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Weak solutions of grain boundary motion model with singularity

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Dedicated to Professor Umberto Mosco on the occasion of his 70th birthday

Abstract: *We consider the grain boundary motion model of Kobayashi-Warren-Carter type, which arises in material sciences. The system, which consists of two nonlinear parabolic PDEs with singularity, is of the phase-field type. In this paper we show the global existence of solutions for our model in a weak variational sense.*

1 – Introduction

In this paper we consider a model for grain boundary motion of the form, denoted by (P):

$$
\text{(P)}\begin{cases}
\eta_t - \kappa\Delta\eta + g(\eta) + \alpha'(\eta)|\nabla\theta| = 0 & \text{a.e. in } Q_T := \Omega\times(0,T),\\
\alpha_0(\eta)\theta_t - \nu\Delta\theta - \operatorname{div}\left(\alpha(\eta)\dfrac{\nabla\theta}{|\nabla\theta|}\right) = 0 & \text{a.e. in } Q_T,\\
\dfrac{\partial\eta}{\partial n} = 0, \quad \theta = 0 & \text{a.e. on } \Sigma_T := \Gamma\times(0,T),\\
\eta(x,0) = \eta_0(x), \quad \theta(x,0) = \theta_0(x) & \text{for a.a. } x\in\Omega,
\end{cases}
$$

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where Ω is a bounded domain in $\mathbf{R}^N$ $(1 \le N \le 3)$ with smooth boundary $\Gamma := \partial\Omega$, $T > 0$ is a fixed finite time, $\kappa > 0$ and $\nu > 0$ are small positive constants, $g(\cdot)$, $\alpha(\cdot)$ and $\alpha_0(\cdot)$ are given functions on $\mathbf{R}$, $\partial/\partial n$ is the outward normal derivative on Γ, and $\eta_0(x)$, $\theta_0(x)$ are given initial data.

Problem (P) of two dimensional grain structure was proposed in Kobayashi *et al.* [12] as a polar coordinate system, where the variable θ is an indicator of the mean orientation of the crystalline and the variable η is an order parameter for the degree of crystalline orientational order: $\eta \equiv 1$ implies a completely oriented state and $\eta \equiv 0$ is a state where no meaningful value of orientation exists. In [12] the system (P) was derived from the free energy functional of the following form:

$$\mathcal{F}(\eta,\theta) := \frac{\kappa}{2}\int_\Omega |\nabla\eta|^2 dx + \int_\Omega \hat{g}(\eta) dx + \frac{\nu}{2}\int_\Omega |\nabla\theta|^2 dx + \int_\Omega \alpha(\eta)|\nabla\theta| dx,$$

where $\hat{g}$ is a primitive of g. Moreover, Kobayashi *et al.* [12] presented some numerical simulations of (P), which showed both grain rotation and shrinkage, in the case where $\hat{g}(\eta) := \frac{1}{2}(1-\eta)^2$, $\alpha_0(\eta) = \alpha(\eta) = \eta^2$ and Ω is a bounded domain in $\mathbf{R}^2$. But, any theoretical results have not been there established.

There are many mathematical models of grain boundary formation. For some related works of grain boundary motions, we refer to [3], [4], [5], [7], [8], [12], [14], [15], [16]. Also, for singular diffusion equations kindred to the second one of (P), we refer to [1], [2], [6], [11].

Recently, system (P) was studied in [9], [10] from the theoretical point of view, when $\alpha_0 \ge \delta(>0)$ on $\mathbf{R}$ for a positive constant δ. More precisely, in [9] the one-dimensional grain boundary model of Kobayashi-Warren-Carter type, with $-\kappa\Delta\eta$ replaced by $-(\sigma\eta_t + \kappa\eta)_{xx}$, $0 < \sigma < \infty$, in the first equation, was discussed and the existence–uniqueness of solutions was proved. Also, in [10] the existence of a global in time solution to (P) was shown in higher dimensional spaces by employing a new method, and its uniqueness was proved in one dimensional space.

The main objective of the present paper is to show the global existence of a weak solution to (P) in the case when $\alpha_0 \ge 0$ on $\mathbf{R}$. In this case we can not expect that the time-derivative of θ exists in the classical sense on the region where $\alpha_0(\eta)$ vanishes. We shall establish a mathematical treatment to such a difficulty.

The plan of this paper is as follows. In Section 2, we mention the main theorem of this paper. In Section 3, we consider the approximate systems to (P). In the final section, we give the existence proof for (P).

Notation. For a general (real) Banach space X we denote by $\|\cdot\|_X$ the norm in X. For $1 \le p \le \infty$ and any positive integer m, we simply write L^p, $W^{m,p}$ and $W_0^{m,p}$ for $L^p(\Omega)$, $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$, respectively, where $W^{m,p}(\Omega)$ is the usual Sobolev space. As usual, $W^{m,2}$ and $W_0^{m,2}$ are denoted by H^m and H_0^m, respectively.

2 – Main result

Throughout this paper, the following assumptions are always made:

(A1) α_0 is a non-negative function in $C^1(\mathbf{R})$ such that

$$\frac{|\alpha_0'(r)|}{\sqrt{\alpha_0(r)+\delta}} \leq M, \quad \forall \delta \in (0,1], \quad \forall r \in [0,1],$$

where M is a positive constant and α_0' is the derivative of α_0.

(A2) α is a non-negative function in $C^1(\mathbf{R})$, whose derivative α' is non-decreasing and bounded on $\mathbf{R}$ such that $\alpha'(0) = 0$. We denote by $L(\alpha)$ the Lipschitz constant.

(A3) g is a Lipschitz continuous function on $\mathbf{R}$ such that $g \leq 0$ on $(-\infty, 0]$ and $g \geq 0$ on $[1, \infty)$. We denote by $\hat{g}$ a primitive of g, and assume that $\hat{g}$ is non-negative on $\mathbf{R}$.

(A4) $\eta_0 \in H^1$ with $0 \leq \eta_0 \leq 1$ a.e. on Ω, and $\theta_0 \in H_0^1$.

Next, we give the notion of weak solutions to (P).

Definition 2.1. A pair $[\eta, \theta]$ of functions $\eta : [0,T] \longrightarrow H^1$ and $\theta : [0,T] \longrightarrow H_0^1$ is a solution to (P) on $[0,T]$, if the following conditions (1)-(4) are satisfied:

(1) $\eta \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$, $\theta \in L^\infty(0,T;H_0^1)$ and $\alpha_0(\eta)\theta \in W^{1,2}(0,T;L^{\frac{3}{2}}) \cap L^2(0,T;W_0^{1,\frac{3}{2}})$.

(2) The following parabolic equation holds:

$$\begin{aligned} &\eta'(t) - \kappa \Delta_N \eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)| = 0 \text{ in } L^2 \\ &\text{for a.a. } t \in (0,T), \end{aligned} \tag{2.1}$$

where $\eta' := \frac{d\eta}{dt}$ and $\Delta_N : D(\Delta_N) := \{z \in H^2;\ \frac{\partial z}{\partial n} = 0 \text{ a.e. on } \Gamma\} \longrightarrow L^2$ is the Laplacian with homogeneous Neumann boundary condition.

(3) For any $z \in H_0^1$ and a.a. $t \in (0,T)$, the following variational inequality holds:

$$\begin{aligned} &\int_\Omega [\alpha_0(\eta)\theta]'(x,t)(\theta(x,t) - z(x))dx - \\ &\quad - \int_\Omega \alpha_0'(\eta(x,t))\eta'(x,t)\theta(x,t)(\theta(x,t) - z(x))dx + \\ &\quad + \nu \int_\Omega \nabla\theta(x,t) \cdot \nabla(\theta(x,t) - z(x))dx + \int_\Omega \alpha(\eta(x,t))|\nabla\theta(x,t)|dx \leq \\ &\leq \int_\Omega \alpha(\eta(x,t))|\nabla z(x)|dx, \end{aligned} \tag{2.2}$$

where $[\alpha_0(\eta)\theta]' := \frac{d}{dt}[\alpha_0(\eta)\theta]$.

(4) $\eta(0) = \eta_0$ and $[\alpha_0(\eta)\theta](0) = \alpha_0(\eta_0)\theta_0$ in L^2.

We should notice that the first and second terms of (2.2) yield

$$\int_\Omega \alpha_0(\eta(x,t))\theta'(x,t)(\theta(x,t)-z(x))dx,$$

if $\theta' := \frac{d\theta}{dt}$ exists in $L^2(0,T;L^2)$.

Our main result of this paper is stated as follows:

Theorem 2.1. *Assume* (A1)-(A4) *hold. Then, there is at least one solution* $[\eta,\theta]$ *of* (P) *in the sense of Definition* 2.1, *and* η *satisfies*

$$0 \le \eta \le 1 \qquad a.e \text{ on } Q_T.$$

Remark 2.1. In [12] some numerical experiments of (P) were tried in the case where $\hat{g}(\eta) := \frac{1}{2}(1-\eta)^2$, $\alpha_0(\eta) = \alpha(\eta) = \eta^2$ and Ω is a bounded domain in $\mathbf{R}^2$. Clearly, assumption (A1) is satisfied for $\alpha_0(\eta) = \eta^2$.

The main idea for the proof of Theorem 2.1 is to discuss the convergence of the following approximate problems $(\mathrm{P})_\delta$ with real parameter $\delta \in (0,1]$, as $\delta \downarrow 0$:

$$(\mathrm{P})_\delta \begin{cases} \eta_\delta' - \kappa\Delta\eta_\delta + g(\eta_\delta) + \alpha'(\eta_\delta)|\nabla\theta_\delta| = 0 & \text{a.e. in } Q_T, \\ \alpha_\delta(\eta_\delta)\theta_\delta' - \nu\Delta\theta_\delta - \operatorname{div}\left(\alpha(\eta_\delta)\dfrac{\nabla\theta_\delta}{|\nabla\theta_\delta|}\right) = 0 & \text{a.e. in } Q_T, \\ \dfrac{\partial\eta_\delta}{\partial n} = 0, \quad \theta_\delta = 0 & \text{a.e. on } \Sigma_T, \\ \eta_\delta(x,0) = \eta_0(x), \quad \theta_\delta(x,0) = \theta_0(x) & \text{for a.a. } x \in \Omega, \end{cases}$$

where $\alpha_\delta(r) := \alpha_0(r) + \delta$ for $r \in \mathbf{R}$.

In the next section we study problem $(\mathrm{P})_\delta$, and give some uniform estimates of solutions $[\eta_\delta, \theta_\delta]$ with respect to $\delta \in (0,1]$. In Section 4 we accomplish the proof of Theorem 2.1. Namely, we show that $[\eta_\delta,\theta_\delta]$ converges in a suitable sense as $\delta \downarrow 0$ and a limit function is a solution to (P).

3 – Approximate problems

We begin by defining the notion of weak solutions to $(\mathrm{P})_\delta$.

Definition 3.1. For each $\delta \in (0,1]$, a pair $[\eta_\delta,\theta_\delta]$ of functions $\eta_\delta : [0,T] \longrightarrow H^1$ and $\theta_\delta : [0,T] \longrightarrow H^1_0$ is a solution to $(\mathrm{P})_\delta$ on $[0,T]$, if the following conditions (1)-(4) are satisfied:

(1) $\eta_\delta \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ and $\theta_\delta \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1_0)$.

(2) The following parabolic equation holds:

$$\eta_\delta'(t) - \kappa\Delta_N\eta_\delta(t) + g(\eta_\delta(t)) + \alpha'(\eta_\delta(t))|\nabla\theta_\delta(t)| = 0 \text{ in } L^2 \quad \text{for a.a. } t \in (0,T). \tag{3.1}$$

(3) For any $z \in H_0^1$ and a.a. $t \in (0,T)$, the following variational inequality holds:

$$\begin{aligned}&(\alpha_\delta(\eta_\delta(t))\theta_\delta'(t), \theta_\delta(t) - z) + \nu\left(\nabla\theta_\delta(t), \nabla\theta_\delta(t) - \nabla z\right) + \\ &+ \int_\Omega \alpha(\eta_\delta(x,t))|\nabla\theta_\delta(x,t)|dx \leq \int_\Omega \alpha(\eta_\delta(x,t))|\nabla z(x)|dx,\end{aligned} \tag{3.2}$$

where $\theta_\delta' := \frac{d\theta_\delta}{dt}$ and $(\cdot,\cdot)$ stands for the usual inner product in L^2 or in $(L^2)^N$.

(4) $\eta_\delta(0) = \eta_0$ and $\theta_\delta(0) = \theta_0$ in L^2.

Here, in order to reformulate (3.2) as an evolution equation governed by subdifferentials, let us introduce a proper, l.s.c. (lower semi-continuous) and convex function $\varphi(\eta_\delta(t);\cdot)$ on L^2, depending on $\eta_\delta \in W^{1,2}(0,T;L^2)$, which is defined by

$$\varphi(\eta_\delta(t); z) := \begin{cases} \dfrac{\nu}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \int_\Omega \alpha(\eta_\delta(t))|\nabla z|dx & \text{if } z \in H_0^1, \\ \infty & \text{otherwise.}\end{cases} \tag{3.3}$$

Then, the variational inequality (3.2) can be rewritten in the following form:

$$\alpha_\delta(\eta_\delta(t))\theta_\delta'(t) + \partial\varphi(\eta_\delta(t);\theta_\delta(t)) \ni 0 \ \text{ in } L^2 \ \text{ for a.a. } t \in (0,T), \tag{3.4}$$

where $\partial\varphi(\eta_\delta(t);z)$ is the subdifferential of $\varphi(\eta_\delta(t);z)$ with respect to z in L^2. This is one of the keys in our approach to problem $(\mathrm{P})_\delta$.

According to the result in [10], we have the following proposition, which is concerned with the existence of solutions to $(\mathrm{P})_\delta$.

Proposition 3.1 (*cf.* [10]). *Assume* (A1)-(A4) *hold. Then, for each* $\delta \in (0,1]$, *there is at least one solution* $[\eta_\delta, \theta_\delta]$ *of* $(\mathrm{P})_\delta$ *in the sense of Definition* 3.1, *and* η_δ *satisfies*

$$0 \leq \eta_\delta \leq 1 \qquad a.e \text{ on } Q_T. \tag{3.5}$$

Moreover, there is a positive constant C_1 *independent of* $\delta \in (0,1]$ *such that*

$$\begin{aligned}&\|\eta_\delta'\|^2_{L^2(0,T;L^2)} + \|\sqrt{\alpha_\delta(\eta_\delta)}\theta_\delta'\|^2_{L^2(0,T;L^2)} + \|\nabla\eta_\delta\|^2_{L^\infty(0,T;L^2)} + \\ &+ \|\theta_\delta\|^2_{L^\infty(0,T;H_0^1)} \leq C_1\left\{\|\nabla\eta_0\|^2_{L^2} + \int_\Omega \hat{g}(\eta_0)dx + \|\theta_0\|^2_{H_0^1} + 1\right\}.\end{aligned} \tag{3.6}$$

Proof. The existence assertion of a solution $[\eta_\delta, \theta_\delta]$ satisfying (3.5) and (3.6) is due to [10, Theorem 2.1]. However, for the completeness, we repeat here a brief proof of (3.6).

For simplicity we omit the subscript $\delta \in (0,1]$ except for α_δ, namely $\eta := \eta_\delta$ and $\theta := \theta_\delta$. Now, we multiply the equation (3.1) by $\eta'(t)$ to get

$$\|\eta'(t)\|_{L^2}^2 + \frac{\kappa}{2}\frac{d}{dt}\|\nabla\eta(t)\|_{L^2}^2 + \frac{d}{dt}\int_\Omega \hat{g}(\eta(t))dx \le \int_\Omega |\alpha'(\eta(t))||\nabla\theta(t)||\eta'(t)|dx \qquad \text{for a.a. } t \in (0,T).$$

Hence, from (A2) and the Schwarz inequality it follows that

$$\|\eta'(t)\|_{L^2}^2 + \kappa\frac{d}{dt}\|\nabla\eta(t)\|_{L^2}^2 + 2\frac{d}{dt}\int_\Omega \hat{g}(\eta(t))dx \le L(\alpha)^2\|\nabla\theta(t)\|_{L^2}^2 \qquad \text{for a.a. } t \in (0,T). \tag{3.7}$$

Next multiply (3.4), which is equivalent to (3.2), by θ' to obtain

$$(\alpha_\delta(\eta(t))\theta'(t), \theta'(t)) + (\theta^*(t), \theta'(t)) = 0 \qquad \text{for a.a. } t \in (0,T), \tag{3.8}$$

where $\theta^*(t) \in \partial\varphi(\eta(t); \theta(t))$ in L^2 for a.a. $t \in (0,T)$. Here, we recall the following inequality:

$$\left|\frac{d}{dt}\varphi(\eta(t);\theta(t)) - (\theta^*(t), \theta'(t))\right| \le L(\alpha)\|\eta'(t)\|_{L^2}\|\nabla\theta(t)\|_{L^2} \qquad \text{for a.a. } t \in (0,T). \tag{3.9}$$

For the detailed proof of (3.9), see [10, Section 3]. Now, using the inequality (3.9), we get from (3.8) that

$$\|\sqrt{\alpha_\delta(\eta(t))}\theta'(t)\|_{L^2}^2 + \frac{d}{dt}\varphi(\eta(t);\theta(t)) \le L(\alpha)\|\eta'(t)\|_{L^2}\|\nabla\theta(t)\|_{L^2} \qquad \text{for a.a. } t \in (0,T). \tag{3.10}$$

Therefore, adding (3.7) and (3.10) and using (3.3), we get

$$\begin{aligned}
&\frac{1}{2}\|\eta'(t)\|_{L^2}^2 + \|\sqrt{\alpha_\delta(\eta(t))}\theta'(t)\|_{L^2}^2 + \\
&\quad + \frac{d}{dt}\left\{\kappa\|\nabla\eta(t)\|_{L^2}^2 + 2\int_\Omega \hat{g}(\eta(t))dx + \varphi(\eta(t);\theta(t))\right\} \le \\
&\le \frac{3}{2}L(\alpha)^2\|\nabla\theta(t)\|_{L^2}^2 \le \\
&\le \frac{3}{2}L(\alpha)^2 \cdot \frac{2}{\nu}\varphi(\eta(t);\theta(t)) \le \\
&\le \frac{3L(\alpha)^2}{\nu}\left\{\kappa\|\nabla\eta(t)\|_{L^2}^2 + 2\int_\Omega \hat{g}(\eta(t))dx + \varphi(\eta(t);\theta(t))\right\}
\end{aligned}$$

for a.a. $t \in (0,T)$. Applying Gronwall's lemma to the above inequality, we obtain (3.6) for some constant $C_1 > 0$ independent of $\delta \in (0,1]$. □

Corollary 3.1. *Let* $u_\delta := \alpha_\delta(\eta_\delta)\theta_\delta$ *and* $v_\delta := \sqrt{\alpha_\delta(\eta_\delta)}\theta_\delta$ *for each* $\delta \in (0,1]$. *Then, there is a positive constant* $C_2(\eta_0,\theta_0)$, *depending only on the initial data* η_0 *and* θ_0, *such that*

$$\|u'_\delta\|^2_{L^2(0,T;L^{\frac{3}{2}})} + \|u_\delta\|^2_{L^2(0,T;W_0^{1,\frac{3}{2}})} \le C_2(\eta_0,\theta_0), \ \forall \delta \in (0,1], \tag{3.11}$$

$$\|v'_\delta\|^2_{L^2(0,T;L^{\frac{3}{2}})} + \|v_\delta\|^2_{L^2(0,T;W_0^{1,\frac{3}{2}})} \le C_2(\eta_0,\theta_0), \ \forall \delta \in (0,1]. \tag{3.12}$$

Proof. We observe that

$$\begin{aligned} \left|[\sqrt{\alpha_\delta(\eta_\delta)}\theta_\delta]'\right| &\le \frac{|\alpha'_\delta(\eta_\delta)|}{2\sqrt{\alpha_\delta(\eta_\delta)}}|\eta'_\delta||\theta_\delta| + \sqrt{\alpha_\delta(\eta_\delta)}|\theta'_\delta| = \\ &= \frac{|\alpha'_0(\eta_\delta)|}{2\sqrt{\alpha_0(\eta_\delta)+\delta}}|\eta'_\delta||\theta_\delta| + \sqrt{\alpha_\delta(\eta_\delta)}|\theta'_\delta| \end{aligned} \tag{3.13}$$

and similarly

$$\left|\nabla[\sqrt{\alpha_\delta(\eta_\delta)}\theta_\delta]\right| \le \frac{|\alpha'_0(\eta_\delta)|}{2\sqrt{\alpha_0(\eta_\delta)+\delta}}|\nabla\eta_\delta||\theta_\delta| + \sqrt{\alpha_\delta(\eta_\delta)}|\nabla\theta_\delta|. \tag{3.14}$$

Since $\|\eta'_\delta\theta_\delta\|_{L^{\frac{3}{2}}} \le \|\eta'_\delta\|_{L^2}\|\theta_\delta\|_{L^6}$, it follows from (3.13) that

$$\begin{aligned} \|[\sqrt{\alpha_\delta(\eta_\delta)}\theta_\delta]'\|_{L^{\frac{3}{2}}} &\le M\|\eta'_\delta\|_{L^2}\|\theta_\delta\|_{L^6} + \|\sqrt{\alpha_\delta(\eta_\delta)}\theta'_\delta\|_{L^{\frac{3}{2}}} \le \\ &\le MC_3\|\eta'_\delta\|_{L^2}\|\theta_\delta\|_{H_0^1} + |\Omega|^{\frac{1}{6}}\|\sqrt{\alpha_\delta(\eta_\delta)}\theta'_\delta\|_{L^2}, \end{aligned}$$

where M is the same constant as in (A1), $|\Omega|$ stands for the volume of Ω and C_3 is a positive constant satisfying $\|z\|_{L^6} \le C_3\|z\|_{H_0^1}$ for all $z \in H_0^1$. Hence

$$\begin{aligned} &\|[\sqrt{\alpha_\delta(\eta_\delta)}\theta_\delta]'\|^2_{L^2(0,T;L^{\frac{3}{2}})} \le \\ &\le 2(M^2C_3^2 + |\Omega|^{\frac{1}{3}})\Big\{\|\theta_\delta\|^2_{L^\infty(0,T;H_0^1)}\|\eta'_\delta\|^2_{L^2(0,T;L^2)} + \\ &\quad + \|\sqrt{\alpha_\delta(\eta_\delta)}\theta'_\delta\|^2_{L^2(0,T;L^2)}\Big\}. \end{aligned} \tag{3.15}$$

Similarly, with $C_4 := \sup_{0\le r\le 1}\sqrt{\alpha_0(r)+1}$, we have by (3.14)

$$\begin{aligned} \|\nabla[\sqrt{\alpha_\delta(\eta_\delta)}\theta_\delta]\|_{L^{\frac{3}{2}}} &\le M\|\nabla\eta_\delta\|_{L^2}\|\theta_\delta\|_{L^6} + \|\sqrt{\alpha_\delta(\eta_\delta)}\nabla\theta_\delta\|_{L^{\frac{3}{2}}} \le \\ &\le MC_3\|\nabla\eta_\delta\|_{L^2}\|\theta_\delta\|_{H_0^1} + C_4|\Omega|^{\frac{1}{6}}\|\nabla\theta_\delta\|_{L^2}, \end{aligned}$$

whence

$$\begin{aligned} &\|\nabla[\sqrt{\alpha_\delta(\eta_\delta)}\theta_\delta]\|^2_{L^2(0,T;L^{\frac{3}{2}})} \le \\ &\le 2(M^2C_3^2 + TC_4^2|\Omega|^{\frac{1}{3}})\|\theta_\delta\|^2_{L^\infty(0,T;H_0^1)}\left(\|\nabla\eta_\delta\|^2_{L^2(0,T;L^2)} + 1\right). \end{aligned} \tag{3.16}$$

Now, by (3.15) and (3.16) it is easy to find a constant $C_2(\eta_0,\theta_0)$ such that (3.12) holds. In a way similar to the case (3.12), we obtain (3.11), too. ∎

4 – Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1 in three steps.

First step. By the uniform estimates (3.5), (3.6), (3.11) and (3.12), we see that $\{\theta_\delta\}$ is bounded in $L^\infty(0,T;H^1_0)$, $\{\eta_\delta\}$ is bounded in $W^{1,2}(0,T;L^2)\cap L^\infty(0,T;H^1)\cap L^\infty(0,T;L^\infty)$, hence is bounded in $L^2(0,T;H^2)$ by (3.1), and $\{u_\delta := \alpha_\delta(\eta_\delta)\theta_\delta\}$ and $\{v_\delta := \sqrt{\alpha_\delta(\eta_\delta)}\theta_\delta\}$ are bounded in $W^{1,2}(0,T;L^{\frac{3}{2}})\cap L^2(0,T;W_0^{1,\frac{3}{2}})$. Since $W_0^{1,\frac{3}{2}} \hookrightarrow L^2 \subset L^{\frac{3}{2}}$ and the first imbedding is compact, it follows from the Aubin's compactness theorem (*cf.* [13]) that we can extract a subsequence $\{\delta_n\}$ from $(0,1]$ with $\delta_n \downarrow 0$ (as $n\to\infty$) and find functions η,θ,χ,ζ such that

$$
\begin{aligned}
&(4.1) && \eta_n := \eta_{\delta_n} \to \eta \text{ in } C([0,T];L^2),\ \text{weakly in } W^{1,2}(0,T;L^2),\\
&(4.2) && \text{weakly in } L^2(0,T;H^2),\\
&(4.3) && \text{and weakly}^* \text{ in } L^\infty(0,T;H^1)\cap L^\infty(0,T;L^\infty),\\
&(4.4) && \theta_n := \theta_{\delta_n} \to \theta \text{ weakly}^* \text{ in } L^\infty(0,T;H^1_0),\\
&(4.5) && u_n := \alpha_{\delta_n}(\eta_{\delta_n})\theta_{\delta_n} \to \chi \text{ in } L^2(0,T;L^2),\\
&(4.6) && \text{and weakly in } W^{1,2}(0,T;L^{\frac{3}{2}})\cap L^2(0,T;W_0^{1,\frac{3}{2}}),\\
&(4.7) && v_n := \sqrt{\alpha_{\delta_n}(\eta_{\delta_n})}\theta_{\delta_n} \to \zeta \text{ in } L^2(0,T;L^2),\\
&(4.8) && \text{and weakly in } W^{1,2}(0,T;L^{\frac{3}{2}})\cap L^2(0,T;W_0^{1,\frac{3}{2}}).
\end{aligned}
$$

Clearly, $\eta \in W^{1,2}(0,T;L^2)\cap L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$, $\theta\in L^\infty(0,T;H^1_0)$, $\chi \in W^{1,2}(0,T;L^{\frac{3}{2}})\cap L^2(0,T;W_0^{1,\frac{3}{2}})\cap L^2(0,T;L^2)$, $\zeta \in W^{1,2}(0,T;L^{\frac{3}{2}})\cap L^2(0,T;W_0^{1,\frac{3}{2}})\cap L^2(0,T;L^2)$, $\eta(0)=\eta_0$ in L^2 and

$$
(4.9) \qquad 0\le\eta\le 1,\quad \text{hence } 0\le\alpha_0(\eta)\le C_5 := \sup_{0\le r\le 1}\alpha_0(r) \ \text{ a.e on } Q_T.
$$

Moreover, by (4.1)-(4.3),

$$
(4.10) \qquad \alpha_{\delta_n}(\eta_{\delta_n}) \to \alpha_0(\eta) \quad \text{in } C([0,T];L^2) \text{ and weakly}^* \text{ in } L^\infty(0,T;L^\infty).
$$

It is easy to see from (4.4), (4.5), (4.7) and (4.10) that

$$
(4.11) \qquad
\begin{aligned}
u_n &= \alpha_{\delta_n}(\eta_{\delta_n})\theta_{\delta_n} \to \alpha_0(\eta)\theta \quad \text{weakly in } L^2(0,T;L^2),\\
v_n &= \sqrt{\alpha_{\delta_n}(\eta_{\delta_n})}\theta_{\delta_n} \to \sqrt{\alpha_0(\eta)}\theta \quad \text{weakly in } L^2(0,T;L^2).
\end{aligned}
$$

Accordingly, we have by (4.11) that $\chi = \alpha_0(\eta)\theta := u$ and $\zeta = \sqrt{\alpha_0(\eta)}\theta := v$. By the way, on account of the Aubin's compactness theorem, (4.6) implies that

$$
(4.12) \qquad u_n \to u = \alpha_0(\eta)\theta \text{ in } C([0,T];L^{\frac{3}{2}}) \text{ and hence } u(0)=\alpha_0(\eta_0)\theta_0 \text{ in } L^2.
$$

Second step. In this step we show

$$\theta_n \to \theta \quad \text{in } L^2(0,T;H_0^1) \text{ (as } n \to \infty), \tag{4.13}$$

which is the key convergence in proving that the pair of functions $[\eta, \theta]$ is a solution of (P) in the sense of Definition 2.1.

Since $[\eta_n, \theta_n]$ is a solution of $(\mathrm{P})_{\delta_n}$, the following variational inequality holds (*cf.* (3.2)):

$$\begin{aligned}&\int_0^T (\alpha_n(\eta_n(t))\theta_n'(t), \theta_n(t)-\theta_m(t))\,dt + \nu\int_0^T (\nabla\theta_n(t), \nabla(\theta_n(t)-\theta_m(t)))\,dt + \\ &+\int_0^T\int_\Omega \alpha(\eta_n(x,t))|\nabla\theta_n(x,t)|dxdt \leq \int_0^T\int_\Omega \alpha(\eta_n(x,t))|\nabla\theta_m(x,t)|dxdt,\end{aligned} \tag{4.14}$$

where $\alpha_n(\cdot) := \alpha_{\delta_n}(\cdot)$ for simplicity. Adding (4.14) and the inequality obtained by exchanging n for m in (4.14), we get:

$$\int_0^T (\alpha_n(\eta_n(t))\theta_n'(t) - \alpha_m(\eta_m(t))\theta_m'(t), \theta_n - \theta_m)dt + \nu\int_0^T \|\nabla(\theta_n - \theta_m)\|_{L^2}^2 dt \leq$$

$$\leq L(\alpha)\int_0^T \|\eta_n - \eta_m\|_{L^2}\|\nabla(\theta_n - \theta_m)\|_{L^2} dt,$$

whence

$$\begin{aligned}&\frac{\nu}{2}\int_0^T \|\nabla(\theta_n - \theta_m)\|_{L^2}^2 dt \leq \\ &\leq \left|\int_0^T (\alpha_n(\eta_n(t))\theta_n'(t) - \alpha_m(\eta_m(t))\theta_m'(t), \theta_n - \theta_m)dt\right| + \\ &\quad + \frac{L(\alpha)^2}{2\nu}\int_0^T \|\eta_n - \eta_m\|_{L^2}^2 dt =: \\ &=: I_1 + I_2.\end{aligned} \tag{4.15}$$

Clearly, we observe from (4.1) that $I_2 \to 0$ as $n, m \to \infty$. As to I_1 we note the following inequality:

$$\begin{aligned}
I_1 \leq & \left| \int_0^T (\sqrt{\alpha_n(\eta_n(t))}\theta_n'(t), \sqrt{\alpha_n(\eta_n(t))}\theta_n(t) - \sqrt{\alpha_n(\eta_n(t))}\theta_m(t))dt \right| + \\
& + \left| \int_0^T (\sqrt{\alpha_m(\eta_m(t))}\theta_m'(t), \sqrt{\alpha_m(\eta_m(t))}\theta_n(t) - \sqrt{\alpha_m(\eta_m(t))}\theta_m(t))dt \right| \leq \\
\leq & \left| \int_0^T (\sqrt{\alpha_n(\eta_n(t))}\theta_n'(t), \sqrt{\alpha_n(\eta_n(t))}\theta_n(t) - \sqrt{\alpha_m(\eta_m(t))}\theta_m(t))dt \right| + \\
& + \left| \int_0^T \left(\sqrt{\alpha_n(\eta_n(t))}\theta_n'(t), \left[\sqrt{\alpha_m(\eta_m(t))} - \sqrt{\alpha_n(\eta_n(t))}\right] \theta_m(t)\right) dt \right| + \\
& + \left| \int_0^T \left(\sqrt{\alpha_m(\eta_m(t))}\theta_m'(t), \left[\sqrt{\alpha_m(\eta_m(t))} - \sqrt{\alpha_n(\eta_n(t))}\right] \theta_n(t)\right) dt \right| + \\
& + \left| \int_0^T (\sqrt{\alpha_m(\eta_m(t))}\theta_m'(t), \sqrt{\alpha_n(\eta_n(t))}\theta_n(t) - \sqrt{\alpha_m(\eta_m(t))}\theta_m(t))dt \right| =: \\
=: & \ I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

In these inequalities, we infer from (3.6) and (4.7) that $I_3 \to 0$ and $I_6 \to 0$ as $n,\ m \to \infty$. On the other hand, we observe from (3.6) and (4.10) with the help of the Lebesgue's dominated convergence theorem and a Sobolev inequality that

$$\begin{aligned}
& \left\| \left[\sqrt{\alpha_m(\eta_m)} - \sqrt{\alpha_n(\eta_n)}\right] \theta_m \right\|^2_{L^2(0,T;L^2)} \leq \\
& \leq \int_0^T \left\| \sqrt{\alpha_m(\eta_m(t))} - \sqrt{\alpha_n(\eta_n(t))} \right\|^2_{L^4} \|\theta_m(t)\|^2_{L^4}\, dt \leq \\
& \leq C_6^2 \|\theta_m\|^2_{L^\infty(0,T;H_0^1)} \int_0^T \left\| \sqrt{\alpha_m(\eta_m(t))} - \sqrt{\alpha_n(\eta_n(t))} \right\|^2_{L^4} dt \longrightarrow \\
& \longrightarrow 0 \qquad \text{as } n,\ m \to \infty,
\end{aligned} \tag{4.16}$$

where C_6 is a positive constant such that

$$\|z\|_{L^4} \leq C_6 \|z\|_{H_0^1}, \quad \forall z \in H_0^1.$$

Therefore, by (3.6) and (4.16) that $I_4 \to 0$ as $n,\ m \to \infty$. Similarly, we see that $I_5 \to 0$ as $n,\ m \to \infty$. Thus, letting $n,\ m \to \infty$ in (4.15), we see that $\nabla(\theta_n - \theta_m) \to 0$ in $L^2(0,T;L^2)$ as $n,\ m \to \infty$. This implies that $\theta_n \to \theta$ in $L^2(0,T;H_0^1)$. Thus (4.13) holds.

Third step. In this step, we accomplish the proof of Theorem 2.1, namely we show that the pair of functions $[\eta, \theta]$ is a solution of (P) in the sense of

Definition 2.1. Conditions (1) and (4) of Definition 2.1 are already seen in the first and second steps (*cf.* (4.1)-(4.9), (4.12)). We shall prove conditions (2) and (3) of Definition 2.1.

Since $[\eta_n, \theta_n]$ is a solution of $(\mathrm{P})_{\delta_n}$, the following equation holds (*cf.* (3.1)):

$$\begin{aligned}&\eta_n'(t) - \kappa\Delta_N\eta_n(t) + g(\eta_n(t)) + \alpha'(\eta_n(t))|\nabla\theta_n(t)| = 0 \ \text{ in } L^2\\ &\text{for a.a. } t \in (0,T).\end{aligned} \tag{4.17}$$

Now, letting $n \to \infty$ in (4.17), we easily see by (4.1), (4.2) and (4.13) that

$$\eta'(t) - \kappa\Delta_N\eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)| = 0 \ \text{ in } L^2 \ \text{ for a.a. } t \in (0,T).$$

Thus (2) of Definition 2.1 has been obtained.

Finally, we show the variational inequality (2.2). By (3.2) we have for each $n = 1, 2, \cdots$, and $z \in L^2(0,T;H_0^1)$

$$\begin{aligned}&\int_0^T (\alpha_n(\eta_n(t))\theta_n'(t), \theta_n(t) - z(t))\,dt + \nu\int_0^T (\nabla\theta_n(t), \nabla\theta_n(t) - \nabla z(t))\,dt +\\ &+\int_0^T\int_\Omega \alpha(\eta_n(x,t))|\nabla\theta_n(x,t)|dxdt \le \int_0^T\int_\Omega \alpha(\eta_n(x,t))|\nabla z(x,t)|dxdt.\end{aligned} \tag{4.18}$$

Here, note that the first term in the left hand side of (4.18) is written in the following form:

$$\begin{aligned}&\int_0^T (\alpha_n(\eta_n(t))\theta_n'(t), \theta_n(t) - z(t))\,dt =\\ &= \int_0^T\int_\Omega u_n'(t)(\theta_n(x,t) - z(x,t))dxdt +\\ &- \int_0^T\int_\Omega \alpha_n'(\eta_n(x,t))\eta_n'(x,t)\theta_n(x,t)(\theta_n(x,t) - z(x,t))dxdt.\end{aligned} \tag{4.19}$$

Here we note from (4.1)-(4.3) that

$$\alpha_n'(\eta_n)\eta_n' = \alpha_0'(\eta_n)\eta_n' \to \alpha_0'(\eta)\eta' \quad \text{weakly in } L^2(0,T;L^2).$$

Also, by (4.13), the compact imbedding $H_0^1 \hookrightarrow L^4$ and the Lebesgue's dominated convergence theorem, we easily see that

$$\theta_n^2 \to \theta^2 \quad \text{and} \quad \theta_n z \to \theta z \quad \text{in } L^2(0,T;L^2).$$

Therefore, by (4.6), (4.13), (4.19) and the above convergences, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T (\alpha_n(\eta_n(t))\theta_n'(t), \theta_n(t) - z(t))dt =\\ &= \int_0^T\int_\Omega u'(x,t)(\theta(x,t) - z(x,t))dxdt -\\ &- \int_0^T\int_\Omega \alpha_0'(\eta(x,t))\eta'(x,t)\theta(x,t)(\theta(x,t) - z(x,t))dt.\end{aligned} \tag{4.20}$$

Hence, passing to the limit as $n \to \infty$ in (4.18), we infer from (4.20) that $[\eta, \theta]$ satisfies that

$$\begin{aligned}
&\int_0^T \int_\Omega u'(x,t)(\theta(x,t) - z(x,t))dxdt- \\
&\quad - \int_0^T \int_\Omega \alpha_0'(\eta(x,t))\eta'(x,t)\theta(x,t)(\theta(x,t) - z(x,t))dxdt+ \\
&\quad + \nu \int_0^T (\nabla\theta(t), \nabla(\theta(t) - z(t)))\, dt + \int_0^T \int_\Omega \alpha(\eta(x,t))|\nabla\theta(x,t)|dxdt \leq \\
&\leq \int_0^T \int_\Omega \alpha(\eta(x,t))|\nabla z(x,t)|dxdt, \qquad \forall z \in L^2(0,T;H_0^1).
\end{aligned}$$

This is equivalent to the statement that (2.2) holds for a.a. $t \in (0,T)$ and all $z \in H_0^1$. Hence, we conclude that $[\eta, \theta]$ is a solution of (P) in the sense of Definition 2.1. Thus, the proof of Theorem 2.1 has been completed.

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