Rendiconti di Matematica, Serie VII
Volume 29, Roma (2009), 83-95

A compactness result for quasilinear elliptic equations by mountain pass techniques

MARIO GIRARDI – MICHELE MATZEU

Dedicated to Umberto Mosco, on the occasion of his 70th birthday
Al mio carissimo Maestro, con affetto e stima – Michele
Al carissimo amico Umberto – Mario

ABSTRACT: *A class of solutions to some quasilinear elliptic equations is considered. Some estimates due to some Mountain Pass techniques allow to obtain a compactness result for this class of solutions, with a suitable continuous dependence on the data.*

1 – Introduction

In [2], [3] a method to solve a quasilinear elliptic problem of the type

$$(*)\qquad \begin{cases} -\Delta u(x) = f(x, u(x), \nabla u(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$

(where Ω is a bounded open subset of $\mathbb{R}^N, N \geq 3$) was proposed. It is based on the consideration of "approximated" Mountain Pass solutions u^m to some semilinear problems associated with the quasilinear one. When the parameter

KEY WORDS AND PHRASES: *Quasilinear elliptic equations – Mountain Pass techniques – Iteration methods – Continuous dependence of the solutions on the data*
A.M.S. CLASSIFICATION: 35J60 – 49R99 – 35R99.
The authors are supported by M.U.R.S.T., Project "Variational Methods and Nonlinear Differential Equations".

m goes to $+\infty$, it is proved that $\{u^m\}$ converges to a classical solution of the quasilinear problem. In the following, an analogous way of approximating a quasilinear problem by a sequence of semilinear ones was applied in order to find periodic solutions of some quasilinear nonautonomous second order Hamiltonian systems (see [4]).

The aim of the present paper is to give a compactness result for the solutions obtained by this method for a class of quasilinear elliptic problems. More precisely, one considers, for any $n \in \mathbb{N}$, a problem of the type

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}^{(n)}(x)\frac{\partial u_n}{\partial x_i}\right) = f_n(x, u_n(x),\, \nabla u_n(x)) & x \in \Omega \\ u_n(x) = 0 & x \in \partial\Omega \end{cases}$$

where $a_{ij}^{(n)} \in C^1(\overline{\Omega})$ $\forall n \in \mathbb{N}$, $i, j = 1, \ldots, N$, are equibounded with respect to n and satisfy the following uniform ellipticity condition

$$\sum_{i,j=1}^{N} a_{i_j}^{(n)}(x)\xi_i\xi_j \geq \lambda|\xi|^2\,, \quad \forall\, \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N\,, \quad x \in \overline{\Omega}\,, \quad \text{for some } \lambda > 0,$$

and f_n satisfies a list of hypotheses of the same type as f in $(*)$, but with some suitable "uniformity" with respect to n. Then one can state that, if $\{f_n\}$ converges to f in a suitable way (in particular, if $\{f_n\}$ converges to f uniformly on each compact subset of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, and $a_{ij}^{(n)} \to a_{ij}$ in $C^1(\overline{\Omega})$), then a subsequence of the solutions u_n, obtained by Mountain Pass techniques as in [3], actually converges to a classical solution u of the limit problem

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) = f(x, u(x)\,,\ \nabla u(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

In order to state the result, we apply, for any $n \in \mathbb{N}$, the approximation procedure proposed in [3], which exhibits a solution u_n of the quasilinear problem as the limit of solutions $\{u_m^n\}_m$ of semilinear ones. The significant originality with respect to the result of [3] is that the suitable "uniformity" of assumptions on f_n with respect to n enable to give some estimates on u_n^m which are uniform not only with respect to m, but also with respect to n. This allows to go to the limit on a subsequence of $\{u_n\}$ and to obtain a classical solution of the limit problem.

1. Let us consider the following problem, for any $n \in \mathbb{N}$,

$$(P_n) \qquad \begin{cases} A_n\, u_n(x) = f_n(x, u_n(x), \nabla u_n(x)) & x \in \Omega \\ u_n(x) = 0 & x \in \partial\Omega \end{cases}$$

where:

- Ω is a sufficiently smooth bounded open subset of $\mathbb{R}^n (N \geq 3)$
- $A_n : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the operator defined as

$$A_n\, v(x) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}^{(n)}(x)\frac{\partial v(x)}{\partial x_i}\right) \quad \forall\, v \in H_0^1(\Omega)\ ,\ x \in \Omega$$

  for some $a_{ij}^{(n)} \in C^1(\overline{\Omega})$, with $a_{ij}^{(n)} = a_{ji}^{(n)}$ $(i, j = 1, \ldots, N)$ satisfying the equiuniform ellipticity condition

  (1) $\sum_{i,j=1}^{N} a_{ij}^{(n)}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N,\ \forall, x \in \overline{\Omega}$, for some $\lambda > 0$

  and the equiboundedness condition

  (2) $\sum_{i,j=1}^{N} a_{ij}^{(n)}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \quad \forall\, x \in \overline{\Omega}$, for some $\Lambda > 0$

- $f_n : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ verifies the following conditions:

  (3) f_n is locally Lipschitz continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, uniformly w.r. to $n \in \mathbb{N}$

  (4) $\lim_{t\to 0} \frac{f_n(x,t,\xi)}{t} = 0$ uniformly w.r. to $\overline{\Omega}$ and to each bounded subset of $\mathbb{R}^N$

  (5) $\exists\, a_1 > 0,\ p \in \left(1, \frac{N+1}{N-1}\right),\ r \in (0,1)$, sufficiently small in dependence of p and N, such that

$$|f_n(x,t,\xi)| \leq a_1(1 + |t|^p)(1 + |\xi|^r) \quad \forall\, x \in \overline{\Omega}\ ,\ t \in \mathbb{R}\ ,\ \xi \in \mathbb{R}^N\ ,\ n \in \mathbb{N}$$

  (6) $\exists\, \vartheta > 2$ such that

$$0 < \vartheta F_n(x,t,\xi) \leq t f_n(x,t,\xi) \quad \forall\, x \in \overline{\Omega},\ t \in \mathbb{R}\backslash\{0\},\ \xi \in \mathbb{R}^N,\ n \in \mathbb{N}$$

  where

$$F_n(x,t,\xi) = \int_0^t f_n(x,\tau,\xi)d\tau$$

  (7) $\exists\, a_2, a_3 > 0$ such that

$$F_n(x,t,\xi) \geq a_2|t|^\vartheta - a_3 \quad \forall\, x \in \overline{\Omega}\ ,\ t \in \mathbb{R},\ \xi \in \mathbb{R}^N,\ n \in \mathbb{N}$$

(8) $\exists\, \overline{R} > 0$, depending on $p, \vartheta, a_1, a_2, a_3, N, |\Omega|$ in an explicite way, such that the *smallest positive* numbers $L'_{\overline{R}}, L''_{\overline{R}}$ for which

$$|f_n(x,t_1,\xi) - f_n(x,t_2,\xi)| \le L'_{\overline{R}}|t_1 - t_2| \quad \forall\, x \in \overline{\Omega},\ |t_1| \le \overline{R},\ |t_2| \le \overline{R},\ |\xi| \le \overline{R}$$

$$|f_n(x,t,\xi_1) - f_n(x,t,\xi_2)| \le L''_{\overline{R}}|\xi_1 - \xi_2| \quad \forall\, x \in \overline{\Omega},\ |t| \le \overline{R},\ |\xi_1| \le \overline{R},\ |\xi_2| \le \overline{R}$$

verify the relation

$$\lambda_1^{-1} L'_{\overline{R}} + \lambda_1^{-1/2} L''_{\overline{R}} < \lambda$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ on $H_0^1(\Omega)$.

The following theorem can be proved (see also [3])

THEOREM 1. *Let* (1), ... ,(8) *be satisfied. Then, for any* $n \in \mathbb{N}$, *there exists a classical solution* u_n *of* (P_n) *such that*

$$\|u_n\|_{H_0^1(\Omega)} \ge c_1 > 0 \quad \forall\, n \in \mathbb{N} \tag{9}$$

and

$$\|u_n\|_{C^2(\overline{\Omega})} \le c_2 \quad \forall\, n \in \mathbb{N} \tag{10}$$

Let us suppose now that

(11) $a_{ij}^{(n)} \to a_{ij}$ in $C^1(\overline{\Omega})$ as $n \to +\infty$, for $i,j = 1,\dots,N$
and consider the following problem:

$$(P) \qquad \begin{cases} Au(x) = f(x, u(x),\ \nabla u(x)), & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$

where A is defined as A_n with $a_{ij}^{(n)}$ replaced by a_{ij} and $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$.

Then the following stability theorem holds:

THEOREM 2. *Let* (1), ... ,(8), (11) *be satisfied and let* u_n *be the solution of* (P_n) *given by Theorem* 1, *for any* $n \in \mathbb{N}$. *Moreover let the following condition be satisfied*:

(12) *For any sequence* $\{v_n\}$ *converging to some* v *in* $C^{1,\beta}(\overline{\Omega})$ *for any* $\beta \in (0,1)$, *one has*

$$f_n(x, v_n(x, \nabla v_n(x)) \to f(x, v(x), \nabla v(x)) \text{ as } n \to +\infty,\ \forall\, x \in \overline{\Omega}.$$

Then $\{u_n\}$ *possesses a subsequence converging in* $C^2(\overline{\Omega})$ *to a nontrivial solution* u *of* (P).

REMARK 1. One can replace assumption (6) with the following one

(6′) There exist some compact subset K of $\mathbb{R} \times \mathbb{R}^N$ and some numbers $\vartheta > 2$ and $c_3 > 0$ such that

$$\begin{cases} 0 < F_n(x,t,\xi) \le c_3 \quad \forall n \in \mathbb{N} \quad \forall (x,t,\xi) \in \overline{\Omega} \times K, & t \neq 0 \\ 0 < \vartheta F_n(x,t,\xi) \le t f_n(x,t,\xi) \quad \forall n \in \mathbb{N} \quad \forall (x,t,\xi) \in \overline{\Omega} \times ((\mathbb{R} \times \mathbb{R}^N) \backslash K), & \\ & t \neq 0. \end{cases}$$

REMARK 2. As a standard example of a function f_n satisfying (1), ... ,(8), one can choose

$$f_n(x,t,\xi) = \varphi(x)|t|^{(p+1)\frac{n+1}{n}}(1 + |\xi|^{\frac{rn}{n+1}})$$

where $\varphi \in \mathrm{Lip}(\overline{\Omega})$, with $\varphi(x) > 0 \quad \forall x \in \overline{\Omega}$.

In order to obtain other examples, one can use the alternative condition (6′) to (6) expressed in Remark 1, by standard truncature arguments at zero and at infinity in the (t, ξ) variable.

REMARK 3. A sufficient condition assuring (12) is that $\{f_n\}$ converges to f uniformly on each compact subset of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. In particular, this condition is verified by the function f_n given in Remark 2 with

$$f(x,t,\xi) = \varphi(x)|t|^{p+1}(1 + |\xi|^r).$$

REMARK 4. Let us note that, if one considers the critical exponent of $H_0^1(\Omega)$, that is

$$2^* = \frac{2N}{N-2}$$

one can check that the number p appearing in condition (5) is less that $2^* - 1$, since $\frac{N+1}{N-1} < \frac{N+2}{N-2}$.

First let us fix $n \in \mathbb{N}$, and consider, for any $w \in H_0^1(\Omega)$, the following semilinear problem

$$(P_n^w) \qquad \begin{cases} A_n\, u_n^w(x) = f_n(x, u_n^w(x),\ \nabla w(x)) & x \in \Omega \\ u_n^w(x) = 0 & x \in \partial\Omega. \end{cases}$$

As A_n is selfadjoint, any weak solution u_n^w of (P_n^w) is a critical point of the functional

$$I_n^w(x) = \frac{1}{2}\int_\Omega (A_n v, v) - \int_\Omega F_n(x, v(x), \nabla w(x)) \quad v \in H_0^1(\Omega).$$

Note that, denoting the H_0^1-norm of v by $\|v\|$, one has, by (1),

$$I_n^w(v) \ge \frac{\lambda}{2}\|v\|^2 - \int_\Omega F_n(x, v(x), \nabla w(x)) \quad \forall v \in H_0^1(\Omega) \tag{13}$$

and, by (2),

$$I_n^w(v) \leq \frac{\Lambda}{2}\,\|v\|^2 - \int_\Omega F_n(x, v_x), \nabla\, w(x)) \quad \forall\, v \in H_0^1(\Omega). \tag{14}$$

One can state that I_n^w has a Mountain Pass critical point $u_n^w \not\equiv 0$ for any w belonging to the following set $C_{R,\alpha}$ with a fixed $R > 0$ and $\alpha \in (0, 1)$

$$C_{R,\alpha} = \{w \in C^{1,\alpha}(\overline{\Omega}) : \|w\|_{C^{1,\alpha}(\overline{\Omega})} \leq R\}.$$

LEMMA 1. *For any $w \in C_{R,\alpha}$, there exist $\rho_R, \sigma_R > 0$, depending on R, but not on w, such that*

$$I_n^w(v) \geq \sigma_R \quad \forall\, v \in H_0^1(\Omega) \text{ with } \|v\| = \rho_R.$$

PROOF. From (4) it follows that, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$F_n(x, v(x), \nabla\, w(x)) < \frac{1}{2}\varepsilon|v(x)|^2 \quad \forall\, x \in \Omega\ ,\ |v(x)| \leq \delta$$

hence, from (5)

$$\begin{aligned}\int_\Omega F_n(x, v(x), \nabla w(x)) &\leq \frac{\varepsilon}{2}\int_\Omega |v(x)|^2 + K(1+R)^r \int_\Omega |v(x)|^{p+1} \leq \\ &\leq K'\left(\frac{\varepsilon}{2} + K(1+R)^r\|v\|^{p-1}\right)\|v\|^2 \quad \forall\, v \in H_0^1(\Omega)\end{aligned}$$

with a positive constant number K' depending on the Poincaré and Sobolev inequalities.

By choosing $\|v\| = \left(\frac{\varepsilon}{2K(1+R)^r}\right)^{\frac{1}{p-1}} = \rho_R$, one gets

$$\int_\Omega F_n(x, v(x), \nabla w(x)) \leq K'\varepsilon\|v\|^2.$$

Therefore the thesis follows from (13) by taking $\varepsilon < \frac{\lambda}{2K'}$ and $\sigma_R = \left(\frac{\lambda}{2} - K'\varepsilon\right)\rho_R^2$.

LEMMA 2. *Let $w \in C_{r,\alpha}$ and let us fix $\tilde{v}$ in $H_0^1(\Omega)\backslash\{0\}$. Then there exists some $\tilde{s} > 0$ independent of w and R such that*

$$I_n^w(s\tilde{v}) \leq 0 \quad \forall\, s \geq \tilde{s}, \tag{15}$$

so $\overline{v} = \tilde{s}\tilde{v}$ *verifies*

$$\|\overline{v}\| > \rho_R \quad I_n^w(\overline{v}) \le 0.$$

Proof. It follows from (7), (13) that

$$I_n^w(s\tilde{v}) \le \frac{\Lambda}{2}s^2\|\tilde{v}\|^2 - a_2|s|^\vartheta \int_\Omega |\tilde{v}|^\vartheta + a_3|\Omega|.$$

As $\vartheta > 2$, one can find $\tilde{s}$ in such a way that (15) holds.

Proposition 1. *Let $w \in C_{R,\alpha}$. Then there exists a Mountain Pass critical point u_n^w for I_n^w on $H_0^1(\Omega)$, that is*

$$I_n^w(u_n^w) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_n^w(\gamma(t)) \tag{16}$$

where

$$\Gamma = \{\gamma \in C^0([0,1];\ H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = \overline{v}\}$$

and

$$I_n^w(u_n^w) \ge \sigma_R > 0,$$

so $u_n^w \not\equiv 0$.

Proof. It is an immediate consequence of the Ambrosetti and Rabinowitz theorem (see [1]), as $I_n^w(0) = 0$, Lemma 1 and Lemma 2 hold, and the Palais–Smale condition is satisfied by I_n^w due to the continuous embedding of $H_0^1(\Omega)$ in $L^p(\Omega)$ (as $p < \frac{N+2}{N-2}$, see Remark 4).

At this point, one can give the

Proof of Theorem 1. Let us proceed by steps.

Step 1. Let $w \in C_{R,\alpha}$ and let u_n^w a Mountain Pass solution of (P_n^w) given by Proposition 1. Then there exists a positive number $c_1(R)$ depending on R but not on w, nor on n, such that

$$\|u_n^w\| \ge c_1(R).$$

Proof. Actually the estimate holds for any critical point u_n^w of I_n^w with $u_n^w \not\equiv 0$ and it does not depend by the Mountain Pass nature of u_n^w. Indeed, putting $v = u_n^w$ in the relation

$$\int_\Omega (A_n u_n^w, v) = \int_\Omega F_n(x, u_n^w(x), \nabla w(x))v(x) \quad \forall\, v \in H_0^1(\Omega)$$

one gets, by using (3),

$$\|u_n^w\|^2 \le \Lambda \int_\Omega F_n(x, u_w^n(x), \nabla w(x)) u_n^w(x). \tag{17}$$

From (4), (5) it follows that, for any $\varepsilon > 0$, there exists a positive number $c_{\varepsilon,R}$, depending on ε and R, but not on w nor on n, such that

$$|F_n(x, u_w^n(x), \nabla w(x))| \le \varepsilon |u_w^n(x)| + c_{\varepsilon,R} |u_w^n(x))|^p \quad \forall\, x \in \overline{\Omega}.$$

Then, by (17),

$$\|u_n^w\|^2 \le \varepsilon \int_\Omega |u_n^w|^2 + \Lambda c_{\varepsilon,R} \int_\Omega |u_n^w|^{p+1}$$

so, by Poincaré and Sobolev inequalities, one gets

$$(1 - c\varepsilon)^2 \|u_n^w\|^2 \le \tilde{c}_{\varepsilon,R} \|u_n^w\|^{p+1}$$

which implies the thesis if one chooses $\varepsilon < \frac{1}{c}$, as $p + 1 > 2$.

Step 2. Let $w \in C_{\mathbb{R},\alpha}$ and let u_n^w a Mountain Pass solution of (P_n^w) given by Proposition 1. Then there exists a positive number C independent of w, R and n such that

$$\|u_n^w\| \le C.$$

Proof. From the characterization (16) of u_n^w, by choosing as γ in Γ the segment line joining 0 with $\overline{v}$ in $H_0^1(\Omega)$, one gets

$$I_n^w(u_n^w) \le \sup_{s \ge 0} I_n^w(s\overline{v})$$

hence, by (7), (2)

$$I_n^w(u_n^w) \le \sup_{s \ge 0} \left\{ \frac{\Lambda^2 s^2}{2} \|\overline{v}\|^2 - a_2 |s|^\vartheta \int_\Omega |v|^\vartheta + a_3 |\Omega| \right\}.$$

As $\vartheta > 2$, this supremum is indeed a maximum and it does not depend on n, R, w, hence

$$I_n^w(u_n^w) \le \text{const.} \quad \forall\, R > 0 \quad w \in C_{R,\alpha} \quad \forall n \in \mathbb{N}.$$

At this point, using the criticality of u_n^w, (6), (17), (2), one gets

$$\frac{1}{2}\|u_n^w\|^2 \le \text{const.} + \frac{1}{\vartheta} \int_\Omega f_n(x, u_n^w(x), \nabla w(x)) u_n^w(x) \le \text{const.} + \frac{1}{\vartheta}\|u_n^w\|^2$$

and the thesis follows from the fact that $\vartheta > 2$.

STEP 3. For any $R > 0$ and any $w \in C_{R,\alpha}$, u_n^w belongs to $C^2(\overline{\Omega})$. Moreover there exist two numbers $\overline{\alpha} \in (0,1)$, $\mu > 0$ such that

$$\|u_n^w\|_{C^{1,\overline{\alpha}}} \leq \mu(1+R)^{r_{p,N}}, \tag{18}$$

with a suitable number $r_{p,N} \in (0,1)$, $\forall\, n \in \mathbb{N}$, $R > 0$, $w \in C_{R,\overline{\alpha}}$.

PROOF. At first, as u_n^w is a solution of the equation

$$A_n u_n^w = f(x, u_n^w(x), \nabla w(x)), \tag{19}$$

in particular u_n^w belongs to L^{2^*}, then $f(x, u_n^w(x), \nabla w(x))$, by (5), the definition of $C_{R,\alpha}$ and the Nemitsky theorem, belongs to $L^{2^*/p}$, therefore, by the Agmon–Douglis–Nirenberg theorem, u_n^w belongs to $H^{2,2^*/p}$. At this point, one applies Morrey or Sobolev embedding theorems. Actually, one has three possibilities:

$$\frac{p}{2^*} - \frac{2}{N} < 0 \tag{20}$$

$$\frac{p}{2^*} - \frac{2}{N} = 0 \tag{21}$$

$$\frac{p}{2^*} - \frac{2}{N} > 0. \tag{22}$$

In case (20), the Morrey's theorem directly yields the α_1-Hölder continuity of u_n^w for some $\alpha_1 \in (0,1)$. In case (21), u_n^w belongs to $L^q(\Omega)$ $\forall\, q \in [1,+\infty)$, so that, by (19), (5) and the Agmon-Douglis-Nirenberg theorem, u_n^w belongs to $H^{2,q}$ for any $q \in [1,+\infty)$, in particular for $q > N$, which still yields the α_1–Hölder continuity of u_n^w. Finally, if (22) holds, putting $q = \left(\frac{p}{2^*} - \frac{2}{N}\right)^{-1}$, the Sobolev embedding theorem implies that u_n^w belongs to $L^q(\Omega)$ with

$$q = \left(\frac{p}{2^*} - \frac{2}{N}\right)^{-1} = \frac{2N}{(N-2)p-4}, \tag{23}$$

so that $f_n(x, u_n^w(x), w(x))$ belongs to $L^{q/p}$ with q given by (23), thus u_n^n belongs to $H^{2,q/p}$. At this point one continues the same argument as above and, after a finite number j of steps, one arrives to some cases analogous to (20) and (21), which yields the α_1-Hölder continuity of u_n^w, otherwise one can define a number analogous to q as

$$\frac{2N}{((N-2)p-4)p-4)p-4\cdots)} \tag{24}$$

with p repeated exactly j times. Actually it is possible to check that the denominator of the number appearing in (24) becomes *not positive* if j is sufficiently large, as a consequence of the fact that $p < \frac{N+2}{N-2}$ (see Remark 4). Let us note $j(p, N)$ the first of these numbers.

At this point one comes back to a situation as (20) and the same argument still implies the conclusion that u_n^w is α_1-Hölder continuous for some $\alpha_1 \in (0, 1)$. Moreover, Step 2, (5) and the very definition of C_{R,α_1} imply that there exists $\mu_0 > 0$ such that

$$\|u_n^w\|_{C^{0,\alpha_1}} \leq \mu_0(1 + R)^{r'_{p,N}} \ , \quad \forall\, n \in \mathbb{N} \quad \forall\, w \in C_{R,\alpha_1} \tag{25}$$

where $r'_{p,N} = r(j(p, N))$ is a suitable positive number. Obviously if r is sufficiently small w.r. to p and $N, r'_{p,N}$ is less than 1.

As for ∇u_n^w, one starts from the fact that it belongs to $H^{1,2^*/p}$, as $u_n^w \in H^{2,2^*/p}$. Now the alternatives analogous to (20), (21), (22) are the following:

$$\frac{p}{2^*} - \frac{1}{N} < 0 \tag{26}$$

$$\frac{p}{2^*} - \frac{1}{N} = 0 \tag{27}$$

$$\frac{p}{2^*} - \frac{1}{N} > 0. \tag{28}$$

At this point, one argues exactly as before for u_n^w, arriving to the same conclusion due to the fact that, in this case, the denominator of the number

$$\frac{4N}{(2N-2)p-4)p-4)p-4)\cdots)}$$

with p repeated j times for some $j \in \mathbb{N}$, becomes *not positive* for j sufficiently large, as a consequence, in this case, of the condition $p < \frac{N+1}{N-1}$. Therefore one can still conclude that $\nabla\, u_n^w$ belong to $C^{0,\alpha_2}(\overline{\Omega})$, for some $\alpha_2 \in (0, 1)$ and that, by Step 2, (5) and the very definition of C_{R,α_2}, there exists some $\mu_1 > 0$ such that

$$\|\nabla\, u_n^w\|_{C^{0,\alpha_2}} \leq \mu_1(1 + R)^{r''_{p,N}}, \quad \forall\, n \in \mathbb{N} \quad \forall\, w \in C_{R,\alpha_2} \tag{29}$$

with a suitable number $r''_{p,N} \in (0, 1)$. Finally, one gets (18), as a consequence of (25) and (29), by choosing $\overline{\alpha} = \min(\alpha_0, \alpha_1)$, $\mu = \max(\mu_0, \mu_1)$ and $r_{p,N} = \min(r'_{p,N}, r''_{p,N})$.

Step 4. There exists some $\overline{R} > 0$ such that

$$w \in C_{\overline{R},\overline{\alpha}} \Rightarrow u_n^w \in C_{\overline{R},\overline{\alpha}} \quad \forall n \in \mathbb{N}. \tag{30}$$

Proof. It is an obvious consequence of (18), by choosing, in $C_{R,\alpha}$, $\alpha = \overline{\alpha}$ and $\overline{R}$ sufficiently large in such a way that

$$\mu(1+\overline{R})^{r_{p,N}} \leq \overline{R}.$$

Note that such a number $\overline{R}$ exists, since $r_{p,N} \in (0,1)$.

At this point, it is very natural to introduce an iterative scheme in the following way. Let $\overline{R}$ and $\overline{\alpha}$ be given by Step 4 and let u_n^0 be arbitrarily fixed in $C_{\overline{R},\overline{\alpha}}$. Let us define u_n^m as a Mountain Pass solution of the following problem, for any $m, n \in \mathbb{N}$:

$$(P_n^m) \qquad \begin{cases} A_n u_n^m(x) = f_n(x, u_n^m(x)\,,\ \nabla u_n^{m-1}(x)) & x \in \Omega \\ u_n^m(x) = 0 & x \in \partial\Omega. \end{cases}$$

Obviously, by Step 4, one has $u_n^m \in C_{\overline{R},\overline{\alpha}}$ for any $m, n \in \mathbb{N}$. Now we are in a position to give the proof of Theorem 1.

Proof of Theorem 1. Let m and n be fixed in $\mathbb{N}$, let $u_n^0 \in C_{\overline{R},\overline{\alpha}}$, with $\overline{R}, \overline{\alpha}$ given by Step 4, and let u_n^m be a Mountain Pass solution of (P_n^m). First of all we prove that the *whole* sequence $\{u_n^m\}_m$ strongly converges in $H_0^1(\Omega)$ for all $n \in \mathbb{N}$. Indeed, using (P_n^m) and (P_n^{m+1}), one gets

$$\int_\Omega A_n u_n^{m+1}(u_n^{m+1} - u_n^m) = \int_\Omega f_n(x, u_n^{m+1}(x), \nabla u_n^m(x))(u_n^{m+1} - u_n^m) \tag{31}$$

$$\int_\Omega A_n u_n^m(u_n^{m+1} - u_n^m) = \int_\Omega f_n(x, u_n^m(x), \nabla u_n^{m-1}(x))(u_n^{m+1} - u_n^m). \tag{32}$$

So, by subtracting (32) from (31) and taking into account (1), (8), one gets

$$\begin{aligned} \lambda \|u_n^{m+1} - u_n^m\|^2 \leq \Bigg(& L'_{\overline{R}} \int_\Omega (u_n^{m+1}(x) - u_n^m(x))^2 + \\ & + L''_{\overline{R}} \int_\Omega |\nabla u_n^m(x) - \nabla u_n^{m-1}(x)|\,|u_n^{m+1}(x) - u_n^m(x)| \Bigg) \leq \\ & \leq \lambda_1^{-1} L'_{\overline{R}} \|u_n^{m+1} - u_n^m\|^2 + \lambda^{-1/2} L''_{\overline{R}} \|u_n^m - u_n^{m-1}\| \|u_n^{m+1} - u_n^m\|, \end{aligned}$$

thus

$$(\lambda - \lambda_1^{-1} L'_{\overline{R}}) \|u_n^{m+1} - u_n\| \leq \lambda_1^{-1/2} L''_{\overline{R}} \|u_n^m - u_n^{m-1}\|$$

then $\{u_n^m\}_m$ is a Cauchy sequence in $H_0^1(\Omega)$, as a consequence of (8), therefore $\{u_n^m\}_m$ strongly converges in $H_0^1(\Omega)$ to some u_n for any $n \in \mathbb{N}$. On the other side, by (3), Step 4, (5) and the Schauder's theorem, one has $u_n^m \in C^{2,\overline{\alpha}}(\overline{\Omega})$ for any $n \in \mathbb{N}$, and

$$\|u_n^m\|_{C^{2,\overline{\alpha}}} \leq \text{const } \forall\, n, m \in \mathbb{N}.$$

Therefore, by the Ascoli–Arzela's theorem, the whole sequence $\{u_n^m\}_m$ converges in $C^2(\overline{\Omega})$ to u_n, which satisfies the estimate

$$\|u_n\|_{C^2} \leq c_2.$$

At this point, it is easy to verify, by the regularity properties of A_n and f_n, that u_n is a classical solution of (P_n). Finally estimate (9) (so the nontriviality of u_n) derives from Step 1 with $c_1 = c_1(\overline{R})$.

Now it is quite easy to give the proof of Theorem 2.

Proof of Theorem 2. First of all, by (10), (5) and the Schauder's theorem, one has

$$\|u_n\|_{C^{2,\beta}} \leq \text{const } \forall\, n \in \mathbb{N} \quad \text{and} \quad \forall\, \beta \in (0,1)$$

then, still by Ascoli–Arzela's theorem, there exists a subsequence $\{u_{n_k}\}_k$ converging in $C^2(\overline{\Omega})$ to some $\overline{u}$, with

$$\|\overline{u}\|_{H_0^1(\Omega)} \geq c_1.$$

Actually $\overline{u}$ is a (nontrivial) solution of problem (P), due to (11) and (12).

Remark 5. Assumption (3) can be weakened for the proof of Theorem 1, 2. Indeed in the proof of Theorem 1 one only uses the Lipschitz continuity with respect the variables t, ξ for $|t| \leq \overline{R}$, $|\xi| \leq \overline{R}$. Similarly, in Theorem 2, one uses this property only for $|t| \leq \tilde{R}$, $|\xi| \leq \tilde{R}$, where $\tilde{R}$ is determined by the use of the Schauder's Theorem in the proof of this theorem.

REFERENCES

[1] A. Ambrosetti – P. H. Rabinowitz: *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (1973), 349-381.

[2] D. De Figueiredo – M. Girardi – M. Matzeu: *Semilinear elliptic equations with dependence on the gradient via Mountain-Pass techniques*, Diff. Integral Equations, **17** (2004), 119-126.

[3] M. Girardi –M. Matzeu: *Positive and negative solution of a quasi–linear equa tion by a Mountain-Pass method and truncature techniques*, Nonlinear Anal., **59** (2004), 199-210.

[4] M. Girardi – M. Matzeu: *Existence of periodic solutions for some second order quasilinear Hamiltonian systems*, Rend. Lincei Mat. Appl., **18** (2007), 1-9.

Lavoro pervenuto alla redazione il 1 dicembre 2008
ed accettato per la pubblicazione il 2 febbraio 2009.
Bozze licenziate il 18 marzo 2009

Indirizzo degli Autori:

Mario Girardi – Dipartimento di Matematica – Università degli Studi di Roma Tre – Largo S. Leonardo Murialdo – 00146 Roma, Italy
E-mail: girardi@mat.uniroma3.it

Michele Matzeu – Dipartimento di Matematica – Università Roma "Tor Vergata" – Viale della Ricerca Scientifica – 00133 Roma, Italy
E-mail: matzeu@mat.uniroma2.it