Rendiconti di Matematica, Serie VII
Volume 29, Roma (2009), 143–152

Uniform approximation of continuous functions on compact sets by biharmonic and bisuperharmonic functions in a biharmonic space

MOHAMED EL KADIRI

Abstract: *We give a characterization of functions that are uniformly approximable on a compact K of a biharmonic space satisfying the axiom D by biharmonic functions or by bisuperharmonic in neighborhoods of K.*

1 – Introduction

Let X be a harmonic space satisfying the axiom D. For any compact $K \subset X$ let us write

$$H_0(K) = \{u_{|K} | u \text{ is harmonic in some open set } \supset K\}$$
$$S_0(K) = \{u_{|K} | u \in \mathcal{C}(U) \cap \mathcal{S}(U) \text{ for some open set } U \supset K\},$$

where $\mathcal{C}(U)$ and $\mathcal{S}(U)$ are respectively the space of real and continuous functions on U and the convex cone of superharmonic functions on U.

Denoting by $C(K)$ the space of continuous functions $K \longrightarrow \mathbf{R}$, and by K' the fine interior of K, we have

Theorem 1.1. *The uniform closure $H(K) = \overline{H_0(K)}$ (resp. $S(K) = \overline{S_0(K)}$) in $C(K)$ consists of all $u \in C(K)$ such that $u_{|K'}$ is finely harmonic (resp. finely superharmonic).*

Key Words and Phrases: *Harmonic function and space – Biharmonic function and space – Fine topology – Finely biharmonic function*
A.M.S. Classification: 31A05, 31A30, 31B05, 31B30.

This result is due to Debiard and Gaveau [7] for the harmonic case in the classical harmonic space $\mathbf{R}^n$ endowed with the harmonic sheaf defined by the Laplace operator, and was extended for both the harmonic and superharmonic cases by Bliedtner and Hansen in [3] for the general setting of a harmonic space satisfying axiom D.

In [2] we have introduced and studied the notion of finely biharmonic pairs and functions in a fine open of a biharmonic space in the sense of Smyrnelis whose associated harmonic spaces satisfy axiom D and having the same fine topology. We have proved that all essential properties of finely harmonic and hyperharmonic extend naturally to this setting.

Our main purpose in this work is to extend the above theorem to the pairs of functions which are uniformly approximable on a compact set K of a biharmonic space X in the sense of Smyrnelis, satisfying axiom D and whose associated harmonic spaces have the same fine topologies, by the restrictions to K of biharmonic pairs of functions on neighborhoods of K. More precisely, let $\mathcal{B}H_0(K)$ (resp. $\mathcal{B}S_0(K)$) the set of restrictions to K of biharmonic (resp. superharmonic) pairs of functions in neighborhoods of K equipped with the norm

$$||(f,g)|| = \sup_{x\in K} |f(x)| + \sup_{x\in K} |g(x)|,$$

then we shall prove that the completion of $\mathcal{B}H_0(K)$ (resp. $\mathcal{B}S_0(K)$)under the norm $||\ ||$ is exactly the space $\mathcal{B}H(K)$ (resp. $\mathcal{B}S(K)$) of pairs of continuous functions on K which are finely biharmonic (resp. superharmonic) in K'.

We recall here that the fine topology on a harmonic space X is the coarsest one making continuous the superharmonic functions in X. We will use the word fine (finely) to distinguish between the notions relative to the initial topology from those relative to the fine topology. The fine topology on a harmonic space has been extensively studied by Fuglede in many papers, where he showed in particular that it has nice properties such as local connectedness which allow him to develop a nice (fine) potential theory on the fine open sets (see [10]).

The word function always means, unless mentionned, function with values in $\overline{\mathbf{R}}$. The order on the set of pairs of functions on a set M is the usual order product:

$$(f,g) \le (h,k) \iff f \le h \ \text{ et } \ g \le k.$$

We also write $(h,k) \ge (f,g)$ instead of $(f,g) \le (h,k)$. If $(f,g) \ge (0,0)$, we simply write $(f,g) \ge 0$.

2 – Biharmonic measures

For the definition of the notion of finely biharmonic functions we need to use the notion of biharmonic measures on fine open sets of a biharmonic space

X. The definition of these measures is based on a result from the general theory of biharmonic spaces of Smyrnelis ([10] and [11]).

Let $(X, \mathcal{H})$ be a biharmonic space in the sense of Smyrnelis [10], we denote by $\mathcal{U}^+(X)$ the convexe cone of hyperharmonic pairs ≥ 0 on X. For every pair $\Phi = (f, g)$ of functions on X, and every subset E of Ω, we denote by Φ^E the reduced pair of Φ on E. We recall that this pair is defined by

$$\Phi^E = \inf\{(u, v) \in \mathcal{U}^+(X); (u, v) \geq \Phi \text{ sur } E\},$$

where the infimum is taken in the sense of the order product. The balayedged pair of Φ on E is denoted by $\widehat{\Phi}^E$ and defined by $\widehat{\Phi}^E = (\widehat{\Phi}_1^E, \widehat{\Phi}_2^E)$, where, for a function h on X, $\widehat{h}$ denotes the l.s.c. (lower semicontinuous) regularization of h, that is, the greatest l.s.c. minorant of h in X. We remark that we have $\Phi^E = (\Phi^+)^E$, where $\Phi^+ = \max(\Phi, 0)$.

As in the theory of harmonic spaces, it is the notion of balayedged of a pair of measures which allows to define the notion of finely bihyperharmonic, bisuperharmonic or biharmonic functions. To that effect we recall the following result ([11], Theorem 7.11 and Theorem 7.12):

Theorem 2.1. *For every pair (σ, τ) of positive Radon measures on X and every subset E of X, there exist three positive Radon measures σ^E, ς^E and τ^E on X such that, for every $\mathcal{H}$-potential $P = (p, q)$, one has*

$$\int^* \widehat{P}_1^E d\sigma = \int^* p d\sigma^E + \int^* q d\varsigma^E,$$
$$\int^* \widehat{P}_2^E d\tau = \int^* q d\tau^E,$$

where $\widehat{P}^E = (\widehat{P}_1^E, \widehat{P}_2^E)$.)

Remark 1. The above relations are true for any pair $P = (p, q) \in \mathcal{U}^+(X)$. This can be easily seen by remarking that every pair $P \in \mathcal{U}^+(X)$ is the supremum of an increasing sequence (P_n) of $\mathcal{H}$-potentials in X.

Remark 2. The measures σ^E and τ^E are just the balayaged of the measures σ and τ with respect to the harmonic spaces associated with the biharmonic space $(X, \mathcal{H})$ (see [2], [11] and the proof of Proposition 2.3 below).

When $\sigma = \tau = \epsilon_x, x \in X$, we denote the corresponding measures σ^E, ς^E and τ^E in the above theorem by μ_x^{CE}, ν_x^{CE} and λ_x^{CE} respectively. These are the measures which allow us to define the notion of finely biharmonic and finely hyperharmonic or superharmonic pairs of functions. Let's recall that these notions have been introduced and studied in [2] for which we refer for more details on them. Note that if ω is a $\mathcal{H}$-regular open of X, then for any $x \in \omega$, μ_x^ω, ν_x^ω and λ_x^ω are just the biharmonic measures of ω at x (see [2]).

Proposition 2.2. ([2], *Prop.* 2.3) *For any subset* E *of* X*, and any* $x \in X$*, one has* $\mu_x^E = {}^1\epsilon_x^{CE}$*, and* $\mu_x^E = {}^2\epsilon_x^{CE}$*, where* ${}^j\epsilon_x^{CE}$ *is the balayaged (or the sweepted out) measure of* ϵ_x *on* CE *in the harmonic space* $(X, \mathcal{H}_j)$*,* $j = 1, 2$.

3 – Finely biharmonic pairs and functions

Let us consider a strong biharmonic space $(X, \mathcal{H})$ (that is, there exists a $\mathcal{H}$-potential (p, q) such that $q > 0$ on X) satisfying the domination axiom D and whose associated harmonic spaces $(X, \mathcal{H}_1)$ and $(X, \mathcal{H}_2)$ have the same fine topology that we will simply call the fine topology of X. This space will be fixed in all the sequel. We will use the word fine (finely) to distinguish between the notions relative to the fine topology from those relative to the topology of X (the initial topology).

Definition 3.1. Let ω be a relatively compact (in the initial topology) fine open subset of X, the triple $(\mu_x^\omega, \nu_x^\omega, \lambda_x^\omega)$ is called the triple of biharmonic measures at x.

For every fine open V we denote by $\partial_f V$ the fine boundary of V and by $\widetilde{V}$ its fine closure. It is well known that if a fine open ω is regular, then the measures ${}^1\epsilon_x^{C\omega}$ and ${}^2\epsilon_x^{C\omega}$ are supported by $\partial_f \omega$ (see [8]). By corollary 2 of Proposition 2.7 in [2] we have the following

Proposition 3.2. *If* ω *is regular, then, for every* $x \in \omega$*, the measures* μ_x^ω*,* ν_x^ω *and* λ_x^ω *are supported by* $\partial_f \omega$.

Let us now recall the definitions of finely hyperharmonic and harmonic pairs of functions studied in [2].

Definition 3.3. A pair (u, v) of functions on a fine open subset U of X is said to be finely hyperharmonic in U if u and v are finely l.s.c. (lower semicontinuous) with values in $]-\infty, +\infty]$ and if the fine topology induced on U has a base $\mathcal{B}$ formed by open sets ω such that $\widetilde{\omega} \subset U$ and

$$u(x) \geq \int^* u d\mu_x^\omega + \int^* v d\nu_x^\omega, v(x) \geq \int^* v d\lambda_x^\omega$$

for every $x \in \omega$.

For more details on $\mathcal{H}$-harmonic and $\mathcal{H}$-hyperharmonic pairs, the reader should be referred to [2] where these notions are extensively studied.

Definition 3.4. A pair (u, v) of functions on a fine open set U of X is said to be finely biharmonic in U if the pairs (u, v) and $(-u, -v)$ are finely hyperharmonic in U.

This definition makes sense because for every $x \in \omega$ the measures μ_x^ω, ν_x^ω and λ_x^ω are supported by $\partial_f \omega$ and the $\mathcal{H}$-polar sets are negligeable for these measures.

A finely hyperharmonic pair (u, v) on a fine open U of X is said to be finely superharmonic if u and v are finite on a dense subset of U. For more details on finely hyperharmonic and related notions see [2].

Lemma 3.5. *For any relatively compact (in the initial topology) finely open set ω of X and any $x \in \omega$, we have $\int d\nu_x^\omega > 0$.*

Proof. It follows easily from [2], Theorem 8.1. and Theorem 8.2, that the pair $(\int d\nu_\cdot^\omega, \int d\lambda_\cdot^\omega)$ is non-negative finely superharmonic, not identically 0 in each finely connected component of ω, hence $\int d\nu_x^\omega > 0$ for any $x \in \omega$. ∎

Let us now consider for a finely open set U the family $D(U)$ of finely continuous functions f on U such that the limit

$$Lf(x) = \lim_{\omega \downarrow x} \frac{f(x) - \int f d\nu_x^\omega}{\int d\nu_x^\omega}$$

along the filter of fine neighborhoods ω of x exists and is finite for every $x \in U$.

Definition 3.6 A continuous function on U is said to be finely biharmonic on U if $f \in D(U)$ and Lf is finely harmonic on U.

The following proposition underlines the link between the notion of finely biharmonic function in the sense of definition 3.6 and the notion of finely biharmonic pair in the sense of definition 3.4:

Proposition 3.7. *If a pair (u, v) is finely biharmonic in a fine open U, then $u \in D(U)$ and $Lu = v$.*

Proof. Let $x \in U$ and $\epsilon > 0$, there exists a fine open $\omega_0 \subset U$, $x \in \omega_0$, such that $|v(x) - v(y)| < \epsilon$ for any $y \in \omega_0$. Then, for any fine open $\omega \subset \tilde{\omega} \subset \omega_0$, $x \in \omega$, we have

$$|u(x) - \int u d\mu_x^\omega - v(x) \int d\nu_x^\omega| < \epsilon \int d\nu_x^\omega$$

and therefore $u \in D(U)$ and $Lu = v$. ∎

Remark. We do not know if the converse of Proposition 3.7 is true or not. In the case where U is an open of the initial topology and v is $\mathcal{H}_2$-harmonic in U, then it can be easyly seen that the pair (u, v) is $\mathcal{H}$-harmonic.

4 – Approximation of continuous functions by biharmonic functions

We begin by recalling the following result on approximation by harmonic functions:

THEOREM 4.1. ([1], *Th.* 1.2) *Let* $(\Omega, \mathcal{K})$ *be a* $\mathcal{P}$*-harmonic space in the sense of Constantinesu- Cornea [6], and* K *a compact set of* Ω *and* (V_n) *a sequence of open subsets of* Ω *such that* $\overline{V}_{n+1} \subset V_n$ *for all* n *and* $\cap_n V_n = K$. *Then for for every continuous function* f *on* Ω *and finely harmonic on* K', *the sequence* $H_f^{V_n}$ *converges uniformly to* f *in* K.

Here H_f^ω is the solution of the Dirichlet problem on ω in the biharmonic space $(\Omega, \mathcal{K})$ for the data function f on $\partial\omega$.

Let us also recall the following result due to Bouleau [4]:

THEOREM 4.2. *Let* $(X, \mathcal{H})$ *be a strong biharmonic space in the sense of Smyrnelis. Then there exists a unique Borel kernel* $\mathcal{V}$ *on* X *such that*

i) For every continuous finite fonction φ *on* X *with compact support, the function* $\mathcal{V}(\varphi)$ *is harmonic outside of the support of* φ.

ii) For every non-negative $\mathcal{H}_2$*-hyperharmonic function* v *on* X, $\mathcal{V}(v)$ *is the smallest non-negative* $\mathcal{H}_1$*-hyperharmonic function on* X *such that the pair* $(\mathcal{V}(v), v)$ *is* $\mathcal{H}$*-hyperharmonic.*

For any domain (open and connected) V of X, we denote by $\mathcal{V}_V$ the kernel associated to the biharmonic space V endowed with the sheaf induced on V by $\mathcal{H}$. For an arbitrary open subset of X, let us denote by $\mathcal{V}_V$ the kernel on V which coincides with $\mathcal{V}_\omega$ on each connected component ω of V.

Let us note that if V is relatively compact, then for any bounded harmonic function k on V the function $\mathcal{V}_V(k)$ is biharmonic in V. In fact, let q be a $\mathcal{H}_2$-potential > 0 on X such that $\mathcal{V}(q) < +\infty$. Then we have $\mathcal{V}_V(k) \leq C\mathcal{V}(q)$ for some constant $C > 0$, hence the pair $(\mathcal{V}(k), k)$ is biharmonic.

Let U be a relatively compact open subset of X. We denote by $H_{(f,g)}^U$ the solution of the Riquier problem on U with boundary data $(f, g) \in \mathcal{C}(\partial U)^2$ and by $H_f^{U,j}$ the solution of the Dirichlet problem in U with boundary data $f \in \mathcal{C}(\partial U)$ in the harmonic space $(X, \mathcal{H}_j)$, $j = 1, 2$.

LEMMA 4.3. ([2], *cor.* 1 *du th.* 8.1) *For every pair* $(f, g) \in \mathcal{C}(\partial U)^2$, *we have*

$$H_{(f,g)}^U = (H_f^{U,1} + \mathcal{V}_U H_g^{U,2}, H_g^{U,2}).$$

Now we can prove the following

THEOREM 4.4. *Let K be a compact set of X and (V_n) a sequence of open subsets of Ω such that $\overline{V}_{n+1} \subset V_n$ for all n and $\cap_n V_n = K$. Then for every pair (f,g) of continuous function on X, finely biharmonic on K', the sequence $H^{V_n}_{(f,g)}$ converges uniformly to (f,g) in K.*

PROOF. Let K be a compact set of X and (f,g) a pair of continuous functions on X, finely biharmonic on K'. It follows from Theorem 4.1 that the sequence $(H^{V_n,2}_g)$ converges uniformly on K to the function g. For every n, put $\mathcal{V}_n = \mathcal{V}_{V_n}$ and $k_n = \mathcal{V}_n(H^{2,V_n}_g)$. The function k_n extends continuously to $\overline{V_n}$ to a function $\bar{k}_n$ such that $\bar{k}_n = 0$ on ∂V_n, because it is the first component of the solution of the fine Riquier problem for the given data $(0,g)$ on ∂V_n. The sequence (k_n) converge uniformly on K to a continuous function k. Let $\bar{k}$ be a continuous function on X which coincides with k on K. The function $f - \bar{k}$ is continuous on K and finely $\mathcal{H}_1$-harmonic in K'. It follows then by Lemma 4.3 and Theorem 4.1 that the sequence $(H^{V_n,2}_{f-\bar{k}_n})$ converges uniformly on K to $f-k$. Let us put $H^{V_n}_{(f,g)} = (H^{V_n,1}_{(f,g)}, H^{V_n,2}_{(f,g)})$, then by Lemma 4.3, we have

$$\begin{aligned} H_(f,g)^{V_n,1} &= H^{V_n,1}_f + \mathcal{V}_n(H^{V_n,2}_g) \\ &= {}^1H^{V_n}_{f-\bar{k}_n} + \mathcal{V}_n({}^2H^{V_n}_g). \end{aligned}$$

Hence the sequence $(H_(f,g)^{V_n,1})$ converges uniformly on K to $f - k + k = f$. This ends the proof. ∎

We say that a function f on an open U of X is biharmonic if there exists a function g on U such that the pair (f,g) is biharmonic on U. It follows from the hypothesis in the definition of biharmonic spaces [10] that the biharmonic pairs are compatible (in the sense that if a pair (o,k) is biharmonic in U, then $k=0$ in U), thus the function k is unique. We then denote it by Af.

Because of the sheaf property of biharmonic functions, the operator A may be interpreted as a local operator on biharmonic functions.

THEOREM 4.5. *Let f be a real function on a compact K of X. Then the following are equivalent:*

1. *There exists a sequence (h_n) of biharmonic functions, each one is defined on an open neighborhood of K, such that (h_n) converges uniformly on K to f and (Ah_n) converges uniformly on K to a contiuous function g.*
2. *f is continuous on K and finely biharmonic on K', and Lf extends continuously on K.*

PROOF. 1. $\Longrightarrow$ 2: Since the pairs (h_n, Ah_n) are finely biharmonic in K' and converge uniformly in K, it follows from the definition of biharmonic pairs that the pair (f, g) is finely biharmonic in K', and clearly continuous in K.

2. $\Longrightarrow$ 1: Let us suppose that f is continuous in K and finely biharmonic in K' and that the function Lf extends continuously to a function g on K. The function $f - \int g d\nu_x^{K'}$ is finely harmonic in K'. On the other hand it follows from Lemma 4.3 and Theorem 4.4 that the function $f - \int g d\nu_x^{K'}$ is continuous in K. Then, by Bliedtner-Hansen's Theorem there exists a sequence (k_n) of functions such that, for every n, k_n is $\mathcal{H}_1$-harmonic on an open neighborhood U_n of K, and (k_n) converges uniformly in K to $f - \int g d\nu_x^{K'}$. On the other hand, the function g is finely $\mathcal{H}_2$-harmonic in K' and extends to a continuous function on K, then by Bliedtner-Hansen's Theorem, there exists a sequence (g_n) of functions such that, for each n, g_n is $\mathcal{H}_2$-harmonic in an open neighborhood V_n of K, and that (g_n) converges uniformly to g in K. For each n, let W_n be an open set of X such that $K \subset \overline{W_n} \subset U_n \cap V_n$. The functions $k_n + \int g_n d\nu^{CW_n}$ are biharmonic on W_n and converge uniformly in K to f, and we have seen that the harmonic functions $A(k_n + \int g_n d\nu^{CW_n}) = g_n$ converge uniformly on K to Lf. This ends the proof of 2) $\Longrightarrow$ 1). ∎

Let us denote by $\mathcal{H}(K)$ the space of continuous functions on K that are finely $\mathcal{H}_2$-harmonic in K', the fine interior of K. By Bliedtner-Hansen's Theorem, the space $\mathcal{H}(K)$ is identical to the one of finely harmonic functions in K' with continuous extension to K. The above theorem can be stated as follows:

THEOREM 4.5$'$. *Let f be a real function on a compact K. Then the following are equivalent:*

1. *There exists a sequence (h_n) of biharmonic functions, each one is defined on an open neighborhood of K, such that (h_n) converges uniformly on K to f and (Ah_n) converges uniformly on K to a contiuous function g.*

2$'$. *f is continuous on K, finely biharmonic on K', and $Lf \in \mathcal{H}_2(K)$.*

COROLLARY 1.. *A function f on U is finely biharmonic if and only if for every point $x \in U$ there exists a compact fine open neighborhood $K \subset U$ and a sequence (h_n) of biharmonic functions in neighborhoods of K such that (h_n) converges uniformly on K to f.*

COROLLARY 2.. *A pair of functions (f, g) on U is finely biharmonic if and only if for every point $x \in U$ there exist a compact fine open neighborhood $K \subset U$ and a sequence (h_n, k_n) of biharmonic pairs of functions in neighborhoods of K such that (h_n) and (k_n) converge uniformly on K respectively to f and g.*

Theorem 4.5 above can also be stated as follows

THEOREM 4.5″. *The uniform closure* $\overline{\mathcal{B}H}_0$ *of* $\mathcal{B}H_0(K)$ *is* $\mathcal{B}H(K)$.

5 – Approximation by bisuperharmonic functions.

For any compact $K \subset X$ we denote by $\mathcal{B}S_0(K)$ the set

$$\{(u_{|K}, v_{|K}) : (u, v) \text{ is finite continuous } \mathcal{H}-\text{superharmonic in some open set} \supset K\}$$

and by $\mathcal{B}S(K)$ the set of continuous functions on K that are finely $\mathcal{H}$-superharmonic on K'.

THEOREM 5.1. *The uniform closure* $\overline{\mathcal{B}S}_0(K)$ *in* $C(K)$ *is identical to* $\mathcal{B}S(K)$.

PROOF. We clearly have $\mathcal{B}S_0(K) \subset \mathcal{B}S(K)$. Let $(f, g) \in \mathcal{B}S(K)$. Then $g \in \mathcal{S}_2(K)$ and hence by Fuglede's theorem there exists a sequence (V_n) of open neighborhoods of K and for each $n \geq 0$ a $\mathcal{H}_2$-superharmonic function g_n on V_n such that the sequence $(g_{n|K})$ converges uniformly to g. For each n let $\mathcal{V}_n$ be the kernel $\mathcal{V}_{V_n}$. Since the sequence $(\mathcal{V}_n(g_n))$ converges uniformly in K, we only have to show that the pointwise limit of $(\mathcal{V}_n(g_n))$ is $\mathcal{V}(g)$. By adding if necessary the second projection of a $\mathcal{H}$-potential to g, we can assume that $g > 0$. Let u be a function $\mathcal{H}_1$-surharmonic in K' such that the pair (u, g) is finely $\mathcal{H}$-surharmonic in K' and let $0 < \epsilon < \inf_K g$, then there exists an integer $n_0 > 0$ such that, for $n > n_0$, one has $g \geq g_n - \epsilon > 0$ in K, hence the pair $(u, g_n - \epsilon)$ is finely $\mathcal{H}$-surharmonic in K'. It follows that $u + \epsilon\mathcal{V}(1) \geq \mathcal{V}(g_n)$ for every $n \geq n_0$, hence $u \geq \mathcal{V}(g)$. Then $\mathcal{V}(g) = \lim \mathcal{V}(g_n)$. □

REFERENCES

[1] CH. BENSOUDA: *Approximation en théorie du Potentiel*, Ph. D. thesis, University of Montreal, 1994.

[2] CH. BENSOUDA – M. EL KADIRI – I. ROUCHDI: *Fonctions finement biharmonic dans un espace biharmonique*, Rend. Math. Appl. (2003).

[3] J. BLIEDTNER – W. HANSEN: *Approximation by continuous potentials*, Potential theory, (Prague, 1987), 53-58, Plenum, New York, 1988.

[4] N. BOULEAU: *Espaces biharmoniques et couplage de processus de Markov*, J. Math. Pures Appl. **59** (1980), 187-240.

[5] M. CHADLI – M. EL KADIRI: *Uniform approximation on compacy sets by biharmonic functions*, Comment. Math. Univ. Carolinae, **3** (2003).

[6] C. Constantinescu – A. Cornea: *Potential Theory on Harmonic spaces*, Springer Verlag Heidelberg, 1972.

[7] A. Debiard – B. Gaveau: *Potentiel fin et algèbre de fonctions analytiques*, J. Funct. Anal. **16** (1974), 289-304.

[8] B. Fuglede: *Finely harmonic functions*, Lecture Notes in Math. 289, Springer, Berlin-Heidelberg-New York, 1972.

[9] L. L. Helms: *Introduction to Potential Theory*, Wiley-Interscience, 1969.

[10] E. P. Smyrnelis: *Axiomatique des fonctions biharmoniques*, 1e section, Ann. Inst. Fourier, **1** (1975), 35-98.

[11] E. P. Smyrnelis: *Axiomatique des fonctions biharmoniques*, 2e section, Ann. Inst. Fourier, **3** (1976), 1–47.

Lavoro pervenuto alla redazione il 29 maggio 2008
ed accettato per la pubblicazione il 28 gennaio 2009.
Bozze licenziate il 01 ottobre 2009

Indirizzo dell'Autore:

Mohamed El Kadiri – B.P. 726 – Salé-Tabriquet – Salé – Morocco
E-mail: elkadiri@fsr.ac.ma