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A note on de Finetti's lower probabilities and belief measures

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Abstract: *De Finetti has introduced lower and upper probabilities as boundary set functions when extending a fixed probability from an atomic algebra to an atomic superalgebra. We discuss the relationship of de Finetti's lower and upper probabilities with belief and plausibility measures, and also investigate the order of their additivity. Finally, two open problems are formulated.*

1 – Introduction

For the sake of simplicity and transparency, in this paper we will deal with a fixed finite space X and algebras $\mathcal{A}$, $\mathcal{B}$ of its subsets. Recall that a probability measure Q on $(X, \mathcal{B})$ is a mapping $Q : \mathcal{B} \to [0,1]$ which is additive, *i.e.*, $Q(A \cup B) = Q(A)+Q(B)$ whenever $A \cap B = \emptyset$, and which fulfils the boundary condition $Q(X) = 1$. The aim of this paper is to recall some ideas of de Finetti [2] concerning the set of all probability measures P on $(X, \mathcal{A})$ such that for a fixed probability measure Q on $(X, \mathcal{B})$, where $\mathcal{B} \subset \mathcal{A}$, P is an extension of Q, *i.e.*, $P|\mathcal{B} = Q$. With no loss of generality, we will suppose that $\mathcal{A} = 2^X$, and we will denote by $\mathcal{P}_Q$ the set of all probabilities P on $(X, 2^X)$ characterized above. Evidently, $\mathcal{B}$ is an algebra generated by atoms $B_1, \ldots, B_m$, and we suppose $m < |X| = \text{card}X$ (or, equivalently, $\max_i |B_i| > 1$), to avoid the trivial case $\mathcal{B} = \mathcal{A}$ and $\mathcal{P}_Q = \{Q\}$.

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In fact, the above described situation occurs often in real world problems, when the observer is not able to distinguish elements belonging to the same atom B_i, and hence we can only estimate the real state of the observed universe X. Note that in Bayesian approaches we have first of all the information on a priori probabilities of atoms $B_1, \dots, B_m$, and to be able to get a complete description of the actual state in A (probability $P(A)$ for some $A \subset X$ not belonging to $\mathcal{B}$) we need an additional information on conditional probabilities $P(A|B_1), \dots, P(A|B_m)$.

The paper is organized as follows. In the next section, we recall lower and upper probabilities as introduced by de Finetti in [2]. In Section 3, Dempster–Shafer theory of belief and plausibility measures is recalled and the relationship with de Finetti's lower and upper probabilities is discussed. Section 4 brings the notion of k-additivity and its application to de Finetti's lower and upper probabilities. Finally, some concluding remarks and two open problems are given.

2 – Lower and upper probabilities

For a fixed probability measure Q on $(X, \mathcal{B})$, and a mapping $\tau : \{1, \dots, m\} \to X$ such that $\tau(i) \in B_i$, $i = 1, \dots, m$, clearly $P_{Q,\tau} : 2^X \to [0,1]$ given by

$$P_{Q,\tau}(A) = \sum_{\tau(i) \in A} Q(B_i) \tag{1}$$

is an element of $\mathcal{P}_Q$. It is not difficult to check that $\mathcal{P}_Q$ is a convex compact subset of the set $[0,1]^{2^X}$ (equipped with standard topology of uniform convergence) and that $P_{Q,\tau}$ are vertices (convexly irreducible elements) of $\mathcal{P}_Q$, *i.e.*,

$$P \in \mathcal{P}_Q \quad \text{if and only if} \quad P = \sum_j \lambda_j \, P_{Q,\tau_j}$$

for a finite system (τ_j), $\lambda_j > 0$ and $\sum\limits_j \lambda_j = 1$.

Though with no additional information we are unable to determine an element $P \in \mathcal{P}_Q$, we can always describe two distinguished set functions $\underline{Q}$, $\overline{Q}$: $\mathcal{A} \to [0,1]$ given by

$$\underline{Q}(A) = \inf \{P(A) \,|\, P \in \mathcal{P}_Q\}\,, \tag{2}$$

$$\overline{Q}(A) = \sup \{P(A) \,|\, P \in \mathcal{P}_Q\}\,. \tag{3}$$

The set function $\underline{Q}$ is called the lower probability, $\overline{Q}$ the upper probability, and it also holds

$$\underline{Q}(A) = \inf \big\{P(A) \,|\, P \quad \text{is a probability on} \quad (X, \mathcal{A}) \ \text{ and } \ P \geq \underline{Q}\big\}\,,$$

$$\overline{Q}(A) = \sup \big\{P(A) \,|\, P \quad \text{is a probability on} \quad (X, \mathcal{A}) \ \text{ and } \ P \leq \overline{Q}\big\}\,.$$

Moreover,

$$\underline{Q} = \inf_{\tau} P_{Q,\tau} \quad \text{and} \quad \overline{Q} = \sup_{\tau} P_{Q,\tau},$$

which implies the next important formulae:

$$\underline{Q}(A) = \sum_{B_i \subseteq A} Q(B_i), \tag{4}$$

$$\overline{Q}(A) = \sum_{B_i \cap A \neq \emptyset} Q(B_i). \tag{5}$$

To characterize the relationship of $\underline{Q}$ and $\overline{Q}$ we need to introduce the notion of duality of set functions.

Definition 1. Let $\mathfrak{C}$ be an algebra of subsets of X and let $G : \mathfrak{C} \to [0,1]$ be a given set function. Then the set function $G^d : \mathfrak{C} \to [0,1]$ defined by

$$G^d(A) = 1 - G(X \setminus A)$$

is called the dual of G.

Evidently, duality is an involution, *i.e.*, $(G^d)^d = G$. Moreover, any probability measure P on $(X, \mathcal{A})$ (Q on $(X, \mathcal{B})$) is invariant with respect to the duality, *i.e.*, $P^d = P$.

Proposition 1. *Under the above notation, $\underline{Q}$ is the dual of $\overline{Q}$, and vice versa, $\underline{Q}^d = \overline{Q}$.*

Proof. For any $A \in \mathcal{A}$, we get directly

$$\underline{Q}^d(A) = 1 - \underline{Q}(X \setminus A) = 1 - \inf_{\tau} P_{Q,\tau}(X \setminus A) = \sup_{\tau} (1 - P_{Q,\tau}(X \setminus A)) =$$
$$= \sup_{\tau} P^d_{Q,\tau}(A) = \sup_{\tau} P_{Q,\tau} = \overline{Q}(A).$$

3 – Belief measures

Formulae (4) and (5) are well known from the Dempster–Shafer theory of belief and plausibility measures. Indeed, for any probability distribution m on $2^X \setminus \{\emptyset\}$, *i.e.*, a set function m such that for each $\emptyset \neq A \subseteq X$, it holds $m(\{A\}) = m(A) \geq 0$ and $\sum_{\emptyset \neq A \subseteq X} m(A) = 1$,

$$Bel_m(A) = \sum_{\emptyset \neq B \subseteq A} m(B) \tag{6}$$

defines a belief measure.

Similarly,

$$Pl_m(A) = \sum_{B\cap A\neq\emptyset} m(B) \tag{7}$$

defines a plausibility measure.

The lower probability $\underline{Q}$ given by (2) is thus a special belief measure characterized by the disjointness of focal elements, *i.e.*, the sets A for which $m(A) > 0$, where m is the corresponding probability distribution. Observe that each belief measure Bel is characterized by $Bel(\emptyset) = 0$, $Bel(X) = 1$ and by ∞–monotonicity, *i.e.*, $Bel : 2^X \to [0,1]$ is a belief measure if and only if for any $k \in \mathbb{N}$, $A_1, \dots A_k \subseteq X$, it holds

$$Bel\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset\neq I\subseteq\{1,\dots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i\in I} A_i\right). \tag{8}$$

For a fixed belief measure Bel on $(X, \mathcal{A})$, the corresponding probability distribution m is just the Möbius transform of Bel, $m = m_{Bel}$, given by

$$m_{Bel}(A) = \sum_{B\subseteq A} (-1)^{|A\setminus B|} Bel(B), \tag{9}$$

which is necessarily non–negative and then $Bel = Bel_{m_{Bel}}$. For more details we recommend [3], [7], [8].

Note that transformations (6) (also called zeta transformation) and (9) are, in fact, inverse transformations of set functions. Observe also that we can rewrite (8) into

$$Bel\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\emptyset\neq I\subseteq\{1,\dots,k\}} (-1)^{|I|} Bel\left(\bigcap_{i\in I} A_i\right) \geq 0 \tag{10}$$

and then, for any $\emptyset \neq A \subseteq X$, putting $k = |A|$ and choosing subsets $A_1, \dots, A_k$ of A such that $|A_1| = \dots = |A_k| = k - 1$, $\bigcup_i A_i = A$ and $\bigcap_i A_i = \emptyset$, one gets on the left-hand side of (10) exactly the formula for $m_{Bel}(A)$.

The above summarized facts bring a new characterization of de Finetti's lower and upper probabilities.

Proposition 2. *A set function $\underline{Q} : 2^X \to [0,1]$ is a lower probability in de Finetti's sense* (2) *if and only if $m_{\underline{Q}} : 2^X \to \mathbb{R}$ given by* (9), *i.e., $m_{\underline{Q}}(A) = \sum_{B\subseteq A} (-1)^{|A\setminus B|} \underline{Q}(B)$ is non–negative and $\{A \subseteq X \mid m_{\underline{Q}}(A) > 0\}$ is a subpartition of X.*

Example 1. For $X = \{1,2\}$ let $H : 2^X \to [0,1]$ be given by

$$H(A) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset. \end{cases}$$

Then $m_H(\emptyset) = 0$, $m_H(\{1\}) = m_H(\{2\}) = 1$ and $m_H(X) = -1$. Though $\{A \subseteq X \,|\, m_H(A) > 0\} = \{\{1\},\{2\}\}$ is a partition of X, H is not a belief measure and thus neither a lower probability of Finetti. Moreover, $G : 2^X \to [0,1]$ given by

$$G(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \dfrac{1}{3} & \text{if } |A| = 1 \\ 1 & \text{if } A = X, \end{cases}$$

fulfils $m_G(\emptyset) = 0$, $m_G(\{1\}) = m_G(\{2\}) = m_G(X) = \frac{1}{3}$, *i.e.*, G is a belief measure, $G = Bel_{m_G}$. However, $\{A \subseteq X \,|\, m_G(A) > 0\} = \{\{1\},\{2\},\{1,2\}\}$ is not a subpartition of X and thus G is not a lower probability of de Finetti.

4 – k–additive belief measures

The additivity of a set function G on $(X, \mathcal{A})$, *i.e.*, the property $G(A \cup B) = G(A) + G(B)$ for all A, $B \in \mathcal{A}$ such that $A \cap B = \emptyset$, has an interesting impact on the corresponding Möbius transform m_G of G (see (9)). Namely, then for each A with $|A| > 1$ it holds $m_G(A) = 0$. Therefore, no lower (upper) probability $\underline{Q}$ ($\overline{Q}$) can be additive, up to the trivial case when $Q(B_i) > 0$ implies $|B_i| = 1$ and thus $\mathcal{P}_Q$ is a singleton, $\underline{Q} = \overline{Q}$.

Grabisch [4] has proposed the next generalization of additivity.

Definition 2. Let $k \in \mathbb{N}$. A set function $G : 2^X \to \mathbb{R}$, $G(\emptyset) = 0$, is k–additive if $m_G(A) = 0$ whenever $|A| > k$. Moreover, if $m_G(B) \neq 0$ for some B with $|B| = k$, then G is called a pure k–additive set function.

Observe that Mesiar [5] has proposed an alternative approach to k–additivity applicable to any measurable space $(X, \mathcal{A})$, which coincides with the Grabisch approach whenever the later is applicable, *i.e.*, if X is finite. Namely, k–additivity of a set function $G : \mathcal{A} \to \mathbb{R}$, $G(\emptyset) = 0$, means that there is an additive set function $F : \mathcal{A}^k \to \mathbb{R}$ (here the product space $(X, \mathcal{A})^k$ is denoted by $(X^k, \mathcal{A}^k)$) such that for all $A \in \mathcal{A}$, $G(A) = F(A^k)$.

For de Finetti's lower probabilities we have the next interesting result.

Proposition 3. *For a given probability Q on $(X, \mathcal{B})$ with X finite and $\mathcal{B} = \sigma(B_1, \ldots, B_m)$, the set function $\underline{Q} : 2^X \to [0,1]$ given by* (2) *(i.e., the lower probability in the sense of de Finetti) is a pure k–additive belief measure with $k = \max\{|B_i| \,|\, Q(B_i) > 0\}$. Moreover, $\overline{Q} : 2^X \to [0,1]$ given by* (3) *is a k–additive plausibility measure.*

Proof. It is enough to define a probability measure F on X^k which is a convex combination of Dirac measures,

$$F = \sum_{i=1}^{m} Q(B_i)\delta_{\boldsymbol{x}_i},$$

where $\boldsymbol{x}_i = (x_{i1}, \dots, x_{ik})$ is some element of X^k such that $\{x_{i1}, \dots, x_{ik}\} = B_i$, $i = 1, \dots, m$. Then $\underline{Q}(A) = F(A^k) = \sum_{B_i \subseteq A} Q(B_i)$. Moreover, if $|B_j| = k$ and $Q(B_j) > 0$ then $m_{\underline{Q}}(B_j) = Q(B_j) > 0$ and due to Mesiar, see [5], $\underline{Q}$ is pure k–additive.

The duality of $\overline{Q}$ and $\underline{Q}$ ensures the k–additivity of $\overline{Q}$, see [5].

5 – Concluding remarks

After recalling the concepts of lower and upper probabilities of de Finetti and belief and plausibility measures from the Dempster–Shafer theory, we have clarified their relationship. Moreover, we have shown the order of additivity k of lower probability $\underline{Q}$ (and upper probability $\overline{Q}$) to be the maximal cardinality of a non-null atom in $\mathcal{B}$.

As an interesting problem in de Finetti's spirit we turn the attention of the readers to the following problems (formulated for dimension two only, the generalization to higher dimensions is obvious).

Open problems.

(i) Let $(X_i, \mathcal{B}_i, Q_i)$, $i = 1, 2$, be two finite probability spaces. Characterize the following two set functions defined on the product space $(X, 2^X)$, $X = X_1 \times X_2$:

$$Q_*(A) = \inf \big\{ P(A) \,|\, P \text{ is a probability on } (X, 2^X) \\ \forall\, B_{1i} \in \mathcal{B}_1,\, B_{2j} \in \mathcal{B}_2 \quad P(X_1 \times B_{2j}) = Q_2(B_{2j}) \\ \text{and} \quad P(B_{1i} \times X_2) = Q_1(B_{1i}) \big\}\,; \tag{11}$$

$$Q^*(A) = \sup \big\{ P(A) \,|\, P \text{ is a probability on } (X, 2^X) \\ \forall\, B_{1i} \in \mathcal{B}_1,\, B_{2j} \in \mathcal{B}_2 \quad P(X_1 \times B_{2j}) = Q_2(B_{2j}) \\ \text{and} \quad P(B_{1i} \times X_2) = Q_1(B_{1i}) \big\}\,. \tag{12}$$

Note that as in the case of the set functions given by (2) and (3), also the functions Q_* and Q^* given by (11) and (12), respectively, are dual, $Q_*^d = Q^*$.

(ii) Recall also that the marginals of Q_* and Q^* are just 1–dimensional lower and upper probabilities $\underline{Q}_1$ and $\underline{Q}_2$, resp. $\overline{Q}_1$ and $\overline{Q}_2$. In the case of determining joint probability measure P when marginal probabilities P_1 and

P_2 are known, copulas [9], [6] are of use. Our second open problem can be formulated in this spirit, asking for identification of the way how from marginal lower (upper) probabilities to determine the joint lower (upper) probability (compare the concept of semicopulas introduced by Bassan and Spizzichino [1]).

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