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Leibniz Inverse Series Relations and Pfaff-Cauchy Derivative Identities

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ABSTRACT: *We establish a new pair of inverse series relations with the connection coefficients being involved in higher derivatives of a fixed analytic function, which are utilized, as a uniform approach, to present new proofs of Pfaff-Cauchy derivative identities and Abel-Hagen-Rothe binomial convolution formulae.*

Denote by D the derivative operator with respect to x. Then the following Leibniz rule for computing higher derivatives of product of two functions $u := u(x)$ and $v := v(x)$ (which are at least n-times differentiable)

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k u D^{n-k} v$$

has been fundamental in calculus. Two very important generalizations due to Pfaff [28] and Cauchy [4] have recently been unearthed by Johnson [26] who investigated also their applications to orthogonal polynomials and multivariate extentions of Hurwitz' type [24].

Motivated by these beautiful derivative identities, this paper will establish a new pair of inverse series relations with the connection coefficients being involved in higher derivatives of a fixed analytic function. Then we shall utilize this inverse

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pair to review the derivative identities of Olver [27], Pfaff [28] and Cauchy [4] by providing direct and new proofs. Finally, we shall sketch applications to Abel formulae, Hagen-Rothe identities as well as Jensen's convolution on binomial coefficients.

1 – Inverse Series Relations

This section will establish the following main theorem about inverse series relations with the connection coefficients being involved in higher derivatives of a fixed analytic function.

THEOREM 1. (New inverse series relations). *Let $\phi := \phi(x)$ be an infinitely differentiable function with respect to x. Then the system of equations*

$$f(n)=\sum_{k=0}^{n}\binom{n}{k}\frac{a+bk}{a+bn}D^{n-k}\phi^{a+bn}g(k),\quad n=0,1,2,\ldots \tag{1a}$$

is equivalent to the system

$$g(n)=\sum_{k=0}^{n}\binom{n}{k}D^{n-k}\phi^{-a-bk}f(k),\quad n=0,1,2,\ldots. \tag{1b}$$

Replacing $g(k)$ by $g(k)/(a+bk)$, we may express the relations displayed in the theorem in the following more symmetrical form:

$$f(n)=\sum_{k=0}^{n}\binom{n}{k}D^{n-k}\frac{\phi^{a+bn}}{a+bn}g(k), \tag{2a}$$

$$g(n)=\sum_{k=0}^{n}\binom{n}{k}D^{n-k}\frac{a+bn}{\phi^{a+bk}}f(k). \tag{2b}$$

PROOF. One implication of this inverse pair is that for every identity of the form (1a) or (1b), there is a companion of the dual identity. To prove each is to prove both. Therefore it is sufficient to show that the latter implies the former. Substituting (1b) into (1a) and then applying the binomial relation

$$\binom{n}{k}\binom{k}{i}=\binom{n}{i}\binom{n-i}{k-i}$$

we can reformulate the double sum expression as follows:

$$\begin{aligned}&\sum_{k=0}^{n}\frac{a+bk}{a+bn}\binom{n}{k}D^{n-k}\phi^{a+bn}\sum_{i=0}^{k}\binom{k}{i}D^{k-i}\phi^{-a-bi}f(i)\\&=\sum_{i=0}^{n}\binom{n}{i}f(i)\sum_{k=i}^{n}\frac{a+bk}{a+bn}\binom{n-i}{k-i}D^{n-k}\phi^{a+bn}D^{k-i}\phi^{-a-bi}.\end{aligned}$$

In order to show that the last double sum reduces to $f(n)$, we have to prove the following orthogonal relation:

$$\sum_{k=i}^{n}\frac{a+bk}{a+bn}\binom{n-i}{k-i}D^{n-k}\phi^{a+bn}D^{k-i}\phi^{-a-bi}=\delta_{i,n}. \tag{3}$$

Splitting the last sum according to the relation

$$\frac{a+bk}{a+bn}\binom{n-i}{k-i}=\binom{n-i}{k-i}-\frac{b(n-i)}{a+bn}\binom{n-i-1}{k-i}$$

and then appealing to the Leibniz rule, we get

$$\begin{aligned}&\sum_{k=i}^{n}\frac{a+bk}{a+bn}\binom{n-i}{k-i}D^{n-k}\phi^{a+bn}D^{k-i}\phi^{-a-bi}\\&=\sum_{k=i}^{n}\binom{n-i}{k-i}D^{n-k}\phi^{a+bn}D^{k-i}\phi^{-a-bi}\\&\quad-\frac{b(n-i)}{a+bn}\sum_{k=i}^{n-1}\binom{n-i-1}{k-i}D^{n-k}\phi^{a+bn}D^{k-i}\phi^{-a-bi}\\&=D^{n-i}\phi^{b(n-i)}-\frac{b(n-i)}{a+bn}D^{n-i-1}\Big\{\phi^{-a-bi}D\phi^{a+bn}\Big\}.\end{aligned}$$

Then the orthogonal relation displayed in (3) follows from the trivial fact

$$\phi^{-a-bi}D\phi^{a+bn}=\frac{a+bn}{b(n-i)}D\phi^{b(n-i)}.$$

Instead, if we substitute (1a) into (1b), then we would get another orthogonal relation

$$\sum_{k=i}^{n}\frac{a+bi}{a+bk}\binom{n-i}{k-i}D^{n-k}\phi^{-a-bk}D^{k-i}\phi^{a+bk}=\delta_{i,n}. \tag{4}$$

However, the proof of this orthogonal relation would be much more tedious.

2 – Derivative Identities

This section will review the four derivative identities due to Olver [27, 1992], Pfaff [28, 1795] and Cauchy [4, 1826] by combining the Leibniz rule with the inverse series relations proved in the last section.

2.1 – Derivative identity due to Olver (1992)

By means of differential operator method, Olver [27, 1992] found the following identity.

THEOREM 2. (Derivative identity). *Let $\phi := \phi(x)$ be a n-times differentiable function with respect to x. Then there holds the following formula:*

$$\frac{a+c}{a+c+n} D^n \phi^{a+c+n} = \sum_{k=0}^{n} \frac{ac}{(a+k)(c+n-k)} \binom{n}{k} D^k \phi^{a+k} D^{n-k} \phi^{c+n-k}.$$

PROOF. Replacing c by $c-n$, we may rewrite the equation as

$$\frac{c(a+c-n)}{(a+c)(c-n)} D^n \phi^{a+c} = \sum_{k=0}^{n} \binom{n}{k} D^{n-k} \phi^{c-k} \frac{ac}{(a+k)(c-k)} D^k \phi^{a+k}.$$

This relation perfectly matches (1b) specified with

$$a \to -c \quad \text{and} \quad \begin{cases} f(k) := \dfrac{ac}{(a+k)(c-k)} D^k \phi^{a+k}, \\[2ex] g(n) := \dfrac{c(a+c-n)}{(a+c)(c-n)} D^n \phi^{a+c}. \end{cases}$$

According to Theorem 1, it suffices to prove the dual relation corresponding to (1a):

$$\frac{ac}{(a+n)(c-n)} D^n \phi^{a+n} = \sum_{k=0}^{n} \binom{n}{k} \frac{c-k}{c-n} D^{n-k} \phi^{n-c} \frac{c(a+c-k)}{(a+c)(c-k)} D^k \phi^{a+c}$$

which is equivalent to the following equation:

$$\frac{a}{a+n} D^n \phi^{a+n} = \sum_{k=0}^{n} \binom{n}{k} \frac{a+c-k}{a+c} D^k \phi^{a+c} D^{n-k} \phi^{n-c}.$$

By invoking the relation

$$\frac{a+c-k}{a+c} \binom{n}{k} = \binom{n}{k} - \frac{n}{a+c} \binom{n-1}{k-1}$$

we may evaluate the last sum through the Leibniz rule as follows:

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{a+c-k}{a+c}D^k\phi^{a+c}D^{n-k}\phi^{n-c}\\
&=\sum_{k=0}^{n}\binom{n}{k}D^k\phi^{a+c}D^{n-k}\phi^{n-c}\\
&\quad-\frac{n}{a+c}\sum_{k=1}^{n}\binom{n-1}{k-1}D^k\phi^{a+c}D^{n-k}\phi^{n-c}\\
&=D^n\phi^{a+n}-\frac{n}{a+c}D^{n-1}\big\{\phi^{n-c}D\phi^{a+c}\big\}\\
&=D^n\phi^{a+n}\Big\{1-\frac{n}{a+n}\Big\}=\frac{a}{a+n}D^n\phi^{a+n}.
\end{aligned}$$

2.2 – Derivative identity due to Pfaff (1795)

Pfaff [28, 1795] found the following generalization of Leibniz' rule with an additional ϕ-function (see also [2, 3], [26, Eq 1.1] and [31]).

Theorem 3. (Derivative identity). *Let $u := u(x)$, $v := v(x)$ and $\phi := \phi(x)$ be three n-times differentiable functions with respect to x. Then there holds the following formula:*

$$D^n\big(uv\big)=\sum_{k=0}^{n}\binom{n}{k}D^{k-1}\big(\phi^k u'\big)D^{n-k}\big(\phi^{-k}v\big).$$

In fact, when $\phi\equiv 1$, this identity becomes the Leibniz formula.

Proof. According to the Leibniz rule, we have the following triple sum expression:

$$\begin{aligned}
&\sum_{k=1}^{n}\binom{n}{k}D^{k-1}\big(\phi^k u'\big)D^{n-k}\big(\phi^{-k}v\big)\\
&=\sum_{k=1}^{n}\binom{n}{k}\sum_{i=1}^{k}\binom{k-1}{i-1}u^{(i)}D^{k-i}\phi^k\sum_{j=k}^{n}\binom{n-k}{j-k}v^{(n-j)}D^{j-k}\phi^{-k}\\
&=\sum_{i=1}^{n}\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}u^{(i)}v^{(n-j)}\sum_{k=i}^{j}\frac{i}{k}\binom{j-i}{k-i}D^{k-i}\phi^kD^{j-k}\phi^{-k}.
\end{aligned}$$

In view of (4), the last sum with respect to k reduces to $\delta_{i,j}$. Applying again the Leibniz rule leads us to the following simplified expression:

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}D^{k-1}\big(\phi^k u'\big)D^{n-k}\big(\phi^{-k}v\big)\\&=uv^{(n)}+\sum_{i=1}^{n}\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}u^{(i)}v^{(n-j)}\delta_{i,j}\\&=\sum_{i=0}^{n}\binom{n}{i}u^{(i)}v^{(n-i)}=D^n\big(uv\big).\end{aligned}$$

2.3 – Derivative identities due to Cauchy (1826)

Replacing v by $\phi^n v$, we recover from Theorem 3 the following formula due to Cauchy (cf. Gould [20] and Johnson [26, Eq 1.3]):

$$D^n\big\{\phi^n uv\big\}=\sum_{k=0}^{n}\binom{n}{k}D^{k-1}\big(\phi^k u'\big)D^{n-k}\big(\phi^{n-k}v\big). \tag{5}$$

Observing the relations

$$\begin{aligned}\phi^n D(uv)&=D\big(\phi^n uv\big)-n\phi^{n-1}\phi' uv,\\ D\big(\phi^{n-k}v\big)&=\phi^{n-k}v'+(n-k)\phi^{n-k-1}\phi' v;\end{aligned}$$

we can carry out through (5) the following computation

$$\begin{aligned}D^{n-1}\big\{\phi^n D(uv)\big\}&=D^n\big(\phi^n uv\big)-nD^{n-1}\big\{\phi^{n-1}\phi' uv\big\}\\&=\sum_{k=0}^{n}\binom{n}{k}D^{k-1}\big(\phi^k u'\big)D^{n-k}\big(\phi^{n-k}v\big)\\&\quad-n\sum_{k=0}^{n-1}\binom{n-1}{k}D^{k-1}\big(\phi^k u'\big)D^{n-k-1}\big(\phi^{n-k-1}\phi' v\big)\\&=\sum_{k=0}^{n}\binom{n}{k}D^{k-1}\big(\phi^k u'\big)D^{n-k-1}\big(\phi^{n-k}v'\big).\end{aligned}$$

This results in another symmetric companion also due to Cauchy [4, 1826].

Theorem 4. (Derivative identity: see [26, Eq 1.4] and [33] also). *Let $u:=u(x)$, $v:=v(x)$ and $\phi:=\phi(x)$ be three n-times differentiable functions with respect to x. Then there holds the following formula:*

$$D^{n-1}\big\{\phi^n D(uv)\big\}=\sum_{k=0}^{n}\binom{n}{k}D^{k-1}\big(\phi^k u'\big)D^{n-k-1}\big(\phi^{n-k}v'\big).$$

For $u = \phi^a$ and $v = \phi^c$, it is not hard to check that this identity reduces to Olver's one displayed in Theorem 2. Vice versa, this theorem is also implied by Theorem 2. This can be shown directly as follows.

PROOF. According to the Leibniz rule, we have the following triple sum expression:

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k} D^{k-1}\big(\phi^k u'\big) D^{n-k-1}\big(\phi^{n-k} v'\big)\\
&=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\frac{i}{k}\binom{k}{i} u^{(i)} D^{k-i}\phi^k \sum_{j=k}^{n}\frac{n-j}{n-k}\binom{n-k}{n-j} v^{(n-j)} D^{j-k}\phi^{n-k}\\
&=\sum_{i=0}^{n}\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i} u^{(i)} v^{(n-j)} \sum_{k=i}^{j}\frac{i(n-j)}{k(n-k)}\binom{j-i}{k-i} D^{k-i}\phi^k D^{j-k}\phi^{n-k}.
\end{aligned}$$

Specifying in Theorem 2

$$a \to i, \quad c \to n-j, \quad n \to j-i, \quad k \to k-i$$

we may evaluate the last sum with respect to k as follows:

$$\sum_{k=i}^{j}\frac{i(n-j)}{k(n-k)}\binom{j-i}{k-i} D^{k-i}\phi^k D^{j-k}\phi^{n-k} = \frac{n+i-j}{n} D^{j-i}\phi^n.$$

Substituting this into the triple sum expression and then changing the summation indices by the replacements $j \to n-j$ and $i+j \to k$, we can reduce, through the Leibniz rule, the triple sum as follows:

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k} D^{k-1}\big(\phi^k u'\big) D^{n-k-1}\big(\phi^{n-k} v'\big)\\
&=\sum_{i=0}^{n}\sum_{j=i}^{n}\frac{n+i-j}{n}\binom{n}{j}\binom{j}{i} u^{(i)} v^{(n-j)} D^{j-i}\phi^n\\
&=\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{i+j}{n}\binom{n}{i+j}\binom{i+j}{i} u^{(i)} v^{(j)} D^{n-i-j}\phi^n\\
&=\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k} D^{n-k}\phi^n \sum_{i=0}^{k}\binom{k}{i} u^{(i)} v^{(k-i)}\\
&=\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k} D^{k}\big(uv\big) D^{n-k}\phi^n = D^{n-1}\Big\{\phi^n D\big(uv\big)\Big\}
\end{aligned}$$

2.4 – Derivative identity of Jensen type due to Pfaff (1795)

Define the binomial convolution on derivatives

$$J_n(u,v) := \sum_{k=0}^{n} \binom{n}{k} D^k\left(\phi^k u\right) D^{n-k}\left(\phi^{-k} v\right).$$

Applying Pfaff's formula displayed in Theorem 3 to the three-terms relation

$$D\left(\phi^k u\right) = \phi^k u' + k\phi^{k-1}\phi' u$$

we can derive recursively the following expression

$$\begin{aligned} J_n(u,v) &= D^n(uv) + nJ_{n-1}(u\phi', v/\phi) \\ &= D^n(uv) + nD^{n-1}\left(\frac{\phi'}{\phi}uv\right) + n(n-1)J_{n-2}(u{\phi'}^2, v/\phi^2) \\ &= n!J_0(u{\phi'}^n, v/\phi^n) + \sum_{k=1}^{n} \frac{n!}{k!} D^k\left\{(\phi'/\phi)^{n-k} uv\right\}. \end{aligned}$$

The finite induction confirms the following formula also due to Pfaff [28, 1795].

THEOREM 5. (Derivative identity of Jensen type: see [26, Eq 1.2] also) *Let $u := u(x)$, $v := v(x)$ and $\phi := \phi(x)$ be three n-times differentiable functions with respect to x. Then there holds the following formula:*

$$\sum_{k=0}^{n} \binom{n}{k} D^k\left(\phi^k u\right) D^{n-k}\left(\phi^{-k} v\right) = \sum_{k=0}^{n} \frac{n!}{k!} D^k\Big\{\left(\phi'/\phi\right)^{n-k} uv\Big\}.$$

This can be verified directly by combining the inverse series relations with the telescoping method.

PROOF. First we show the following special case with $u = \phi^a$ and $v = \phi^c$:

$$\sum_{k=0}^{n} \binom{n}{k} D^k \phi^{a+k} D^{n-k} \phi^{c-k} = \sum_{k=0}^{n} \frac{n!}{k!} D^k \left\{\phi^{a+c}\left(\frac{\phi'}{\phi}\right)^{n-k}\right\}. \tag{6}$$

According to Theorem 1, it suffices to prove the following dual relation:

$$D^n\phi^{a+n}=\sum_{k=0}^{n}\frac{c-k}{c-n}\binom{n}{k}D^{n-k}\phi^{n-c}\sum_{i=0}^{k}\frac{k!}{i!}D^i\left\{\phi^{a+c}\Big(\frac{\phi'}{\phi}\Big)^{k-i}\right\}.$$

Denote the last double sum by $\mathcal{S}$. Reversing the internal summation order by $i\to k-i$ and then interchanging the double sum order, we can reformulate $\mathcal{S}$ as follows:

$$\mathcal{S}=\sum_{i=0}^{n}\frac{n!}{(n-i)!}\sum_{k=i}^{n}\frac{c-k}{c-n}\binom{n-i}{k-i}D^{n-k}\phi^{n-c}D^{k-i}\left\{\phi^{a+c}\left(\frac{\phi'}{\phi}\right)^{i}\right\}.$$

Writing

$$\frac{c-k}{c-n}\binom{n-i}{k-i}=\binom{n-i}{k-i}+\frac{n-i}{c-n}\binom{n-i-1}{k-i}$$

and then applying the Leibniz rule, we can proceed

$$\begin{aligned}\mathcal{S}-n!\phi^a{\phi'}^n&=\sum_{i=0}^{n-1}\frac{n!}{(n-i)!}\left\{D^{n-i}\left(\phi^{a+n-i}{\phi'}^i\right)-(n-i)D^{n-i-1}(\phi i^{a+n-i-1}{\phi'}^{i+1})\right\}\\&=n!\sum_{i=0}^{n-1}\left\{\frac{D^{n-i}\left(\phi^{a+n-i}{\phi'}^i\right)}{(n-i)!}-\frac{D^{n-i-1}\left(\phi^{a+n-i-1}{\phi'}^{i+1}\right)}{(n-i-1)!}\right\}\\&=D^n\phi^{a+n}-n!\phi^a{\phi'}^n\end{aligned}$$

where the last line is justified by the telescoping method. This proves (6).

According to the definition of $J_n(u,v)$, we have again a triple sum expression:

$$\begin{aligned}J_n(u,v)&=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}u^{(i)}D^{k-i}\phi^k\sum_{j=0}^{n-k}\binom{n-k}{j}v^{(j)}D^{n-k-j}\phi^{-k}\\&=\sum_{i,j:0\le i+j\le n}\binom{n}{i+j}\binom{i+j}{i}u^{(i)}v^{(j)}\sum_{k=i}^{n-j}\binom{n-i-j}{k-i}D^{k-i}\phi^kD^{n-k-j}\phi^{-k}\\&=\sum_{\ell=0}^{n}\binom{n}{\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}u^{(i)}v^{(\ell-i)}\sum_{k=0}^{n-\ell}\binom{n-\ell}{k}D^k\phi^{k+i}D^{n-k-\ell}\phi^{-k-i}.\end{aligned}$$

Applying (6) to the last sum and then recalling the Leibniz rule, we can reduce the triple sum just displayed as

$$\begin{aligned}J_n(u,v)&=\sum_{\ell=0}^{n}\binom{n}{\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}u^{(i)}v^{(\ell-i)}\sum_{k=0}^{n-\ell}\frac{(n-\ell)!}{k!}D^k\Big(\frac{\phi'}{\phi}\Big)^{n-k-\ell}\\&=\sum_{\ell=0}^{n}\binom{n}{\ell}D^\ell(uv)\sum_{k=0}^{n-\ell}\frac{(n-\ell)!}{k!}D^k\Big(\frac{\phi'}{\phi}\Big)^{n-k-\ell}.\end{aligned}$$

Finally the last expression can further be simplified, by letting $k + \ell = m$, as follows:

$$J_n(u,v) = \sum_{m=0}^{n} \frac{n!}{m!} \sum_{k=0}^{m} \binom{m}{k} D^k \left(\frac{\phi'}{\phi}\right)^{n-m} D^{m-k}(uv)$$
$$= \sum_{m=0}^{n} \frac{n!}{m!} D^m \left\{ (\phi'/\phi)^{n-m}\, uv \right\}.$$

This completes the proof of Pfaff's identity of Jensen type.

3 – Generalized Binomial Convolution Formulae

Just as done by Johnson [26, §3], the derivative identities proved in the last section can be used to recover the convolution formulae of Abel and Hagen-Rothe as well as Jensen.

3.1 – Abel identities

Taking $u = e^{ax}$, $v = e^{cx}$ and $\phi = e^{bx}$ in (5) and Theorem 4, we derive immediately the Abel formulae (cf. [1], [5, 12], [15, §3.1] and [29, §1.5]):

$$\text{(7a)} \qquad \frac{(a+c+bn)^n}{n!} = \sum_{k=0}^{n} \frac{a}{a+bk} \frac{(a+bk)^k}{k!} \frac{\{c+b(n-k)\}^{n-k}}{(n-k)!},$$

$$\text{(7b)} \qquad \frac{a+c}{a+c+bn} \frac{(a+c+bn)^n}{n!} = \sum_{k=0}^{n} \frac{a}{a+bk} \frac{(a+bk)^k}{k!} \frac{c}{c+b(n-k)} \frac{\{c+b(n-k)\}^{n-k}}{(n-k)!}.$$

Furthermore, Theorem 5 under the same setting leads us to the following convolution formula of Jensen type (cf.[6] and [18, 21])

$$\text{(8)} \qquad \sum_{k=0}^{n} \binom{n}{k} (a+bk)^k (c-bk)^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!} (a+c)^k b^{n-k}.$$

3.2 – Hagen-Rothe identities

Similarly, taking $u = (1+x)^a$, $v = (1+x)^c$ and $\phi = (1+x)^b$ in (5) and Theorem 4, we derive the Hagen-Rothe identities (cf. [5, 12], [16, 17], [23, §5.4] and [30, 32]):

$$\text{(9a)} \qquad \binom{a+c+bn}{n} = \sum_{k=0}^{n} \frac{a}{a+bk} \binom{a+bk}{k} \binom{c+b(n-k)}{n-k},$$

$$\text{(9b)} \qquad \frac{a+c}{a+c+bn} \binom{a+c+bn}{n} = \sum_{k=0}^{n} \frac{a}{a+bk} \binom{a+bk}{k} \frac{c}{c+b(n-k)} \cdot \binom{c+b(n-k)}{n-k}.$$

In addition, applying the same setting to Theorem 5 recovers Jensen's binomial convolution formula (cf.[6], [18, 19] and [25]):

$$\sum_{k=0}^{n} \binom{a+bk}{k} \binom{c-bk}{n-k} = \sum_{k=0}^{n} \binom{a+c-k}{n-k} b^k. \tag{10}$$

3.3 – Gould-Hsu inverse series relations

For two arbitrary complex sequences $\{a_k\}$ and $\{b_k\}$, define the polynomials

$$\varphi(x;0) = 1 \quad \text{and} \quad \varphi(x;n) = \prod_{k=0}^{n-1} (a_k + x b_k) \quad \text{for} \quad n = 1, 2, \ldots. \tag{11}$$

Gould and Hsu [22] found the following very general inverse pair:

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi(k;n) g(k), \tag{12a}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a_k + k b_k}{\varphi(n;k+1)} f(k). \tag{12b}$$

Even though this inverse pair is very different from that displayed in Theorem 1, we can check without difficulty that as special cases, they contain the following reciprocal relations in common:

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a+bk)^n g(k), \tag{13a}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a+bk}{(a+bn)^{k+1}} f(k). \tag{13b}$$

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a+bk)_n g(k), \tag{14a}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a+bk+k}{(a+bn)_{k+1}} f(k). \tag{14b}$$

These inverse series relations have been employed by Chu and Hsu [14] to verify directly Abel-Hagen-Rothe identities through the binomial theorem

$$(a+c)^n = \sum_{k=0}^{n} \binom{n}{k} a^k c^{n-k}$$

and the Chu-Vandermonde convolution formula of binomial coefficients

$$\binom{a+c}{n} = \sum_{k=0}^{n} \binom{a}{k} \binom{c}{n-k}.$$

Further applications of Gould-Hsu inversions to classical hypergeometric summation formulae as well as their generalizations can be found in Chu [7, 8, 9, 10, 11, 13].

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