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Schauder estimates for a system of equations of mixed type

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Dedicated to Umberto Mosco

Abstract: *We study a linear problem of mixed type and we prove existence, uniqueness results and coercive estimates in Hölder spaces. Moreover we establish weighted estimates in Hölder spaces and a stability result for a non-linear system of mixed type.*

0 – Introduction

We consider the initial-boundary value nonlinear problem

$$
\begin{aligned}
&u_t + \mathcal{A}\left(x, \frac{\partial}{\partial x}\right) u + \mathcal{R}(u) = f, \quad x \in \Omega \subset \mathbb{R}^n,\ t \in (0,T) \\
&B\left(x, \frac{\partial}{\partial x}\right) v\Big|_{x\in\partial\Omega} = 0, \quad u(x,0) = u_0(x),
\end{aligned}
\tag{0.1}
$$

in a bounded, smooth, n-dimensional domain Ω, for the vector field $u = (v, w)$, $v = (v_1, \dots, v_{m_1})$, $w = (w_1, \dots, w_{m_2})$, $m_1,\, m_2 \in \mathbb{N},\ m_1 + m_2 = m$.

The data of the problem, $f(x,t)$ and $u_0(x)$, have a similar structure: $f = (g, h)$, $g = (g_1, \dots, g_{m_1})$, $h = (h_1, \dots, h_{m_2})$, $u_0 = (v_0, w_0)$, $v_0 = (v_{01}, \dots, v_{0m_1})$,

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$w_0 = (w_{01}, \dots, w_{0m_2})$. Here $\mathcal{A}$ is a matrix differential operator of the form

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}\left(x, \dfrac{\partial}{\partial x}\right) & \ell_1(x) \\ \ell_2(x) & \ell(x) \end{pmatrix} \tag{0.2}$$

where $\ell_1(x)$, $\ell_2(x)$, $\ell(x)$ are $m_1 \times m_2$, $m_2 \times m_1$, $m_2 \times m_2$ matrices, respectively, $\mathcal{L}(x, \frac{\partial}{\partial x}) = (\mathcal{L}_{ij}(x, \frac{\partial}{\partial x}))_{i,j=1,\dots,m_1}$ is an $m_1 \times m_1$ matrix second order elliptic operator, and $\mathcal{R}(u) = (\mathcal{P}(u), \mathcal{Q}(u))$ is a vector field of non-linear terms. We assume that $\mathcal{P} = (\mathcal{P}_i(v, \nabla v, D^2 v, w))_{i=1,\dots,m_1}$, $\mathcal{Q} = (\mathcal{Q}_k(v, w))_{k=1,\dots,m_2}$ and $\mathcal{P}_i$, $\mathcal{Q}_k$ are polynomials of degree $h \geq 2$ with respect to their arguments.

By $B(x, \frac{\partial}{\partial x}) = (B_{ij}(x, \frac{\partial}{\partial x}))_{i,j=1,\dots,m_1}$ we mean an $m_1 \times m_1$ matrix operator of the boundary conditions. For simplicity we assume that the B_{ij} are first order operators and that the coefficients of the operators $\mathcal{A}$ and B are sufficiently regular. The initial data should satisfy the compatibility conditions (see (1.2)).

This type of problems comes from biological applications (see [11], Chapter 13, [12], Chapter 1, Chapter 13 and [9]) and from ecological applications as studies of forestry ecosystems (see [3], [4], [5], [8] and [15]). Moreover similar systems have been used to treat various problems in physics (see [6], and [13]).

A simpler version of problem (0.1) is introduced in [12], Chapter 13, as a model of rabies epidemics. The population consists of two types of foxes: infective ones $I(x, t)$ and susceptibles $S(x, t)$, which interact with each other. The vector $u = (v, w)$ is the couple (I, S) and the system (0.1) reads

$$\begin{cases} I_t - D\triangle I - rIS + aI = 0, \\ S_t + rIS = 0, \end{cases} \tag{0.3}$$

where D, r, a are (positive) constants with biological meaning: $1/a$ is the life expectance of an infective fox, r is a measure of transmission efficiency of the disease from infectives to susceptibles, D is the diffusion coefficient of infected foxes. Finally, the initial data I_0, S_0 and zero Neumann boundary condition on the infected foxes are given (*i.e.*, the migration of cubs seeking their own territory is excluded).

Coercive Schauder type estimates for linear parabolic systems have been first estabished by Solonnikov in [14] and extended by Belonosov in [1] to weighted Hölder spaces. Stability results have been obtained for a large class of non-linear parabolic systems, by Belonosov and Višnevskiǐ ([2]): they also follow from an abstract approach of Henry ([7]).

Recently, Mulone and Solonnikov ([10]) have studied stability and instability of a stationary solution for problem (0.1) (in the case of more general boundary conditions) and proved a linearization principle in Sobolev-Slobodetskii spaces with an exponential weight.

Their paper ([10]) concerns the problem in two spatial dimensions and a large class of non-linear operators $\mathcal{P}$ and $\mathcal{Q}$. In the present paper we study problem (0.1) in n spatial dimensions and for a larger class of non-linear operators *i.e.*, we admit a fully non-linear operator $\mathcal{P}$ (see Theorem 3.1).

First we study a linear problem of mixed type (see (2.1)), and we prove an existence, uniqueness result and coercive estimates in the Hölder spaces for the solution (see Theorem 2.1). Assuming suitable conditions on the spectrum of the operator A and on the eigenvalues of "$-\ell$" (see (2.2)) we establish weighted estimates in the Hölder spaces (see Theorem 2.2). These estimates, that we think are interesting in themselves, are "crucial" for proving the stability result for the non-linear problem (0.1) (see Theorem 3.1).

1 – Preliminaries

We consider the linear initial-boundary value problem

$$\begin{aligned} &v_t + L\left(x, \frac{\partial}{\partial x}\right) v = g, \quad x \in \Omega \subset \mathbb{R}^n,\ t \in (0,T) \\ &B\left(x, \frac{\partial}{\partial x}\right) v\Big|_{x\in\partial\Omega} = 0, \quad v(x,0) = v_0(x) \end{aligned} \tag{1.1}$$

in a bounded n-dimensional domain Ω, for the vector field $v = (v_1, \dots, v_{m_1})$, $m_1 \in \mathbb{N}$. The data of the problem, $g(x,t)$ and $v_0(x)$, have a similar structure: $g = (g_1, \dots, g_{m_1})$, $v_0 = (v_{01}, \dots, v_{0m_1})$; $L(x, \frac{\partial}{\partial x}) = (L_{ij}(x, \frac{\partial}{\partial x}))_{i,j=1,\dots,m_1}$ is an $m_1 \times m_1$ matrix second order elliptic operator.

By $B(x, \frac{\partial}{\partial x}) = (B_{ij}(x, \frac{\partial}{\partial x}))_{i,j=1,\dots,m_1}$ we mean the $m_1 \times m_1$ matrix operator of the boundary conditions. For simplicity we assume that the B_{ij} are first order operators. The coefficients of the operators L and B are sufficiently regular.

From now on we assume that problem (1.1) is a well-posed parabolic problem, *i.e.* the operator $\frac{\partial}{\partial t} + L$ is parabolic in the sense of Petrovskii and the operators $(\frac{\partial}{\partial t} + L, B)$ satisfy the Lopatinskii condition (see [14]).

In particular, we can have

$$\begin{aligned} L\left(x, \frac{\partial}{\partial x}\right) &= -\mathrm{diag}(D_1, \dots, D_m)\Delta,\ D_i > 0, \\ B\left(x, \frac{\partial}{\partial x}\right) &= I\frac{\partial}{\partial n} \qquad \text{(the Neumann condition).} \end{aligned}$$

We introduce standard Hölder spaces $C^\alpha(\overline{\Omega})$, $C^{\alpha,\alpha/2}(\overline{Q_T})$, where $Q_T = \Omega \times (0,T)$, and also the spaces $C^{\alpha,0}(\overline{Q_T})$, $C^{0,\alpha/2}(\overline{Q_T})$ with the norms

$$\begin{aligned} \|u\|_{C^{\alpha,0}(\overline{Q_T})} &= \sup_{t<T} \|u(\cdot,t)\|_{C^\alpha(\overline{\Omega})}, \\ \|u\|_{C^{0,\alpha/2}(\overline{Q_T})} &= \sup_{\overline{\Omega}} \|u(x,\cdot)\|_{C^{\alpha/2}[0,T]}, \end{aligned}$$

moreover, we define the weighted spaces $C_b^{\alpha,\alpha/2}(\overline{Q_T})$, $C_b^{\alpha,0}(\overline{Q_T})$, $C_b^{0,\alpha/2}(\overline{Q_T})$, $b>0$, with the norms

$$\begin{aligned}\|u\|_{C_b^{\alpha,\alpha/2}(\overline{Q_T})} &= \|e^{bt}u\|_{C^{\alpha,\alpha/2}(\overline{Q_T})},\\ \|u\|_{C_b^{\alpha,0}(\overline{Q_T})} &= \|e^{bt}u\|_{C^{\alpha,0}(\overline{Q_T})},\\ \|u\|_{C_b^{0,\alpha/2}(\overline{Q_T})} &= \|e^{bt}u\|_{C^{0,\alpha/2}(\overline{Q_T})}.\end{aligned}$$

In the proof of our results we will use the following Proposition (see [14]).

PROPOSITION 1.1. *If the coefficients of $L\in C^{\alpha}(\overline{\Omega})$, the coefficients of $B\in C^{\alpha+1}(\partial\Omega)$, $g\in C^{\alpha,\alpha/2}(\overline{Q_T})$, $v_0\in C^{\alpha+2}(\overline{\Omega})$, and the compatibilitity condition*

$$B\left(x,\frac{\partial}{\partial x}\right)v_0|_S=0 \tag{1.2}$$

is satisfied, then problem (1.1) *has a unique solution $v\in C^{2+\alpha,1+\alpha/2}(\overline{Q_T})$, and*

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_T})}\le c(T)\Big(\|g\|_{C^{\alpha,\alpha/2}(\overline{Q_T})}+\|v_0\|_{C^{\alpha+2}(\overline{\Omega})}\Big). \tag{1.3}$$

Moreover $v(x,t)$ satisfies the inequality

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_T})}\le c_1\Big(\|g\|_{C^{\alpha,\alpha/2}(\overline{Q_T})}+\|v_0\|_{C^{\alpha+2}(\overline{\Omega})}\Big)+c_2\sup_{\overline{Q_T}}|v(x,t)| \tag{1.4}$$

where the constants c_1 and c_2 are independent of T.

2 – The linear problem

In this section our objective is to establish existence, uniqueness results and coercive Schauder estimates for the solution of problem (2.1). These estimates, which in our opinion are of interest in themselves, are "crucial" for proving the stability result for the non linear problem (0.1) (see Theorem 3.1).

We consider the linear initial-boundary value problem

$$\begin{aligned}&u_t+A\left(x,\frac{\partial}{\partial x}\right)u=f,\quad x\in\Omega\subset\mathbb{R}^n,\ t\in(0,T)\\ &B\left(x,\frac{\partial}{\partial x}\right)v\Big|_{x\in\partial\Omega}=0,\quad u(x,0)=u_0(x)\end{aligned} \tag{2.1}$$

in a bounded n-dimensional domain Ω, for the vector field $u=(v,w)$, $v=(v_1,\dots,v_{m_1})$, $w=(w_1,\dots,w_{m_2})$, $m_1,m_2\in\mathbb{N}$, $m_1+m_2=m$. The data of the problem, $f(x,t)$ and $u_0(x)$, have a similar structure: $f=(g,h)$, $g=$

$(g_1, \ldots, g_{m_1})$, $h = (h_1, \ldots, h_{m_2})$, $u_0 = (v_0, w_0)$, $v_0 = (v_{01}, \ldots, v_{0m_1})$, $w_0 = (w_{01}, \ldots, w_{0m_2})$.

By A we mean a matrix differential operator of the form

$$A = \begin{pmatrix} L\left(x, \dfrac{\partial}{\partial x}\right) & \ell_1(x) \\ \ell_2(x) & \ell(x) \end{pmatrix} \tag{2.2}$$

where $\ell_1(x)$, $\ell_2(x)$, $\ell(x)$ are $m_1 \times m_2$, $m_2 \times m_1$, $m_2 \times m_2$ matrices, respectively, and $L(x, \frac{\partial}{\partial x}) = (L_{ij}(x, \frac{\partial}{\partial x}))_{i,j=1,\ldots,m_1}$ is an $m_1 \times m_1$ matrix second order elliptic operator. By $B(x, \frac{\partial}{\partial x}) = (B_{ij}(x, \frac{\partial}{\partial x}))_{i,j=1,\ldots,m_1}$ we mean the $m_1 \times m_1$ matrix operator of the boundary conditions. For simplicity we assume that B_{ij} are the first order operators. The coefficients of the operators A and B are sufficiently regular.

We recall that problem (1.1) is a well-posed parabolic problem, *i.e.* the operator $\frac{\partial}{\partial t} + L$ is parabolic in the sense of Petrovskii and the operators $(\frac{\partial}{\partial t} + L, B)$ satisfy the Lopatinskii condition (see [14]).

Our first result is

Theorem 2.1. *If the coefficients of $A \in C^{\alpha}(\overline{\Omega})$, the coefficients of $B \in C^{\alpha+1}(\partial\Omega)$, $v_0 \in C^{2+\alpha}(\overline{\Omega})$, $w_0 \in C^{\alpha}(\overline{\Omega})$, $g \in C^{\alpha,\alpha/2}(\overline{Q_T})$, $h \in C^{\alpha,0}(\overline{Q_T})$, $\alpha \in (0,1)$, and if compatibility condition (1.2) is satisfied, then problem (2.1) has a unique solution $u = (v, w)$ with $v \in C^{2+\alpha,1+\alpha/2}(\overline{Q_T})$, $w, w_t \in C^{\alpha,0}(\overline{Q_T})$, and the following estimate holds*:

$$\begin{aligned} &\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_T})} + \|w\|_{C^{\alpha,0}(\overline{Q_T})} + \|w_t\|_{C^{\alpha,0}(\overline{Q_T})} \leq \\ &\leq c(T)\Big(\|g\|_{C^{\alpha,\alpha/2}(\overline{Q_T})} + \|h\|_{C^{\alpha,0}(\overline{Q_T})} + \|v_0\|_{C^{\alpha+2}(\overline{\Omega})} + \|w_0\|_{C^{\alpha}(\overline{\Omega})}\Big). \end{aligned} \tag{2.3}$$

The proof of Theorem 2.1 is based on the analysis of the Cauchy problem

$$w_t(x,t) + \ell(x)w(x,t) = \varphi(x,t), \quad w|_{t=0} = w_0(x), \quad x \in \Omega, \ t \in (0,T) \tag{2.4}$$

and of the parabolic initial-boundary value problem (1.1).

We start by proving the following proposition

Proposition 2.1. *If $w_0 \in C^{\alpha}(\overline{\Omega})$, $\ell \in C^{\alpha}(\overline{\Omega})$ and $\varphi \in C^{\alpha,0}(\overline{Q_T})$, then problem (2.4) has a unique solution $w \in C^{\alpha,0}(\overline{Q_T})$, possessing time derivative $w_t \in C^{\alpha,0}(\overline{Q_T})$, and satisfying the inequality*

$$\|w\|_{C^{\alpha,0}(\overline{Q_T})} + \|w_t\|_{C^{\alpha,0}(\overline{Q_T})} \leq c(T)\Big(\|w_0\|_{C^{\alpha}(\overline{\Omega})} + \|\varphi\|_{C^{\alpha,0}(\overline{Q_T})}\Big). \tag{2.5}$$

Proof. It is well known that the solution of problem (2.4) is given by the formula

$$w(x,t) = V(t)w_0(x) + \int_0^t V(t-\tau)\varphi(x,\tau)d\tau, \tag{2.6}$$

where $V(t) = e^{-t\ell}$ is the resolving operator of problem (2.4). It can be expressed by means of the contour integral:

$$V(t)w_0(x) = \frac{1}{2\pi i}\int_\Gamma e^{\sigma t}(\ell(x)+\sigma I)^{-1}w_0(x)d\sigma \equiv w'(x,t), \tag{2.7}$$

where Γ is a simple closed contour enclosing the set $\mathcal{S}$ of all the eigenvalues of $-\ell(x)$ for all $x \in \Omega$. Indeed,

$$w'_t(x,t) + \ell(x)w'(x,t) = \frac{1}{2\pi i}\int_\Gamma e^{\sigma t}w_0(x)d\sigma = 0$$

and

$$w'(x,0) - w_0(x) = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_n}((\ell(x)+\sigma I)^{-1} - \sigma^{-1}I)w_0(x)d\sigma = 0,$$

where Γ_n is a sequence of the contours expanding towards infinity. Estimate (2.5) follows easily from (2.6) and (2.7). Proposition 2.1 is proved, let us note that in general, $c(T)$ grows exponentially, as $T \to +\infty$.

Proof of Theorem 2.1. We start by the reduction of the problem (2.1) to the problem (1.1) with an additional non-local lower order term in the system of equations for v. We consider w as a solution of (2.4) with $\varphi = h - \ell_2(x)v$, which yields the following expression for w:

$$w(x,t) = V(t)w_0(x) + \int_0^t V(t-\tau)h(x,\tau)d\tau - \int_0^t V(t-\tau)\ell_2(x)v(x,\tau)d\tau.$$

When we plug this expression into the first m_1 equations of the system (2.1), we obtain the initial-boundary value problem for v:

$$\begin{aligned} &v_t(x,t) + L\left(x,\frac{\partial}{\partial x}\right)v(x,t) - \ell_1(x)\int_0^t V(t-\tau)\ell_2(x)v(x,\tau)d\tau = \\ &= d(x,t), \quad (x,t)\in Q_T, \\ &B\left(x,\frac{\partial}{\partial x}\right)v\Big|_{x\in\partial\Omega} = 0, \;\; v(x,0) = v_0(x), \end{aligned} \tag{2.8}$$

where

$$d(x,t) \equiv g(x,t) - \ell_1(x)\left(V(t)w_0(x) + \int_0^t V(t-\tau)h(x,\tau)d\tau\right). \tag{2.9}$$

Let us show that $d \in C^{\alpha,\alpha/2}(\overline{Q_T})$. By (2.5), we have

$$\begin{aligned}
\|d\|_{C^{\alpha,0}(\overline{Q_T})} &\le \|g\|_{C^{\alpha,0}(\overline{Q_T})} + c\Big(\|w_0\|_{C^\alpha(\overline{\Omega})} + \|h\|_{C^{\alpha,0}(\overline{Q_T})}\Big),\\
\|d\|_{C^{0,\alpha/2}(\overline{Q_T})} &\le \|g\|_{C^{0,\alpha/2}(\overline{Q_T})} +\\
&\qquad + c\left\|V(t)w_0(x) + \int_0^t V(t-\tau)h(x,\tau)d\tau\right\|_{C^{0,1}\overline{Q_T})} \le\\
&\le \|g\|_{C^{0,\alpha/2}(\overline{Q_T})} + c\left(\sup_{\overline{\Omega}}|w_0(x)| + \sup_{\overline{Q_T}}|h(x,t)|\right).
\end{aligned}$$

Hence

$$\|d\|_{C^{\alpha,\alpha/2}(\overline{Q_T})} \le \|g\|_{C^{\alpha,\alpha/2}(\overline{Q_T})} + c\Big(\|w_0\|_{C^\alpha(\overline{\Omega})} + \|h\|_{C^{\alpha,0}(\overline{Q_T})}\Big). \tag{2.10}$$

The problem (2.8) differs from (1.1) by the presence of the integral operator of the Volterra type

$$\mathcal{I}v = -\ell_1(x)\int_0^t V(t-\tau)\ell_2(x)v(x,\tau)d\tau$$

in the system of equations. Let us estimate the norm $\|\mathcal{I}v\|_{C^{\alpha,\alpha/2}(\overline{Q_t})}$, for any $t \leqslant T$. We have

$$\begin{aligned}
\|\mathcal{I}v\|_{C^{\alpha,0}(\overline{Q_t})} &\le c(t)\|v\|_{C^{\alpha,0}(\overline{Q_t})},\\
\|\mathcal{I}v\|_{C^{0,\alpha/2}(\overline{Q_t})} &\le c(T)\|\mathcal{I}v\|_{C^{0,1}(\overline{Q_t})} \le c(T)\sup_{\overline{Q_t}}|v(x,\tau)|,
\end{aligned}$$

hence

$$\|\mathcal{I}v\|_{C^{\alpha,\alpha/2}(\overline{Q_t})} \le c(T)\|v\|_{C^{\alpha,0}(\overline{Q_t})} \tag{2.11}$$

and also

$$\|\mathcal{I}v\|_{C^{\alpha,\alpha/2}(\overline{Q_t})} \le \epsilon\|v\|_{C^{2+\alpha,0}(\overline{Q_t})} + c(\epsilon,T)\sup_{\overline{Q_t}}|v(x,\tau)| \tag{2.12}$$

for arbitrarily small $\epsilon > 0$. This inequality allows one to prove the solvability of problem (2.8) by successive approximations, using the Gronwall lemma, according to the following scheme.

We define v_1 as the solution of the problem

$$\begin{aligned} &\frac{\partial v_1}{\partial t} + L(x, \frac{\partial}{\partial x}) v_1 = d(x,t), \quad x \in \Omega,\ t \in (0,T),\\ &B(x, \frac{\partial}{\partial x}) v_1|_{x \in \partial\Omega} = 0, \qquad v_1(x,0) = v_0(x) \end{aligned}$$

and for $m \geqslant 1$

$$\begin{aligned} &\frac{\partial v_{m+1}}{\partial t} + L\left(x, \frac{\partial}{\partial x}\right) v_{m+1} = \ell_1(x) \int_0^t V(t-\tau)\ell_2(x) v_m(x,\tau) d\tau + d(x,t),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega,\ t \in (0,T),\\ &B\left(x, \frac{\partial}{\partial x}\right) v_{m+1}|_{x \in \partial\Omega} = 0, \quad v_{m+1}(x,0) = v_0(x). \end{aligned}$$

Set

$$\xi_{m+1} = v_{m+1} - v_m, \ \ m \geqslant 1$$

and

$$\xi_1 = v_1.$$

The function ξ_{m+1} is a solution of the problem

$$\begin{aligned} &\frac{\partial \xi_{m+1}}{\partial t} + L\left(x, \frac{\partial}{\partial x}\right) \xi_{m+1} = -\mathcal{I}\xi_m, \quad x \in \Omega,\ t \in (0,T),\\ &B\left(x, \frac{\partial}{\partial x}\right) \xi_{m+1}|_{x \in \partial\Omega} = 0, \qquad \xi_{m+1}(x,0) = 0. \end{aligned}$$

From Proposition 1.1 (see (1.3)) and (2.12) we obtain

$$\|\xi_{m+1}\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_t})} \leq \epsilon \|\xi_m\|_{C^{2+\alpha,0}(\overline{Q_t})} + c(\epsilon,T) \sup_{\overline{Q_t}} |\xi_m(x,\tau)|, \ \ m \geqslant 1,$$

which implies

$$\begin{aligned} \sum_{m=0}^{M} \|\xi_{m+1}\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_t})} \leqslant \varepsilon \sum_{m=1}^{M} \|\xi_m\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_t})} + \|\xi_1\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_t})} +\\ + c(\varepsilon,T) \int_0^t \sum_{m=1}^{M} \|\xi_m\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_\tau})} d\tau. \end{aligned}$$

Using the Gronwall lemma, we obtain

$$\sum_{m=1}^{M} \|\xi_m\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_t})} \leqslant c(\varepsilon,T) \|\xi_1\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_t})}.$$

We deduce that the sequence $\{v_m\}$ converges in the Hölder space $C^{\alpha+2,\alpha/2+1}(\overline{Q_T})$ to a solution of problem (2.8). Consequently also problem (2.1) admits a solution. The uniqueness of the solution follows from inequalities (1.3) and (2.12) applied to the difference of two solutions v and v' of (2.8). These inequalities yield

$$\|v - v'\|_{C^{\alpha+2,\alpha/2+1}(\overline{Q_t})} \leq c(\varepsilon, T)t\|v - v'\|_{C^{\alpha+2,\alpha/2+1}(\overline{Q_t})},$$

hence there exists a positive time T_0 such that $v = v'$ for $t < T_0$. In a finite number of steps we prove that problem (2.8) (and then also problem (2.1)) admits one and only one solution in the space $C^{\alpha+2,\alpha/2+1}(\overline{Q_T})$ for any fixed time T. Moreover, the estimate

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q_t})} \leq c(T)\Big(\|d\|_{C^{\alpha,\alpha/2}(\overline{Q_t})} + \|v_0\|_{C^{2+\alpha}(\overline{\Omega})}\Big) \tag{2.13}$$

of the solution is obtained. Inequality (2.3) follows from estimates (2.13), (2.10) and (2.5). Theorem 2.1 is proved.

Our second result is w

Theorem 2.2. *Assume all the hypotheses of Theorem* 2.1. *Moreover, suppose that the following conditions are satisfied*

(a) *the eigenvalues of* $-\ell$, *i.e. the solutions of the equation* $\det(\ell(x) + \sigma I) = 0$ *have negative real parts for all* $x \in \Omega$;

(b) *the spectrum of the operator* A *defined in the space of functions* $u = (v, w)$ *satisfying the boundary conditions* $Bv|_{x\in\partial\Omega} = 0$ *is located in the left half plane* $\mathrm{Re}\lambda < 0$.

Then the solution $u = (v, w)$ *of problem* (2.1) *is such that* $v \in C_b^{2+\alpha,1+\alpha/2}(\overline{Q_T})$, $w \in C_b^{\alpha,0}(\overline{Q_T})$, $w_t \in C_b^{\alpha,0}(\overline{Q_T})$, *and*

$$\begin{aligned} &\|v\|_{C_b^{2+\alpha,1+\alpha/2}(\overline{Q_T})} + \|w\|_{C_b^{\alpha,0}(\overline{Q_T})} + \|w_t\|_{C_b^{\alpha}(\overline{Q_T})} \leq \\ &\leq c\Big(\|g\|_{C_b^{\alpha,\alpha/2}(\overline{Q_T})} + \|h\|_{C_b^{\alpha,0}(\overline{Q_T})} + \|v_0\|_{C^{2+\alpha}(\overline{\Omega})} + \|w_0\|_{C^{\alpha}(\overline{\Omega})}\Big) \end{aligned} \tag{2.14}$$

with the constant c independent of T. *Here* $T \leq +\infty$ *and b is a suitable positive number.*

Proof. In [10] it is shown that (assuming conditions (a) and (b)) the solution of the problem (2.1) with $f = 0$ satisfies the inequality

$$\|u(\cdot, t)\|_{L_2(\Omega)} \leq ce^{-\beta_0 t}\|u_0\|_{L_2(\Omega)}$$

with a suitable $\beta_0 > 0$ independent of T. This inequality is equivalent to

$$\|e^{-tA}\|_{L_2(\Omega)\to L_2(\Omega)} \leq ce^{-\beta_0 t}, \tag{2.15}$$

where A is the operator (2.2) defined on the set of vector fields $u = (v, w)$ such that $v \in W_2^2(\Omega)$, $w \in L_2(\Omega)$ and $Bv|_{\partial\Omega} = 0$.

It follows that the solution of a non-homogeneous problem (2.1), that is given by the formula

$$u = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} f(\tau)d\tau,$$

satisfies the inequality

$$\|e^{\beta t}u(\cdot,t)\|_{L_2(\Omega)} \le c\Big(\|u_0\|_{L_2(\Omega)} + \sup_{\tau<t}\|e^{\beta\tau}f(\cdot,\tau)\|_{L_2(\Omega)}\Big) \tag{2.16}$$

with $\beta \in (0, \beta_0)$ and with the constant c independent of T (see also [7]).

We will prove (2.14) using this inequality. We go back to the formula $w = V(t)w_0$ for the solution of the problem (2.4) with $\varphi = 0$. If all the eigenvalues of the matrix $-\ell(x)$ are located in the left half-plane of the complex plane $\mathbb{C}$, then the contour Γ can be drawn in the same half-plane, and in this case

$$\|w(\cdot,t)\|_{C^\alpha(\overline{\Omega})} \le ce^{-\beta_1 t}\|w_0\|_{C^\alpha(\overline{\Omega})}$$

with a suitable $\beta_1 > 0$. For the solution of the non-homogeneous problem (2.4) we obtain

$$\|w\|_{C_\beta^{\alpha,0}(\overline{Q_T})} + \|w_t\|_{C_\beta^{\alpha,0}(\overline{Q_T})} \le c\Big(\|w_0\|_{C^\alpha(\overline{\Omega})} + \|\varphi\|_{C_\beta^{\alpha,0}(\overline{Q_T})}\Big), \tag{2.17}$$

where $\beta \in (0, \beta_1)$ and c is independent of T.

Now we choose $b \in (0, min\{\beta_0, \beta_1\})$. Taking into account (2.17), we prove the weighted estimates for d defined in (2.9), namely,

$$\begin{aligned}
\|d\|_{C_b^{\alpha,0}(\overline{Q_T})} &\le \|g\|_{C_b^{\alpha,0}(\overline{Q_T})} + c\Big(\|w_0\|_{C^\alpha(\overline{\Omega})} + \|h\|_{C_b^{\alpha,0}(\overline{Q_T})}\Big),\\
\|d\|_{C_b^{0,\alpha/2}(\overline{Q_T})} &\le \|g\|_{C_b^{0,\alpha/2}(\overline{Q_T})} + c\Big(\|w_0\|_{C^\alpha(\overline{\Omega})} + \|h\|_{C_b^{\alpha,0}(\overline{Q_T})}\Big),\\
\|d\|_{C_b^{\alpha,\alpha/2}(\overline{Q_T})} &\le \|g\|_{C_b^{\alpha,\alpha/2}(\overline{Q_T})} + c\Big(\|w_0\|_{C^\alpha(\overline{\Omega})} + \|h\|_{C_b^{\alpha,0}(\overline{Q_T})}\Big),
\end{aligned}$$

with the constant c independent of T.

Next, we apply Proposition 1.1 (see inequality (1.4)) to the solution $v_b(x,t) = v(x,t)e^{bt}$ of the problem

$$\begin{aligned}
&\frac{\partial v_b}{\partial t} + L\left(x, \frac{\partial}{\partial x}\right)v_b + e^{tb}\mathcal{I}v = e^{tb}d + bv_b,\\
&B\left(x, \frac{\partial}{\partial x}\right)v_b|_{x\in\partial\Omega} = 0, \;\; v_b(x,0) = v_0(x).
\end{aligned}$$

Proceeding as previously and using assumption (a), we prove that

$$\|e^{tb}\mathcal{I}v\|_{C^{\alpha,\alpha/2}(\overline{Q_T})} \le c\|v\|_{C_b^{\alpha,0}(\overline{Q_T})} \tag{2.18}$$

with the constant c independent of T. Hence we obtain (see (1.4))

$$\begin{aligned}
&\|v\|_{C_b^{2+\alpha,1+\alpha/2}(\overline{Q_T})} \le \\
&\le c_1\Big(\|w_0\|_{C^\alpha(\overline{\Omega})} + \|v_0\|_{C^{2+\alpha}(\overline{\Omega})} + \|h\|_{C_b^{\alpha,0}(\overline{Q_T})} + \|g\|_{C_b^{\alpha,\alpha/2}(\overline{Q_T})}\Big) + \\
&\quad + c_2\|v\|_{C_b^{\alpha,0}(\overline{Q_T})}
\end{aligned} \tag{2.19}$$

with constants independent of T.

Estimates (2.19) and (2.17) (with $\varphi = h - \ell_2 v$) yield

$$\begin{aligned}
&\|w\|_{C_b^{\alpha,0}(\overline{Q_T})} + \|w_t\|_{C_b^{\alpha,0}(\overline{Q_T})} + \|v\|_{C_b^{2+\alpha,1+\alpha/2}(\overline{Q_T})} \le \\
&\le c\Big(\|w_0\|_{C^\alpha(\overline{\Omega})} + \|v_0\|_{C^{2+\alpha}(\overline{\Omega})} + \\
&\quad + \|h\|_{C_b^{\alpha,0}(\overline{Q_T})} + \|g\|_{C_b^{\alpha,\alpha/2}(\overline{Q_T})}\Big) + \\
&\quad + \epsilon\|e^{bt}v\|_{C^{2+\alpha,0}(\overline{Q_T})} + c(\epsilon)\sup_{\tau<t}\|e^{b\tau}v(\cdot,\tau)\|_{L_2(\Omega)}.
\end{aligned}$$

When we choose ϵ sufficiently small and estimate the norm $\|e^{b\tau}v\|_{L_2(\Omega)}$ by inequality (2.16), we obtain (2.14) and complete the proof of Theorem 2.2.

Remark 2.1 Theorem 2.1 and Theorem 2.2 hold true under more general assumptions concerning the operator B (in particular, we can choose $B = I$, *i.e.*, the Dirichlet boundary conditions), but then compatibility condition (1.2) should be modified.

3 – The nonlinear problem

In this section, we apply Theorems 2.1 and 2.2 to the analysis of the nonlinear problem discussed in the Introduction (see (0.1)). More precisely we consider the mixed type problem

$$\begin{gathered}
u_t + \mathcal{A}\left(x, \frac{\partial}{\partial x}\right)u + \mathcal{R}(u) = f, \quad x \in \Omega \subset \mathbb{R}^n,\ t \in (0,T) \\
B\left(x, \frac{\partial}{\partial x}\right)v\Big|_{x\in\partial\Omega} = 0, \quad u(x,0) = u_0(x),
\end{gathered} \tag{3.1}$$

where $\mathcal{A}$ is a linear operator of the same type as A (see (2.2)) and $\mathcal{R}(u) = (\mathcal{P}(u), \mathcal{Q}(u))$ is a vector field of nonlinear terms.

We assume that $\mathcal{P} = (\mathcal{P}_i(v, \nabla v, D^2 v, w))_{i=1,\dots,m_1}$, $\mathcal{Q} = (\mathcal{Q}_k(v, w))_{k=1,\dots,m_2}$ and $\mathcal{P}_i$, $\mathcal{Q}_k$ are polynomials of degree $h \geq 2$ with respect to their arguments. In this section, we assume that the datum $f(x)$ is independent of t and it belongs to the Hölder space $C^\alpha(\overline{\Omega})$.

We refer to the Introduction and to the references cited there for a discussion of (mixed type) systems modelling biological phenomena, ecological studies and physical problems.

Assume that problem (3.1) has a regular, stationary solution $u_\sigma = (v_\sigma, w_\sigma)$, *i.e.*,

$$\mathcal{A}\left(x, \frac{\partial}{\partial x}\right) u_\sigma + \mathcal{R}(u_\sigma) = f(x), \quad x \in \Omega, \quad B\left(x, \frac{\partial}{\partial x}\right) v_\sigma\Big|_{x \in \partial\Omega} = 0. \tag{3.2}$$

We are interested in the problem of stability of this solution. We perturb it at the initial time $t = 0$ by a small $u_0' = (v_0', w_0')$ and consider the evolution problem for the perturbation $u' \equiv u - u_\sigma$ with $u' = (v', w')$, which can be written in the form

$$\begin{gathered} u_t' + A\left(x, \frac{\partial}{\partial x}\right) u' + R(u') = 0, \quad x \in \Omega \subset \mathbb{R}^n,\ t > 0, \\ B\left(x, \frac{\partial}{\partial x}\right) v'\Big|_{x \in \partial\Omega} = 0, \quad u'(x, 0) = u_0(x), \end{gathered} \tag{3.3}$$

where

$$A\left(x, \frac{\partial}{\partial x}\right) u' = \mathcal{A}(x, \frac{\partial}{\partial x})u' + \delta\mathcal{R}(u_\sigma)u', \tag{3.4}$$

$$\delta\mathcal{R}(u_\sigma)u' = \frac{d}{ds}\mathcal{R}(u_\sigma + su')\Big|_{s=0}, \tag{3.5}$$

$$R(u') = \mathcal{R}(u_\sigma + u') - \mathcal{R}(u_\sigma) - \delta\mathcal{R}(u_\sigma)u' = (P(u'), Q(u')). \tag{3.6}$$

We will prove the following stability result.

Theorem 3.1. *Assume that the operator A, defined in* (3.4) *and the operator B satisfy all the assumptions of Theorem* 2.2, *including* (a), (b). *There exists $\delta > 0$ such that if*

$$\|v_0'\|_{C^{\alpha+2}(\overline{\Omega})} + \|w_0'\|_{C^\alpha(\overline{\Omega})} \leq \delta, \tag{3.7}$$

then problem (3.3) *has a unique solution $u' = (v', w')$, $v' \in C_b^{2+\alpha, 1+\alpha/2}(\overline{Q_\infty})$, $w' \in C_b^{\alpha,0}(\overline{Q_\infty})$, $w_t' \in C_b^{\alpha,0}(\overline{Q_\infty})$, and the following estimates hold*

$$\begin{aligned} &\|v'\|_{C_b^{2+\alpha,1+\alpha/2}(\overline{Q_\infty})} + \|w'\|_{C_b^{\alpha,0}(\overline{Q_\infty})} + \|w_t'\|_{C_b^\alpha(\overline{Q_\infty})} \leq \\ &\leq c\Big(\|v_0'\|_{C^{2+\alpha}(\overline{\Omega})} + \|w_0'\|_{C^\alpha(\overline{\Omega})}\Big). \end{aligned} \tag{3.8}$$

Proof. The proof is based on inequality (2.14) and on the estimate of non-linear terms.

The components P_i, Q_k of P and Q that are sums of homogeneous polynomials of degree $j = 2, \dots , h$ with respect to v', $\nabla v'$, $D^2 v'$, w' and v', w', respectively, satisfy the inequalities

$$\|P_i\|_{C_b^{\alpha,\alpha/2}(\overline{Q_\infty})} \le c \sum_{j=2}^{h} \Big(\|v'\|_{C_b^{2+\alpha,1+\alpha/2}(\overline{Q_\infty})} + \|w'\|_{C_b^{\alpha,\alpha/2}(\overline{Q_\infty})} \Big)^j,$$
$$\|Q_k\|_{C_b^{\alpha,0}(\overline{Q_\infty})} \le c \sum_{j=2}^{h} \Big(\|v'\|_{C_b^{\alpha,0}(\overline{Q_\infty})} + \|w'\|_{C_b^{\alpha,0}(\overline{Q_\infty})} \Big)^j.$$

As a consequence, we have

$$\sum_{i=1}^{m_1} \|P_i\|_{C_b^{\alpha,\alpha/2}(\overline{Q_\infty})} + \sum_{k=1}^{m_2} \|Q_k\|_{C_b^{\alpha,0}(\overline{Q_\infty})} \le c \sum_{j=2}^{h} V^j,$$

where

$$V(u') = \|v'\|_{C_b^{2+\alpha,1+\alpha/2}(\overline{Q_\infty})} + \|w'\|_{C_b^{\alpha,0}(\overline{Q_\infty})} + \|w'_t\|_{C_b^{\alpha,0}(\overline{Q_\infty})}.$$

The solution of (3.3) can be constructed by successive approximations according to the following scheme. We define u_1 as the solution of the linear problem

$$\frac{\partial u_1}{\partial t} + A\left(x, \frac{\partial}{\partial x}\right) u_1 = 0, \quad x \in \Omega,\ t \in (0,T),$$
$$B\left(x, \frac{\partial}{\partial x}\right) v_1|_{x\in\partial\Omega} = 0, \quad u_1(x,0) = u'_0(x)$$

and for $m \geqslant 1$

$$\frac{\partial u_{m+1}}{\partial t} + A\left(x, \frac{\partial}{\partial x}\right) u_{m+1} = -R(u_m), \quad x \in \Omega,\ t \in (0,T),$$
$$B\left(x, \frac{\partial}{\partial x}\right) v_{m+1}|_{x\in\partial\Omega} = 0, \qquad u_{m+1}(x,0) = u'_0(x).$$

It is clear that the u_m are defined for all m and that the $V_m = V(u_m)$ satisfy the inequalities

$$V_{m+1} \le c \sum_{j=2}^{h} V_m^j + F \equiv \Phi(V_m),$$

where

$$F = c_1\Big(\|v_0'\|_{C^{2+\alpha}(\overline{\Omega})} + \|w_0'\|_{C^{\alpha}(\overline{\Omega})}\Big).$$

We will show that, if the norm F is small, then the norms V_m are uniformly bounded.

Let ξ_0 be a (positive) root of the equation

$$1 - \Phi'(\xi) = 1 - c\sum_{j=2}^{h} j\xi^{j-1} = 0. \tag{3.9}$$

It is clear that this root is unique. The equation $\xi = \Phi(\xi)$ has a solution if and only if $\xi_0 - \Phi(\xi_0) \geq 0$. Since

$$\xi_0 - \Phi(\xi_0) = c\sum_{j=2}^{h}(j-1)\xi_0^j - F,$$

this is the case if

$$c_1\Big(\|v_0'\|_{C^{2+\alpha}(\overline{\Omega})} + \|w_0'\|_{C^{\alpha}(\overline{\Omega})}\Big) \leq c\sum_{j=2}^{h}(j-1)\xi_0^j,$$

and this condition is satisfied if δ in (3.7) is sufficiently small. Let ξ_1 be the root of $\xi = \Phi(\xi)$ such that $0 \leq \xi_1 \leq \xi_0$. If $V_m \leq \xi_1$, then

$$V_{m+1} \leq c\sum_{j=2}^{h} V_m^j + F \leq c\sum_{j=2}^{h}\xi_1^j + F = \xi_1. \tag{3.10}$$

Since $V_1 \leq F \leq \xi_1$, (3.10) holds for all m, *i.e.*, successive approximations are uniformly bounded in the norm $V(\cdot)$. We note that the value of ξ_1 depends in a continuous way on F (*i.e.* on δ) and then we can make the value of ξ_1 as small as we need (by choosing δ sufficently small).

In order to show that the sequence $\{u_m\}$ converges to the solution of (3.3) we consider the difference $U_{m+1} \equiv u_{m+1} - u_m$ with $U_{m+1} = (\zeta_{m+1}, \eta_{m+1})$. The function U_{m+1} is the solution of the problem

$$\begin{aligned} &\frac{\partial U_{m+1}}{\partial t} + A\left(x, \frac{\partial}{\partial x}\right)U_{m+1} = R(u_{m-1}) - R(u_m), \quad x \in \Omega,\ t \in (0,T),\\ &B\left(x, \frac{\partial}{\partial x}\right)\zeta_{m+1}|_{x\in\partial\Omega} = 0, \qquad\qquad U_{m+1}(x,0) = 0. \end{aligned}$$

Let $X_{m+1} = V(U_{m+1})$. Repeating the above arguments, we can prove that

$$X_{m+1} \leqslant c\xi_1 X_m,\ m \geqslant 2. \tag{3.11}$$

Now we set $Y_n = \sum_{m=2}^{n} X_m$ and we derive from (3.11) that

$$Y_{n+1} - Y_2 \leqslant c\xi_1 Y_n,\ n \geqslant 2 \tag{3.12}$$

and

$$Y_n \leqslant \frac{Y_2}{1 - c\xi_1},\ n \geqslant 3 \tag{3.13}$$

if ξ_1 is so small that $c\xi_1 < 1$.

From estimate (3.13) we deduce that the sequence $\{u_m\}$ converges in the Hölder space $C_b^{\alpha+2,\alpha/2+1}(\overline{Q}_\infty)$ to a solution of (3.3). The uniqueness proof follows from the estimate of the difference of two solutions of (3.3), as above. Thus, Theorem 3.1 is proved.

Remark 3.1. Theorem 3.1 holds true under more general assumptions concerning the operators $\mathcal{P} = (\mathcal{P}_i(v, \nabla v, D^2 v, w))_{i=1,\dots,m_1}$ and $\mathcal{Q} = (\mathcal{Q}_k(v, w))_{k=1,\dots,m_2}$ where $\mathcal{P}_i$, $\mathcal{Q}_k$ are polynomials of degree $h \geq 2$ with respect to their arguments. In particular we may assume that the polynomials $\mathcal{P}_i$, $\mathcal{Q}_k$ have coefficients belonging to the Hölder space $C^\alpha(\overline{\Omega})$ with respect to the space variable x and independent of t.

REFERENCES

[1] V. S. Belonosov: *Estimates of the solutions of parabolic systems in weighted Hölder classes and some their applications*, Adv. Math. USSR Sbornik, **110** (1979), 163-188.

[2] V. S. Belonosov – M. P. Višnevskiĭ: *On the stability of stationary solutions of nonlinear parabolic systems*, Adv. Math. USSR Sbornik, **33** (1977), 465-484.

[3] L. H. Chuan – A. Yagi: *Dynamical systems for forest kinematic model*, Math. Sci. Appl., **16** (2006), 393-409.

[4] L. H. Chuan – T. Tsujikawa – A. Yagi: *Asyptotic behavior of solutions for forest kinematic model*, Funkcial Ekvac., **49** (2006), 427-449.

[5] L. H. Chuan – T. Tsujikawa – A. Yagi: *Stationary solutions for forest kinematic model*, preprint.

[6] S. V. Gurevich – H. U. Bödeker – A. S. Moskalenko – A. W. Liehr – H. G. Purwins: *Drift bifurcation of dissipative solutions due to change of shape: experiment and theory*, Physica D, **199** (2004), 115-128.

[7] D. Henry: *Geometric theory of semilinear parabolic equations*, Lectures Notes Vol. 840, Spriger New York, 1981.

[8] Yu. A. Kutznetsov – M. Ya. Antonovsky – V. N. Biktashev – A. Aponima: *A crossdiffusion model of forest boundary dynamics*, J. Math. Biol., **32** (1994), 219-232.

[9] D. S. Jones – B. D. Sleeman: *Differential Equations and mathematical biology*, Allen and Unwin, London, 1983.

[10] G. Mulone – V. A. Solonnikov: *Linearization principle for the system of equation of a mixed type*, to appear on Nonlinear Analysis.

[11] J. D. Murray: *Mathematical biology I. An introduction*, Third edition. Interdisciplinary Applied Mathematics 17, Springer-Verlag, New York, 2002.

[12] J. D. Murray: *Mathematical biology II. Spatial models and biomedical applications*, Third edition. Interdisciplinary Applied Mathematics 18, Springer-Verlag, New York, 2003.

[13] C. V. Pao: *Nonlinear parabolic and elliptic equations*, Plenum Press, New York, 1992.

[14] V. A. Solonnikov: *On boundary value problems for linear parabolic systems of differential equations of general form*, Proc. Math. Inst. Steklov, **83** (1965), 3-162.

[15] L. Wu – Y. Lin: *The stability of steady states for a model with diffusion and spatial average*, J. Math. Anal. Appl., **232** (1999) 259-271.

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