$A_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega\Big[S_n'(u_1)A(t, x)Du_1 - S_n'(u_2)A(t, x)Du_2\Big]\cdot \tag{3.6}$$
$$\cdot DT_\sigma^+\Big(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)\Big)\, dx\, ds\, dt$$

$$B_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega S_n''(u_1)A(t, x)Du_1Du_1T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2)\, dx\, ds\, dt + \tag{3.7}$$
$$-\frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega S_n''(u_2)A(t, x)Du_2Du_2T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2))\, dx\, ds\, dt$$

$$C_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega[\Phi_{S_n}(u_1) - \Phi_{S_n}(u_2)]DT_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2))\, dx\, ds\, dt \tag{3.8}$$

$$D_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega[f_1S_n'(u_1) - f_2S_n'(u_2)]T_\sigma^+(b_{S_n}(x, u_1) - b_{S_n}(x, u_2))\, dx\, ds\, dt. \tag{3.9}$$

In the sequel we pass to the limit in (3.5) when σ tends to 0 and then n tends to $+\infty$. Upon application of Lemma 2.4 of [9], the first term in the right hand side of (3.5) is derived as

$$\begin{aligned}
&\frac{1}{\sigma}\int_0^T\int_0^t\left\langle\frac{\partial(b_{S_n}(x,u_1)-b_{S_n}(x,u_2))}{\partial t};T_\sigma^+(b_{S_n}(x,u_1)-b_{S_n}(x,u_2))\right\rangle ds\,dt=\\
&=\frac{1}{\sigma}\int_Q\tilde{T}_\sigma^+(b_{S_n}(x,u_1)-b_{S_n}(x,u_2))\,dx\,dt+\\
&\quad-\frac{T}{\sigma}\int_\Omega\tilde{T}_\sigma^+(b_{S_n}(x,u_0^1)-b_{S_n}(x,u_0^2))\,dx
\end{aligned}\tag{3.10}$$

where $\tilde{T}_\sigma^+(r)=\int_0^r T_\sigma^+(s)\,ds$.

Due to the assumption $u_0^1\le u_0^2$ a.e. in Ω and the monotone character of $b_{S_n}(x,\ .)$ and $T_\sigma(.)$, we have

$$\int_\Omega \tilde{T}_\sigma^+\Big(b_{S_n}(x,u_0^1)-b_{S_n}(x,u_0^2)\Big)\,dx=0.\tag{3.11}$$

It follows from (3.5), (3.10) and (3.11) that

$$\frac{1}{\sigma}\int_Q\tilde{T}_\sigma^+\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\,dx\,dt+A_n^\sigma=B_n^\sigma+C_n^\sigma+D_n^\sigma\tag{3.12}$$

for any $\sigma>0$ and any $n>0$.

Step 2. In this step, we study the behaviors of the terms A_n^σ, B_n^σ, C_n^σ and D_n^σ when σ tends to 0 and $n\to+\infty$. More precisely, we prove the following estimates,

$$\lim_{n\to+\infty}\ \underline{\lim}_{\sigma\to0}A_n^\sigma\ge0,\tag{3.13}$$

$$\underline{\lim}_{n\to+\infty}\ \underline{\lim}_{\sigma\to0}B_n^\sigma=0,\tag{3.14}$$

$$\underline{\overline{\lim}}_{\sigma\to0}C_n^\sigma=0\quad\text{for all } n,\tag{3.15}$$

$$\underline{\lim}_{n\to+\infty}\ \overline{\lim}_{\sigma\to0}D_n^\sigma\le0.\tag{3.16}$$

Proof of (3.13)

$$A_n^\sigma=\frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega\Big[S_n'(u_1)A(t,x)Du_1-S_n'(u_2)A(t,x)Du_2\Big]$$
$$DT_\sigma^+\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\,dx\,ds\,dt.$$

To establish (3.13) we first write A_n^σ, as follows

$$
\begin{aligned}
A_n^\sigma =&\int_Q \frac{(T-t)}{\sigma}\Big[S_n'(u_1)\Big(\frac{\partial b(x,u_1)}{\partial s}\Big)^{\frac12}A(t,x)^{\frac12}Du_1+\\
&-S_n'(u_2)\Big(\frac{\partial b(x,u_2)}{\partial s}\Big)^{\frac12}A(t,x)^{\frac12}Du_2\Big]^2(T_\sigma^+)'(b_{S_n}(x,u_1)-b_{S_n}(x,u_2))\,dx\,dt+\\
&-\int_Q \frac{(T-t)}{\sigma}\Big[\Big(\frac{\partial b(x,u_1)}{\partial s}\Big)^{\frac12}-\Big(\frac{\partial b(x,u_2)}{\partial s}\Big)^{\frac12}\Big]^2A(t,x)DS_n(u_1)DS_n(u_2)\cdot\\
&\cdot(T_\sigma^+)'\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\,dx\,dt+\int_Q\frac{(T-t)}{\sigma}\Big[S_n'(u_1)A(t,x)Du_1+\\
&-S_n'(u_2)A(t,x)Du_2\Big]\Big[\int_{u_2}^{u_1}S_n'(s)\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\,ds\Big]\cdot\\
&\cdot(T_\sigma^+)'\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\,dx\,dt.
\end{aligned}
\tag{3.17}
$$

We denote by C_n the compact subset $[-n-1,n+1]$ of $\mathbb{R}$, and remark that due to (2.2) and (3.1), there exist a positive real numbers λ_n and β_n such that

$$
\begin{aligned}
&\Big|\Big(\frac{\partial b(x,z_1)}{\partial s}\Big)^{\frac12}-\Big(\frac{\partial b(x,z_2)}{\partial s}\Big)^{\frac12}\Big|\le\\
&\le\frac{\beta_n}{2\sqrt{\lambda_n}}\,|z_1-z_2|\ \text{ for all } z_1,\ z_2 \text{ lying in } C_n,
\end{aligned}
\tag{3.18}
$$

for almost every x in Ω.

Due to the definition of $b_{S_n}(x,r)$, we have

$$
\Big|b_{S_n}(x,s)-b_{S_n}(x,r)\Big|=\Big|\int_r^s S_n'(z)\frac{\partial b(x,z)}{\partial z}\,dz\Big|\ge\lambda_n\Big|S_n(s)-S_n(r)\Big| \tag{3.19}
$$

for almost every x in Ω, and $\forall\ s,\ r\in\mathbb{R}$.

As a consequence it follows that for $\sigma<n$ and if s and r are real numbers such that $\Big|S_n(s)-S_n(r)\Big|\le\sigma$, then both $S_n(s)$ and $S_n(r)$ belong to concave or to convex branch of S_n. For $\sigma<n$, we then have:

$$
\min\Big(S_n'(s),\ S_n'(r)\Big)|r-s|\le\Big|S_n(s)-S_n(r)\Big|
$$

for all real numbers such that $\Big|S_n(s)-S_n(r)\Big|\le\sigma$.

From the above inequality and since $\|S_n'\|_{L^\infty(\mathbb{R})}=1$ we deduce that

$$
\Big|S_n(s)-S_n(r)\Big|\le\sigma<n\Longrightarrow S_n'(s)S_n'(r)|s-r|\le\Big|S_n(s)-S_n(r)\Big|.
$$

Due to the definition of T_σ^+, it follows that

$$S_n'(s)S_n'(r)|s-r|(T_\sigma^+)'(S_n(s)-S_n(r)) \leq \sigma\ (T_\sigma^+)'(S_n(s)-S_n(r)) \tag{3.20}$$

for all numbers s and r.

Recalling that $supp(S_n') \subset [-(n+1),\ n+1]$, inequalities (3.18) and (3.19) lead to:

$$\begin{aligned}
&\Big|\int_Q \frac{(T-t)}{\sigma}\Big[\Big(\frac{\partial b(x,u_1)}{\partial s}\Big)^{\frac{1}{2}} - \Big(\frac{\partial b(x,u_2)}{\partial s}\Big)^{\frac{1}{2}}\Big]^2 \cdot \\
&\quad \cdot (T_\sigma^+)'\Big(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)\Big) A(t,x) DS_n(u_1) DS_n(u_2)\, dx\, dt\Big| \leq \\
&\leq \frac{T\beta_n}{2\sqrt{\lambda_n}} \int_Q \Big|\Big(\frac{\partial b(x,u_1)}{\partial s}\Big)^{\frac{1}{2}} - \Big(\frac{\partial b(x,u_2)}{\partial s}\Big)^{\frac{1}{2}}\Big| \chi_{\{u_1\neq u_2\}} \cdot \\
&\quad \cdot (T^+_{\frac{\sigma}{\lambda_n}})'\Big(S_n(u_1) - S_n(u_2)\Big)\Big|A(t,x) DT_{n+1}(u_1) DT_{n+1}(u_2)\Big|\, dx\, dt.
\end{aligned}$$

The term just above is easily shown to converge to zero as σ goes to zero since the function

$$\begin{aligned}
&\Big|\Big(\frac{\partial b(x,u_1)}{\partial s}\Big)^{\frac{1}{2}} - \Big(\frac{\partial b(x,u_2)}{\partial s}\Big)^{\frac{1}{2}}\Big| \chi_{\{u_1\neq u_2\}} \cdot \\
&\quad \cdot (T^+_{\frac{\sigma}{\lambda_n}})'\Big(S_n(u_1) - S_n(u_2)\Big)\Big|A(t,x) DT_{n+1}(u_1) DT_{n+1}(u_2)\Big|
\end{aligned}$$

converges to zero almost everywhere in Q as σ goes to zero and (due to (3.1)) is bounded by the $L^1(Q)$-function $2\|\frac{\partial b(x,s)}{\partial s}\|_{L^\infty(\Omega\times C_n)}|A(t,x)DT_{n+1}(u_1)DT_{n+1}(u_2)|$. We remark that

$$\begin{aligned}
&\Big|\int_Q \frac{(T-t)}{\sigma}[S_n'(u_1)A(t,x)Du_1 - S_n'(u_2)A(t,x)Du_2](T_\sigma^+)'(b_{S_n}(x,u_1)+ \\
&\quad - b_{S_n}(x,u_2))\Big[\int_{u_2}^{u_1} S_n'(s)\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\, ds\Big]\, dx\, dt\Big| \leq \\
&\leq \Big|\int_Q \frac{(T-t)}{\sigma}\Big|S_n'(u_1)A(t,x)Du_1 - S_n'(u_2)A(t,x)Du_2\Big|(T_\sigma^+)'(b_{S_n}(x,u_1)+ \\
&\quad - b_{S_n}(x,u_2))\Big\|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big\|_{L^\infty(C_n)}\Big|S_n(u_1) - S_n(u_2)\Big|\, dx\, dt \leq \\
&\leq T\int_Q \Big|S'(u_1)A(t,x)DT_{n+1}(u_1) - S_n'(u_2)A(t,x)DT_{n+1}(u_2)\Big|\chi_{\{u_1\neq u_2\}} \cdot \\
&\quad \cdot (T^+_{\frac{\sigma}{\lambda_n}})'\Big(S_n(u_1) - S_n(u_2)\Big)\|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\|_{L^\infty(C_n)}\, dx\, dt.
\end{aligned} \tag{3.22}$$

The term just above is easily shown to converge to zero as σ goes to zero since the function

$$\Big|S'(u_1)A(t,x)DT_{n+1}(u_1) - S'_n(u_2)A(t,x)DT_{n+1}(u_2)\Big|\chi_{\{u_1\neq u_2\}}\cdot$$
$$\cdot\,(T^+_{\frac{\sigma}{\lambda_n}})'\Big(S_n(u_1)-S_n(u_2)\Big)\Big\|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big\|_{L^\infty(C_n)}$$

converges to zero almost everywhere in Q as σ goes to zero and is bounded by the $L^1(Q)$-function

$$\Big|S'(u_1)A(t,x)DT_{n+1}(u_1) - S'_n(u_2)A(t,x)DT_{n+1}(u_2)\Big|B_n(x)$$

since $\|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\|_{L^\infty(C_n)} \le B_n(x)\in L^2(\Omega)$ (see (2.2)).

From the above analysis we conclude that (3.13) holds true.

Proof of (3.14). We have

$$(3.23)\qquad \begin{aligned}|B^\sigma_n| \le T&\int_{\{n\le|u_1|\le n+1\}} A(t,x)Du_1Du_1\,dx\,dt+\\ &+T\int_{\{n\le|u_2|\le n+1\}} A(t,x)Du_2Du_2\,dx\,dt.\end{aligned}$$

As a consequence of (2.8), letting n go to infinity in the above estimates of B^σ_n shows that (3.14) holds true.

Proof of (3.15).

$$C^\sigma_n = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega \Big[\Phi_{S_n}(u_1)-\Phi_{S_n}(u_2)\Big]DT^+_\sigma\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\,dx\,ds\,dt.$$

To establish (3.15), let us remark that for all s, r in $\mathbb{R}$, the following inequality holds true

$$(3.24)\qquad \Big\|\Phi_{S_n}(s)-\Phi_{S_n}(r)\Big\|_{\mathbb{R}^N} \le \|\Phi'\|_{L^\infty(C_n)}{}^N\Big|S_n(s)-S_n(r)\Big|$$

indeed, since $suppS'_n\subset[-n-1,\ n+1]$ and Φ' is assumed to be locally Lipschitz continuous, it follows that

$$\Big|\Phi_{S_n}(s)-\Phi_{S_n}(r)\Big| \le \Big|\int_r^s S'(z)\Phi'(z)\,dz\Big| \le \|\Phi'\|_{L^\infty(C_n)^N}\Big|S_n(s)-S_n(r)\Big|.$$

With the help of (3.24) the term C_n^σ may be estimated as follows

$$
\begin{aligned}
|C_n^\sigma| \leq T\|\Phi'\|_{L^\infty(C_n)}{}^N \int_{\{0\leq\left(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\right)\leq\sigma\}} \frac{\Big|S_n(u_1)-S_n(u_2)\Big|}{\sigma}\cdot \\
\cdot\Big|DT_\sigma^+\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\Big|\,dx\,dt \leq \\
\leq T\|\Phi'\|_{L^\infty(C_n)}{}^N \int_{\{0\leq\left(S_n(u_1)-S_n(u_2)\right)\leq\frac{\sigma}{\lambda_n}\}} \frac{\Big|S_n(u_1)-S_n(u_2)\Big|}{\sigma}\cdot \\
\cdot\Big|DT_\sigma^+\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\Big|\,dx\,dt.
\end{aligned}
$$

It yields

$$
|C_n^\sigma| \leq \frac{T}{\lambda_n}\|\Phi'\|_{L^\infty(C_n)^N}\int_Q \Big|DT_\sigma^+\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\Big|\,dx\,dt,
$$

which in turn implies (3.15) since $DT_\sigma^+\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)$ converges to zero in $L^1(Q)$ as σ goes to zero.

Proof of (3.16).

$$
D_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega \Big[f_1S_n'(u_1)-f_2S_n'(u_2)\Big]T_\sigma^+\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\,dx\,ds\,dt.
$$

We have as n tends to $+\infty$,

$$
f_1S_n'(u_1)-f_2S_n'(u_2)\to f_1-f_2 \text{ strongly in } L^1(Q). \tag{3.25}
$$

Letting σ tends to 0, we have $\frac{1}{\sigma}T_\sigma^+(t)$ goes to $sg_0^+(t)$, for all $t\in\mathbb{R}$. For $n>0$ fixed, we have

$$
\lim_{\sigma\to 0} D_n^\sigma = \int_0^T\int_0^t\int_\Omega \Big(f_1-f_2\Big)sg_0^+\Big(b_{S_n}(x,u_1))-b_{S_n}(x,u_2))\Big)\,dx\,ds\,dt.
$$

Since $f_1\leq f_2$ a.e. in Q and $sg_0^+(t)\geq 0$ for all t in $\mathbb{R}$, then shows that (3.16) holds true. In view of the definition of $\tilde{T}_\sigma^+$ and T_n^σ, we have

$$
\begin{aligned}
&\lim_{n\to+\infty}\lim_{\sigma\to 0}\frac{1}{\sigma}\int_0^T\int_\Omega \tilde{T}_\sigma^+\Big(b_{S_n}(x,u_1)-b_{S_n}(x,u_2)\Big)\,dx\,dt = \\
&= \int_Q \Big(b(x,u_1)-b(x,u_2)\Big)^+\,dx\,dt.
\end{aligned} \tag{3.26}
$$

In view of estimates (3.11), (3.12), (3.13), (3.14), (3.15), (3.16) and (3.26) we have

$$\int_Q \Big(b(x,u_1) - b(x,u_2)\Big)^+ \, dx\, dt \le 0,$$

so that $b(x,u_1) \le b(x,u_2)$ a.e. in Q which in turn implies that $u_1 \le u_2$ a.e. in Q, Theorem 3.0.4 is then established.

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