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The Levi problem on Stein spaces with singularities. A survey

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Dedicated to the memory of Giovanni Bassanelli

ABSTRACT: *We discuss the well-known open problems : the Local Steiness Problem and the Union Problem.*

1 – A brief history of the smooth case

In 1910 E. E. Levi [Lev] noticed that a domain of holomorphy Ω in $\mathbb{C}^n$, with smooth C^2 boundary, should satisfy some pseudocovexity condition on the boundary points. More precisely he showed that if ρ is a C^2 defining function for the boundary $\partial\Omega$ of Ω then the associated quadratic form $L\rho$ (we shall call it, as usual, the Levi form of ρ) is necessarily positive semi-definite on the holomorphic tangent space $T_z(\partial\Omega) := \{w \in \mathbb{C}^n \mid \Sigma_i^n w_i \partial\rho/\partial z_i(z) = 0\}$ for any point $z \in \partial\Omega$.

O. Blumenthal [Blu] raised the important and difficult question on the validity of the converse of this statement, i.e. if a domain $\Omega \subset \mathbb{C}^n$ with smooth pseudoconvex boundary is necessarily a Stein domain. This problem, called also the Levi problem, was open for a long time, untill 1953, when K. Oka [O] solved it completely in the affirmative (an indepedent proof of this result was also obtained by F. Norguet [No] and by H.J. Bremermann [Brem]). More generally, K. Oka considered unbranched Riemann domaines $\pi : \Omega \to \mathbb{C}^n$ (i.e. π is lo-

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cally biholomorphic) and proved that Ω is Stein iff $-\log d$ is a plurisubharmonic function on Ω where d denotes the boundary distance on Ω. Note that Riemann unbranched domains over $\mathbb{C}^n$ appear naturally as domains of existence of families of holomorphc functions defined on open subsets in $\mathbb{C}^n$. Oka's result shows in particular that the Steiness of Ω is a local property of its boundary. To be more precise we shall call a holomorphic map $p : \Omega \to X$ of complex manifolds (or, more generally, of complex spaces) a Stein morphism if every point $x \in X$ has a neighborhood $V = V(x)$ such that $p^{-1}(V)$ is Stein. For example, if we consider the inclusion map $i : \Omega \to X$ of an open subset Ω of X, then Ω is called locally Stein (in X) iff the map i is a Stein morphism, or equivalently each point $x \in \partial\Omega$ has a neighborhood $V = V(x)$ such that $V \cap \Omega$ is Stein. Oka's theorem can therefore be stated as follows: an open subset $\Omega \subset \mathbb{C}^n$ is Stein if and only if i is a Stein morphism, or, more generally, a Riemann unbranched morphism $\pi : \Omega \to \mathbb{C}^n$ is Stein iff π is a Stein morphism. If $\pi : \Omega \to \mathbb{C}^n$ is a Stein morphism which has discrete fibers (even finite) but π is not assumed to be locally biholomorphic (i.e. it is a branched Riemann domain) then Ω could be non Stein (see [F 2], [C-D4]).

Having in mind Oka's result, H. Cartan at the important Colloque sur les fonctions de plusieurs variables held in Brussels in 1953 [Car] raised the following problem: let X be a Stein manifold and $\Omega \subset X$ a locally Stein open subset. Does it folow that Ω is itself Stein? (the Local Steiness Problem in the smooth case, the manifold case). A positive answer to this question has been given by H. Grauert and F. Doquier [D-G]. Additionally, they solved the more general case of Riemann unbranched domains over Stein manifolds by proving: If $\pi : \Omega \to X$ is a Riemann unbranched domain over a Stein manifold X and if π is a Stein morphism, then Ω is itself Stein. Grauert's method [D-G] is essentilally based on the following result: if $X \subset \mathbb{C}^n$ is a complex closed submanifold then there exists an open neighborhood V of X (which can be chosen to be Stein) and a holomorphic retract $\rho : V \to X$. This result can be used to reduce the general case of Riemann unbranched domains over Stein manifolds to the case of Riemann unbranched domains spread over $\mathbb{C}^n$. Such a holomorphic retract does not exist if X has singularities. In fact Rossi [Ro] showed that the existence of a holomorphic retract onto X as above, even of a neighborhood of X minus a point, onto X minus a point, implies the smoothness of X. Therefore, in order to study the Levi problem in the case of Stein spaces with singularities, one needs other methods and new ideas.

2 – The most important questions related to the Levi problem for singular Stein spaces

The main open problem, related to the Levi problem in the singular case (note that, due to its importance, a new section regarding the Levi problem in the singular case was introduced in the 2000 AMS classification 32C55: The Levi problem in complex spaces; generalizations) is the Local Steiness Problem, or the singular Levi problem, which can be stated as follows:

Question 1

Let X be a Stein space and $D \subset X$ an open subset which is locally Stein. Does it follow that D is itself Stein?

A more general question, in Oka's context of Riemann unbranched domains, is the following:

Question 1′

Let X be a Stein space and $\pi : \Omega \to X$ a Riemann unbranched domain such that π is a Stein morphism. Does it follow that Ω is itself Stein?

If π is the inclusion map, one gets the particular case of the Question 1.

Another importat open problem on Stein spaces with singularities is the Union Problem:

Question 2

Let X be a Stein space and $D = \bigcup_{n\in\mathbb{N}} D_n$ an increasing union of Stein open sets. Does it follow that D is itself Stein ?

The answer to the Union Problem is yes if $X = \mathbb{C}^n$ [B-S] (one does not need necessarily the solution to the Levi problem, but the distance to the boundary is very important in the proof) and more generally if X is a Stein manifold using Grauert's result on the existence of the holomorphic retraction [D-G].

3 – The state of the art for the problems Q1, Q1′, Q2 for isolated singularities

In 1964 in [A-N] A. Andreotti and R. Narasimhan solved Q 1 for Stein spaces with isolated singularities. Therefore they proved that a locally Stein open subset D of a Stein space X with isolated singularities is itself Stein. Their method is of "projective" type, namely they realize D as a suitable finite union of Riemann unbranched domains $\Phi_j : D_j \to \mathbb{C}^n$, $n = \dim X$, $D_j \subset X$, and, using the corresponding boundary distances d_j on D_j, they are able to construct, by a patching technique, a strongly plurisubharmonic continuous exhaustion function $\phi : D \to \mathbb{R}$, and consequently, by a result due to H. Grauert [G 1], generalized in the singular case by R. Narasimhan [Nar], it follows that D is a Stein space.

This "projective" method cannot be applied to the more general case of Q1', i.e. for Riemann unbranched domains over Stein spaces with isolated singularities because one cannot control the behaviour of the constructed function in the vertical direction, i.e. in the fiber direction of the map π. However it was proved by M. Colţoiu and K. Diederich [C-D4] that the answer to Q1' is also positive for isolated singularities, i.e. one has:

THEOREM 3.1.*Let X be a Stein space with isolated singularities and π : $\Omega \to X$ a Riemann unbranched domain such that π is a Stein morphism. Then Ω is itself Stein.*

For the proof, in order to avoid the difficulties in the construction of a function with nice behaviour (Levi form bounded from below) in the vertical direction, in [C-D4] it is considered the pull-back of the given Riemann domain on a resolution of singularities $\tau : \tilde{X} \to X$ and the existence theorem due to M. Colţoiu and N. Mihalache [C-M2] of a strongly plurisubharmonic function $\phi : \tilde{X} \to [-\infty, \infty)$ which is $-\infty$ exactly on the exceptional set of the desingularization $\tilde{X}$ (for the basic theory of exceptional sets see [G 2]). Finally one uses a classical patching technique for strongly plurisubharmonic functions with bounded differences. A wrong proof of Theorem 3.1. was given by V. Vajaitu [V] which is based on his lemma 3.3. which does not hold.

As for Q 2 this problem (the Union Problem) is unsolved even for Stein spaces with isolated singularities. The most general result which is known to hold is the following, due to M. Colţoiu and M. Tibar [C-T 2]:

THEOREM 3.2. *Let X be a Stein space of dimension 2 and let $D = \bigcup_{n\in\mathbb{N}} D_n$ be an increasing sequence of Stein open sets. Then D satisfes the discrete Kontinuitätssatz (the disk property).*

We recall that a complex space D satisfies the discrete Kontinuitätssatz (the disk property) if for any sequence of maps $\varphi_\nu : \bar{\Delta} \to D$ (where Δ is the open unit disk in $\mathbb{C}$) which are holomorphic in Δ and continuous on $\bar{\Delta}$, if $\cup\varphi_\nu(\partial\Delta)$ is relatively compact in D, then $\cup\varphi_\nu(\bar{\Delta})$ is relatively compact in D.

The proof of the above Theorem is essentially based on the recent classification of 2 dimensional normal singularities due to M. Colţoiu and M. Tibar [C-T 1]. Namely, let (X, x_0) be a germ of a 2 dimensional normal singularity and denote by K the associated singularity link, i.e. in some local embedding, K is the intersection of the boundary of a small ball (centered in x_0) with X. If the fundamental group $\pi_1(K)$ is a finite group, then it is well-known that (X, x_0) is a quotient singulaity, therefore the universal covering of $X \setminus \{x_0\}$, for small X, is a ball minus a point (this singularity is of "concave" type). The main result in [C-T 1] asserts that if $\pi_1(K)$ is an infinite group, then the universal covering of $X \setminus \{x_0\}$, for small X, is a Stein manifold (we can say that this singularity is of "convex" type). The proof of this assertion is divided in two steps:

Step 1 . It is assumed that the homology group $H_1(K, \mathbb{Z})$ is an infinite group. In this case, using a suitable infinite "necklace" (Nori string) and a patching technique it is constructed a Stein covering of $X \setminus \{x_0\}$, for small X.

Step 2. The general case when $\pi_1(K)$ is an infinite group is reduced to the previuos step using some results from the classification theory of real 3-dimensional compact manifolds in order to cover finitely sheeted the link K by another 3-manifold with infinite first $\mathbb{Z}$-homology group (which will be the link of another singularity and Step 1 can be applied to this new singularity).

In connection with the disk property (discrete Kontinuitätssatz) of increasing unions of Stein domains contained in a 2 dimensional complex space X it is important to note that an arbitrary increasing union of Stein manifolds (not contained in a Stein space) might not have the disk property as it was proved by J.E. Fornaess [F 1]. If the discrete Kontinuitätssatz condition is replaced by the continuous Kontinuitätssatz (the parameter indexing the discs continuously is $t \in \mathbb{R}$) then, obviously, from the definition, it follows that an arbitrary union of Stein manifolds satisfies the continuous Kontinuitätssatz.

4 – The state of the art for arbitrary singularities

First let us briefly recall the notion of envelope of holomorphy for a domain D contained as an open subset in a Stein space X (for arbitrary complex spaces see e.g. [C-D2]). The domain D is said to have an envelope of holomorphy, say $\tilde{D}$, if $\tilde{D}$ is a Stein space, $D \subset \tilde{D}$ as an open complex subspace, and every holomorphic function on D extends uniquely to a holomorphic function on $\tilde{D}$ (note that, if the envelope of holomorphy exists, then it is unique). If X is a Stein manifold then it is well-known that D has always an envelope of holomorphy $\tilde{D}$ and $\tilde{D}$ can be realized as an unbranched Riemann domain $\pi : \tilde{D} \to X$. If X has singularities (even isolated and normal) the problem of the existence of envelopes of holomorphy is more difficult. In his well-known article " Remarkable pseudoconvex manifolds " [G 3] H. Grauert constructed an example of a 3 dimensional normal Stein sapce X, with an isolated singularity, and an open subset $D \subset X$ (which is the complement of a hypersurface) such that D has not an envelope of holomorphy (Grauert's example is also discussed in detail in [Su]). J. Bingener [Bi] also constructed another example with similar properties using Nagata's counterexample to the Hilbert 14-th problem [Nag]. However his proof is quite involved. Other counterexamples have been obtained in [C-D1] using the hypersurface section problem (see also [Col 3]) or in [C-D3] using the non-separation of the topology of the cohomology group $H^1(D, \mathcal{O})$.

In [C-D2] it is proved the following result:

Theorem 4.1. *Let X be a Stein space and $D \subset\subset X$ a locally Stein open subset. Then the following two conditions are equivalent:*
1) *D is Stein*
2) *D has an envelope of holomorphy*

An analogous result is shown for increasing unions of Stein domains contained in a Stein space. The proof is essentially based on the following theorem due to Fornaess and Narasimhan [F-N], proved using L^2 estimates: Let $D \subset\subset X$

be a locally Stein open subset contained in a normal Stein space X. Then for every point $x_0 \in (\partial D) \cap Reg(X)$ and for every sequence of points $x_n \in D$, $x_n \to x_0$, there exists a holomorphic function $f \in \mathcal{O}(D)$ which is unbounded on $\{x_n\}$.

Concerning the envelopes of holomorphy in normal Stein spaces of dimension 2 it seems to the author that the answer to the following problem is unknown: Let X be a normal Stein space of dimension 2 and $D \subset X$ an open subset. Is it true that D has an envelope of holomorphy ?

Let us recall in this context, the following question raised by H. Grauert and R. Remmert [G-R]: Let X be a normal Stein space of dimension 2 and $D \subset X$ a domain of holomorpy. Does it follow that D is itself Stein? Under some additional topological assumptions (e.g. D is locally simply connected near $Sing(X)$) it follows directly from Colţoiu-Tibar classification of normal 2 dimensional singularities [C-T 1] that D as above is Stein.

For non-isolated singularities in dimension 3 in [C-D2] it was proved the following result: Let X be a Stein space of dimension 3 and $H \subset X$ a hypersurface (i.e. a closed analytic subset of codimension 1). If $D = X \setminus H$ is locally Stein then D is itself Stein.

It is not known if an analogous result as above holds if the condition "locally Stein" is replaced by "an increasing union of Stein open sets".

There is a strong connection between the Steiness condition of a locally Stein D and the question of the separation of the cohomology group $H^1(D, \mathcal{O})$. Namely in [J] it is shown that an open subset D of a Stein space is Stein if it is locally Stein and if $H^1(D, \mathcal{O})$ is separated (an analogous result holds for increasing unions of Stein open sets). Therefore it is interesting to decide if the following is true: if X is a (normal) Stein space and $H \subset X$ is a hypersurface, $D := X \setminus H$, does it follow that $H^1(D, \mathcal{O})$ is separated ?

The question of the separation of $H^i(X \setminus A, \mathcal{F})$, $A \subset X$ closed analytic subset, $\mathcal{F} \in Coh(X)$ was studied in detail by Siu and Trautmann [S-T 1],[S-T 2], Trautmann [T] (sufficient conditions). However in [C-D3] it was constructed a normal Stein space of dimension 3, having only one singular point, and a hypersurface $H \subset X$ such that for $D = X \setminus H$ the cohomology group $H^1(D, \mathcal{O})$ is not separated. It would be also interesting to consider closed analytic subsets $A \subset \mathbb{C}^n$ and to study if the cohomology groups $H^i(X \setminus A, \mathcal{O})$ are separated (for $i = 1$ the answer is yes [T]). One interesting example in this context is $A \subset \mathbb{C}^6$ where $\dim A = 3$ and A is the cone over the Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$ and to study the vanishing of the cohomology groups in degree 3 (see also W. Barth [Ba] and M. Colţoiu [Col 4]). In view of this discussion it would be important to decide the answer to the question: given a Stein space, $dimX \geq 4$, and $H \subset X$ a hypersurface (closed analytic subset of codimension 1) such that $D := X \setminus H$ is locally Stein, does it follow that D is itself Stein ?

5 – The connection between Levi's problem and the hypersurface section problem

Related to the Local Steiness Problem, is the following question "The hypersurface section problem" considered for the first time by J. E. Fornaess and R. Narasimhan [F-N] (under some additional cohomological vanishing assumptions).

This problem can be stated as follows:

QUESTION Let X be a Stein space, $dim X \geq 3$, and $D \subset X$ an open subset such that the intersection $D \cap H$ is Stein for any hypersurface $H \subset X$. Does it follow that D is Stein ?

A counterexample of dimension 3, with X normal, having only one singular point, and D is the complement of a hypersurface $A \subset X$, has been constructed in [C-D1] (see also [Col 3]). An important tool in this construction are the line bundles which are topologically trivial but none of their power is analytically trivial (which were studied for the first time by H. Grauert [G 3]) and a result of R. R. Simha [Sim] about the Steiness of the complement of a curve in a 2 dimensional normal Stein space. Later, H. Brenner [Bren] obtained another 3 dimensional counterexample X, but with X non-normal, using "forcing equations" and the result of R. R. Simha, with a construction which is more algebraic then geometric. He communicated to the author that the normalization of his counterexample works also for the hyperintersection problem and has only one singular point, but the computation to prove that the normalization has only one singular point is quite involved (polynomials of degree 12).

The connection between the "Local Steiness Problem" LSP (The Levi Problem) and the "Hypersurface section problem" HSP is the following: HSP implies LSP, which follows by induction on $\dim(X)$ since LSP is known to hold if $\dim(X) = 2$ ([A-N]). However, as we already remarked, HSP does not hold if $\dim(X) = 3$. It would be interesting to construct counterexamples to HSP with $\dim(X) \geq 4$. This is a much more difficult problem than the 3-dimensional case (absence of a Simha type result if dim > 2). As remarked in [Col 3] in order to construct a counterexample X with $\dim(X) \geq 4$, it suffices to construct a compact projective algebraic space M, $\dim(M) \geq 3$, and an open subset $U \subset M$ such that:

1) U is not Stein, but the intersection $T \cap U$, of U with every hypersurface $T \subset M$, is Stein
2) U is weakly pseudoconvex, i. e. U admits a smooth plurisubharmonic exhaustion function

If X is a Stein manifold the answer to HSP is known to be affirmative. For $X = \mathbb{C}^n$ this problem was solved by P. Lelong [L], and the general case of Stein manifolds follows easily from this case.

6 – Some final remarks and conjectures

Let $\pi : X \to Y$ be a proper holomorphic map of complex spaces. We recall that π is called relatively ample if there is a holomorphic line bundle $p : L \to X$ over X such that the restriction of L to the fibers of π is an ample (positive) line bundle.

In connection with the Local Steiness Problem we raise the following:

Question A

Let $\pi : X \to Y$ be a proper holomorphic map which is relatively ample and assume that Y is a Stein space. Let $W \subset X$ be an open subset such that the restriction of π to W is a Stein morphism. Does it follow that W is Stein ?

A negative answer to this question would imply a counterexample to the Local Steiness Problem. This follows easily from the results of relative contraction [K-S]. Let us remind in this context the Serre question [Se]: if $\pi : E \to B$ is a locally trivial holomorphic fibration with Stein base B and Stein fiber F, does it follow that the total space E is itself Stein ? The first counterexample was obtaned by H. Skoda [Sk] having as fiber $\mathbb{C}^2$ (studied also by J.P. Demailly in [Dem] and J.P. Rosay [R]) and with a bounded Stein domain in $\mathbb{C}^2$ as fiber by Coeuré and Loeb [C-L]. It would be interesting to study if it is possible to obtain counterexamples to the Serre problem so that the automorphisms of the Stein fiber extend to automorphisms of some algebraic compactification of the fiber and the resulting projection map, after compactifying the fiber, is a projective morphism. Then one would get, according to the previous discussion (question A), a counterexample to the Local Steiness Problem.

A particular case of Question A is the Levi problem in a product:

Question B

Let Y be a Stein space (even a smooth Stein curve) and M a projective algebraic manifold. Consider the canonical projection $\pi : M \times Y \to Y$ and let $W \subset M \times Y$ be an open subset such that the restriction of π to W is a Stein morphism. Does it follow that W is Stein ?

If $\dim(M) = 1$ and Y is a Stein manifold the answer to Question B is affirmative [Bru], [Mats].

Due to this remark, and taking also into account the fact (already mentioned) that the complement $D := X \setminus A$ (with X Stein, $\dim(X) = 3$, $A \subset X$ hypersurface) is Stein if D is assumed to be locally Stein, it is natural to make the following:

Conjecture 1

The Levi Problem (i.e. the Local Steiness Problem) holds if $\dim(X) = 3$.

Question B is also related to the following problem: let X be a Stein space with an isolated singularity and let Y be a Stein manifold. Denote $Z = X \times Y$

and let $U \subset Z$ be a locally Stein open subset. Does it follow that U is itself Stein ?

The main difficulty in the Levi problem in the singular case is the lack of a boundary distance d such that $-\log d$ is plurisubharmonic for Stein open subsets. If $U \subset X$ is locally Stein one can cover the boundary ∂D of D by Stein open sets V_i,$i \in \mathbb{N}$, such that $V_i \cap D$ is Stein for each $i \in \mathbb{N}$, therefore there exist plurisubharmonic exhaustion functions $\phi_i : V_i \cap D \to \mathbb{R}$. If D is locally hyperconvex (i.e. it admits locally, negative plurisubharmonic exhaustion functions) then it is possible (see e.g. [C-M3]) to achieve that the differences $\phi_i - \phi_j$ are bounded (by composing the given ϕ_i with suitable convex increasing functions), and consequently, by a simple patching technique, one gets a strongly plurisubharmonic exhaustion $\phi : D \to \mathbb{R}$ (which implies that D is Stein, by the results of H. Grauert [G 1] and R. Narasimhan [Nar]). If D is not locally hyperconvex, and it is assumed only locally Stein, it seems that it is not possible to get the plurisubharmonic local functions ϕ_i with bounded differences (if we compose them with non-convex increasing functions then, locally, their Levi form loses one positive eigenvalue, and in the patching process one gets only 2-completeness with corners, not even 2-completeness).

Remark 6.1. If $D \subset X$ is locally Stein and X is a Stein space, it follows by a bumping technique (see [C-M1]) that the cohomology groups $H^i(X, \mathcal{O})$ vanish if $i \geq 2$. If, additionally, it is assumed that $H^1(D, \mathcal{O}) = 0$ then one gets immedialtely, by using the Koszul complex, that D is Stein. Similarly it is known (see [Mar], [Sil]) that an arbitrary increasing union of Stein spaces $\{X_n\}_{n\in\mathbb{N}}$ is itself Stein if $H^1(X, \mathcal{O}) = 0$ (in fact it suffices to assume that $H^1(X, \mathcal{O})$ is separated.

Remark 6.2. By using a bumping argument one can easily see that another equivalent statement to L.S.P. (of Oka's glueing lemma type) is the following: Let X be a Stein space and $D \subset X$ an open subset. Assume that there exists $f \in \mathcal{O}(X)$ and real constants $a < b$ such that $D \cap \{Ref < b\}$ and $D \cap \{Ref > a\}$ are Stein. Does it follow that D is itself Stein ? (see also [A-N] concerning Oka's Heftungslemma).

Concerning the Union Problem we already mentioned [C-T 2] that for any Stein space X, with $dim(X) = 2$, and for any open subset $U \subset X$ such that $U = \bigcup_{n\in\mathbb{N}} U_n$, $U_n \subset U_{n+1}$, U_n is Stein for every $n \in \mathbb{N}$, if follows that U satisfies the discrete Kontinuitätssatz (the disk property). On the other hand any such increasing union $U \subset X$, with X Stein, normal, 2 dimensional, is a domain of holomorphy, therefore if the Grauert and Remmert problem [G-R], already mentioned, has a positive answer, then it would follow that U is Stein.

Therefore it is natural to make the following:

Conjecture 2
The Union Problem holds if $\dim(X) = 2$.

Let us make also some remarks concerning the Union Problem for Stein spaces with isolated singularities, which is an open question. Suppose that M is a projective algebraic manifold and $U = \bigcup_{n\in\mathbb{N}} U_n$ is an increasing union of Stein open sets. By considering a negative line bundle over M, and denoting X the Stein space (in fact affine algebraic) obtained by contractiong the null section to a point, we easily see, using some arguments involving $\mathbb{C}^*$ fibrations, that if the Union Problem has a positive answer for isolated singularities, then necessarily U is itself Stein (i. e. the Union Problem holds for projective algebraic manifolds). For example it would be interesting to consider the case when U is the complement of a divsor A and A is the limit (in the Hausorff sense) of a sequence of divisors A_n whose complements are Stein for every $n \in \mathbb{N}$. Does it follow that U is itself Stein ?

For other problems and discussions concerning the singular Levi problem the reader is advised to consult the papers of M. Colţoiu [Col 1], [Col 2], J. E. Fornaess and R. Narasimhan [F-N] and the survey of Y-T. Siu [Siu].

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