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$C^{1,\alpha}$ and Glaeser type estimates

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Abstract: *In this paper we are concerned with gradient estimates for viscosity solutions of fully nonlinear second order elliptic equations of the form $F(D^2u) = f$ with bounded right-hand side. We generalize to a nonlinear setting the results of Yan Yan Li and Louis Nirenberg [12] about the so-called Glaeser estimate. We improve also some qualitative results in this direction contained in [4].*

1 – Introduction and results

The Glaeser's inequality, see [7], states that for any non-negative function $u \in C^2([-R, R])$ with $|u''(t)| \leq M$ for all $t \in [-R, R]$, the following estimates hold:

$$|u'(0)| \leq \sqrt{2u(0)M} \quad \text{if} \quad M \geq \frac{2u(0)}{R^2}$$

$$|u'(0)| \leq \left(\frac{u(0)}{R} + \frac{R}{2}M\right) \quad \text{if} \quad M < \frac{2u(0)}{R^2}$$

The issue of bounding the first derivative of u in terms of u and u'' is a classical one since the works of Hadamard [9], Kolmogorov [10] and Landau [11]. For more recents results in this direction see [5], [13] and [14].

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In the recent paper [12], Y.Y.Li and L.Nirenberg showed the validity of the Glaeser inequality for smooth non-negative functions with bounded Laplacian in an $n-$dimensional ball. More precisely, they proved that if $0 \le u \in C^2(B_R) \cap C^0(\overline{B}_R)$ is any solution of the Poisson equation

$$\Delta u = f$$

in the ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, then

$$\begin{cases} |Du(x)| \le C\sqrt{u(0)M} & \text{if} \quad 2|x| \le \sqrt{\frac{u(0)}{M}} \le R \\ |Du(x)| \le C\left(\frac{u(0)}{R} + MR\right) & \text{if} \quad 2|x| \le R \le \sqrt{\frac{u(0)}{M}} \end{cases} \tag{1.1}$$

where $M = \sup_{x \in \overline{B}_R} |f(x)|$. The constant C in the above inequalities depends only on the dimension n and $C = \sqrt{2}$ is optimal for $n = 1$.

Extensions to non-negative strong solutions u of linear second-order uniformly elliptic equations of the form

$$\text{Tr}\left(A(x)D^2u\right) = f$$

are also treated in the paper [12]. In this more general case, the constants appearing in the corresponding Glaeser type inequalities depend on the maximum and minimum eigenvalue of the positive definite matrix $A(x)$ and the moduli of continuity of the matrix entries as well as on n.

Our aim here is to present some new results concerning the validity of Glaeser type estimates for non-negative continuous functions u satisfying in the viscosity sense the fully nonlinear uniformly elliptic equation

$$F(D^2u) = f \quad \text{in} \ B_R \tag{1.2}$$

for bounded and continuous right-hand side f. Here F is a real-valued function defined on $\mathcal{S}^n$, the space of real $n \times n$ symmetric matrices.

Our leading assumption on F is uniform ellipticity, namely that for real numbers $\Lambda \ge \lambda > 0$ the following structural condition holds:

$$\lambda \text{Tr}(Y^+) - \Lambda \text{Tr}(Y^-) \le F(X+Y) - F(X) \le \Lambda \text{Tr}(Y^+) - \lambda \text{Tr}(Y^-) \tag{1.3}$$

for all $X, Y \in \mathcal{S}^n$. Here we have used the standard decomposition of a symmetric matrix $Y \in S^n$ in the form $Y = Y^+ - Y^-$ with $Y^+Y^- = 0$ and $Y^\pm \ge 0$, in the sense of the partial ordering induced by semidefinite positiveness.

Note that $Y \to \lambda \text{Tr}(Y^+) - \Lambda \text{Tr}(Y^-)$ and $Y \to \Lambda \text{Tr}(Y^+) - \lambda \text{Tr}(Y^-)$ are, respectively, the minimal and maximal Pucci operators usually denoted by $P^-_{\lambda,\Lambda}$, $P^+_{\lambda,\Lambda}$, see [2].

Condition (1.3) appears therefore as a nonlinear version of the standard notion of uniform ellipticity for linear second order operators. Observe that for $\Lambda = \lambda = 1$ the class of functions F fulfilling condition (1.3) reduces to the single operator $F = \Delta$.

Relevant examples of operators satisfying condition (1.3) are, of course, the Pucci operator themselves, the Bellman operators

$$F(X) = \inf_{i \in I} \operatorname{Tr}(A_i X)$$

where A_i, $i \in I$, is a family of positive definite symmetric matrices such that

$$\lambda |\xi|^2 \le A\xi \cdot \xi \le \Lambda |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n, \tag{1.4}$$

and the Isaacs operators

$$F(X) = \sup_{j \in J} \inf_{i \in I} \operatorname{Tr}(A_{i,j} X)$$

where $A_{ij}, i \in I$, $j \in J$, is a double-indexed family of symmetric matrices satisfying (1.4). We shall also assume for simplicity that

$$F(0) = 0 \tag{1.5}$$

Our main results are the following:

Theorem 1.1. *Assume that F satisfies* (1.3) *and* (1.5). *If $u \in C^0(\overline{B}_R)$ is a non-negative viscosity solution of $F(D^2 u) = f$ in B_R with $f \in C^0(\overline{B}_R)$, then u is differentiable at any $x \in B_R$ and the inequalities* (1.1) *hold true with $M = \sup_{\overline{B}_R} |f|$ and some positive constant C depending only on n and the ellipticity constants λ, Λ.*

This result is based on the Harnack inequality for non-negative viscosity solutions and the $C^{1,\alpha}$ regularity of continuous (not necessarily non-negative) solutions of equation (1.2). Let us state below this result of independent interest.

Theorem 1.2. *Assume that F satisfies* (1.3) *and* (1.5). *If $u \in C^0(\overline{B}_R)$ is a viscosity solution of $F(D^2 u) = f$ in B_R with $f \in C^0(B_R)$, then $u \in C^{1,\alpha}(B_R)$ for some $\alpha \in (0,1)$ and*

$$\|u\|_{L^\infty(B_{R/2})} + R\,\|Du\|_{L^\infty(B_{R/2})} + R^{1+\alpha}\,[Du]_{\alpha, B_{R/2}} \le C\,\left(\|u\|_{L^\infty(B_R)} + R^2\,M\right) \tag{1.6}$$

where $M = \sup_{\overline{B}_R} |f|$ and $C > 0$ depend only on n and the ellipticity constants λ and Λ.

Here $[h]_{\alpha,\Omega}$ denotes the Hölder seminorm

$$[h]_{\alpha,\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|h(x)-h(y)|}{|x-y|^\alpha}$$

Theorem 1.2 can be regarded as a consequence of Theorem 2 of Caffarelli [1] and Corollary 5.7 of [2], where $C^{1,\alpha}$- estimates are established for solutions of the homogeneous equation $F(D^2u) = 0$.

Should we adopt the slightly stronger notion of L^n-viscosity solutions which require testing subsolutions and supersolutions on $\varphi \in W^{2,n}_{loc}$ rather than on $\varphi \in C^2$, see [1], [3], then the assumption of continuity of f can be dropped and it is enough to assume $f \in L^\infty$ in order that Theorems 1.1 and 1.2 remain true for L^n-viscosity solutions.

Rather precise information about the constant C in the statement of Theorem 1.1 can be obtained under some symmetry assumption on F.

A mapping $F : \mathcal{S}^n \to \mathbb{R}$ is reflection-invariant with respect to the unit vector $\nu \in \mathbb{R}^n$ if

$$F(RXR) = F(X) \quad \text{for all} \quad X \in \mathcal{S}^n \tag{1.7}$$

where R is the reflection matrix with respect to the hyperplane of equation $\nu \cdot x = 0$.

If, for instance, $\nu = (0, \dots, 0, 1)$ then

$$R = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$$

where $\mathbb{I}_{n-1}$ is the $(n-1)$-dimensional identity matrix.

By suitably exploiting reflection-invariance properties of F we obtain the following form of the gradient estimate:

THEOREM 1.3. *Assume that F satisfies* (1.3) *and* (1.5). *Assume also that F is reflection-invariant with respect to n linearly independent unit vectors $\nu_1, \dots, \nu_n$.*

If $u \in C^0(\overline{B}_R)$ is a viscosity solution of $F(D^2u) = f$ with $f \in C^0(\overline{B}_R)$, then

$$(\tfrac{\mu}{n})^{\frac{1}{2}} |Du(0)| \le \tfrac{n}{R}\,(1+\tfrac{\Lambda}{\lambda})^{\frac{1}{2}} \sup_{\overline{B}_R} |u| + \tfrac{R}{2\lambda}\,(1+\tfrac{\Lambda}{\lambda})^{-\frac{1}{2}} \sup_{\overline{B}_R} |f| \tag{1.8}$$

where μ is the least eigenvalue of the positive definite matrix $S \in \mathcal{S}^n$ with entries $S_{ij} = \nu_i \cdot \nu_j$.

It is easy to check that if the directions ν_j in the statement of Theorem (1.3) are mutually orthogonal then $\mu = 1$. Instead, if $F(X)$ depends only on the eigenvalues of X, then we obtain the better estimate

$$|Du(0)| \leq \tfrac{n}{R}\,(1+\tfrac{\Lambda}{\lambda})^{\frac{1}{2}} \sup_{B_R}|u| + \tfrac{R}{2\lambda}\,(1+\tfrac{\Lambda}{\lambda})^{-\frac{1}{2}} \sup_{B_R}|f| \tag{1.9}$$

In the special case $\lambda = \Lambda = 1$, $F(X) = \mathrm{TrX}$, the above reduces to a well-known estimate for classical solutions of the Poisson equation $\Delta u = f$, see [8].

2 – A few facts about viscosity solutions

Let us recall for the convenience of the reader the notion of viscosity solution of equation (1.2) with $f \in C^0(B_R)$ and F as above.

A function $u \in C^0(B_R)$ a viscosity subsolution of (1.2) if for all $x_0 \in B_R$ and $\varphi \in C^2(B_R)$ such that $u - \varphi$ has a local maximum at x_0

$$F(D^2\varphi(x_0)) \geq f(x_0)$$

On the other hand, u is a viscosity supersolution of (1.2) if for all $x_0 \in \Omega$ and $\varphi \in C^2(B_R)$ such that $u - \varphi$ has a local minimum at x_0

$$F(D^2\varphi(x_0)) \leq f(x_0)$$

A viscosity solution of $F(D^2u) = f$ is both a subsolution and a supersolution.

Let us point out that classical solutions $u \in C^2(B_R)$ of equation (1.2) are also viscosity solutions and, conversely, if $u \in C^2(B_R)$ is a viscosity solution of (1.2) then u is a classical solution of the same equation.

For a general review of the theory of viscosity solutions of fully nonlinear second order elliptic equations we refer to [6] and [2].

A fundamental tool in the regularity theory for viscosity solutions is the Harnack inequality, see [1],[2]. Here we quote it in an appropriate version:

THEOREM 2.1. *Assume that F satisfies* (1.3) *and* (1.5). *If $u \in C^0(B_r)$ is a non-negative viscosity solution of equation $F(D^2u) = f$ with $f \in C^0(B_r)$, then*

$$\sup_{B_{\frac{3}{4}r}} u \leq C\left(\inf_{B_{\frac{3}{4}r}} u + r^2\|f\|_{L^\infty(B_r)}\right) \tag{2.1}$$

3 – Proofs

Proof of Theorem 1.2. Take first $r = 1$ and let $|x_0| < r_1 < 1$ so that $\overline{B}_{\rho_0}(x_0) \subset B_{\frac{1+r_1}{2}}$ for $\rho_0 = \frac{1-r_1}{4}$. For any $\alpha \in (0,1)$ we have of course

$$\sup_{0<\rho\leq\rho_0} \rho^{-\alpha} \, \|f\|_{L^n(B_\rho(x_0))} \leq \|f\|_{L^\infty(B_1)} < +\infty \tag{3.1}$$

for $\rho \in (0, \rho_0)$. Note that here the finiteness of the Morrey norm on the left-hand side follows from the continuity of f, which is the natural assumptions in our viscosity framework. Considering instead L^n-viscosity solutions we could bypass both continuity and boundedness assumption on f. In this case, for instance, $\sup_{x\in B_1} |f(x)||x|^{1-\alpha'} < +\infty$ should be sufficient.

Furthermore, since F satisfies (1.3) and (1.5), by an important regularity result due to L. Caffarelli, see [2], Corollary 5.7, any continuous viscosity solution $w \in C^0(B_r)$ of the homogeneous equation $F(D^2 w) = 0$ is $C^{1,\overline{\alpha}}(B_r)$ for some $\overline{\alpha} \in (0,1)$ and

$$\|w\|_{L^\infty(B_{r/2})} + r\|Dw\|_{L^\infty(B_{r/2})} + r^{1+\overline{\alpha}}[Dw]_{\overline{\alpha},B_{r/2}} \leq C \, \|w\|_{L^\infty(B_r)} \tag{3.2}$$

with $C > 0$ depending only on n, λ and Λ.

Thanks to (3.1), (3.2) we can apply Theorem 2 of [1] to deduce that u is differentiable at x_0, its gradient satisfies the inequality

$$|Du(x_0)| \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\right) : \, = \overline{C}_{\alpha'} \tag{3.3}$$

and

$$|u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \leq \overline{C}_{\alpha'} \, |x - x_0|^{1+\alpha'}, \quad |x - x_0| \leq \rho_0 \tag{3.4}$$

for some $\alpha' \in (0,1)$.

Our aim now is to evaluate the oscillation of Du on a compact subset K of B_{r_1}.

At this purpose, for $\varepsilon \in (0,1)$ to be chosen in the sequel, take $\rho_1 > 0$ such that $\rho_1^{\frac{1}{1-\varepsilon}} = \frac{1}{2}\,\rho_0$. Hence,

$$\overline{B}_{\frac{1}{2}\rho_1^{\frac{1}{1-\varepsilon}}}(x_1) \subset B_{\frac{1}{2}\rho_0}(x_0).$$

for any $x_1 \in K$ with $|x_1 - x_0| < \frac{\rho_1}{4}$. Consider then

$$x_1 = x_0 + h\,e, \quad |e| = 1, \quad 0 < h < \frac{\rho_1}{4}$$

and set, if $Du(x_1) \neq Du(x_0)$,

$$\nu = \frac{Du(x_1) - Du(x_0)}{|Du(x_1) - Du(x_0)|}$$

Choose now $x \in B_{\frac{1}{2}\rho_1}(x_1)$ in such a way that

$$x = x_0 + h^{1-\varepsilon}\,\nu$$

By the choice above, $x \in B_{\rho_0}(x_0) \cap B_{\rho_0}(x_1)$. Since $x_1 \in K$, inequality (3.4) still holds if we replace x_0 with x_1, leading to

$$\begin{aligned}&|u(x_1) - u(x_0) + Du(x_1)\cdot(x - x_1) - Du(x_0)\cdot(x - x_0)|\\ &\le \overline{C}_\alpha\,\left(|x - x_0|^{1+\alpha'} + \overline{C}_{\alpha'}\,|x - x_1|^{1+\alpha'}\right)\end{aligned}$$

and, consequently,

$$|Du(x_1) - Du(x_0)|\,h^{1-\varepsilon} \le |Du(x_1)|\,h + |u(x_1) - u(x_0)| + 2\overline{C}_{\alpha'}h^{(1+\alpha')(1-\varepsilon)} \tag{3.5}$$

Observe that inequalities (3.3) and (3.4) hold uniformly on K so that, dividing (3.5) by $h^{1-\varepsilon}$, we get

$$\begin{aligned}|Du(x_1) - Du(x_0)| &\le \overline{C}_{\alpha'}\,h^\varepsilon + \frac{|u(x_1) - u(x_0)|}{|x_1 - x_0|}\,h^\varepsilon + 2\,\overline{C}_{\alpha'}\,h^{\alpha'(1-\varepsilon)}\\ &\le 2\,\overline{C}_{\alpha'}\,h^\varepsilon + 2\,C_{\alpha'}\,h^{\alpha(1-\varepsilon)}\end{aligned}$$

Finally, choosing $\varepsilon = \frac{\alpha'}{1+\alpha'}$, we obtain the desired oscillation estimate

$$|Du(x_1) - Du(x_0)| \le 4\,\overline{C}_{\alpha'}\,h^{\frac{\alpha'}{1+\alpha'}} = 4\,\overline{C}_{\alpha'}\,|x_1 - x_0|^{\frac{\alpha'}{1+\alpha'}} \tag{3.6}$$

for $x_0, x_1 \in K$.

This, with the uniformly boundedness of u and Du in K shows that $u \in C^{1,\alpha}(K)$ with $\alpha = \frac{\alpha'}{1+\alpha'}$, and the estimate (1.6) follows in the present case $r = 1$.

If $r \neq 1$, let us consider the scaled function $\tilde{u}(y) = u(ry)$, $|y| \le 1$. FIt is easy to check that $\tilde{u}$ is a continuous (up to the boundary) viscosity solution of the equation

$$G(D^2\tilde{u}(y)) = g(y) = r^2 f(ry)$$

in $|y| < 1$, where $G(Y) = r^2 F(r^{-2}Y)$ is uniformly elliptic with the same ellipticity constants λ and Λ as F.

Therefore, the $C^{1,\alpha}$-estimate (1.6) already proved for $r = 1$ applies to $\tilde{u}$ yielding (1.6) in its scaled form. ∎

Proof of Theorem 1.1. Let $|x_0| \le \frac{r}{2}$, $0 < r < R$. Applying Theorem 1.2 in the ball $B_{r/8}(x_0)$, we get in particular

$$r|Du(x)| \le C\,\left(\|u\|_{L^\infty(B_{r/4}(x_0))} + r^2\,\|f\|_{L^\infty(B_{r/4})}\right)$$

$$\leq C \left(\|u\|_{L^\infty(B_{\frac{3}{4}r})} + r^2 \|f\|_{L^\infty(B_{\frac{3}{4}r})} \right)$$

for $|x - x_0| \leq \frac{r}{8}$. Then,

$$|Du(x)| \leq C \left(\frac{1}{r} \|u\|_{L^\infty(B_{\frac{3}{4}r})} + r \|f\|_{L^\infty(B_{\frac{3}{4}r})} \right) \tag{3.7}$$

for $|x| \leq \frac{r}{2}$. Hence, by the Harnack inequality (2.1), we have

$$|Du(x)| \leq C \left(\frac{1}{r} u(0) + r \|f\|_{L^\infty(B_{\frac{3}{4}r})} \right), \quad |x| \leq \frac{r}{2}$$

The assertion follows by optimizing with respect to $r \in (0, R)$ the right-hand side of the above inequality. □

The proof of Theorem 1.3 is performed using a different technique, based on the Maximum Principle and, more precisely, on comparison results between viscosity sub and supersolutions. A major step in the proof is contained in the next Lemma which can be seen as the extension to our nonlinear setting of a well- known result for the Poisson equation, see [8], Theorem 3.9.

LEMMA 3.1. *Assume that F satisfies* (1.3) *and* (1.5). *Assume also that F is reflection invariant along some direction ν, see* (1.7). *If $u \in C^0(\overline{B}_d)$ is a viscosity solution of the equation $F(D^2u) = f$ with $f \in C(\overline{B}_d)$, then*

$$|D_\nu u(0)| \leq \tfrac{n}{d} \left(1 + \tfrac{\Lambda}{\lambda}\right)^{\frac{1}{2}} \sup_{\overline{B}_d} |u| + \tfrac{d}{2\lambda} \left(1 + \tfrac{\Lambda}{\lambda}\right)^{-\frac{1}{2}} \sup_{\overline{B}_d} |f| \tag{3.8}$$

where $D_\nu u(0)$ is the directional derivative of u at $x = 0$ along the direction ν.

PROOF OF LEMMA 3.1. Let $u \in C^0(\overline{B}_d)$ be a viscosity solution of the equation

$$F(D^2u) = f$$

in B_d. By Theorem 1.2, u is differentiable at $x = 0$. Setting $M = \sup_{\overline{B}_d} |f|$, we have

$$-M \leq F(D^2u) \leq M \tag{3.9}$$

in B_d in the viscosity sense.

Up to a rotation, which does not change the ellipticity constants of F and the L^∞-norm of f, we can suppose that the invariance direction is the x_n-axis.

Consider the open cylinder

$$K := \{(x', x_n) \in \mathbb{R}^n \mid |x'| < \overline{d}\sqrt{\Lambda},\ |x_n| < \overline{d}\sqrt{\lambda}\}$$

so that $\overline{K} \subset B_d$. Set $x^* := (x', -x_n)$ and the reflected function

$$u^*(x) := u(x^*) \quad x \in K_+ = K \cap \{x_n > 0\}$$

It turns out that $u^* \in C^0(\overline{K}_+)$ and $F(D^2u^*) = f^*$ in the viscosity sense in K_+.

Indeed, if $x_0 \in K_+$ and $\varphi \in C^2(K_+)$ touches u^* from above at x_0, then φ^* touches u from above at $x_0^* \in B_d$. Therefore,

$$F(D^2\varphi^*(x_0^*)) \geq f(x_0^*) \geq -M$$

By reflection invariance $F(D^2\varphi(x_0)) = F(D^2\varphi^*(x_0^*)) \geq -M$. A similar argument for supersolutions yields the inequality

$$F(D^2u^*) \leq M \tag{3.10}$$

in K_+ in the viscosity sense. Setting

$$\tilde{u} = \frac{u - u^*}{2} \tag{3.11}$$

and using Theorem 5.3 in [2], we get

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2\tilde{u}) \geq -M, \quad \mathcal{P}^-_{\lambda,\Lambda}(D^2\tilde{u}) \leq M \tag{3.12}$$

in K_+ in the viscosity sense. Note that the argument of the above mentioned Theorem 5.3 of [2], concerning sub and supersolutions of the homogeneous equation $F(D^2u) = 0$, still holds for a nonzero constant right-hand side.

Next, we construct a smooth barrier function:

$$\Phi(x) = \frac{N}{\overline{d}^2}\left[\frac{|x'|^2}{\Lambda} + \frac{x_n}{\sqrt{\lambda}}\left(n\overline{d} - (n-1)\frac{x_n}{\sqrt{\lambda}}\right)\right] + \frac{M}{2}\frac{x_n}{\sqrt{\lambda}}\left(\overline{d} - \frac{x_n}{\sqrt{\lambda}}\right)$$

where $N = \sup_K |u|$. A simple computation shows that

$$\mathrm{P}^+_{\lambda,\Lambda}(D^2\Phi) = \Lambda\frac{2N}{\Lambda\overline{d}^2}(n-1) - \lambda\left(\frac{2N}{\lambda\overline{d}^2}(n-1) + \frac{M}{\lambda}\right) = -M$$

By (3.12) we obtain then

$$\mathrm{P}^+_{\lambda,\Lambda}(D^2(\tilde{u} - \Phi)) \geq 0 \geq \mathrm{P}^-_{\lambda,\Lambda}(D^2(\tilde{u} + \Phi)) \quad \text{in} \quad K_+ \tag{3.13}$$

On the other hand, for $x \in \partial K_+$, we have

$$\Phi(x) = \frac{N}{\overline{d}^2}\frac{|x'|^2}{\Lambda} \geq 0 \geq \tilde{u} \quad \text{on} \quad x_n = 0$$

$$\Phi(x) \geq N \geq |\tilde{u}| \quad \text{on} \quad |x'| = \overline{d}\sqrt{\Lambda} \quad \text{and} \quad x_n = \overline{d}\sqrt{\lambda}$$

This provides the boundary conditions

$$\tilde{u} - \Phi \leq 0 \leq \tilde{u} + \Phi \quad \text{on } \partial K_+$$

which, coupled with equations (3.13), by the Maximum Principle yield

$$\tilde{u} - \Phi \leq 0 \leq \tilde{u} + \Phi \quad \text{in } K_+$$

Taking now $x' = 0$ and dividing by $x_n > 0$, we get

$$\frac{|u(0,x_n) - u(0,-x_n)|}{2x_n} \leq \frac{N}{\overline{d}^2}\left(n\overline{d} - (n-1)\frac{x_n}{\sqrt{\lambda}}\right) + \frac{M}{2\sqrt{\lambda}}\left(\overline{d} - \frac{x_n}{\sqrt{\lambda}}\right)$$

Letting $x_n \to 0^+$ we conclude that

$$|u_{x_n}(0)| \leq \frac{n}{\overline{d}} \sup_K |u| + \frac{\overline{d}}{2\sqrt{\lambda}} \sup_K |f|$$

From this it is immediate to derive the validity of inequality (3.8). ∎

Proof of Theorem 1.3. This is matter of elementary vector calculus. In fact, let $\nu_k = (\nu_{k1}, \ldots, \nu_{kn})$, $k = 1, \ldots, n$, then

$$\sum_{k=1}^{n} |Du(0) \cdot \nu_k|^2 = \sum_{k=1}^{n} \left(Du(0) \cdot \sum_{i=1}^{n} \nu_{ki} e_i\right) \left(Du(0) \cdot \sum_{j=1}^{n} \nu_{kj} e_j\right)$$

$$= \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} \nu_{ki}\nu_{kj}\right) u_{x_i}(0)\, u_{x_j}(0) = \sum_{i,j=1}^{n} S_{ij}\, u_{x_i}(0)\, u_{x_j}(0) \geq \mu\, |Du(0)|^2$$

and the statement easily follows from (3.8). ∎

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