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Can the integral curves of ODEs be accepted as orbits of an autonomous force field?

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Abstract: *It is shown that the two-parametric set of all solutions of any linear ordinary differential equation (ODE) of the second order* $y'' + a(x)y' + b(x)y = f(x)$ *(solvable by quadratures or not) can become a set of orbits traced by a material point of unit mass, in the presence of at least one autonomous force field (conservative or not)* $\mathbf{F}\langle X(x,y), Y(x,y)\rangle$*, for adequate initial conditions. The field* $\mathbf{F}$ *(except for a multiplicative constant* F_0*) is determined by quadratures on the grounds of the coefficients* $a(x)$*,* $b(x)$*,* $f(x)$ *which specify the given ODE. We give some appropriate examples.*

1 – Introduction

The following version of the inverse problem of Dynamics has been studied in the past: A monoparametric family of planar curves is given in the *"solved for the parameter c"* form

$$\varphi(x,y) = c \tag{1}$$

To find all autonomous force fields

$$X = X(x,y), \qquad Y = Y(x,y) \tag{2}$$

in the presence of which a material point of unit-mass can trace all the curves (1) in the Cartesian plane Oxy as orbits, for adequate initial conditions, of course.

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As reported by Whittaker [1], the force $\mathbf{F}\langle X, Y\rangle$ (which may or may not be conservative) was first given by Dainelli [2] (1880). We know that there exist infinitely many such force fields and that infinitely many out of these are also conservative. Each conservative field is associated with a different distribution of the total energy along the various members of the family. Szebehely's PDE [3] (1974) relating potentials and families of orbits, traced with a preassigned energy-dependence function $E = E(\varphi(x, y))$, renewed interest in this old problem.

The *"slope function"*

$$\gamma(x, y) = \frac{\varphi_y}{\varphi_x} \tag{3}$$

comes to an one-to-one correspondence with the given family (1). Actually, it is the function $\gamma(x, y)$ (and not the function $\varphi(x, y)$ in (1)) which enters into the basic formulae of this inverse problem (see Bozis [4]). For monoparametric families (1) this means that the *"mother differential equation"* of the first order, in the solved form

$$y' = -\frac{1}{\gamma(x,y)} \tag{4}$$

(solvable or not by quadratures, depending on the given $\gamma(x, y)$) can replace the given family (1).

The picture changes altogether if the preassigned geometrical information regarding the orbits is increased [5]. The problem not only ceases to have infinitely many solutions but it may no longer possess even a single solution. This is what happens e.g. if we seek a force field which can create a two-parametric set of planar curves $\varphi(x, y, c^*) = c$, given in advance. In general, no solution exists, unless the pertinent slope function $\gamma(x, y, c^*)$, corresponding to the family $\varphi(x, y, c^*) = c$, satisfies certain conditions. These conditions were found by Bozis [4] for families of planar orbits in the Cartesian plane and recently by Kotoulas [6] for two-parametric families of orbits lying on a surface with given metric. So or otherwise, inverse problems of this type are put *with the equations of the orbits (or, at least, with adequate slope functions* $\gamma(x, y, c^*)$) given in advance.

In the present study, we shall consider an inverse problem of different character: Neither the equation $\varphi(x, y, c^*) = c$ of the two-parametric family will be given in advance nor its pertinent slope function $\gamma = \gamma(x, y, c^*)$. Instead, it will be assumed that its *"mother differential equation"* is given, i.e. *a second order ordinary differential equation (ODE) whose solutions are the orbits.* The essential difference lies in that *the orbits themselves may not even be known,* as the mother differential equation may not be analytically solvable.

We examine in detail the case of a *linear* second order ODE, given in the solved for y'' form. The procedure however may be extended to nonlinear, solved for y'', second order ordinary differential equations, provided that we have some information regarding the dependence of y'' on the first order derivative y'.

2 – Basic facts for planar autonomous force fields

The velocity components at any point (x, y) of any orbit (1) are given by

$$\dot{x} = g(x,y)\varphi_y, \qquad \dot{y} = -g(x,y)\varphi_x \tag{5}$$

where $g(x, y)$ is an arbitrary function. A second derivation in time of the equations (5) provides the pertinent force components X, Y in terms of the functions φ, g and first order derivatives of these (see [7]). In place of φ and g , we can introduce the slope function γ [4], [8] and the kinetic energy T and obtain the two equations

$$2T = \frac{(1+\gamma^2)}{\Gamma}(X + \gamma Y) \tag{6}$$

and

$$Y_y - X_x + \frac{1}{\gamma}\, X_y - \gamma Y_x = \lambda X + \mu Y \tag{7}$$

The positional functions λ and μ in (7) are given by

$$\lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma} \quad \text{and} \quad \mu = \lambda\gamma + \frac{3\Gamma}{\gamma} \tag{8}$$

where

$$\Gamma = \gamma\gamma_x - \gamma_y \tag{9}$$

Equation (7) is a necessary and sufficient condition for the autonomous force $\mathbf{F}\langle X, Y\rangle$. to be compatible with the family (1) traced with velocity (5). For any admissible pair $\langle X, Y\rangle$, equation (6) serves not only to determine the kinetic energy of the moving point but also to establish the region of the xy plane

$$\frac{X + \gamma Y}{\Gamma} \geq 0 \tag{10}$$

where *real motion* is allowed to take place [9]. If, for a given family $\gamma(x, y)$, the pair $\langle X, Y\rangle$ satisfyies (7), so does the pair $\langle k_0 X,\ k_0 Y\ \rangle$. All these pairs will be considered as *one* solution. The sign of the real constant k_0 decides whether the orbits of the family will be lying in the interior or the exterior of the region (10).

If the family (1) depended on *two* parameters, then the second parameter, say c^*, would appear in the function φ and, consequently, in γ, λ and μ. It is then easily seen that, in general, there would not exist pairs of functions (2) satisfying the PDE (7), because the force components *must be independent of the parameters c and c^** and, in particular, *of the additional parameter c^*,* first inserted in the left hand side of (1).

3 – Analysis for second order linear ODEs

Consider the linear, second order, ordinary differential equation (ODE)

$$y'' + a(x)y' + b(x)y = f(x). \tag{11}$$

The equation is then identified by the three sufficiently smooth functions $a(x)$, $b(x)$, $f(x)$. The totality of its solutions

$$y = y(x, c_1, c_2). \tag{12}$$

constitutes a set of planar curves depending on two parameters.

We put the following question: Is there a planar *autonomous* force field (2) which can produce as real orbits, traced by a unit-mass material point, *all the members* of the two-parameter set (12)? We shall show that, contrary to what is generally expected, the answer in this case *is always affirmative.* In fact, given the ODE (11),*no matter if we can or we cannot obtain its general solution* , we can determine *at least one* force field (2) which can produce as real orbits all the set of curves (12).

We proceed as follows: Differentiating in x both members of (4), we obtain, in view of (9),

$$y'' = \frac{\Gamma}{\gamma^3} \tag{13}$$

Now, in view of (4) and (13), we can write the given differential equation (11) as

$$\Gamma = \gamma^3 \left(\frac{a}{\gamma} - by + f \right) \tag{14}$$

In so doing, we managed to express Γ in terms of x, y and γ. Taking into account that the positional function Γ depends on x and on y explicitly and through $\gamma(x, y)$, we calculate derivatives of Γ in x and y and we proceed to the calculation of the coefficients λ and μ appearing in our basic equation (7). There results:

$$\lambda = 3(yb - f)\gamma - 2a + \frac{(yb' - f')\gamma - (b + a')}{(f - yb)\gamma + a} \tag{15}$$

and

$$\mu = \frac{(yb' - f')\gamma^2 - (b + a')\gamma}{(f - yb)\gamma + a} + a\gamma \tag{16}$$

where primes denote derivatives in x. Inserting (15) and (16) into (7), we obtain the cubic in γ algebraic equation

$$s_3\gamma^3 + s_2\gamma^2 + s_1\gamma + s_0 = 0 \tag{17}$$

where the coefficients $s_j (j = 0, 1, 2, 3)$ are given by

$$
\begin{aligned}
s_3 &= 3(f - yb)^2 X - [a(f - yb) - (f' - yb')]\, Y - (f - yb) Y_x \\
s_2 &= [5a(f - yb) + (f' - yb')]\, X + (b + a' - a^2) Y + \\
&\quad - (f - yb) X_x - a Y_x + (f - yb) Y_y \\
s_1 &= (2a^2 + a' + b) X - a X_x + (f - yb) X_y + a Y_y
\end{aligned}
$$

(18) $$s_0 = a X_y.$$

The slope function γ in (17) depends on *one of the two* integration constants c_1, c_2 , introduced by the general solution (12) of the given ODE (11), as this general solution will be brought to the "solved for one constant" form (1). On the other hand, the above coefficients $s_j (j = 0, 1, 2, 3)$ *must be independent of the integration constants.* In fact, they are expressed in (18) in terms of the coefficients a, b, f of the given ODE (11) and the force components X, Y which depend only on the position coordinates x, y. Consequently, for (17) to hold true *for all solutions* of (11), we must have

(19) $$s_3 = s_2 = s_1 = s_0 = 0.$$

Our problem then is to find if, given the ODE (11), the four PDEs (19) in the two unknown functions $X(x, y)$, $Y(x, y)$ are compatible. To this end we start with the last equation $s_0 = aX_y = 0$ of the system (18), which happens to be the simplest one. We distinguish two cases: A (with $a \neq 0$) and B (with $a = 0$), to be studied separately.

3.1 – The case $a \neq 0$

For this case, we find it convenient to introduce the following notation:

(20) $$I = \exp\left(\int \frac{b}{a} dx\right), \qquad J = \exp\left(\int a dx\right), \qquad \Theta = \int I^2 J^{-3} dx.$$

The functions I and J are determined up to a multiplicative constant (I_0 and J_0 , respectively) and the integral Θ up to an additive constant Θ_0. It will appear that these constants are usually superfluous as they are absorbed by other integration constants entering into the calculations.

Since $a \neq 0$ and $s_0 = 0$, the $X-$component of the required force field depends merely on x, i.e. $X = X(x)$ and, because of that, as seen from the third expression (18), so does the term Y_y in $s_1 = 0$. From $s_1 = 0$ there results

(21) $$Y(x, y) = y\left[X' - \left(2a + \frac{a' + b}{a}\right) X\right] + \epsilon(x)$$

where $\epsilon(x)$ is arbitrary and $X' = \frac{dX}{dx}$. So, in this case, the last two equations (19) served to inform us that X is independent of y and that Y is linear in y. We are left to deal with the first two equations of the system (19), having in addition at our disposal the arbitrary function $\epsilon(x)$. Using (21), we write the two equations $s_3 = 0$ and $s_2 = 0$ respectively as

$$s_{30} + ys_{31} + y^2 s_{32} = 0 \qquad \text{and} \qquad s_{20} + ys_{21} = 0. \tag{22}$$

The functions $s_{30}, s_{31}, s_{32}, s_{20}, s_{21}$ (computed by MATHEMATICA but not recorded here) depend only on x. In fact they are expressed in terms of X, X', X'' and in terms of a, b, f, ϵ and first order derivatives of these. Apparently, it must be

$$s_{30} = s_{31} = s_{32} = 0 \qquad \text{and} \qquad s_{20} = s_{21} = 0. \tag{23}$$

In particular, for $f \neq 0$, from $s_{30} = 0$ we determine the component

$$X = \frac{(a\epsilon + \epsilon')f - \epsilon f'}{3f^2}. \tag{24}$$

The result (24) obliges us to distinguish two subcases (A_1 : $f \neq 0$ and $\mathrm{A}_2 : f = 0$), corresponding to nonhomogeneous and homogeneous ODEs (11), respectively.

3.1.1 – Subcase A_1: $a \neq 0,\ f \neq 0$.

Using X, as given by (24), we come to see that the rest of the equations (23) are all satisfied if we select

$$\epsilon = \epsilon_0 f J^2. \tag{25}$$

Then, in accordance with (24) and (21), we obtain the unique (except for the multiplicative constant ϵ_0) solution

$$X = \epsilon_0 a J^2 \quad \text{and} \quad Y = \epsilon_0 (f - yb) J^2. \tag{26}$$

Thus: For $a \neq 0$, $f \neq 0$, all solutions of (11) (either known or not) are orbits of a material point moving in the force field (26).

3.1.2 – Subcase A_2: $a \neq 0,\ f = 0$.

The coefficient $s_{30} = 0$, X cannot be determined from (24) and we turn attention to s_{20}. The calculations lead to $s_{20} = -a\left[a\epsilon' + (a^2 - a' - b)\epsilon\right] = 0$ (which is of course satisfied for $\epsilon = 0$). Since $a \neq 0$, with $\epsilon \neq 0$, we must select

$$\epsilon = \tilde{\epsilon}_0 a I J^{-1}, \qquad (\tilde{\epsilon}_0 \neq 0,\ \text{constant}). \tag{27}$$

So, we examine two subsubcases (A_{2a} and A_{2b} below).

$\mathbf{A_{2a}}$: $a \neq 0$, $f = 0$, $\epsilon = 0$.

Then, not only $s_{30} = s_{20} = 0$ but also $s_{31} = 0$ whereas, in order to make $s_{21} = 0$ and $s_{32} = 0$ it is required that the X-component satisfies the two equations

$$\frac{X'}{X} = 2a + \frac{a'}{a} \qquad \text{and} \qquad \frac{X''}{X} = 4a^2 + 6a' + \frac{a''}{a}. \tag{28}$$

It can be checked that the equations (28) *are compatible* and that they lead to

$$X = X_0 a J^2, \qquad (X_0 = const.) \tag{29}$$

From (21) we find

$$Y(x, y) = -y X_0 b J^2. \tag{30}$$

Remark 1. We notice that the result (29) coincides with the first of the equations (26) and that (30) can be taken from the second formula (26), if we allow this formula (in spite of the way it was derived) to be applicable for $f = 0$ also. Up to this point then, it is all summarized in the following:

Proposition 1. *For $a \neq 0$, all solutions of the ODE $y'' + ay' + by = f$ can become orbits of a material point moving in the presence of the force field given by the equations* (26).

$\mathbf{A_{2b}}$: $a \neq 0$, $f = 0$, $\epsilon \neq 0$.

With formulae (21) and (27) giving respectively $Y(x, y)$ and $\epsilon(x)$, besides $s_{20} = 0$, it is also $s_{30} = 0$. The equation $s_{21} = 0$ leads to a linear in X, second order, ODE to be satisfied by $X(x)$. This equation reads

$$X'' + s_1 X' + s_0 X = 0 \tag{31}$$

where

$$s_1 = -\left(a + \frac{2(a' + b)}{a}\right) \qquad \text{and} \qquad s_0 = \frac{2a'}{a^2}(a' + b) - \frac{a''}{a} + 4b - 2a^2 - a' \tag{32}$$

For any solution $X(x)$ of (31), the corresponding $Y(x, y)-$component is then found with the aid of (21). There remain the two equations $s_{31} = 0$ and

$s_{32} = 0$. The equation $s_{31} = 0$ requires that the coefficients a and b of the given equation (11) satisfy the relation

$$\left(\frac{b}{a}\right)' = \left(\frac{b}{a}\right)^2 \tag{33}$$

Working with (33), we see that $s_{32} = 0$ is also valid if $X(x)$ is taken to satisfy the same ODE (31). Obviously it is highly desirable to solve (31). To this end, inspired by the result (29), we observe that $X_1 = aJ^2$ is a *particular solution* of the linear homogeneous equation (31). On this basis, we transform (31) from second to first order ODE and we readily find a second, linearly independent, solution. As can directly be verified, it is $X_2 = aJ^2\Theta$, with Θ given in (20). So, the general solution of (31) is

$$X(x) = aJ^2(c_1 + c_2\Theta) \tag{34}$$

where c_1, c_2 are constants. Working with (34) and using (27), we find the component $Y(x, y)$ from (21). It is

$$Y(x, y) = y\left[-(c_1 + c_2\Theta)bJ^2 + c_2 aI^2J^{-1}\right] + \tilde{\epsilon}_0 aIJ^{-1} \tag{35}$$

REMARK 2. The force field given by (34) and (35) is not unique but depends on three constants: $c_1 + c_2\Theta_0$, c_2 and $\tilde{\epsilon}_0$. If (in spite of the assumption $\epsilon \neq 0$ which we made to derive them) we allow these formulae to be applicable for $\epsilon = 0$, we cover also the previous case $\mathbf{A_{2a}}$. Indeed, for $c_1 = X_0$, $c_2 = 0$, $\tilde{\epsilon}_0 = 0$, the formulae (34) and (35) reduce to (29) and (30). However, not to be disregarded is the fact that, for $\tilde{\epsilon}_0 \neq 0$, (34) and (35) are valid only when the coefficients a and b are related by (33).

On the other hand, the equation (33) implies that either $b = \frac{a}{(x_0 - x)}$, $x_0 =$ *const.* , or $b = 0$ ($a =$arbitrary), So we can state the following:

PROPOSITION 2. *For $a \neq 0$ and any constant x_0, all solutions of the homogeneous ODE $y'' + ay' + \frac{a}{x_0 - x}y = 0$ are orbits in any of the fields given by the formulae* (34) *and* (35)*, applied for $b = \frac{a}{(x_0 - x)}$, implying $I = \frac{I_0}{x_0 - x}$.*

In particular for $b = 0$, we deal with the equation $y'' + ay' = 0$ whose general solution is

$$y = \tilde{c}_1 + \tilde{c}_2 \int J^{-1} dx. \tag{36}$$

Then formulae (34) and (35) must be applied for $b = 0$, $I = I_0$, $J = \exp(\int a dx)$, $\Theta = I_0^2 \int J^{-3} dx$. We conclude that

Proposition 3. *For any function $a = a(x)$, all orbits* (36) *are created by any member of the two-parametric set of force fields*

$$X(x) = aJ^2(c_1 + k_1 \int J^{-3}dx), \qquad Y = aJ^{-1}(k_1 y + k_2) \tag{37}$$

where the parameters are: $k_1 = c_2 I_0^2$, $k_2 = \tilde{\epsilon}_0 I_0$.

Comment: A general conclusion from the study of the Case **A** ($a \neq 0$) is the following: No distinction should be made between homogeneous and nonhomogeneous ODEs (11). The Proposition 1 covers both cases. Only if $b = 0$ or if $\frac{b}{a} = \frac{1}{x_0 - x}$, such a distinction is necessary. Then, to the homogeneous ($f = 0$) ODEs (11) there corresponds *a broader set of force fields* given by the formulae (34) and (35), in the sense that these formulae, applied (formally) for $c_1 = \epsilon_0$, $c_2 = 0$, $\tilde{\epsilon}_0 = 0$, lead to the formulae (26), with $f = 0$ of course.

3.2 – The case $a = 0$

The given equation (11) reduces to

$$y'' + b(x)y = f(x) \tag{38}$$

and, as seen from (18), the last equation (19) is satisfied identically. The $X-$component is now allowed to depend both on x and y. Two subcases have to be distinguished: (B_1: at least one of the coefficients b, f is not zero and B_2: both b and f are zero)

3.2.1 – Subcase B_1: $a = 0$, $b^2 + f^2 \neq 0$

Since $a = 0$, it is also $a' = 0$ and, in view of the third expression (18), the equation $s_1 = 0$ is satisfied for

$$X(x, y) = (f - yb)K(x) \tag{39}$$

where $K(x)$ is at the moment arbitrary. Inserting (39) into the other two equations (19) $s_2 = 0$, $s_3 = 0$, we obtain two expressions (E_1), (E_2) (not recorded here) giving respectively Y_x and Y_y in terms of Y. It can be checked that these expressions are compatible, provided that

$$K'' + 3bK = 0. \tag{40}$$

Now from (E_1) we obtain

$$Y(x, y) = (f - yb)\ [L(x) + yK'(x)] \tag{41}$$

where $L(x)$ is a new arbitrary function of x. Inserting (41) into (E_2) , we find

$$L' = 3fK. \tag{42}$$

In conclusion we have:

Proposition 4. *The set of all force fields defined by the equations* (39) *and* (41) *can create all orbits-solutions of the ODE $y'' + by = f$, provided that the one-variable functions K and Lsatisfy the equations* (40) *and* (42).

Notice that two integration constants enter into the field through K and one through L.

3.2.2 – Subcase B_2: $a = 0$, $b^2 + f^2 = 0$

Equation (11) now becomes

$$y'' = 0 \tag{43}$$

and its general solution

$$y = c_1 x + c_2 \tag{44}$$

is a two-parameter family of straight lines with "slope" $\gamma(x, y) = -\frac{1}{c_1}$ to which, according to (9), there corresponds $\Gamma = 0$. So, the coefficients λ and μ of our basic equation (7) become indeterminate and the system (19) is meaningless. In fact, all its four equations are satisfied identically.

Yet, it is understood that not any autonomous force field (2) can produce as orbits all the straight lines (44) but only those whose direction coincides with the direction of these lines, meaning that

$$\gamma(x, y) = -\frac{X(x, y)}{Y(x, y)}. \tag{45}$$

Inserting (44) into $\Gamma = 0$, we obtain

$$X^2 Y_x - Y^2 X_y = XY(X_x - Y_y) \tag{46}$$

Proposition 5. *All the straight lines* (44) *can be traced by any of the force fields* $\langle X, Y \rangle$ *, satisfying the condition* (46).

4 – Conservativeness and Examples

Case A: In all cases A, examined in section 3, we had $X = X(x)$, therefore the force field would be conservative if

$$Y_x = 0. \tag{47}$$

A_1**:** For the force field (26), the condition (47) leads to

$$f = f_0 J^{-2}, \quad b = b_0 J^{-2} \tag{48}$$

$(f_0, b_0$ *constants*$)$. This allows us to conclude that:

For any $a(x)$, all solutions of an ODE of the form

$$y'' + ay' + b_0 J^{-2} y = f_0 J^{-2} \tag{49}$$

are orbits generated by the potential

$$V(x,y) = -\epsilon_0 \left[f_0 y - \frac{1}{2} b_0 y^2 + \int a J^2 dx \right] \tag{50}$$

Case $\mathbf{A_{2a}}$: For (30), the condition (47) gives $b = b_0 J^{-2}$. According to the Remark 1 (preceding the Proposition 1 in subsection 3.1.2), the formula (50) can be applied for $f_0 = 0$ also. Thus, all solutions of the homogeneous equation

$$y'' + ay' + b_0 J^{-2} y = 0 \tag{51}$$

are orbits in the potential (50) for $f_0 = 0$.

Case $\mathbf{A_{2b}}$: For $Y(x,y)$, as given by (35), to satisfy (47) we must have

$$\left[-(c_1 + c_2\Theta) b J^2 + c_2 a I^2 J^{-1}\right]_x = 0 \qquad \text{and} \qquad (aIJ^{-1})_x = 0. \tag{52}$$

The above equations lead respectively to

$$b' + 2ab = 0 \qquad \text{and} \qquad b = a^2 - a'. \tag{53}$$

Besides (53), the two coefficients a and b must satisfy also (33). It can be shown that this occurs only if $a = -\frac{1}{x+x_0}$, $b = 0$. To ease the algebra, we shall restrict ourselves to the selection $x_0 = 0$ of the additive to x constant in the expression for a. So, we shall deal with the specific ODE

$$y'' - \frac{1}{x} y' = 0 \tag{54}$$

for which $a = -\frac{1}{x}$, $b = f = 0$. From (20) we obtain

$$I = I_0, \qquad J = \frac{J_0}{x}, \qquad \Theta = \frac{I_0^2}{4J_0^3} x^4 + \Theta_0 \tag{55}$$

and, from (34) and (35), we obtain X and Y. Finally, we find the set of (separable in x, y) potentials

$$V(x,y) = \frac{k_1}{x^2} + k_2 x^2 + 4k_2 y^2 + k_3 y \tag{56}$$

where the constants k_1, k_2, k_3 are related to I_0, J_0, Θ_0 and to the integration constants c_1, c_2 inherited from the ODE (31) and to $\tilde{\epsilon}_0$ of (27) as follows:

$$k_1 = -\frac{1}{2}J_0^2(c_1 + c_2\Theta_0), \qquad k_2 = \frac{c_2 I_0^2}{8J_0}, \qquad k_3 = \frac{\tilde{\epsilon}_0 I_0}{J_0}. \tag{57}$$

As we happen to know the general solution

$$y = c_1^* x^2 + c_2^* \tag{58}$$

of the ODE (54), we solve (58) either for c_1^* or for c_2^* and, with the aid of (3) and (9), we express γ and Γ in terms of x, y and c_2^* or c_1^* respectively. Then, from Szebehely's (1974) equation

$$E = V - \frac{(1+\gamma^2)(V_x + \gamma V_y)}{2\Gamma} \tag{59}$$

as modified by Bozis (1995), we find the total energy

$$E = c_2^*(4c_2^* k_2 + k_3) - 4c_1^{*2} k_1 - \frac{(8c_2^* k_2 + k_{3)}}{4c_1^*} \tag{60}$$

of the moving material point.

In conclusion: *Any potential of the form (56) can create the two-parametric family (58) with energy dependence function given by (60) along each orbit* $\langle c_1^*, c_2^* \rangle$ *of the family.*

Case $\mathbf{B_1}$: The force field $\langle X, Y \rangle$ is given by the equations (39) and (41) with the aid of $K(x)$ and $L(x)$ which satisfy (40) and (42). For the field to be conservative we must have $X_y = Y_x$ and this leads to a quadratic expression in y which must be identically equal to zero i.e. to three additional restrictions for K and L. In all we must have

$$\begin{aligned} &bK' = k_1, \qquad bL = fK' + k_2, \qquad (fL)' + bK = 0, \\ &K'' + 3bK = 0, \qquad L' = 3fK \end{aligned} \tag{61}$$

(k_1, k_2 constants). One solution of the four equations (61) which we found is

$$K = 0, \qquad L = L_0 = \text{constant} \tag{62}$$

associated with $b = b_0^2 = \mathit{const.}$, $f = f_0 = \mathit{const.}$ and $k_1 = 0$, $k_2 = b_0^2 L_0$. In view of (39),(41) and (62) there results the (one-dimensional) potential

$$V(x,y) = -L_0\left(f_0 y - \frac{1}{2}b_0^2 y^2\right) \tag{63}$$

which admits as orbits the two-parametric family

$$y = \frac{f_0}{b_0^2} + c_1 \cos(b_0 x) + c_2 \sin(b_0 x) \tag{64}$$

of the ODE $y'' + b_0^2 y = f_0 (f_0, b_0$ constants). The family (64) is traced with total energy

$$E = \frac{L_0 \left\{ b_0^2 + b_0^4 (c_1^2 + c_2^2) - f_0^2 \right\}}{2 b_0^2} \tag{65}$$

and, for $L_0 > 0$, all its members are lying in the entire plane Oxy.

Case $\mathbf{B_2}$: For conservative force fields, equation (46) becomes

$$(V_{yy} - V_{xx}) V_x V_y = (V_y^2 - V_x^2) V_{xy}. \tag{66}$$

All potentials $V(x, y)$ satisfying (66) create the families of straight lines (44) (see [10]).

Example 1. For $x > 0$ let us consider the ODE

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = \frac{4}{x^3} \tag{67}$$

From our viewpoint, the equation (67) is of the type A_1, with $a = -\frac{1}{x}$, $b = \frac{1}{x^2}$, $f = \frac{4}{x^3}, I = \frac{1}{x}$, $J = \frac{1}{x}$, $\Theta = \frac{x^2}{2}$. According to the formulae (26), the force field $X = \frac{\epsilon_0}{x^3}$, $Y = \frac{\epsilon_0 (xy - 4)}{x^5}$, for adequate initial conditions, produces as orbits all the solutions of (67). In fact, we can check directly that this pair $\langle X, Y \rangle$ is compatible (in the sense that the equation (7) is satisfied identically) with the general solution $y = \frac{1}{x} + c_1 x + c_2 x \ln x$ of (67), which we happen to know.

Example 2. For $x > 0$ the solutions of

$$y'' + \frac{1}{x} y' = 0 \tag{68}$$

are again known and are given by $y = c_1 + c_2 \ln x$. Yet, we shall not use this fact but only in case we want to verify our results. We shall only use the equation (68) itself, which is homogeneous. So, the case is classified as A_2, with $a = \frac{1}{x}$, $b = 0$. The condition (33) is satisfied and the case is A_{2b}. It is $I = I_0 = const.$, $J = x$, $\Theta = -\frac{I_0^2}{2x^2}$ and from (34) and (35) we find: $X(x) = c_1 x - \frac{c_2 I_0^2}{2x}$, $Y(x, y) = \frac{I_0^2 c_2 y + \epsilon_0 I_0}{x^2}$.

Example 3. The ODE

$$y'' - \frac{4}{x^2} = x \tag{69}$$

is of the type B_1, with $b = -\frac{4}{x^2}$, $f = x$. So, equation (40) reads $K'' - \frac{12}{x^2}K = 0$ and its general solution is $K = k_1 x^4 + \frac{k_2}{x^3}$. Equation (42) gives $L = L_0 + \frac{k_1 x^6}{2} - \frac{3k_2}{x}$. From equations (39) and (41) we obtain

$$\begin{aligned} X(x,y) &= \frac{(x^3+4y)}{x^5}(k_1 x^7 + k_2), \\ Y(x,y) &= \frac{(x^3+4y)}{2x^6}\left[\left(k_1 x^7 + 2L_0 x - 6k_2\right)x^3 + 2y\left(4k_1 x^7 - 3k_2\right)\right]. \end{aligned} \tag{70}$$

As the general solution

$$y = c_1 x^{\rho_1} + c_2 x^{\rho_2} + \frac{1}{2}x^3, \qquad \left(\rho_1, \rho_2 = \frac{1 \pm \sqrt{17}}{2}\right) \tag{71}$$

of (69) is known, we can verify the compatibility of (70) and (71).

5 – Concluding Comments

The backbone of the present study is the cubic equation (17). It resulted from the basic equation (7) combined with information regarding the dependence of y'' on y' (or on γ, as given by the equation (13)). This helped to interrelate the family to the coefficients a, b, f of the given ODE (11) by the expression (14). From equation (17) then there stemmed the system of the four partial differential equations (19) in the two unknown functions $X(x,y), Y(x,y)$. The compatibility of these equations was studied in detail.

Our study was limited to *linear ODEs*, i.e. to a case in which y'' is expressed linearly in terms of y' *in the form indicated by equation (11).* However, even if the *linear dependen*ce of y'' on y' was of the more general form

$$y'' = A(x,y)y' + B(x,y), \tag{72}$$

the pertinent equation (17) would be again *cubic* in γ. The analysis, of course, would not be facilitated, as it did in our case, by the fact that the function A depends merely on x and that the function B is linear in y. The result also would be qualitatively different. *In general,* we would not expect the two-parametric set of solutions of (72) to be consistent with an autonomous force field. In fact, for the existence of such a field, the functions $A(x,y)$ and $B(x,y)$ would have to satisfy certain differential conditions. The question is open to detailed study.

For *nonlinear* second order ODEs more general than (72) (e.g. algebraic with y' of higher degree) the corresponding equation (17) would be algebraic of degree higher than three. The number of equations in the pertinent system (19) would be greater than four.

The basic result of the paper is that autonomous force fields, in the framework of the inverse problem of Dynamics, may be determined not necessarily on the grounds of given two-parametric families but from their "mother equations" which may not even be solvable by quadratures. In other words, we face and we solve this version of the inverse problem of Dynamics (not only having but also) not having at our disposal the equation

$$\varphi(x, y, c^*) = c \tag{73}$$

of the two-parametric family, not even the corresponding slope function

$$\gamma = \gamma(x, y, c^*). \tag{74}$$

In fact our results may be interpreted also as follows: the force field (found according to the case to which the given ODE is classified) may be used as a "mechanical device" to construct the solutions of the equation (11).

The ODE (11), of course, can always be solved numerically. All solutions of (11) starting from a definite point (x_0, y_0) of the Cartesian plane with a definite direction $y_0' = -\frac{1}{\gamma(x_0,y_0)}$ and various velocity magnitudes constitute a monoparametric family (1), associated with a definite value of c^* in (73). The family becomes two-parametric if, starting at the same point (x_0, y_0), we allow the direction y_0' to vary. The slope function (74) becomes numerically known if we record $\gamma(x, y, c^*)$ at all points (x, y), as the material point travels on each orbit.

In the three examples of section 4 we managed to present *solvable* ODEs so that we can check our findings on the grounds of the basic equation (7), as we did already. There exist of course many second order linear ODEs (some of them "famous", as e.g. Bessel's, Legendre's, Chebychev's etc) for which this direct verification cannot be effectuated. In spite of this, we pretend that there exist always *appropriate autonomous force fields* which can produce as orbits their solutions. We cannot, of course, ensure that *an appropriate potential* $V(x, y)$ also exists always. Thus, e.g. , for Bessel's equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{k_0^2}{x^2}\right) y = 0 \tag{75}$$

we find, in view of (29) and (30), the unique force field $X = X_0 x$, $Y = X_0(k_0^2 - x^2)y$, which, however, is not conservative.

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