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Partial results on extending the Hopf Lemma

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Dedicated with affection to Umberto Mosco on his 70th birthday

Abstract: *We proved in [1] a generalization of the Hopf Lemma in one dimension. In this paper we present two conjectures as possible extensions to higher dimensions, and give a very partial answer.*

1 – Introduction

In [1], Theorem 3, the authors proved, in one dimension, a generalization of the Hopf Lemma, and the question arose if it could be extended to higher dimensions. In this paper we present two conjectures as possible extensions, and give a very partial answer. We write this paper to call attention to the problem.

The one dimensional result of [1] was

Theorem 1. *Let $u \geq v$ be positive C^3, C^2 functions respectively on $(0,b)$ which are also in $C^1([0,b])$. Assume*

$$u(0) = \dot{u}(0) = 0 \tag{1}$$

and

$$\text{either } \dot{u} > 0 \text{ on } (0,b) \text{ or } \dot{v} > 0 \text{ on } (0,b).$$

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Main condition:

(2) $$\textit{whenever } u(t) = v(s) \textit{ for } 0 < t \le s < b, \ \textit{there } \ddot{u}(t) \le v''(s),$$

(*here* $\cdot = \frac{d}{dt}$, $' = \frac{d}{ds}$).

Then

(3) $$u \equiv v \quad \textit{on } [0, b].$$

The proof given in [1] is somewhat roundabout. In the Appendix we present a more direct one, but it is still a bit tricky. In [1], it was assumed that u is of class C^2 on $(0, b)$, but its proof there actually required that u be of class C^3.

Turn now to higher dimensions. Let $u \ge v$ be C^∞ functions of (t, y), $y \in \mathbb{R}^n$, in

$$\Omega = \{(t, y) \mid 0 < t < 1, |y| < 1\},$$

and C^∞ in the closure of Ω. Assume that

(4) $$u > 0, \ v > 0, \ u_t > 0 \quad \text{in } \Omega$$

and

(5) $$u(0, y) = 0 \quad \text{for } |y| < 1.$$

We impose a main condition:

(6) $$\begin{aligned}&\text{whenever } u(t, y) = v(s, y) \text{ for } 0 < t \le s < 1, |y| < 1,\\ &\text{there } \Delta u(t, y) \le \Delta v(s, y).\end{aligned}$$

Under some additional conditions we wish to conclude that

(7) $$u \equiv v.$$

Here are two conjectures, in decreasing strength, which would extend Theorem 1. In each, we consider u and v as above.

Conjecture 1. Assume, in addition, that

(8) $$u_t(0, 0) = 0.$$

Then (3) holds:

$$u \equiv v.$$

Conjecture 2. In addition to (8) assume that

(9) $$u(t, 0) \text{ and } v(t, 0) \text{ vanish at } t = 0 \text{ of finite order.}$$

Then

$$u \equiv v.$$

We have not succeeded in proving them. What we present here is a partial answer to Conjecture 2: Here let k, l be the orders of the first $t-$derivative of u, v respectively at the origin which are not zero. Clearly $k \leq l$.

THEOREM 2. *In addition to the conditions of Conjecture 2, we assume the annoying condition*

$$\nabla_y u_{tt}(0,0) = 0. \tag{10}$$

Then $u \equiv v$ provided $k = 2$ or 3.

For $k < 3$ the proof is simple, but not that for $k = 3$.

We will always use Taylor series expansions for u, v, in t,

$$\begin{aligned} u &= a_1(y)t + a_2(y)\frac{t^2}{2!} + a_3(y)\frac{t^3}{3!} + \cdots, \\ v &= b_1(y)t + b_2(y)\frac{t^2}{2!} + b_3(y)\frac{t^3}{3!} + \cdots. \end{aligned} \tag{11}$$

The conditions on u and v are as follows

$$0 \leq u(t) - v(t) = (a_1 - b_1)t + (a_2 - b_2)\frac{t^2}{2!} + (a_3 - b_3)\frac{t^3}{3!} + \cdots \tag{12}$$

where

$$u(t,y) = v(s,y), \quad t \leq s,$$

i.e.

$$a_1(y)t + a_2(y)\frac{t^2}{2!} + a_3(y)\frac{t^3}{3!} + \cdots = b_1(y)s + b_2(y)\frac{s^2}{2!} + b_3(y)\frac{s^3}{3!} + \cdots, \tag{13}$$

there

$$\begin{aligned} 0 \geq \Delta u - \Delta v = (a_2 - b_2) + t(\Delta a_1 + a_3) - s(\Delta b_1 + b_3) + \\ + \frac{t^2}{2}(\Delta a_2 + a_4) - \frac{s^2}{2}(\Delta b_2 + b_4) + \cdots. \end{aligned} \tag{14}$$

We first present the proof of the more difficult case $k = 3$. It takes up Sections 2-5. In Section 6 we treat the case $k = 2$.

2 –

STEPS OF THE PROOF. We are assuming $k = 3$. The proof consists of two steps:

STEP A. This consists in proving

THEOREM 3. *Under the conditions of Theorem* 2*, where* $k = 3$*, we have*

$$l = 3, \text{ and } b_3(0) = a_3(0). \tag{15}$$

STEP B. In this step we consider our condition

$$u(t, y) = v(s, y) \text{ for } 0 \leq t \leq s. \tag{16}$$

Since $u_t > 0$ for $t > 0$, we may solve this for $t = t(s, y)$. Assuming that u is not identically equal to v, for

$$\tau(s, y) = s - t(s, y) \tag{17}$$

we derive, from (6), an elliptic differential inequality for $\tau(s, y)$. Using a comparison function we prove that

$$\tau(s, 0) \geq \epsilon s \text{ for some } 0 < \epsilon \text{ small.} \tag{18}$$

On the other hand, for $y = 0$, we have, by (15) and (11),

$$u(t, 0) = v(s, 0)$$

i.e. after dividing by $a_3(0)$,

$$t^3 + \text{higher order terms} = s^3 + \text{higher order terms}.$$

Hence

$$t(s, 0) = s + \text{higher order terms}.$$

But this contradicts (18), and the proof of Theorem 2 is then complete.

For $k = 3$, we will first present the proof of Step B; it seems more interesting to us.

3 –

Proof of (18) in case $k = 3$. Here we assume that (15) holds, *i.e.*

$$b_3(0) = a_3(0) > 0$$

and first derive the elliptic inequality for $\tau(s, y) = s - t(s, y)$, where $t(s, y)$ is the solution of

$$u(t(s, y), y) = v(s, y). \tag{19}$$

Differentiating this we find, setting $v_i = \partial_{y_i} v$,

$$\begin{aligned} v_s &= u_t t_s, & v_{ss} &= u_t t_{ss} + u_{tt} t_s^2, \\ v_i &= u_t t_i + u_i, & v_{ii} &= u_t t_{ii} + 2u_{ti} t_i + u_{tt} t_i^2 + u_{ii}. \end{aligned}$$

Hence

$$0 \le \Delta v(s, y) - \Delta u(t, y) = u_t \Delta t + 2u_{ti} t_i + u_{tt}(|\nabla t|^2 - 1).$$

In terms of $\tau = s - t$, this becomes, after dividing by u_t,

$$F(\tau) := \Delta\tau - \frac{u_{tt}}{u_t}(|\nabla\tau|^2 - 2\tau_s) + 2\frac{u_{ti}}{u_t}\tau_i \le 0. \tag{20}$$

This is the differential inequality for τ.

We will consider this in the region

$$D = \{(s, y) \mid s > K|y|^2\}, \quad K \text{ large, near the origin,} \tag{21}$$

and use a comparison function:

$$h = s + s^{1+\delta} - C|y|^2, \quad \delta = \frac{1}{4}, C = K + 1. \tag{22}$$

Near the origin we have

$$h(s, y) \le 0 \ \text{ where } s = K|y|^2. \tag{23}$$

We assume now that v is not identically equal to u near the origin and argue by contradiction.

Observe first that if $v(\bar{s}, \bar{y}) = u(\bar{s}, \bar{y})$ for some $\bar{y}$ and some $\bar{s} > 0$ then $\tau(\bar{s}, \bar{y}) = 0$. But near $(\bar{s}, \bar{y})$, $\tau \ge 0$ satisfies the inequality (20), which is elliptic there. By the strong maximum principle, it would follow that $\tau \equiv 0$ there. Then, again by the strong maximum principle $\tau \equiv 0$ everywhere, *i.e.* $v \equiv u$ near the origin, for $t \ge 0$. Contradiction.

Thus we may assume that $\tau > 0$ for $s > 0$.

The basic result of this section is

Lemma 1. *For $0 < \epsilon$ small, $\tau \ge \epsilon h$ in D near the origin.*

Once the lemma is proved, it follows that $\tau(s,0) \geq \epsilon s$ for $0 < s$ small, *i.e.*, (18) holds, and Step B would be complete.

PROOF OF LEMMA 1. Choose positive $\epsilon \leq 1/10$, so small that on $D \cap \{s = c\}$, c to be fixed – where τ is positive, and hence bounded away from zero –

$$\tau \geq \epsilon h, \tag{24}$$

ϵ depends on c.

In view of (23) it follows then that

$$\tau - \epsilon h \geq 0, \text{on the boundary of } \; G = D \cap \{s < c\}.$$

We now use the maximum principle, suitably to show that

$$\tau \geq \epsilon h \quad \text{in } G. \tag{25}$$

Completing the proof of Lemma 1. We argue by contradiction.

Suppose $\tau - \epsilon h$ has a negative minimum at some point $(\bar{s}, \bar{y})$ in G. There, of course,

$$\tau < \epsilon(s + s^{1+\delta}) < 2\epsilon s,$$

and so

$$t = s - \tau \geq (1 - 2\epsilon)s \geq \frac{4}{5}s. \tag{26}$$

At $(\bar{s}, \bar{y})$, $\nabla\tau = \epsilon\nabla h$ and

$$\Delta\tau \geq \epsilon\Delta h.$$

Therefore, there, ϵh satisfies the inequality

$$\Delta(\epsilon h) - \frac{u_{tt}}{u_t}(\epsilon^2|\nabla h|^2 - 2\epsilon h_s) + 2\epsilon\frac{u_{ti}}{u_t}h_i \leq 0$$

i.e. after dividing by ϵ,

$$\begin{aligned} F[\epsilon h] = \Delta h - \frac{u_{tt}}{u_t}\left\{\epsilon[(1 + (1+\delta)s^\delta)^2 + 4C^2|y|^2] - 2 - 2(1+\delta)s^\delta\right\} + \\ - 4C\frac{u_{ti}y_i}{u_t} \leq 0. \end{aligned} \tag{27}$$

For small ϵ and c (which may depend on K),

(28) the expression $\{\;\}$ in (27) is negative.

We will choose K to ensure that

$$u_{tt}(t(\bar{s},\bar{y}),\bar{y}) \geq 0. \tag{29}$$

We have

$$u_{tt} = a_2 + a_3 t + \cdots . \tag{30}$$

Since $a_3(0) > 0$, near the origin,

$$a_3(t,y) \geq \frac{a_3(0)}{2}. \tag{31}$$

Recall that $u_t > 0$, *i.e.*

$$0 < a_1 + t a_2 + \frac{t^2}{2} a_3 + \cdots . \tag{32}$$

Thus $a_1 \geq 0$ and $a_1 = O(|y|^2)$. By (10), and it is only here that (10) is used,

$$|a_2| \leq A|y|^2 \tag{33}$$

for some $A > 0$.

Now, still at $(\bar{s},\bar{y})$, and for $t = t(\bar{s},\bar{y})$, we have

$$\begin{aligned} u_{tt} = a_2 + a_3 t + \cdots &\geq \frac{a_3(0)}{2} t - A|y|^2 + O(t^2) \geq \\ &\geq \frac{a_3(0)}{4} t - A|y|^2 \qquad \text{(for } c \text{ small)} \geq \\ &\geq \frac{a_3(0)}{5} s - A|y|^2 \end{aligned}$$

by (26). We require

$$K \geq \frac{5A}{a_3(0)}.$$

Then (29) holds:

$$u_{tt} \geq 0,$$

(we may suppose $K > 1$.)

Consequently, from (27) we find

$$\Delta h - \frac{4C}{u_t} u_{ti} y_i \leq 0 \qquad \text{at } (\bar{s},\bar{y}). \tag{34}$$

Next, by a well known elementary inequality which uses the fact that the second order derivatives in y of u_t are bounded in absolute value we have, for some constant B,

$$|u_{ti}| \leq B\sqrt{u_t} \qquad \forall\ i.$$

So

$$M := \frac{4C}{u_t}|u_{ti}y_i| \leq \frac{4CB|y|}{\sqrt{u_t}}. \tag{35}$$

Now, recall, $t = t(\bar{s}, \bar{y})$,

$$u_t = a_1 + a_2 t + \frac{a_3 t^2}{2} + \cdots \geq t\left(a_2 + \frac{a_3 t}{2} + \cdots\right) \geq t\left(-A|y|^2 + \frac{a_3(0)}{4}t\right)$$

by (33), for t small. So

$$u_t \geq t\left(-\frac{A}{K}s + \frac{a_3(0)}{4}t\right) \geq \frac{4}{5}s\left(-\frac{A}{K}s + \frac{a_3(0)}{5}s\right)$$

by (26). Hence

$$u_t \geq \frac{a_3(0)}{10}s^2 \tag{36}$$

provided

$$\frac{A}{K} \leq \frac{a_3(0)}{100}. \tag{37}$$

Inserting (36) in (35) we find

$$M = \left|\frac{4C}{u_t}\sum u_{ti}y_i\right| \leq \frac{L|y|}{s} \tag{38}$$

where

$$L = \frac{4\sqrt{10}CB}{\sqrt{a_3(0)}}.$$

Thus, by (21),

$$M \leq \frac{L}{\sqrt{Ks}}.$$

We now insert this in (34) and, computing Δh, we find

$$\delta(1+\delta)s^{\delta-1} - 2nC \leq \frac{4\sqrt{10}}{\sqrt{a_3(0)}}\frac{K+1}{\sqrt{K}}\frac{B}{\sqrt{s}}.$$

But for $\delta = 1/4$, and c restricted still further if necessary, we see that this is impossible.

Remark 1. Our use of the maximum principle is somewhat unusual. Normally, one would prove that $F[\epsilon h]$, in (27) is positive in G; in fact we do not know how to prove that. But, as we see, it suffices only to show that it is positive at $(t(\bar{s}, \bar{y}), \bar{y})$.

4 – Step A

4.1 – We turn now to Step A

Let

$$\hat{a}_i(y) \text{ be the lowest order terms of } a_i(y) \tag{39}$$

in its Taylor expansion; $\hat{a}_i$ is a homogeneous polynomial. We know that

$$\deg \hat{a}_1, \deg \hat{b}_1, \deg(\hat{a}_2 - \hat{b}_2) \geq 2, \tag{40}$$

since, by (14), $\hat{a}_2 - \hat{b}_2$ is non-positive.

Our aim is to prove, in this and the next section, that if $k = 3$ then

$$l = 3 \text{ and } b_3(0) = a_3(0). \tag{41}$$

We will constantly use (12)-(14).

Proof that if $l = 3$ then $b_3(0) = a_3(0)$. Since $u \geq v > 0$ in Ω, necessarily

$$a_3(0) \geq b_3(0) > 0.$$

In (13) set $y = 0$ and solve for $t = t(s)$. Clearly

$$t = \left(\frac{b_3(0)}{a_3(0)}\right)^{\frac{1}{3}} s + O(s^2).$$

Inserting this value for $t(s)$ in (14) we find, by looking at the coefficients,

$$0 \geq \left(\frac{b_3(0)}{a_3(0)}\right)^{\frac{1}{3}} (\Delta\hat{a}_1(0) + a_3(0)) - (\Delta\hat{b}_1(0) + b_3(0)),$$

i.e.

$$(b_3)^{\frac{1}{3}} \Delta\hat{a}_1 - (a_3)^{\frac{1}{3}} \Delta\hat{b}_1 + (b_3)^{\frac{1}{3}} a_3 - (a_3)^{\frac{1}{3}} b_3 \leq 0, \qquad \text{at } y = 0. \tag{42}$$

Since $a_3 \geq b_3 > 0$ at $y = 0$, we infer that

$$(b_3)^{\frac{1}{3}} \Delta\hat{a}_1 - (a_3)^{\frac{1}{3}} \Delta\hat{b}_1 \leq 0, \qquad \text{at } y = 0. \tag{43}$$

Now $\hat{a}_1 \geq \hat{b}_1 \geq 0$. This implies $\Delta\hat{a}_1(0) \geq \Delta\hat{b}_1(0) \geq 0$. If both $= 0$ then (42) implies $a_3(0) = b_3(0)$.

Then, since $\Delta\hat{a}_1(0) > 0$, it follows that

$$\Delta\hat{b}_1(0) > 0. \tag{44}$$

In particular, $\deg \hat{b}_1 = \deg \hat{a}_1 = 2$.

Next, at a point y where $\hat{b}_1(y) > 0$, take

$$s = K\hat{a}_1(y), \ \ K \text{ large.}$$

Then from (13) we solve for $t = t(s)$ and find, looking at terms of various degrees in y,

$$t = K\hat{b}_1(y) + \circ(|y|^2).$$

Insert this in (14); we obtain, looking at terms of second degree in y, and using the fact that K is arbitrarily large,

$$0 \geq \hat{b}_1(y)(\Delta\hat{a}_1(0) + a_3(0)) - \hat{a}_1(y)(\Delta\hat{b}_1(0) + b_3(0)). \tag{45}$$

Since the right hand side is a homogeneous quadratic, its Laplacian is ≤ 0, *i.e.*

$$0 \geq \Delta\hat{b}_1(\Delta\hat{a}_1 + a_3(0)) - \Delta\hat{a}_1(\Delta\hat{b}_1 + b_3(0)),$$

so

$$a_3(0)\Delta\hat{b}_1 - b_3(0)\Delta\hat{a}_1 \leq 0.$$

Using (43) it follows, then, that

$$a_3^{\frac{2}{3}} b_3^{\frac{1}{3}} \Delta\hat{a}_1 \leq b_3 \Delta\hat{a}_1$$

which implies (41):

$$b_3(0) = a_3(0).$$

From now on we assume $l > 3$ and prove that this is impossible.

4.2 – The case $l > 3$

Claim 1. In this case

$$b_1 = O(|y|^4). \tag{46}$$

Proof. Suppose not, then $\hat{b}_1$ has degree 2 since by the positivity of v, $\hat{b}_1 \geq 0$. $\hat{a}_1$ also has degree 2 since $a_1 \geq b_1$. The proof above of (45) still works, and yields

$$0 \geq \hat{b}_1(\Delta\hat{a}_1 + a_3(0)) - \hat{a}_1\Delta\hat{b}_1. \tag{47}$$

Taking trace we find

$$0 \geq \Delta\hat{b}_1 a_3(0)$$

i.e. $\hat{b}_1 = 0$ – recall that $\hat{b}_1 \geq 0$. Contradiction. The claim is proved.

Next, set $y = 0$ and solve for $t(s)$ in (13). We find

$$t = \left(\frac{6}{l!}\frac{b_l(0)}{a_3(0)}\right)^{1/3} s^{l/3} + \circ(s^{l/3}).$$

Inserting this in (14) we find, at $y = 0$, since $\Delta\hat{b}_1 = 0$,

$$0 \geq \left(\frac{6}{l!}\right)^{\frac{1}{3}} \left(\frac{b_l}{a_3}\right)^{1/3} s^{l/3}(\Delta a_1 + a_3) - s^2(\Delta b_2 + b_4) + \circ(s^{l/3} + s^2).$$

Consequently

$$l \geq 6.$$

We shall make use of the following

Lemma 2. *Let $v \geq 0$ be given by* (11) *and assume that l is the order of the first $t-$derivative of v which is > 0 at the origin. Let m be the first value of i* (*if it exists*) *such that*

$$\deg \hat{b}_i = 1.$$

Suppose that for some j, $1 \leq j \leq (l+4)/3$,

$$\deg \hat{b}_i \geq 3 \quad \textit{for } i < j.$$

Then

$$m \geq \frac{l+j}{2}. \tag{48}$$

Proof. Clearly $j \leq m < l$. At some y, $\hat{b}_m(y) < 0$. Then, at that y, if we set

$$s = |y|^a, \qquad 0 < a \text{ to be chosen,}$$

we have, since $v \geq 0$,

$$0 \leq \sum_{i<j} \frac{1}{i!} b_i(y) s^i + \sum_{j \leq i \leq m-1} \frac{1}{i!} b_i(y) s^i + \sum_{m \leq i \leq l-1} \frac{1}{i!} b_i(y) s^i + O(s^l). \tag{49}$$

In case $j = 1$ we find

$$0 \leq -\frac{1}{2m!}\hat{b}_m s^m = O(|y|^2 s) + O(s^l). \tag{50}$$

Suppose that (48) does not hold, *i.e.*

$$m < \frac{l+1}{2}.$$

Then there exists $a > 0$ such that deg LHS of (50) $<$ deg of each term on RHS of (50). One easily verifies this using the fact that

$$\frac{1}{l-m} < \frac{1}{m-1}.$$

But then (50) is impossible.

In case $j > 1$ we find from (49) and the fact that $\hat{b}_1 = O(|y|^4)$, that

$$0 \le -\frac{\hat{b}_m(y)|y|^{am}}{2m!} \le O(|y|^{4+a}) + O(|y|^{3+2a}) + O(|y|^{2+ja}) + O(|y|^{la}). \tag{51}$$

Suppose that (48) does not hold, *i.e.*

$$m < \frac{l+j}{2}. \tag{52}$$

Claim. There exists $a > 0$ such that the degree of LHS of (51) $<$ the degree of each term on RHS of (51).

If so, (52) is impossible.

Proof of Claim. The claim asserts the existence of $a > 0$ such that

$$\begin{cases} 1 + ma < 4 + a, \textit{i.e. } a < \dfrac{3}{m-1}, & \\ 1 + ma < 3 + 2a, \textit{i.e. } a < \frac{2}{m-2} & \text{if } m > 2, \\ 1 + ma < 2 + ja, \textit{i.e. } a < \dfrac{1}{m-j} & \text{if } m > j, \\ 1 + ma < la, \text{ i.e. } a > \dfrac{1}{l-m}. & \end{cases} \tag{53}$$

If $m = 2$, the second and third inequalities automatically hold, so does the third if $m = j$. Otherwise it says that

$$a < \frac{1}{m-j}.$$

One easily verifies using (52) that

$$\frac{1}{l-m} < \begin{cases} \dfrac{3}{m-1}, & \text{if } m = j = 2, \\ \min\left\{\dfrac{3}{m-1}, \dfrac{2}{m-2}\right\} & \text{if } m = j \ge 3, \\ \min\left\{\dfrac{3}{m-1}, \dfrac{2}{m-2}, \dfrac{1}{m-j}\right\} & \text{if } m > j. \end{cases}$$

It follows that the required a exists. Hence, Lemma 2 is proved.

5 –

We come now to a crucial step.

Proposition 1. *If $l \geq 3i$, $l > 3$, $i \geq 1$, then*

$$\deg \hat{b}_i \geq 3.$$

Using the proposition we may now give the

Completion of the proof of Theorem 3. At $y = 0$, if we solve (13) for t we find as before,

$$t = As^{l/3} + \circ(s^{l/3}),$$

where

$$A = \left(\frac{6}{l!}\frac{b_l}{a_3}\right)^{1/3}.$$

Inserting this in (14) and using Proposition 1 we see that

$$0 \geq As^{l/3}(\Delta a_1 + a_3) + O(s^{[l/3]+1}).$$

But this is impossible, and Theorem 3 is proved.

Proof of Proposition 1. By Lemma 2,

$$\deg \hat{b}_i > 1 \quad \text{for } i < \frac{l}{2} + 1.$$

Suppose the proposition is false. Then there is a first $j \leq l/3$ such that

$$\deg \hat{b}_j = 2.$$

We will show that this is impossible.

By (46), $j \geq 2$.

Claim. $\hat{b}_j \geq 0$.

If not, at some y, $\hat{b}_j(y) < 0$. Then, setting

$$s = |y|^a,$$

we have, using Lemma 2, and (46),

$$0 < -\frac{\hat{b}_j |y|^{ja}}{2j!} = O(|y|^{4+a}) + O(|y|^{2a+3}) + O(|y|^{1+a(l+j)/2}) + O(|y|^{al}). \tag{54}$$

Setting $a > 1/j$ but very close to $1/j$, we see that the degree in y of LHS of (54) $<$ the degree of each term on RHS of (54), *i.e.* (here we use $j \leq l/3$)

$$2 + ja < \min\{4 + a, 2a + 3, 1 + a(l + j)/2, al\}. \tag{55}$$

But then (54) is impossible. The claim is proved.

We now distinguish two cases.

CASE 1. $\deg \hat{a}_1 = 2$. We have $\hat{b}_j \geq 0$.

Fix y so that $\hat{b}_j(y) > 0$; since $\hat{a}_1$ cannot vanish on an open set we may also ensure that $\hat{a}_1(y) > 0$.

As before, set $s = |y|^a$, with $a > 1/j$ but very close to $1/j$, so that (55) holds. Then, as before, in the expression for v the term

$$J = \frac{1}{j!}\hat{b}_j(y)s^j = \frac{1}{j!}\hat{b}_j(y)|y|^{aj} \tag{56}$$

has degree smaller than that of any other term.

Consequently we may solve (13) first, and find

$$t = \frac{\hat{b}_j(y)}{j!\hat{a}_1(y)}|y|^{aj} + \circ(|y|^{aj}).$$

Inserting these values for s and t in (14) we find

$$0 \geq \frac{|y|^{aj}}{j!}\frac{\hat{b}_j}{\hat{a}_1}(\Delta\hat{a}_1 + a_3(0)) - \frac{|y|^{aj}}{j!}\Delta\hat{b}_j + \circ(|y|^{aj}),$$

i.e.

$$0 \geq \hat{b}_j(\Delta\hat{a}_1 + a_3(0)) - \hat{a}_1\Delta\hat{b}_j.$$

As before, taking trace, we conclude that $\hat{b}_j = 0$. Contradiction.

CASE 2. $\deg \hat{a}_1 > 2$. Then $\deg \hat{a}_1 \geq 4$.

Still take $s = |y|^a$, with $a > 1/j$ but very close to $1/j$, so that (55) holds. We still have that in the expression for v, the term J in (56) has degree smaller than that of every other term. To solve (13) for t, we note that the leading terms of $u(t, y)$ are now

$$u(t, y) = a_1(y)t + \frac{1}{2}a_2(y)t^2 + \frac{1}{6}a_3(y)t^3 + \cdots = O(|y|^4t) + O(|y|^2t^2) + a_3(0)t^3 + \cdots,$$

where we have used $\deg \hat{a}_2 \geq 2$ which follows from Lemma 2. Thus

$$t = \left(\frac{6}{a_3(0)}J\right)^{\frac{1}{3}} + \circ(|y|^{\frac{2+aj}{3}}).$$

Inserting these values for s and t in (14) we find

$$0 \geq ta_3(0) - \frac{s^j}{j!}\Delta \hat{b}_j + \circ(|y|^{\frac{2+aj}{3}}) + \circ(|y|^{aj}).$$

It follows, since $(2+aj)/3 < aj$, that $0 \geq a_3(0)$, a contradiction.

The proof of Proposition 1 in case $\deg \hat{a}_1 > 2$ is complete. Theorem 3 is proved.

6 – Proof of Theorem 2 in case $k = 2$

The proof has again Step A and Step B. *i.e.* we first prove that

$$l = 2 \text{ and } b_2(0) = a_2(0), \tag{57}$$

and then if u is not identically equal to v, using the differential inequality (20) for τ, and the same comparison function h of (22) we derive a contradiction.

The proof of (57) is trivial: from (12),

$$a_2(0) - b_2(0) \geq 0$$

while from (14), at $t = 0$, the opposite inequality holds.

Turn now to the equation for τ. We follow the argument of Section 3. We have to prove that $\tau - \epsilon h$ cannot have a negative minimum in G. To do this we have to check, as before that $F[\epsilon h]$ in (27) is positive at a possible minimum point $(\bar{s}, \bar{y})$, *i.e.*

$$\delta(1+\delta)\bar{s}^{-\delta-1} - 2nC - \frac{u_{tt}}{u_t}\{\quad\} - \frac{4Cu_{ti}\bar{y}_i}{u_t} > 0. \tag{58}$$

The term $\{\quad\} < 0$, and $u_{tt} = a_2 + O(t) > 0$, since $a_2(0) > 0$. In addition,

$$M = \frac{4C}{u_t}|u_{ti}\bar{y}_i| \leq \frac{4C\sqrt{\sum |u_{ti}|^2}|\bar{y}|}{u_t}.$$

Now

$$u_t = a_1 + a_2 t + \cdots \geq \frac{1}{2}a_2(0)t > \frac{2}{5}a_2(0)s$$

by (26). Thus, since $s > K|y|^2$,

$$M \leq \frac{10C|\nabla^2 u|}{a_2(0)\sqrt{K}\sqrt{s}}.$$

We conclude that (recall $C = K + 1$),

$$F[\epsilon h] \geq \delta(1+\delta)s^{\delta-1} - 2nC - \text{constant} \cdot \frac{\sqrt{K}}{\sqrt{s}} > 0$$

since $\delta = 1/4$. (40) is proved, and the proof of Theorem 2 for $k = 2$ is complete.

7 – Appendix. A simple proof of Theorem 1

We treat only the case:

$$\dot{u} > 0 \quad \text{on } (0, b). \tag{59}$$

We have to prove that

$$u \equiv v. \tag{60}$$

The proof proceeds in two steps:

Step A. (60) holds in case

$$v'(s) \geq 0. \tag{61}$$

Step B. Necessarily,

$$v'(s) \geq 0.$$

Step A. Proof of (60) if $v' \geq 0$.
We have

$$u(t) = v(s),$$

since $u' > 0$, for $t > 0$, we may solve for $t = t(s)$. Here $\cdot = \frac{d}{dt}$, $' = \frac{d}{ds}$. Then

$$v' = \dot{u}t'.$$

Compute

$$(v'^2 - \dot{u}^2)' = 2v'v'' - 2\dot{u}\ddot{u}t' = 2v'(v'' - \ddot{u}) \geq \\ \geq 0 \tag{62}$$

by our main condition (2). But at the origin,

$$v'^2 - \dot{u}^2 = 0,$$

so

$$v'^2 - \dot{u}^2 = \dot{u}^2(t'^2 - 1) \geq 0.$$

Hence

$$t'^2 \geq 1.$$

Since $t' \geq 0$ somewhere for s arbitrarily small, it follows that $t' \geq 1$, *i.e.* $t \geq s$. But then $t \equiv s$ and so $u \equiv v$.

Step B. Proof that $v' \geq 0$.

(i) We use part of an argument of [1]:

$$\ddot{u}(t) \text{ is a function of } t$$

but since $\dot{u} > 0$ it may be written as a function of u, *i.e.*

$$\ddot{u} = f(u), \tag{63}$$

with, however, f an unknown function. f is continuous on an interval $[0, m]$ for some $m > 0$, and of class C^1 on $(0, m]$, since u is of class C^3 for $t > 0$.

The main condition (2):

$$\ddot{u}(t) \leq v''(s) \quad \text{whenever } u(t) = v(s),\ t \leq s,$$

is equivalent to the inequality

$$v'' \geq f(v). \tag{64}$$

We have $u \geq v$ and both vanish, with their first derivatives at the origin. But we cannot apply the Hopf Lemma to $(u - v)$ because f is not known to be Lipschitz near the origin.

Lemma 3. *If $v(s) = u(s)$ for some $s > 0$, then*

$$v \equiv u.$$

Proof. We use a differential inequality which holds for $\tau = s - t(s)$. Namely, we have

$$\begin{aligned} v' &= \dot{u}t', \\ v'' &= \dot{u}t'' + \ddot{u}t'^2 = -\dot{u}\tau'' + \ddot{u}(1 - \tau')^2. \end{aligned}$$

So

$$0 \leq v'' - \ddot{u} = -\dot{u}\tau'' + \ddot{u}(\tau'^2 - 2\tau').$$

Now if $u(s) = v(s)$ for some $s > 0$, then, there, $\tau = 0$. But $\tau \leq 0$. By the strong maximum principle it would follow that $\tau \equiv 0$, *i.e.* $v \equiv u$.

To prove that $v' \geq 0$ we argue by contradiction. Suppose $v' < 0$ somewhere.

(ii) We cannot have $v' \geq 0$ on an interval $(0, c)$, for if this holds, by Step A, we would have

$$v \equiv u \quad \text{on } (0, c).$$

By Lemma 3, we would have

$$v \equiv u \quad \text{everywhere.}$$

So, arbitrarily near the origin there are points where $v' < 0$. But then there must be an interval (a, c), $0 < a < c < b$ on which

$$v' < 0 \text{ and } v'(a) = 0.$$

On this interval, by (62),

$$(v'^2 - \dot{u}^2)' \leq 0.$$

Hence

$$v'(s)^2 - \dot{u}(t(s))^2 \leq -\dot{u}^2(t(a)) \qquad \text{on } (a, c)$$

and, consequently,

$$\dot{u}(t(a)) \leq \dot{u}(t(s)) \qquad \text{for } \; a < s < c.$$

It follows that

$$\ddot{u}(t(a)) \geq 0.$$

By our main condition, then

$$v''(a) \geq \ddot{u}(t(a)) \geq 0.$$

Now we cannot have $v''(a) > 0$ since $0 = \dot{v}(a) > \dot{v}(s)$ for $a < s < c$. Thus

$$v''(a) = 0, \quad \text{and so } \ddot{u}(t(a)) = 0. \tag{65}$$

(iii) We now make use of (63) and (64). By (63),

$$0 = f(u(t(a))) = f(v(a)).$$

Hence, by (64), on (a, c),

$$v''(s) \geq f(v(s)) = f(v(s)) - f(v(a)) = f'(\xi)(v(s) - v(a))$$

for some ξ in $(v(s), v(a))$.

But $v(s) - v(a)$ has its maximum at a. We may apply the classical Hopf Lemma to infer that

$$v'(a) < 0.$$

This contradicts the fact that $v'(a) = 0$.

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