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On the geometry of four-dimensional Walker manifolds

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Abstract: *We study the curvature properties of a large class of four-dimensional Walker metrics. Several interesting examples are found, in particular as regards local symmetry, conformal flatness and Einstein-like metrics. Studiamo le proprietà di curvatura di un'ampia classe di metriche di Walker di dimensione quattro. Vengono messi in evidenza diversi comportamenti interessanti, in particolare per quanto riguarda esempi localmente simmetrici, conformemente piatti e muniti di metriche "Einstein-like".*

1 – Introduction

The study of the curvature properties of a given class of pseudo-Riemannian manifolds is necessary to our knowledge of these spaces. In particular, this study makes possible a comparison between Riemannian results and their pseudo-Riemannian analogues, which develops our understanding of which properties are more strictly related to the signature of the metric tensor and which ones are more general.

While extensive studies have been made about the curvature of Riemannian manifolds, the corresponding study in pseudo-Riemannian settings is relatively recent, and several interesting cases have still to be investigated. Some examples in this direction may be found in [1], [2], [3], [4], [5], [6], [8] and references therein.

In this context, the study of manifolds endowed with Walker structures (that is, admitting parallel degenerate plane fields) is particularly relevant. In fact, several results suggest that Walker structures are responsible of many of the

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basic differences between the Riemannian geometry and the pseudo-Riemannian one.
We recall that a four-dimensional pseudo-Riemannian manifold $\mathcal{M}$ of signature (2.2) is said to be a *Walker manifold* if it admits a parallel totally isotropic 2-plane field. Such a manifold is locally isometric to (U, g_f), where U is an open subset of $\mathbb{R}^4[x_1, x_2, x_3, x_4]$ and the metric is given, with respect to the coordinate vector fields $\partial_i := \frac{\partial}{\partial_{x_i}}$, by

$$g(\partial_1, \partial_3) = g(\partial_2, \partial_4) = 1, \qquad g(\partial_i, \partial_j) = g_{ij}(x_1, x_2, x_3, x_4) \text{ for } i, j = 3, 4.$$

Curvature properties of four-dimensional Walker metrics satisfying $g_{34} = 0$ were investigated in [6], while examples with commuting curvature operators were classified in [2] assuming $g_{33} = g_{44} = 0$. The aim of this paper is to characterize several geometric properties of Walker metrics of the latter type as determined by their curvature. So, we shall consider Walker metrics of the form

$$g_f = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & f \\ 0 & 1 & f & 0 \end{pmatrix}, \tag{1.1}$$

for an arbitrary smooth function $f = f(x_1, x_2, x_3, x_4)$, defined on an open subset U of $\mathbb{R}^4$.

We shall characterize Walker metrics (1.1) which are Ricci-parallel or Einstein-like, conformally flat, locally symmetric. In this way, we determine some wide classes of four-dimensional pseudo-Riemannian manifolds, satisfying some required geometrical properties. These examples have not a Riemannian counterpart, because the same properties turn out to be much more restrictive in the Riemannian case.

The paper is organized in the following way. In Section 2, we shall describe the curvature of Walker metrics. In Section 3 and 4, Einstein-like and locally symmetric Walker metrics (1.1) will be respectively classified.

2 – Curvature of Walker metrics (1.1)

We denote by ∇ the Levi Civita connection of a pseudo-Riemannian metric g and by R its curvature tensor, taken with the sign convention

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]. \tag{2.1}$$

In the sequel, following the same notation used in [2], we put $f_{/i} := \partial_i(f)$, $f_{/ij} := \partial_i\partial_j(f)$ and $f_{/ijk} := \partial_i\partial_j\partial_k(f)$, for all indices i, j, k. Standard calculations give that the only possible non-vanishing Christoffel symbols Γ_{ij}^k, are the following ones:

$$\begin{cases} \Gamma_{13}^2 = \Gamma_{14}^1 = -\Gamma_{34}^3 = \frac{1}{2} f_{/1}, & \Gamma_{23}^2 = \Gamma_{24}^1 = -\Gamma_{34}^4 = \frac{1}{2} f_{/2}, \\ \Gamma_{33}^2 = f_{/3}, & \Gamma_{44}^1 = f_{/4}, \\ \Gamma_{34}^1 = \frac{1}{2} f f_{/2}, & \Gamma_{34}^2 = \frac{1}{2} f f_{/1}. \end{cases}$$

Correspondingly, the possibly non-vanishing covariant derivatives of coordinates vector fields are given by

$$\begin{aligned} &\nabla_{\partial_1}\partial_3 = \frac{f_{/1}}{2}\,\partial_2, \qquad \nabla_{\partial_1}\partial_4 = \frac{f_{/1}}{2}\,\partial_1, \\ &\nabla_{\partial_2}\partial_3 = \frac{f_{/2}}{2}\,\partial_2, \qquad \nabla_{\partial_2}\partial_4 = \frac{f_{/2}}{2}\,\partial_1, \\ &\nabla_{\partial_3}\partial_3 = f_{/3}\,\partial_2, \qquad \nabla_{\partial_4}\partial_4 = f_{/4}\,\partial_1, \\ &\nabla_{\partial_3}\partial_4 = \frac{f f_{/2}}{2}\,\partial_1 + \frac{f f_{/1}}{2}\,\partial_2 - \frac{f_{/1}}{2}\,\partial_3 - \frac{f_{/2}}{2}\,\partial_4. \end{aligned} \tag{2.2}$$

Using (2.2) into (2.1), we can completely determine the curvature tensor of g_f by calculating $R(\partial_i, \partial_j)\partial_k$ for all indices i, j, k. Then, taking into account (1.1), we can determine all curvature components of the $(0,4)$-curvature tensor $R(X,Y,Z,W) = g_f(R(X,Y)Z,W)$ with respect to $\{\partial_i\}$. Via long but routine calculations, we obtain that the possibly non-vanishing components are given by

$$\begin{cases} R_{1334} = \frac{1}{4}\left(f_{/1}f_{/2} - 2f_{/13}\right), & R_{1314} = -\frac{1}{2} f_{/11}, \\ R_{1434} = \frac{1}{4}\left(-f_{/1}^2 + 2f_{/14}\right), & R_{1324} = -\frac{1}{2} f_{/12}, \\ R_{2334} = \frac{1}{4}\left(f_{/2}^2 - 2f_{/23}\right), & R_{1423} = -\frac{1}{2} f_{/12}, \\ R_{2434} = \frac{1}{4}\left(-f_{/1}f_{/2} + 2f_{/24}\right), & R_{2324} = -\frac{1}{2} f_{/22}, \\ R_{3434} = \frac{1}{2}\left(-f f_{/1}f_{/2} + 2f_{/34}\right). \end{cases} \tag{2.3}$$

Next, we can calculate the components $\varrho_{ij} = \varrho(\partial_i, \partial_j)$ with respect to $\{\partial_i\}$ of the Ricci tensor ϱ of $\mathcal{M}$, defined as the contraction of the curvature tensor. We obtain

(2.4)

$$\varrho = \begin{pmatrix} 0 & 0 & \frac{1}{2} f_{/12} & \frac{1}{2} f_{/11} \\ 0 & 0 & \frac{1}{2} f_{/22} & \frac{1}{2} f_{/12} \\ \frac{1}{2} f_{/12} & \frac{1}{2} f_{/22} & f_{23} - \frac{1}{2} f_{/2}^2 & \frac{1}{2}\left(f_{/1}f_{/2} - f_{/13} - f_{/24}\right) + f f_{/12} \\ \frac{1}{2} f_{/11} & \frac{1}{2} f_{/12} & \frac{1}{2}\left(f_{/1}f_{/2} - f_{/13} - f_{/24}\right) + f f_{/12} & f_{/14} - \frac{1}{2} f_{/1}^2 \end{pmatrix}$$

Now, formulas (1.1) and (2.4) also determine the components of the Ricci operator Q given by $g_f(Q(X),Y) = \varrho(X,Y)$ with respect to ∂_i. Explicitly, we obtain:

$$Q = \begin{pmatrix} \varrho_{13} & \varrho_{23} & \varrho_{33} - f\varrho_{23} & \varrho_{34} - f\varrho_{13} \\ \varrho_{14} & \varrho_{13} & \varrho_{34} - f\varrho_{13} & \varrho_{44} - f\varrho_{14} \\ 0 & 0 & \varrho_{13} & \varrho_{14} \\ 0 & 0 & \varrho_{23} & \varrho_{13} \end{pmatrix} \tag{2.5}$$

From (2.5) it easily follows that the Ricci eigenvalues of g_f are the solutions of

$$[(\varrho_{13} - \lambda)^2 - \varrho_{23}\varrho_{14}]^2 = 0.$$

If $\varrho_{23}\varrho_{14} < 0$ (equivalently, $f_{11}f_{22} < 0$ by (2.4)), then Q has complex conjugates eigenvalues and so, is not diagonalizable.

If $\varrho_{23}\varrho_{14} = 0$ (that is, $f_{11}f_{22} = 0$), then $\lambda = \varrho_{13} = \frac{1}{2}f_{12}$ is the only Ricci eigenvalue. In this case, by (2.5) it easily follows that the corresponding eigenspace is not four-dimensional (and so, Q is not diagonalizable), unless $\varrho_{14} = \varrho_{23} = \varrho_{33} = \varrho_{44} = \varrho_{34} - f\varrho_{13} = 0$, that is, by (2.4), if f satisfies

$$\begin{aligned} f_{/11} = f_{/22} = 2f_{/23} - (f_{/2})^2 = 2f_{/14} - (f_{/1})^2 \\ = f_{/1}f_{/2} - f_{/13} - f_{/24} + ff_{/12} = 0. \end{aligned} \tag{2.6}$$

If $\varrho_{23}\varrho_{14} > 0$ (equivalently, $f_{11}f_{22} > 0$ by (2.4)), then Q admits the eigenvalues $\lambda = \rho_{13} + \varepsilon\sqrt{\varrho_{23}\varrho_{14}}$, where $\varepsilon = \pm 1$, each of multiplicity 2. In this case, it is easily seen by (2.5) that Q is not diagonalizable, unless

$$\varrho_{14}\,(\varrho_{33} - f\varrho_{23}) + 2\varepsilon\sqrt{\varrho_{23}\varrho_{14}}\,(\varrho_{34} - f\varrho_{13}) + \varrho_{23}\,(\varrho_{44} - f\varrho_{14}) = 0, \quad \varepsilon = \pm 1,$$

that is,

$$\varrho_{34} - f\varrho_{13} = \varrho_{14}(\varrho_{33} - f\varrho_{23}) + \varrho_{23}\,(\varrho_{44} - f\varrho_{14}) = 0.$$

By (2.4), equations above are equivalent to requiring that the defining function f satisfies

$$\begin{aligned} f_{/1}f_{/2} - f_{/13} - f_{/24} + ff_{/12} = f_{/11}(2f_{/23} - (f_{/2})^2) \\ + f_{/22}(2f_{/14} - (f_{/1})^2) - 2ff_{/11}f_{/22} = 0. \end{aligned} \tag{2.7}$$

Note that (2.6) implies (2.7). Hence, we can state the following

PROPOSITION 2.1. *A Walker metric* (1.1) *has a diagonalizable Ricci operator only if its defining function f satisfies* (2.7).

We can now calculate the covariant derivative $\nabla\varrho$ of the metric (1.1). By using (2.2) and (2.4), we prove the following

PROPOSITION 2.2. *The nonvanishing components* $\nabla_i\varrho_{jk} = (\nabla_{\partial_i}\varrho)(\partial_j, \partial_k)$ *of the covariant derivative* $\nabla\varrho$ *of a Walker metric* (1.1), *are given by*

$$\nabla_1\varrho_{13} = \nabla_1\varrho_{24} = \nabla_2\varrho_{14} = \frac{f_{/112}}{2}, \quad \nabla_1\varrho_{23} = \nabla_2\varrho_{13} = \nabla_2\varrho_{24} = \frac{f_{/122}}{2},$$
$$\nabla_1\varrho_{14} = \frac{f_{/111}}{2}, \quad \nabla_2\varrho_{23} = \frac{f_{/222}}{2},$$
$$\nabla_1\varrho_{33} = f_{/123} - f_{/2}f_{/12} - \frac{f_{/1}f_{/22}}{2}, \quad \nabla_1\varrho_{44} = f_{/144} - \frac{3}{2}f_{/1}f_{/11},$$

$$\nabla_2\varrho_{33} = f_{/223} - \frac{3}{2}f_{/2}f_{/22}, \quad \nabla_2\varrho_{44} = f_{/124} - f_{/1}f_{/12} - \frac{f_{/2}f_{/11}}{2},$$
$$\nabla_3\varrho_{13} = \frac{1}{4}\left(2f_{/123} - f_{/1}f_{/22}\right), \quad \nabla_3\varrho_{14} = \frac{1}{4}\left(2f_{/113} + f_{/2}f_{/11}\right),$$
$$\nabla_3\varrho_{23} = \frac{1}{4}\left(2f_{/223} - f_{/2}f_{/22}\right), \quad \nabla_3\varrho_{24} = \frac{1}{4}\left(2f_{/123} + f_{/1}f_{/22}\right),$$
$$\nabla_3\varrho_{33} = f_{/233} - f_{/2}f_{/23} - f_{/3}f_{/22}, \quad \nabla_4\varrho_{44} = f_{/144} - f_{/1}f_{/14} - f_{/4}f_{/11},$$
$$\nabla_4\varrho_{13} = \frac{1}{4}\left(2f_{/124} + f_{/2}f_{/11}\right), \quad \nabla_4\varrho_{14} = \frac{1}{4}\left(2f_{/114} - f_{/1}f_{/11}\right),$$
$$\nabla_4\varrho_{23} = \frac{1}{4}\left(2f_{/224} + f_{/1}f_{/22}\right), \quad \nabla_4\varrho_{24} = \frac{1}{4}\left(2f_{/124} - f_{/2}f_{/11}\right),$$

$$\nabla_1\varrho_{34} = \frac{1}{2}\left(f_{/2}f_{/11} - f_{/113} - f_{/124}\right) + f_{/1}f_{/12} + ff_{/112},$$
$$\nabla_2\varrho_{34} = \frac{1}{2}\left(f_{/1}f_{/22} - f_{/123} - f_{/224}\right) + f_{/2}f_{/12} + ff_{/122},$$
$$\nabla_3\varrho_{34} = \frac{1}{4}\left(f_{/2}f_{/13} + ff_{/2}f_{/12} - ff_{/1}f_{/22} - f_{/2}f_{/24}\right) + \frac{1}{2}\left(f_{/3}f_{/12} - f_{/133} - f_{/234}\right) + ff_{/123} + f_{/1}f_{/23},$$
$$\nabla_3\varrho_{44} = \frac{1}{2}\left(-3f_{/1}f_{/13} + ff_{/1}f_{/12} - ff_{/2}f_{/11} - f_{/1}f_{/24}\right) + f_{/134} + f_{/2}f_{/14},$$
$$\nabla_4\varrho_{33} = \frac{1}{2}\left(-3f_{/2}f_{/24} + ff_{/2}f_{/12} - ff_{/1}f_{/22} - f_{/2}f_{/13}\right) + f_{/234} + f_{/1}f_{/23},$$
$$\nabla_4\varrho_{34} = \frac{1}{4}\left(f_{/1}f_{/24} + ff_{/1}f_{/12} - ff_{/2}f_{/11} - f_{/1}f_{/13}\right) + \frac{1}{2}\left(f_{/4}f_{/12} - f_{/134} - f_{/244}\right) + ff_{/124} + f_{/2}f_{/14}.$$

3 – Einstein-like and conformally flat Walker metrics g_f

Einstein-like metrics were introduced and first studied by A. Gray [7] in the Riemannian framework as natural generalizations of Einstein metrics. Since they are defined through conditions on the Ricci tensor, their definition extends at once to the pseudo-Riemannian case. A pseudo-Riemannian manifold (M, g)

(i) belongs to class $\mathcal{P}$ if and only if its Ricci tensor ϱ is *parallel*, that is,

$$(\nabla_X \varrho)(Y, Z) = 0, \tag{3.1}$$

for all vector fields X, Y, Z tangent to M.

ii) belongs to class $\mathcal{A}$ if and only if its Ricci tensor ϱ is *cyclic-parallel*, that is,

$$(\nabla_X \varrho)(Y, Z) + (\nabla_Y \varrho)(Z, X) + (\nabla_Z \varrho)(X, Y) = 0, \tag{3.2}$$

for all vector fields X, Y, Z tangent to M. (3.2) is equivalent to requiring that ϱ is a *Killing tensor*, that is,

$$(\nabla_X \varrho)(X, X) = 0. \tag{3.3}$$

iii) belongs to class $\mathcal{B}$ if and only if its Ricci tensor is a *Codazzi tensor*, that is,

$$(\nabla_X \varrho)(Y, Z) = (\nabla_Y \varrho)(X, Z). \tag{3.4}$$

Let us denote by $\mathcal{E}$ and by $\mathcal{C}$ the class of Einstein manifolds and manifolds with constant scalar curvature, respectively; besides, let $\mathcal{P}$ denote the class of manifolds with parallel Ricci tensor. Then we have $\mathcal{E} \subset \mathcal{P} = \mathcal{A} \cap \mathcal{B} \subset \mathcal{A} \cup \mathcal{B} \subset \mathcal{C}$.

Recently, Einstein-like pseudo-Riemannian metrics have been studied by several authors. Some examples can be found in [1],[3],[4],[5].

Let g_f be a Walker metric described by (1.1). Several curvature properties which we shall study in this section and in the next one, force the defining function f to be of the following special form:

$$f(x_1, x_2, x_3, x_4) = x_1 p(x_3, x_4) + x_2 q(x_3, x_4) + s(x_3, x_4), \tag{3.5}$$

where p, q, s are C^∞ real valued functions. More precisely, as concerns Einstein-like Walker metrics (1.1), the following result is easily obtained by applying (3.1), (3.3) and (3.4) respectively to the components of $\nabla \varrho$ described in Proposition 2.2:

Theorem 3.1. *A Walker metric g_f described by* (1.1)

i) *is Ricci-parallel if and only if f is of the form* (3.5)*, where p and q are C^∞ real valued functions satisfying*

$$\begin{aligned} &p^2 = 2p_{/4} + l(x_3), && q^2 = 2q_{/3} + h(x_4), \\ &qp_{/3} - qq_{/4} - 2(p_{/33} + q_{/34}) + 4pq_{/3} = 0, && pq_{/4} - pp_{/3} - 2(p_{/34} + q_{/44}) + 4qp_{/4} = 0, \\ &3pp_{/3} + pq_{/4} - 2p_{/34} - 2qp_{/4} = 0, && 3qq_{/4} + qp_{/3} - 2q_{/34} - 2pq_{/3} = 0, \end{aligned} \tag{3.6}$$

for two arbitrary smooth functions h and l.

ii) *belongs to class $\mathcal{A}$ if and only if*

$$f_{/233} - f_{/2}f_{/23} - f_{/3}f_{/22} = 0, \qquad f_{/144} - f_{/1}f_{/14} - f_{/4}f_{/11} = 0. \tag{3.7}$$

iii) *belongs to class $\mathcal{B}$ if and only if f is of the form* (3.5)*, where p and q satisfy*

$$3qp_{/3} + 5qq_{/4} - 2p_{/33} - 6q_{/34} = 0, \qquad 3pq_{/4} + 5pp_{/3} - 2q_{/44} - 6p_{/34} = 0. \tag{3.8}$$

The classification of Einstein Walker metrics (1.1) was given in [2], were the following result was proved:

Theorem 3.2. *A Walker metric g_f described by* (1.1) *is Einstein if and only if f is of the form* (3.5)*, where p and q are C^∞ real valued functions satisfying*

$$p^2 = 2p_{/4}, \qquad q^2 = 2q_{/3}, \qquad pq = p_{/3} + q_{/4}. \tag{3.9}$$

In this case, g_f is Ricci-flat.

A comparison between (3.6) and (3.9) shows at once that Ricci-parallel Walker metrics (1.1) which are not Einstein form a quite large class, depending on two arbitrary non-vanishing one-variable functions h and l. Note that an irreducible Ricci-parallel Riemannian manifold is necessarily Einstein [7].

Next, as it is well known, a pseudo-Riemannian manifold (M, g), of dimension $n \geq 4$, is *conformally flat* if and only if its Weyl curvature tensor vanishes, that is,

$$\begin{aligned} R(X,Y,Z,W) = &\frac{1}{n-2}(g(X,Z)\varrho(Y,W) + \varrho(X,Z)g(Y,W) \\ &- g(X,W)\varrho(Y,Z) - g(Y,Z)\varrho(X,W)) \\ &- \frac{\tau}{(n-1)(n-2)}\left(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\right), \end{aligned} \tag{3.10}$$

for all vector fields X, Y, Z, W tangent to M, where τ denotes the scalar curvature.

In our case, applying (3.10) to the curvature and Ricci components described by (2.3) and (2.4) respectively, a direct calculation leads to prove the following

Theorem 3.3. *A Walker metric g_f given by* (1.1) *is conformally flat if and only if its defining function f is of the special form* (3.5)*, where p, q, s satisfy*

$$p_{34} = pp_3, \qquad q_{34} = qp_3, \qquad s_{34} = sp_3, \qquad p_3 = q_4. \tag{3.11}$$

4 – Locally symmetric Walker metrics (1.1)

A symmetric space is a connected pseudo-Riemannian manifold whose geodesic symmetries are isometries. A manifold is said to be locally symmetric if it is isometric to a symmetric space. A well-known characterization states that a pseudo-Riemannian manifold (M, g) is locally symmetric if and only if $\nabla R = 0$. In particular, a locally symmetric space is Ricci-parallel.

Consider now a Walker metric g_f given by (1.1). When g_f is Ricci-parallel, we know by Theorem 3.1 that its defining function f is of special form (3.5) and satisfies (3.6). Long but routine calculations show that for such a metric g_f, the possibly non-vanishing components $\nabla_k R_{ijlm} = (\nabla_{\partial_k} R)(\partial_i, \partial_j, \partial_l, \partial_m)$ of the covariant derivative of R, are given by

$$\begin{aligned}
\nabla_1 R_{3434} &= -\nabla_4 R_{1334} = -\nabla_4 R_{2434} = \frac{1}{2}[2p_{/34} - pp_{/3} - pq_{/4}],\\
\nabla_2 R_{3434} &= \nabla_3 R_{2434} = \nabla_3 R_{1334} = \frac{1}{2}[2q_{/34} - qp_{/3} - qq_{/4}],\\
\nabla_3 R_{3434} &= x_1[p_{/334} - p(pq_{/3} + qp_{/3}) + qp_{/34} - p_{/3}q_{/4}]\\
&\quad + x_2[q_{/334} - q(pq_{/3} + qp_{/3}) + qq_{/34} - q_{/4}q_{/3}]\\
&\quad + s_{/334} - s(pq_{/3} + qp_{/3}) + qs_{/34} - q_{/4}s_{/3},\\
\nabla_4 R_{3434} &= x_1[p_{/344} - p(pq_{/4} + qp_{/4}) + pp_{/34} - p_{/3}p_{/4}]\\
&\quad + x_2[q_{/344} - q(pq_{/4} + qp_{/4}) + pq_{/34} - p_{/3}q_{/4}]\\
&\quad + s_{/344} - s(pq_{/4} + qp_{/4}) + ps_{/34} - p_{/3}s_{/4}
\end{aligned}$$

and the ones obtained by them using the symmetries of ∇R. Again taking into account the fact that g_f is Ricci -parallel, we then have that g_f is locally symmetric if and only if f satisfies (3.5), (3.6) and

$$\begin{cases}
2p_{/34} - pp_{/3} - pq_{/4} = 0,\\
2q_{/34} - qp_{/3} - qq_{/4} = 0,\\
p_{/334} - p(pq_{/3} + qp_{/3}) + qp_{/34} - q_{/4}p_{/3} = 0,\\
q_{/344} - q(pq_{/4} + qp_{/4}) + pq_{/34} - q_{/4}p_{/3} = 0,\\
s_{/334} - s(pq_{/3} + qp_{/3}) + qs_{/34} - q_{/4}s_{/3} = 0,\\
s_{/344} - s(pq_{/4} + qp_{/4}) + ps_{/34} - p_{/3}s_{/4} = 0.
\end{cases} \tag{4.1}$$

Using both (3.6) and (4.1), by standard calculations we obtain the following

Theorem 4.1. *A Walker metric g_f described in* (1.1) *is locally symmetric if and only if f is of the special form* (3.5)*, where p, q and s are $\mathcal{C}^\infty$ real valued functions satisfying one of the following sets of conditions*

i) *$q{=}ap$ and*

$$p_{/3} = \frac{a}{2}p^2 + k, \qquad p_{/4} = \frac{1}{2}p^2 + \frac{k}{a},$$
$$s_{334} + aps_{/34} - p_{/3}s_{/3} - 2app_{/3}s = 0, \; s_{/344} + ps_{/34} - p_{/3}s_{/4} - 2pp_{/3}s = 0,$$

for two real constants $a \neq 0$ and k.

ii) *$q = 0$, $p = p(x_4)$, $s_{/34} = G(x_4)$ and*

$$p' = \frac{1}{2}p^2 + \alpha, \qquad G' + pG = 0,$$

for a real constant α.

iii) *$p = 0$, $q = q(x_3)$, $s_{/34} = H(x_3)$ and*

$$q' = \frac{1}{2}q^2 + \beta, \qquad H' + qH = 0,$$

for a real constant β.

iv) *$p = q = 0$ and $s_{/34} = \gamma$ is a real constant.*

It is now easy to compare conditions listed in Theorem 4.1 with the ones characterizing Einstein and conformally flat Walker metrics (1.1), listed in Theorems 3.2 and 3.3, respectively. In this way, we prove the following

Theorem 4.2. *A Walker metric g_f described by* (1.1) *is Einstein locally symmetric if and only if f is of the special form* (3.5)*, where p, q and s are C^∞ real valued functions satisfying one of the following sets of conditions:*

i) *$p = -2(a_0 + ax_3 + x_4)^{-1}$, $q = ap$ and $s_{/334} + aps_{/34} - p_{/3}s_{/3} - 2app_{/3}s = 0$, $s_{/344} + ps_{/34} - p_{/3}s_{/4} - 2pp_{/3}s = 0$, for two real constants $a \neq 0$ and k.*

ii) *$p = -2(x_4 + a_0)^{-1}, q = 0$ and $s_{/34} = a_1(x_4 + a_0)^2$, with a_1 a real constant.*

iii) *$p = 0, q = -2(x_3 + a_0)^{-1}$and $s_{/34} = a_1(x_3 + a_0)^2$, with a_1 a real constant.*

iv) *$p = q = 0$ and $s_{/34} = \gamma$ is a real constant.*

THEOREM 4.3. *A Walker metric g_f described in* (1.1) *is conformally flat locally symmetric if and only if f is of the special form* (3.5), *where p, q and s are C^∞ real valued functions satisfying one of the following sets of conditions:*

i) $q = ap$ *and*

$$p_{/3} = \frac{a}{2}p^2 + k, p_{/4} = \frac{1}{2}p^2 + \frac{k}{a}, s_{/34} = sp_{/3},$$

for two real constants $a \neq 0$ and k.

ii) $q = 0$, $s_{/34} = 0$ *and* $p = p(x_4)$ *satisfies* $p' = \frac{1}{2}p^2 + \alpha$, *for a real constant* α.

iii) $p = 0$, $s_{/34} = 0$ *and* $q = q(x_3)$ *satisfies* $q' = \frac{1}{2}q^2 + \beta$, *for a real constant* β.

iv) $p = q = s_{/34} = 0$.

REMARK 2.1. Using Proposition 2.1, it is easily seen that among locally symmetric Walker metrics g_f described in Theorem 4.1, the ones of case iv) have a diagonalizable Ricci operator, while in cases i), ii) and iii), the Ricci operator is diagonalizable only in the special cases $k = 0$, $\alpha = 0$ and $\beta = 0$, respectively. In particular, most of the conformally flat Lorentzian Walker metrics listed in Theorem 4.3 do not have diagonalizable Ricci operator and so, do not have any correspondance with the known Riemannian examples.

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