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Geodesically Complete Lagrangians on Manifolds

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Abstract: *In this paper we prove the "Geodesic Connectivity" of a large class of Lagrangian Functions defined on a differentiable manifold. The study is carried on by means of "Convex neighborhoods" of suitable associated "Finsler Metrics". Hence, these metrics are useful to solve the problem considered here, too. This result strengthens the conjecture, once considered very promising and now almost forgotten, that the use of "Finsler Metrics" simplify the study of Lagrangian Functions.*

1 – Introduction

Let M be a C^∞–differentiable n–dimensional manifold endowed with a C^∞–differentiable Lagrangian Function $L : \mathbb{R} \times TM \to \mathbb{R}$, being TM the tangent bundle of M and $\mathbb{R}$ the field of real numbers.

One of the most important problems of both Mathematics and Physics is to obtain all the curves of M, which are critical points of the action integral:

$$\mathcal{H}(\gamma) = \int_a^b L(t, \gamma(t), \dot{\gamma}(t))dt \ , \tag{1.1}$$

being $\gamma : [a, b] \to M$, with $a, b \in \mathbb{R}$ and $a < b$, a piecewise C^∞–differentiable curve and $\dot{\gamma}(t)$ the vector tangent to γ in $\gamma(t)$, for any $t \in [a, b]$.

We shall call *geodesic of* L these critical points.

During the first part of the past century, many mathematicians and physicists (see, *e.g.*, [8]) argued that the study of the action $\mathcal{H}$ would be determined

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in a simpler way by means of "Finsler Geometry", since the action $\mathcal{H}$, module a reparametrization of the involved curves, has the same geodesics of the action of the homogeneous function:

$$F(t_0, \dot{t}_0, x, \dot{x}) = L(t_0, x, \frac{\dot{x}}{\dot{t}_0})\dot{t}_0,$$

defined for any $(t_0, \dot{t}_0, x, \dot{x}) \in A$, with $A = \{(t_0, \dot{t}_0, x, \dot{x}) \in \mathbb{R}^2 \times TM | \dot{t}_0 \neq 0\}$.

Now, the previous conjecture is almost forgotten, because of the supposed lack of covariance of the classical Calculus of Variations, even if the transformation, which associates the homogeneous function F to the Lagrangian Function L, is quite natural (*e.g.*, it is used in [11] to introduce the Hamiltonian Function as the generalized momentum with respect to the time variable).

In this paper, we shall show this choice simplify the study of the "sufficient condition", too. In fact, we achieve a sufficient condition for the Geodesic Connectedness of M, with respect to L, by using the function F.

In order to obtain this result, we prove many properties of F, which imply corresponding properties of L and we quote only two among them (see Remarks 3.1 and 3.2).

The problem of geodesic connectivity of manifolds is one of the most studied and until now it is solved only for particular cases (see, *e.g.*, [6] and [9] for the Lagrangian Functions determined by Riemannian metrics and Lorentzian metrics, respectively). Also the case of Finslerian Metrics is well known (see, *e.g.*, [4]). However, the methods introduced in [4] can not be applied here, because in this paper the whole tangent bundle TM of M if used and in any case calculations would be difficult.

In [3], we proved that a large class of connections can be used in order to obtain the results of [4], and that by replacing the bundle TM with the bundle $T'M = TM - \sigma(M)$, being $\sigma : M \to TM$ the zero section, calculations could be simplified. From the beginning of our study, it was clear that the methods introduced in [3], can be used even if one replaces $\sigma(M)$ with larger sets and that calculations could be simplified, by using the results of [1] and [2], too.

In order to continue, we need some observations.

Let $x_0, x_1 \in M$. In the Riemannian and Finslerian cases, if a geodesic $\gamma : [a, b] \to M$ having $\gamma(a) = x_0$ and $\gamma(b) = x_1$ exists, then a curve $\gamma' : [a', b'] \to M$, for any $a', b' \in \mathbb{R}$, with $a' < b'$, obtained by setting $\gamma'(t) = \gamma(\alpha t + \beta)$, for any $t \in [a', b']$, with α and β suitable real numbers, is a geodesic, too. As soon as the Lagrangian function L depends explicitly on $t \in \mathbb{R}$, the problem considered here splits into three quite different problems listed below.

i) Find the conditions which allow the existence of $a, b \in \mathbb{R}$, with $a < b$, such that there exists a geodesic $\gamma : [a, b] \to M$, having $\gamma(a) = x_0$ and $\gamma(b) = x_1$.
ii) By using the same notations of *i*), one can ask to find the Lagrangian Functions having the property that a and b can be chosen in an arbitrary way into $\mathbb{R}$, only subject to the condition $a < b$.

iii) Moreover, if Problem i) allows more than one solution and in any case for Problem ii), one may ask to find the values of a and b which minimize the action of L.

In the Riemannian case, the geodesic completeness at a point implies the geodesic completeness at any point. The proof of this property can be partially applied to Lagrangian Functions and it implies a surprising solution for the Problem ii) (see below).

We set $N = \mathbb{R} \times M$ and we identify $T\mathbb{R}$ with the trivial bundle $pr_1 : \mathbb{R}^2 \to \mathbb{R}$, being pr_1 the canonical projection on the first factor. Then we shall consider the set $A = \mathbb{R} \times (\mathbb{R} - \{0\}) \times TM \subseteq T(\mathbb{R} \times M)$. The canonical projection of the tangent bundle TN onto N induces a fiber bundle structure $\pi : A \to N$ and we shall denote by A_y the fiber over y, for any $y \in N$.

Obviously, we need to impose some conditions on L in order to obtain a sufficiently regular function F, by means of which the problem of geodesic connectedness of L can be solved by means of geometrical methods (one can refer to [9] in order to see the difficulties encountered if the energy functional is not bounded from below). To write these conditions, we introduce the following notations.

If (U, φ) is a local chart of M, we denote by $(TU, T\varphi)$ the local chart canonically induced on A, with $TU = \pi^{-1}U$.

We set

$$\begin{aligned}
&\varphi = (x^1, \dots, x^n), \quad T\varphi = (x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n), \\
&e_\alpha = \frac{\partial}{\partial x^\alpha}, \quad \varepsilon_\beta = \frac{\partial}{\partial \dot{x}^\beta}, \qquad e^\gamma = dx^\gamma, \\
&\varepsilon^\delta = d\dot{x}^\delta, \quad \text{for any} \quad \alpha, \beta, \gamma, \delta \in \{1, \dots, n\}
\end{aligned}$$

Definition 1.1 With these notations, we shall say that L is a *Positive Lagrangian* (*P–Lagrangian*, in short) if and only if:

1_P) There exists $a \in \mathbb{R}$, with $a > 0$, such that

$$L(t, x, \dot{x}) > a \ , \quad \forall (x, \dot{x}) \in TM \ , \quad t \in \mathbb{R}.$$

2_P) The tensor field $H = H_{\alpha\beta} e^\alpha \otimes e^\beta$ along the mapping $\widetilde{\pi} : (t, x, \dot{x}) \in \mathbb{R} \times TM \mapsto (t, x) \in \mathbb{R} \times M$, with

$$H_{\alpha\beta} = \frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta}, \quad \forall \alpha, \beta \in \{1, \dots, n\}$$

is semidefined positive and has constant rank equal to n.

Condition 2_P) leaves out many interesting Lagrangian Functions. In any case, the problem cannot be avoided. In fact, even in the semi-Riemannian case one can obtain results on the existence of geodesics only by imposing strong conditions on both the topology of M and the metric (as, *e.g.*, in [9]). The condition 1_P) looks stronger, than it really is. In fact, it requires only that L is bounded from below, because one can replace L with $L+c$, being c any constant, since additive constants do not play any role in the Calculus of Variations.

When L was studied by means of a "Finsler Metric", the following condition was required (see, *e.g.* [8]):

$$L(t,x,\rho\dot{x}) \neq \rho L(t,x,\dot{x}) \ , \quad \forall t,\rho \in \mathbb{R}, \quad \forall (x,\dot{x}) \in TM, \text{ with } \rho \neq 0.$$

It is easy to see that condition 2_P) implies this classical condition only in a partial way. In fact, in our case the equality:

$$L(t,x,\rho\dot{x}) = \rho L(t,x,\dot{x}),$$

can hold for any $t \in \mathbb{R}$, any $(x,\dot{x}) \in TM$ and only for isolated values of ρ, under the Assumption 2_P).

If L is a P-Lagrangian then we consider the restriction of F to the subset $A^+ = \{(t_0,x,\dot{t}_0,\dot{x}) \in A | \dot{t}_0 > 0\}$ and on this subset F is sufficiently regular and allows us to prove all the needed properties relative to the "convex neighborhoud", by using proofs analogous to the corresponding ones of the Riemannian and the Finslerian cases (see, *e.g.*, [6] and [3]).

The completeness condition, which follows, is necessary to study the problem of geodesic connectivity.

DEFINITION 1.2 We say that L is a *Geodesically Complete* P–Lagrangian at $(t_0,x_0) \in \mathbb{R}\times$ M(GCP*–Lagrangian* at (t_0,x_0)) if and only if:

C) Any geodesic $\gamma : I = [0,1] \to M$ of L, such that $\gamma(t_0) = x_0$ is the restriction to I of a geodesic of L defined on the whole $\mathbb{R}$.

Finally, we set the following:

DEFINITION 1.3 We say that L is a *Geodesically Complete P–Lagrangian* (*GCP–Lagrangian*) if and only if:

GC) There exist a sequence $\{t_n\}_{n\in\mathbb{N}}$ and $x_0 \in M$ such that $\lim_{n\to\infty} t_n = -\infty$ and L is a GCP–Langrangian at (t_n,x_0), for any $n \in \mathbb{N}$.

We have:

THEOREM 1.1. *If L is a GCP–Lagrangian at $(t_0,x_0) \in \mathbb{R}\times M$ (M arcwise connected), then for any $t_1,t_2 \in \mathbb{R}$, with $t_2 > t_1 > t_0$, and for any $x_1,x_2 \in M$ there exists at least a geodesic $\gamma : [t_1,t_2] \to M$ of L, such that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$.*

THEOREM 1.2. *If L is GCP–Lagrangian, then in the previous Theorem t_1 and t_2 can be arbitrarily chosen, subject only to the condition $t_1 < t_2$.*

Our proofs follow in the closest way the corresponding proofs used in the Riemannian and the Finslerian cases (we need only a lemma, which has no correspondence in the standard cases, and the proofs of some theorems of Section 4 are almost different, from the correspondent proofs of Riemannian and Finslerian cases) and this fact can be useful for the proof of further results. We chose to follow the book [6] for the Riemannian case and [3] for the Finslerian one.

2 – P – Lagrangians and FL – Connections

Let M be a C^∞ – differentiable n–dimensional manifold. We set $N = \mathbb{R}\times M$ and we shall use all the notations introduced in the previous Section.

We shall denote by $\pi^+ : A^+ \to N$ the mapping induced by the projection $\pi : A \to N$. Then, $\pi^+ : A^+ \to N$ is a differentiable bundle over N and A^+_y will denote the fiber over $y \in N$. Moreover, a curve $\sigma = (\tau^0, \tau) : [a, b] \to N$ is said to be *admissible* if and only if $\dot\sigma(t) \in A^+$, (equivalently $\dot\tau^0(t) > 0$), for any $t \in [a, b]$.

Then the restriction to A^+ of the tensor field H defined in the Introduction, is a tensor field along π^+. We recall that L is said to be *regular* if and only if the tensor field H has maximal rank. Moreover, if L is a P–Lagrangian, then H is non degenerate and the following generalized Schwarz inequality holds:

$$H_{(t,x,\dot x)}(X,Y)^2 \le H_{(t,x,\dot x)}(X,X)H_{(t,x,\dot x)}(Y,Y), \qquad \forall x \in M \quad \forall \dot x, Y, X \in T_xM \quad \forall t \in \mathbb{R}$$

We associate to any chart (U, φ) of M, the chart $(\tilde U, \tilde\varphi)$ of $\mathbb{R}\times M$, with $\tilde U = \mathbb{R}\times U$ and $\tilde\varphi = (x^0, x^1, \dots, x^n)$, being $\varphi = (x^1, \dots, x^n)$ and $x^0 : \mathbb{R} \to \mathbb{R}$ the identity map. We shall call $(\tilde U, \tilde\varphi)$ *canonical chart of* N. Moreover, we shall denote by $(A^+\tilde U, A^+\tilde\varphi)$ the chart of A^+ induced by the chart $(T\tilde U, T\tilde\varphi)$ of TN. Small Latin indices will run from 0 to n and we shall use the convections of the Introduction for these charts, too. Now, we set $h = h_{ij}e^i \otimes e^j$, with:

$$\begin{aligned}
h_{00}(y,\dot y) &= \left(\frac{\partial^2 F}{(\partial \dot x^0)^2}\right)_{(y,\dot y)} = \frac{1}{(\dot x^0)^3} H_z(\dot x, \dot x),\\
h_{\alpha 0}(y,\dot y) = h_{0\alpha}(y,\dot y) &= \left(\frac{\partial^2 F}{\partial \dot x^0 \partial \dot x^\alpha}\right)_{(y,\dot y)} = -\frac{1}{(\dot x^0)^2} H_{\alpha\beta}(z)\dot x^\beta,\\
h_{\alpha\beta}(y,\dot y) &= \left(\frac{\partial^2 F}{\partial \dot x^\alpha \partial \dot x^\beta}\right)_{(y,\dot y)} = \frac{1}{\dot x^0} H_{\alpha\beta}(z),
\end{aligned}$$

for any $(x^0, x, \dot x^0, \dot x) = (y, \dot y) \in A^+$, having set $\dot x = \dot x^\alpha e_\alpha$ with respect to the local chart $(U\varphi)$ of M and $z = (x^0, x, \dot x/\dot x^0)$. Then the metric tensor $g = g_{ij}e^i \otimes e^j$ of the associated "Finsler Metric" has the standard expression:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial \dot x^i \partial \dot x^j} = \frac{\partial F}{\partial \dot x^i}\frac{\partial F}{\partial \dot x^j} + F\frac{\partial^2 F}{\partial \dot x^i \partial \dot x^j}$$

As in the Finsler case, g is defined positive if and only if h is semi–defined positive and has rank n (see, *e.g.*, [12]). We have:

$$h(Y,Y) = \frac{1}{(\dot{x}^0)^3} H_z(\dot{x},\dot{x})(X^0)^2 - \frac{2}{(\dot{x}^0)^2} H_z(\dot{x},X)X^0 + \frac{1}{\dot{x}^0} H_z(X,X),$$

for any $(y,\dot{y}) = (x^0, x, \dot{x}^0, \dot{x}) \in A^+$, with $z = (x^0, x, \dot{x}/\dot{x}^0)$ and $Y = (X^0, X)$, being $X \in T_x M$ and $X^0 \in \mathbb{R}$. Then both previous conditions are fulfilled by h, since H has rank equal to n and the generalized Schwarz identity holds, being H positive defined.

Now we recall that the local expression of the Euler-Lagrange system of ordinary differential equations of L is:

$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right)_{(t,\gamma,\dot{\gamma})}\right] - \left(\frac{\partial L}{\partial x^\alpha}\right)_{(t,\gamma,\dot{\gamma})} = 0,$$

where $\gamma : [a,b] \to M$ is a C^2–differentiable curve, while, the corresponding system, for the function F, is:

$$\frac{d}{dt}\left[\left(\frac{\partial F}{\partial \dot{x}^i}\right)_{(\tau^0,\tau,\dot{\tau}^0,\dot{\tau})}\right] - \left(\frac{\partial F}{\partial x^i}\right)_{(\tau^0,\tau,\dot{\tau}^0,\dot{\tau})} = 0$$

where $\tau : [a,b] \to M$ is a C^2–differentiable curve and $\tau^0 : [a,b] \to \mathbb{R}$ a strictly increasing C^2–differentiable function. Then we have:

THEOREM 2.1. *If a C^2–differentiable curve $(\tau^0,\tau) : [a,b] \to N$, with $a,b \in \mathbb{R}$, $a < b$ and τ^0 strictly increasing, is a geodesic of F, then the curve $(\widetilde{\tau}^0,\tau) : [a,b] \to N$ is a geodesic of F, for any strictly increasing function $\widetilde{\tau}^0 : [a,b] \to \mathbb{R}$. Moreover, (τ^0,τ) is a geodesic of F if and only if the curve $\gamma = \tau \circ (\tau^0)^{-1}$ is a geodesic of L.*

The previous Theorem is well known and its proof is straightforward (see, *e.g.*, [11]). The local expression of the Euler-Lagrange equation of F^2 is:

$$\frac{d}{dt}\left[\left(\frac{\partial F^2}{\partial \dot{x}^i}\right)_{(\tau,\tau^0,\tau,\dot{\tau})}\right] - \left(\frac{\partial F^2}{\partial x^i}\right)_{(\tau^0,\tau,\dot{\tau}^0,\dot{\tau})} = 0, \tag{2.1}$$

with the obvious meaning of the used symbols and we have:

THEOREM 2.2. *Let $\sigma = (\tau^0,\tau) : I \to \mathbb{R} \times M$ be a C^2–differentiable curve. Then, σ is a geodesic of F^2 if and only if σ is a geodesic of F and the mapping $\phi = (\tau^0)^{-1}$ is given by*

$$\phi(t) = k \int_0^t L(s,\gamma(s),\dot{\gamma}(s))ds + k_1, \tag{2.2}$$

being $k > 0$ a suitable constant, $k_1 \in \mathbb{R}$ and $\gamma = \tau \circ \phi$.

The proof of this Theorem is straightforward, too. The main consequence of the previous Theorems is that the critical points of L and the critical points of F^2 coincide, module reparametrization determined by a first order differential equation.

Now, under our assumptions, the function F determines the tensor field g along π^+ which is not degenerate and positive defined on the open subset $A^+ \subseteq TN$. As a consequence, we can follow [2].

Hence, we shall denote by $V = V(A^+)$ the vertical sub bundle of TA^+; *i.e.*, V is the subset of TA^+ containing all the vectors tangent to the fibers of A^+. Then, we fix two tensor fields of type (1,1) defined on A^+, which, we shall denote by P and Q, respectively. We shall suppose that the local expressions of these tensor fields, with respect to the chart $(A^+\tilde{U}, A^+\tilde{\varphi})$ induced on A^+ by a canonical chart $(\tilde{U}, \tilde{\varphi})$ are:

$$P = \delta^i_j e_i \otimes e^j - P^{\hat{i}}_j \varepsilon_{\hat{i}} \otimes e^j \quad \text{and} \quad Q = \delta^{\hat{i}}_{\hat{j}} \varepsilon_{\hat{i}} \otimes \varepsilon^{\hat{j}} + P^{\hat{i}}_j \varepsilon_{\hat{i}} \otimes e^j ,$$

where $P^{\hat{i}}_j : A^+\tilde{U} \to \mathbb{R}$ are positively homogeneous of degree one, C^∞–differentiable functions (see, *e.g.*, [12]).

Moreover, we shall denote by G the vertical lift of g (see, *e.g.*, [13]); *i.e.* $G(Y_1, Y_2) = g((\pi^+)_* Y_1, (\pi^+)_* Y_2)$, where $(\pi^+)_* : TA^+ \to TN$ is the total differential of the map $\pi^+ : A^+ \to N$. Then G is a tensor field of A^+ semidefined positive and having constant rank equal to $n+1$. Following [2], we can easily prove:

Proposition 2.1. *There exists a connection ∇ on A^+ such that:*

i) $C^2_1(P \otimes T) = 0$*, where T is the torsion tensor of ∇ and C^2_1 is the contraction of the first contravariant index with the second covariant index of $P \otimes T$ (see, e.g.,* [7]*).*

ii) *Let us denote by $\mathcal{X}(A^+)$ the Lie algebra of vector fields on A^+. Then for each $X \in \mathcal{X}(A^+)$, having $\pi^+_*(X_{(y,\dot{y})}) = \dot{y}$, for any $(y,\dot{y}) \in A^+$, it results*

$$(\nabla_X G)(X, Y) = (\nabla_Y G)(X, X) = 0 \ , \quad \forall Y \in \mathcal{X}(A^+)$$

iii) *Let $\mathcal{I}(N)$ (respectively $\mathcal{I}_{\pi^+}$) be the tensor algebra of C^∞–differentiable tensor fields defined on N (along π^+) and $\mathcal{X}_{\pi^+} = \mathcal{I}^1_{0\pi^+}$ the $\mathcal{F}(A^+)$–module of C^∞–differentiable vector fields along π^+, having denoted by $\mathcal{F}(A^+)$ the ring of C^∞–differentiable functions on A^+. Then, the mapping $\nabla' : \mathcal{X}_{\pi^+} \times \mathcal{I}(N) \to \mathcal{I}_{\pi^+}$ defined by setting*

$$\nabla'_X Y = \pi^+_*(\nabla_{X^c} Y^c) \ , \quad \forall X, Y \in \mathcal{X}(N);$$

being c the complete lift (*see, e.g.,* [13]), *is a non-linear connection with three indices* (*see* [2]).

A connection ∇ verifying i), ii) and iii) of the previous proposition is called *FL–connection* (*Finsler Lagrange Connection*). Moreover, the non-linear connection ∇' defined by iii) of the same proposition is called *BL–connection* (*Berwald Lagrange Connection deduced from* ∇).

We recall that: A *non-linear connection* $\widetilde{\nabla}$ (*with three indices*) can be regarded as an $\mathbb{R}$–bilinear mapping $\widetilde{\nabla} : \mathcal{X}_{\pi^+} \times \mathcal{I}(N) \to \mathcal{I}_{\pi^+}$ such that:

1) $\widetilde{\nabla}_{fX}(kY) = fX(k)Y + fk\widetilde{\nabla}_X Y, \forall X \in \mathcal{X}_{\pi^+}, \quad \forall Y \in \mathcal{X}(N),\ \forall f, k \in \mathcal{F}(N)$.
2) $\widetilde{\nabla}$ commutes with all the contractions.
3) For each $X \in \mathcal{X}_{\pi^+}$, with X positively homogeneous of degree ρ, $\widetilde{\nabla}_X Y$ is positively homogeneous of degree ρ, for any $Y \in \mathcal{X}(M)$.

As in [2], we also get:

Proposition 2.2. *Let ∇ be a FL–connection and let ∇' the BL–connection deduced from ∇ and $\sigma = (\tau^0, \tau) : [a, b] \to N$ be an admissible curve. Let us set $Y = (\tau, \dot{\tau}) : [a, b] \to TA^+$. Then, σ is a path of ∇' if and only if there exists a vector field Z along Y such that*

$$(\nabla_{\dot{Y}} \dot{Y})(t) = Q_{Y(t)}(Z_{Y(t)}), \quad \forall t \in [a, b],$$

or equivalently if and only if σ is a geodesic of F^2.

Again, following [2], it is easy to prove that the set of all the FL–Connections can be obtained in the following way. Let $\widetilde{\nabla}$ be any connection on A^+ and $(\tilde{U}, \tilde{\varphi})$ be a standard chart of N. With respect to the chart $(A^+\tilde{U}, A^+\tilde{\varphi})$ induced by $(\tilde{U}, \tilde{\varphi})$ on A^+, we set:

$$\begin{aligned}
\nabla_{e_j} e_k &= \widetilde{\Gamma}^{1i}_{jk} e_i + \widetilde{\Gamma}^{5\hat{i}}_{jk} \varepsilon_{\hat{i}}, \quad \nabla_{\varepsilon_{\hat{j}}} e_k = \widetilde{\Gamma}^{2i}_{\hat{j}k} e_i + \widetilde{\Gamma}^{6\hat{i}}_{\hat{j}k} \varepsilon_{\hat{i}} \\
\nabla_{e_j} \varepsilon_{\hat{k}} &= \widetilde{\Gamma}^{3i}_{j\hat{k}} e_i + \widetilde{\Gamma}^{7\hat{i}}_{j\hat{k}} \varepsilon_{\hat{i}}, \quad \nabla_{\varepsilon_{\hat{j}}} \varepsilon_{\hat{k}} = \widetilde{\Gamma}^{4i}_{\hat{j}\hat{k}} e_i + \widetilde{\Gamma}^{8\hat{i}}_{\hat{j}\hat{k}} \varepsilon_{\hat{i}}
\end{aligned}$$

The $8(n+1)^3$ functions $(\widetilde{\Gamma}^{1i}_{jk}, \widetilde{\Gamma}^{2i}_{\hat{j}k}, \widetilde{\Gamma}^{3i}_{j\hat{k}}, \widetilde{\Gamma}^{4i}_{\hat{j}\hat{k}}, \widetilde{\Gamma}^{5\hat{i}}_{jk}, \widetilde{\Gamma}^{6\hat{i}}_{\hat{j}k}, \widetilde{\Gamma}^{7\hat{i}}_{j\hat{k}}, \widetilde{\Gamma}^{8\hat{i}}_{\hat{j}\hat{k}})$ are the local components of $\widetilde{\nabla}$. By using the local components of G and P and the previous functions, we obtain the following new $8(n+1)^3$ functions defined on $A^+\tilde{U}$:

$$\begin{aligned}
2\Gamma^{1i}_{jk} &= g^{im}(\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{kj} + P^{\hat{h}}_m \hat{\partial}_{\hat{h}} g_{jk} \\
&\quad - P^{\hat{h}}_j \hat{\partial}_{\hat{h}} g_{mk} - P^{\hat{h}}_k \hat{\partial}_{\hat{h}} g_{mj}), \\
\Gamma^{2i}_{\hat{j}k} &= 0, \quad \Gamma^{3i}_{j\hat{k}} = 0, \quad \Gamma^{4i}_{\hat{j}\hat{k}} = 0, \\
\Gamma^{5\hat{i}}_{jk} &= P^{\hat{i}}_t \widetilde{\Gamma}^{1t}_{jk} - P^{\hat{i}}_t \Gamma^{1t}_{jk} + \widetilde{\Gamma}^{5\hat{i}}_{jk}, \quad \Gamma^{6\hat{i}}_{\hat{j}k} = P^{\hat{i}}_t \widetilde{\Gamma}^{2t}_{\hat{j}k} + \widetilde{\Gamma}^{6\hat{i}}_{\hat{j}k}, \\
\Gamma^{7\hat{i}}_{j\hat{k}} &= P^{\hat{i}}_t \widetilde{\Gamma}^{3t}_{j\hat{k}} + \widetilde{\Gamma}^{7\hat{i}}_{j\hat{k}}, \quad \Gamma^{8\hat{i}}_{\hat{j}\hat{k}} = P^{\hat{i}}_t \widetilde{\Gamma}^{4t}_{\hat{j}\hat{k}} + \widetilde{\Gamma}^{8\hat{i}}_{\hat{j}\hat{k}},
\end{aligned}$$

here we used the notations $\partial_i f = \dfrac{\partial f}{\partial x^i}$ and $\hat{\partial}_{\hat{i}} f = \dfrac{\partial f}{\partial \dot{x}^{\hat{i}}}$, for any $f \in \mathcal{F}(A^+)$ and any $i, \hat{i} \in \{0, \dots, n\}$.

The previous functions are the local components of a connection ∇ of A^+. It is easy to prove that ∇ verifies i), ii) and iii) of the previous Definition.As a consequence, ∇ is a FL–Connection.

The previous connection ∇ can be used in order to determine all the FL–Connections. In fact, let ∇^1 be a further connection and let us denote by $C \in \mathcal{I}^1_2(A^+)$ the tensor field defined by setting

$$C(X,Y) = \nabla_X Y - \nabla^1_X Y, \quad \forall X, Y \in \mathcal{X}(A^+)$$

There exists a unique tensor field, $\widetilde{C}$, of type $(1,2)$ along π^+, such that

$$\widetilde{C}(X,Y) = \pi_* C(X^c, Y^c), \quad \forall X, Y \in \mathcal{X}(M).$$

The tensor field $\widetilde{C}$ is called the *tensor field associated to* C. Now we denote by d' the so called "derivation along the fibers", which is defined by setting $d'f = (\hat{\partial}_{\hat{i}} f) e^{\hat{i}}$, for any $f \in \mathcal{F}(A^+)$ and is extended to the whole tensor algebra $\mathcal{I}_{\pi^+}$ in the well known way (see, *e.g.*, [5]). Then we have:

PROPOSITION 2.3. *Under the previous assumptions the connection* ∇^1 *is a FL–Connection if and only if* C *verifies the following conditions:*

i) $\widetilde{C}_{(x,Z)}(Z_x, Z_x,) = 0$, $\quad \forall (x,Z) = Z_x \in A^+$.

ii) $C^1_1(Z_x \otimes C^1_1((d'F \otimes \widetilde{C})_{(x,Z)})) = 0, \quad \forall (x,Z) = Z_x \in A^+$.

iii) $PC(X, QY) = 0$, $\quad \forall X, Y \in \mathcal{X}(A^+)$.

iv) PC *is symmetric with respect to the two lower indices and it is positive homogeneous of degree 0.*

In the literature on Finsler spaces, one can find a lot of tensor fields having the properties i)–iv) (see, *e.g.*, [12] and [10] and [2]) and by following the known examples many other tensors of the same kind can be constructed. From $\widetilde{C}$, one can obtain many tensors like C by following methods, which are analogous to the ones previously used for the construction of the connection ∇^1. Hence, we omit them for the sake of brevity (see [2]).

3 – Convex neighborhoods

If $y \in N$ we shall set $B_\rho(0_y) = \{\dot{y} \in A^+_y | F(y, \dot{y}) < \rho\}$. The set $B_\rho(0_y)$ is an open subset of A^+_y and it will be called *the open L–indicatrix pointed at* 0_y and having ρ as its *radius*. Analogously, the *closed indicatrix pointed at* 0_y and having $\rho > 0$ as its *radius* is the set $\overline{B}_\rho(0_y) = \{\dot{y} \in A^+_y | F(y, \dot{y}) \le \rho\}$ and it is the closure of the open L–indicatrix $B_\rho(0_y)$ into A^+. $\overline{B}_\rho(0_y)$ and $B_\rho(0_y)$ are both convex, that is for any X, Y elements of $\overline{B}_\rho(0_y)$ ($B_\rho(0_y)$) the vector $tX+(1-t)Y$ belongs to $\overline{B}_\rho(0_y)$ ($B_\rho(0_y)$), for any $t \in I$. The vector 0_y belongs to $\overline{B}_\rho(0_y)$ and does not belong to $B_\rho(0_y)$.

From now on we denote by ∇ a FL-Connection and by ∇' the BL-connection deduced from ∇. Then, as in the Finslerian case (see, *e.g.*, [1]), the local expression of the equation of the geodesics of F^2 is independent from ∇. In fact, it is:

$$\ddot{\tau}^i = -\Gamma^i_{jk}\dot{\tau}^j\dot{\tau}^k = -\frac{1}{2}g^{im}(2\partial_j g_{mk} - \partial_m g_{jk})\dot{\tau}^j\dot{\tau}^k$$

with the obvious meaning of the used symbols.

As in the case of standard connections, by using Theorem 2.2, one can easily see that:

LEMMA 3.1. *For any $y \in N$, there exists a real number $\eta = \eta_y > 0$ such that for any $Y \in B_\eta(0_y)$ the geodesic $c_Y = (\tau^0, \tau)$, having $c_Y(0) = y$ and $\dot{c}_Y(0) = Y$, is an admissible curve defined on the interval* $[0, 2[$.

We observe that the constant curve $c_{0_y} = (\tau^0, \tau_1) : I \to N$, having $c_{0_y}(t) = y$, for any $t \in I$, with $y \in N$, is neither a geodesic of F^2, nor a geodesic of L and it is not an admissible curve, because τ^0 is not strictly increasing. In the following, for the sake of brevity, we shall consider this curve as a "degenerate geodesic" of F^2.

The previous Lemma allows us to define *the exponential mapping* exp_y : $B_\eta(0_y) \to N$, by setting $\exp_y(Y) = c_Y(1)$, being $c_Y : [0, 2[\to N$ the geodesic having $c_Y(0) = y$ and $\dot{c}_Y(0) = Y$, for any $Y \in B_\eta(0_y)$.

We also set $\widetilde{A} = \cup_{y \in N} B_{\eta_y}(0_y)$. Then, $\widetilde{A}$ is an open subset of A^+ and we can consider the mapping $exp : \widetilde{A} \to N$ defined by setting $exp(y, Y) = exp_y(Y)$, for any $(y, Y) \in \widetilde{A}$. From the properties of the dependence from the initial conditions of the solutions of ordinary differential equations, we obtain:

PROPOSITION 3.1. *The following assertions hold:*

i) *The map exp_y is C^∞–differentiable, for any $y \in N$.*

ii) *The map exp is C^∞–differentiable.*

iii) *For any $y \in N$ the real number η_y can be chosen in such a way that the mapping $C : \widetilde{A} \to N \times N$, defined by setting $C(y, Y) = (y, \exp_y(Y))$, for any $Y \in B_{\eta_y}(0_y)$, for any $y \in N$ is a diffeomorphism onto the open subset $C(\widetilde{A})$ of $N \times N$.*

iv) *The zero section of N is contained into the boundary of $\widetilde{A}$ and for any infinitesimal sequence $\{Y_n\}_{n \in \mathbb{N}}$ of elements of A^+_y we have $\lim_{n\to\infty} exp_y(Y_n) = y$ and $\lim_{n\to\infty}(Dexp_y)_{Y_n} = K$, for any $y \in N$, being K the Kronecker tensor field of N.*

The previous property allows us to define the exponential mapping on the zero section by setting $exp_y(0_y) = y$, for any $y \in N$.

We also set $B_\rho(y) = exp_y(B_\rho(0_y))$, for any $0 < \rho \leq \eta_y$, then y is a boundary point of $B_\rho(y)$, for any $y \in N$.

Moreover, we set $(Dexp_y)_{0_y} = K$, for any $y \in N$. Then exp_y and $Dexp_y y$ are continuous maps at 0_y.

Finally, for any admissible curve $\sigma : [a, b] \to N$ we define the *length of* σ by setting:

$$\mathcal{L}(\sigma) = \int_a^b F(\sigma(t), \dot{\sigma}(t))dt.$$

It follows that the curve $c(t) = exp_y(tY)$, with $t \in I$, $y \in N$ and $Y \in A_y$, such that exp_y is defined, is the geodesic of F having $c(0) = y$ and $\dot{c}(0) = Y$, being unique the geodesic, which have the previous initial conditions.

Hence, $(Dexp_y)_{tY} Y = \dot{c}(t) \in A^+$, for any $t \in I$, because of Lemma 3.1. Following [3], one can easily see:

Theorem 3.1. *For any $y \in N$ there exist two real numbers $\varepsilon = \varepsilon(y)$ and $\eta = \eta(y)$, with $0 < \varepsilon < \eta$, such that:*

i) *For any $p, q \in B_\varepsilon(y)$ with $p = (p^0, p_1)$, $q = (q^0, q_1)$ and $q^0 > p^0$, there exists a unique geodesic $c_p^q : I \to N$ such that $c_p^q(0) = p$, $c_p^q(1) = q$, $c_p^q(I) \subseteq B_\eta(y)$ and $L(c_p^q) < \eta$.*

ii) *For any $q \in B_\varepsilon(y)$ there exists a unique geodesic $c_y^q : I \to N$ such that $c_y^q(0) = y$, $c_y^q(1) = q$ and $c_y^q(I) \subseteq B_\eta(y) \cup \{y\}$ and $\mathcal{L}(c_y^q) < \eta$.*

The following Lemma is the equivalent of the Gauss Lemma of the Riemannian case (see, *e.g.*, [6]) and its proof can be easily obtained as in [3].

Lemma 3.2.

i) *Let $c : I \to N$ be a geodesic of F^2. Then, $F(c(t), \dot{c}(t)) = F(c(0), \dot{c}(0))$, for any $t \in I$.*

ii) *For any $y \in N$, $Y_1 \in B_\varepsilon(0_y)$ (ε as in the previous Theorem) and $Y_2 \in T_y N$ such that $g_{(y,Y_1)}(Y_1, Y_2) = 0$ it results*

$$g_{(exp_y(tY_1),(Dexp_y)_{tY_1} Y_1)}((Dexp_y)_{tY_1} Y_1, (Dexp_y)_{tY_1} Y_2) = 0,$$

for any $t \in I$.

Remark 3.1. We consider the assumptions of the previous Lemma and a canonical chart $(\tilde{U}, \tilde{\varphi})$ of N such that $y = (y^0, y_1) \in U$. We also consider two vectors $Y_1 \in A_y^+$ and $Y_2 \in T_y N$, with $Y_h = Y_h^i e_i$ $(h = 1, 2)$. Then a simple calculation shows that the orthogonality condition $g_{(y,Y_1)}(Y_1, Y_2) = 0$ is equivalent to:

$$\left[L(x^0, x, \frac{\dot{x}}{Y_1^0}) - \frac{1}{Y_1^0} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right)_{(x^0, x, \frac{\dot{x}}{Y_1^0})} Y_1^\alpha \right] Y_2^0 + \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right)_{(x^0, x, \frac{\dot{x}}{Y_1^0})} Y_2^\alpha = 0,$$

having set $\dot{x} = Y_1^\alpha e_\alpha$. Let $c_y^q = (\tau^0, \tau) : I \to N$ be a geodesic of the previous Lemma. We denote by $\gamma = \tau \circ (\tau^0)^{-1} : [y^0, q^0] :\to M$ the corresponding geodesic of L. Moreover, the vector field $Y = (Dexp_y)_{tY_1} Y_2$ defined along the curve c_y^q defines a vector field $\widetilde{Y}$ along γ, by setting $\widetilde{Y}(t) = Y((\tau^0)^{-1}(t)) \in T_{(t,\gamma(t))}N$, for any $t \in [y^0, q^0]$. In this case the previous equation can be written as:

$$\left[L(t, \gamma(t), \dot{\gamma}(t)) - \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right)_{(t,\gamma(t),\dot{\gamma}(t))} \dot{\gamma}^\alpha(t) \right] \widetilde{Y}_2^0(t)$$
$$+ \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right)_{(t,\gamma(t),\dot{\gamma}(t))} \widetilde{Y}_2^\alpha(t) = 0.$$

Since L is not degenerate, we can consider the momentum p_α conjugate to $\dot{x}^\alpha$, for any $\alpha \in \{1, \ldots, n\}$. Then, the Hamiltonian Function, which coincides with opposite of the generalized momentum p_0 related to the time variable (see, *e.g.*, [11]) is:

$$H(t, x, p_\alpha) = \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right)_{(t,x,\dot{x})} \dot{x}^\alpha - L(t, x, \dot{x}) = -p_0$$

Then ii) of the previous Lemma and (3.1) asserts that if $p_i \tilde{Y}_2^i(t_0) = 0$ is true at the instant $t_0 \in [y^0, q^0]$, then the same relation holds for any $t \in [y^0, q^0]$.

If $L(t, x, \dot{x})$ is the norm of a Riemannian metric, in this case L is a P–Lagrangian and (3.1) is equivalent to:

$$\tilde{Y}_2^0(t) = 2 \frac{|\tilde{Y}(t)|}{|\dot{\gamma}(t)|} \cos\theta(t)$$

being $\gamma : [y^0, q^0] \to M$ a not constant geodesic of h, $\theta(t)$ the angle between $\dot{\gamma}(t)$ and $\tilde{Y}(t)$, for any $t \in [y^0, q^0]$, and where $|\cdot|$ is the norm defined by h the Riemannian tensor. Hence, if the previous relation holds in $t_0 \in [y^0, q^0]$ it holds for any $t \in [y^0, q^0]$.

By means of the Taylor expansion of F, as in the Finsler case (see [5]), one can see:

Lemma 3.3. *For any $y \in N$, for any $Z \in A_y^+$ and any $Y \in T_yN$ such that $Z+Y \in A_y^+$ from $g_{(y,Z)}(Y,Z)=0$, it follows $F(y,Z+Y) \geq F(y,Z)$ and equality holds if and only if $Y = 0_y$.*

The previous Lemma implies:

Theorem 3.2. *Let $y \in N$ and $\tilde{b}(s) \in A_y^+$ such that $\tilde{b}(0) = 0_y$, $\tilde{b}$ is piecewise differentiable on $[0,s_1]$ and such that the function $F(y,\tilde{b}(s))$ is strictly increasing. Suppose that the curve $b(s) = exp_y\tilde{b}(s)$ exists for any $s \in [0,s_1]$. Finally let us put $c_y^{b(s)}(t) = exp_y(t\tilde{b}(s)$, for any $t \in I$ and $s \in [0,s_1]$.*

With these assumptions we have:

i) $\mathcal{L}(c_y^{b(s_1)}) \leq \mathcal{L}(b)$

ii) *If $\tilde{b}(s) = f(s)\tilde{b}(s_1)$, for any $s \in [0,s_1]$, with $f : [0,s_1] \to \mathbb{R}$ strictily increasing and piecewise C^∞–differentiable, then $\mathcal{L}(c_y^{b(s_1)}) = \mathcal{L}(b)$*

iii) *If $(Dexp_y)_{t\tilde{b}(s)}$ has maximal rank for any $s \in [0,s_1]$ and $\mathcal{L}(c_y^{b(s_1)}) = \mathcal{L}(b)$, then $\tilde{b}(s) = f(s)\tilde{b}(s_1)$, for any $s \in [0,s_1]$, with $f : [0,s_1] \to \mathbb{R}$ piecewise differentiable function and f strictly increasing.*

Theorem 3.3. *The geodesics of Theorem 3.1 are the shortest curves joining their end points.*

Remark 3.2. The previous theorem assures us that the “short” geodesics of L minimize the action $\mathcal{H}$.

From now on, we shall suppose that M is arcwise connected, for the sake of brevity. Let $y = (y^0, y_1)$ and $q = (q^0, q_1)$ be two points of N, with $q_0 > y_0$ and let us denote by $\mathcal{C}_y^q$ the set of all admissible curves joining y to q and defined on I. We observe that for any curve $\gamma : I \to M$, such that $\gamma(0) = y_1$ and $\gamma(1) = q_1$, the curve $\tau = (\tau^0, \gamma) : I \to N$, with $\tau^0 : I \to \mathbb{R}$ defined by setting $\tau^0(t) = (q^0 - y^0)t + y^0$, for any $t \in I$, is admissible and such that $\tau(0) = y$ and $\tau(1) = q$. Hence we have $\mathcal{C}_y^q \neq \emptyset$. Then, the following definition makes sense:

Definition 3.1 For any $y = (y^0, y_1), q = (q^0, q_1)$ points of N, with $q_0 > y_0$, we set

$$d(y,q) = \inf_{\tau \in \mathcal{C}_y^q} \{\mathcal{L}(\tau)\}$$

and we call $d(y,q)$ *almost–distance between y and q*. We also set $d(y,y) = 0$, for any $y \in N$.

We denote by P the subset of $N \times N$ defined as follows. $(y,q) \in P$, if and only if $y = (y^0, y_1)$, $q = (q^0, q_1)$ and either $y^0 < q^0$, or $y = q$. The mapping $d : P \to \mathbb{R}$ is not a true distance, but it verifies the following basic property.

Theorem 3.4. *Let* $x = (x^0, x_1)$, $y = (y^0, y_1)$ *and* $z = (z^0, z_1)$ *be three elements of* N, *such that* $x^0 < y^0 < z^0$, *then*

$$d(x, z) \leq d(x, y) + d(y, z). \tag{3.2}$$

Remark 3.3. The Theorem 3.2 and the Lemma 3.2 imply that for any geodesic $c_y^q : I \to N$ such that $c_y^q(0) = y$ and $c_y^q(1) = q \in B_\varepsilon(y)$ (ε as in Lemma 3.2), if $d(y, q) = \sigma < \varepsilon$, then $\mathcal{L}(c_y^q) = F(c_y^q(0), \dot{c}_y^q(0)) = \sigma$. Hence, $q \in \partial B_\sigma(y)$ if and only if $\dot{c}_y^q(0) \in \partial B_\sigma(0_y)$, for any $0 < \sigma < \varepsilon$.

Let $y \in N$, let $\{u_i\}_{0 \leq i \leq n}$ be a basis of T_yN, let $u : \mathbb{R}^n \to T_yN$ be the linear isomorphism defined by this basis and let ε such that exp_y has maximal rank on $B_\varepsilon(0_y)$. Then, the mapping $\varphi = u^{-1} \circ exp_y^{-1} : B_\sigma(y) \to \varphi(B_\sigma(0_y))$ and $\varphi^{-1} : \varphi(B_\sigma(0_y)) \to B_\sigma(y)$ are C^∞–differentiable mappings, for any $0 < \sigma < \varepsilon$. The chart $(B_\sigma(y), \varphi)$, obtained in this way is such that $y \in \partial B_\sigma(y)$, and it can be a non–standard chart of N, for any $y \in N$ and $0 < \sigma < \varepsilon$. Then, following [3], one can easily prove:

Theorem 3.5. *Let* $y \in N$, *and let* $\eta > 0$ *be a real number such that* exp_y *has maximal rank on* $B_\eta(0_y)$. *Then, there exists* $\varepsilon_0 \in]0, \eta[$ *such that for any* $\varepsilon \in]0, \varepsilon_0[$, *for any non constant geodesic* $c : [0, a] \to N$, *having* $c(t_0) \in \partial B_\varepsilon(y)$ *and* $\dot{c}(t_0) \in T\partial B_\varepsilon(y)$ *for some* $t_0 \in]0, a[$, *and for any* $\lambda \in]0, 1[$ *there exists* $\alpha = \alpha(\varepsilon, \lambda)$ *such that*

$$d(y, c(t)) \geq d(y, c(t_0)) + \lambda(t - t_0)^2 = \varepsilon + \lambda(t - t_0)^2, \qquad \forall t \in]t_0 - \alpha, t_0 + \alpha[.$$

4 – Complete P–Lagrangians

In this Section we shall suppose that the P–Lagrangian L is geodesically complete at a point $y = (y^0, y_1) \in N$. Hence, we shall suppose that condition 1_C) of the Introduction holds in y. Moreover, we shall use all the results of the previous Sections and all the symbols introduced there.

First we observe that if $\gamma : \mathbb{R} \to M$ is a geodesic of L such that $\gamma(0) = y_1$, then the mapping φ defined by Theorem 2.2 is defined on the whole $\mathbb{R}$. As a consequence, its inverse $\tau^0 = \varphi^{-1}$ and the curve $\tau_1 = \gamma \circ \tau^0$ are defined on the whole $\mathbb{R}$. Moreover, if we suppose that $\tau^0(0) = y^0$ and we set $\tau^0(1) = q^0 > y^0$ then, formal intregral of τ^0 is:

$$\tau^0(t) = \frac{q^0 - y^0}{\displaystyle\int_0^1 \frac{1}{L(\tau^0(s), \tau(s), \dfrac{\dot{\tau}(s)}{\dot{\tau}^0(s)})} ds} \int_0^t \frac{1}{L(\tau^0(s), \tau(s), \dfrac{\dot{\tau}(s)}{\dot{\tau}^0(s)})} ds + y^0,$$

and we have

$$\dot{\tau}^0(t) = \frac{q^0 - y^0}{\int_0^1 \dfrac{1}{L(\tau^0(s), \tau(s), \dfrac{\dot{\tau}(s)}{\dot{\tau}^0(s)})} ds} \cdot \frac{1}{L(\tau^0(t), \tau(t), \dfrac{\dot{\tau}(t)}{\dot{\tau}^0(t)})},$$

for any $t \in \mathbb{R}$. Hence, it results:

$$Y^0 = \dot{\tau}^0(0) = \frac{q^0 - y^0}{\int_0^1 \dfrac{1}{L(\tau^0(s), \tau(s), \dfrac{\dot{\tau}(s)}{\dot{\tau}^0(s)})} ds} \cdot \frac{1}{L(y^0, y_1, \dfrac{Y_1}{Y^0})}, \tag{4.1}$$

having set $\dot{\tau}(0) = Y_1$. The previous identity implies:

$$Y^0 = 0 \quad \Leftrightarrow \quad q^0 = y^0 \tag{4.2}$$

Now, we can prove the unique Lemma, which has no counterpart in the Riemannian case.

LEMMA 4.1. *Let $y = (y^0, y_1) \in N$ and $\varepsilon > 0$ be a real number such that the exponential mapping has maximal rank on $B_{2\varepsilon}(0_y)$. Then, for any $p = (p^0, p_1) \in N$, having $p^0 > y^0$, there exists $\varepsilon' < \varepsilon$ such that the exponential mapping has maximal rank on $B_{2\varepsilon'}(0_y)$ and for any $q = (q^0, q_1) \in \partial B_{\varepsilon'}(y)$ it results $q^0 < p^0$.*

PROOF. If $q = (q^0, q_1) \in \partial B_{\varepsilon'}(y)$, because of (4.1) and Remark 3.2, for any $\varepsilon' < \varepsilon$, we have:

$$\varepsilon' = L(y^0, y_1, \frac{Y_1}{Y^0}) Y^0 = \frac{q^0 - y^0}{\int_0^1 \dfrac{1}{L(\tau^0(s), \tau(s), \dfrac{\dot{\tau}(s)}{\dot{\tau}^0(s)})} ds}.$$

Then, by the mean value theorem we get:

$$q^0 = y^0 + \varepsilon' L(\tau^0(t_1), \tau(t_1), \frac{\dot{\tau}(t_1)}{\dot{\tau}^0(t_1)}) \leq y_0 + \varepsilon' a, \tag{4.3}$$

being t_1 a suitable element of $]0, 1[$ and a the constant of the Condition 1_C. Then, if $y^0 + \varepsilon' a < p^0$, we have $q^0 < p^0$. Hence, the assertion follows by taking

$$\varepsilon' < \frac{p^0 - y^0}{a}$$

$\square$

Then, we have

LEMMA 4.2. *Let $\rho > 0$ such that exp_y is defined on the set $B_\rho(0_y)$. Then, for every $p = (p^0, p_1) \in N$, having $p^0 > y^0$ and $d(y,p) < \rho$, can be joined to y_0 by a not necessarily unique geodesic of F.*

PROOF. Let us consider $\varepsilon' > 0$ as in the previous Lemma. Then, for any $q = (q^0, q_1) \in \partial B_{\varepsilon'}(y_0)$ we have $q^0 < p^0$, where inequality follows from the previous Lemma. Moreover, if $(Y^0, Y_1) = \dot{c}_y^q(0)$, being c_y^q the geodesic joining y to q, we have also $Y^0 \neq 0$, since $q^0 \neq y^0$, by using (4.3). Then the assertion follows as in [3].

□

The previous lemma implies that:

THEOREM 4.1. *Every point $p \in N$ can be joined to y by a geodesic of F, or equivalently, if we depart at the instant y^0, we can travel from y_1 to any point p_1 of M following a geodesic path and can arbitrarily chose the instant $p^0 > y^0$ in which the point p_1 is reached.*

Now we consider the distance function $d : P \to \mathbb{R}$ of Definition 3.1.

DEFINITION 4.1 Let $y \in N$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called a *L–Sequence at* y if $(x_n, x_{n+1}) \in P$, for any $n \in \mathbb{N}$ and $x_0 = y$. An L–sequence $\{x_n\}_{n\in\mathbb{N}}$ at y is called a *Cauchy L-Sequence at y* if and only if for any $\varepsilon > 0$ there exists $n_0 > 0$ such that for any $m > n > n_0$ it results $d(x_n, x_m) < \varepsilon$. The manifold M is said to be *L–Complete at* y if and only if any Cauchy L–sequence at y converges.

THEOREM 4.2. *If L is a GCP–Lagrangian at y, then M is L–Complete at y.*

PROOF. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy L–Sequence. Then for any $n \in \mathbb{N}$ we set $d(y, x_n) = \rho_n$. The sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. In fact, for any $n, m \in \mathbb{N}$, we can suppose $m > n$ and we have $|\rho_m - \rho_n| = |d(y, x_m) - d(y, x_n)| \leq d(x_n, x_m)$. As a consequence, the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ converges and we set $\lim_{n\to\infty} \rho_n = \rho \in \mathbb{R}$. Moreover, for any $n \in \mathbb{N}$, there exists a geodesic $c_n = c_y^{x_n} : \mathbb{R} \to N$ such that $c_n(0) = y$. We set $\dot{c}_n(0) = X_n$ and we can suppose $F(y, X_n) = 1$, since the construction does not depend from the linear changes of the parameter, then $c_n(\rho_n) = x_n$, for any $n \in \mathbb{N}$. As in the proof of the previous Lemma the sequence $\{X_n\}_{n\in\mathbb{N}}$, converges to $X \in A_y^+$ and $F(y, X) = 1$. Then, as in the Riemannian case, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} c_n(\rho_n) = c(\rho)$, being $c : \mathbb{R} \to N$ the geodesic defined by $c(0) = y$ and $\dot{c}(0) = X$. Hence, the assertion follows.

□

THEOREM 4.3. *Let $p = (p^0, p_1)$ be an element of N, such that $y^0 < p^0$. If M is L–Complete at y, then L is a GCP–Lagrangian at p. As a consequence, any point $q = (q^0, q_1) \in N$, with $q^0 > p^0$, can be joined to p by a geodesic of F^2.*

PROOF. Let $X \in A_p^+$ be a non zero vector, and $c : J \to N$ be a geodesic, being J the maximal interval on which c is defined, with $0 \in J$, and such that $c(0) = p$, $\dot{c}(0) = X$. Again, we can suppose that $F(p, X) = 1$. We suppose, by absurd, that $\sup_{t\in J}\{t\} < +\infty$ and we consider a strictly increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = \sup_{t\in J}\{t\}$. It is easy to see that the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by setting $x_0 = y$ and $x_n = c(t_n)$, for any $n \in \mathbb{N}$, with $n \geq 1$, is an L-sequence at y. Moreover, it results:

$$d(c(t), c(t')) \leq \mathcal{L}(c_{|[t,t']}) = t - t'$$

for any $t, t' \in J$, with $t > t'$. As a consequence, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy L–sequence at y and hence it converges. Then, the assertion follows as in the Riemannian case. $\square$

Then, both Theorems of the Introduction follow in an easy way.

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