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A convection-diffusion elliptic system

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ABSTRACT: *We study a convection-diffusion elliptic system, with Dirichlet boundary conditions. In some cases, we will prove that we have more informations (with respect to the case of a single equation) about the summability of the solutions and of their gradients, thanks to the structure of our system.*

1 – Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N > 2$, and

$$f \in L^p(\Omega),\ p > \frac{rN}{2r+N-2}; \quad g \in L^q(\Omega),\ q > \frac{sN}{2s+N-2}; \tag{1}$$

$$r,\ s > \max\Big(\frac{N}{2} - 1, 1\Big). \tag{2}$$

Note that $p,\ q > \max(N/4, 1)$.

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Here we study the following nonlinear system

$$
\begin{cases}
-\Delta u = f(x) - a \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(u|z|^s) & \text{in } \Omega, \\
-\Delta z = g(x) - b \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(z|u|^r) & \text{in } \Omega, \\
u = z = 0 & \text{on } \partial\Omega
\end{cases}
\tag{3}
$$

where $a,\ b \in \mathbb{R}$.

In the case of a single linear equation like

$$
\begin{cases}
-\mathrm{div}(M(x)\nabla u - u\,E(x)) + B(x)\nabla u + \mu u = f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\tag{4}
$$

existence results can be found in [8], [2], [3]; nonlinear problems are studied in [6] (see also [1]) and the G-convergence is studied in [5].

The main difficulty in the study of our problem is due to noncoercivity of the differential operator. Our approach hinges on the Divergence Theorem and on the Dirichlet boundary condition.

In some cases, we will prove that we have more informations (with respect to the case of a single equation) about the summability of the solutions and of their gradients, thanks to the structure of our system.

We recall the following definition

$$
T_n(s) = \begin{cases} s, & \text{if } \ |s| \le n, \\ n\frac{s}{|s|}, & \text{if } \ |s| > n, \end{cases}
$$

we set $f_n(x) = T_n(f(x))$, $g_n(x) = T_n(g(x))$ and we consider the approximate problems

$$
\begin{cases}
-\Delta u_n = f_n(x) - a \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left[\frac{1}{n} + T_n(u_n)\right]\left[\frac{1}{n} + |T_n(z_n)|\right]^s\right) & \text{in } \Omega, \\
-\Delta z_n = g_n(x) - b \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left[\frac{1}{n} + T_n(z_n)\right]\left[\frac{1}{n} + |T_n(u_n)|\right]^r\right) & \text{in } \Omega, \\
u_n = z_n = 0 & \text{on } \partial\Omega
\end{cases}
\tag{5}
$$

Note that a weak solution u_n of (5) exists thanks to Schauder fixed point theorem. Moreover, since for every fixed n, the right hand sides belong to $W^{-1,\infty}(\Omega)$, every u_n, z_n is bounded (see [8]), so that we can use as test functions nonlinear compositions of u_n, z_n. Furthermore we need the term $\frac{1}{n}$ in the right hand sides in order to apply the Divergence Theorem; if $r,\ s \ge 2$ $\frac{1}{n}$ is not necessary (in (5) and in the choice of the test functions).

2 – Existence

Theorem 2.1. *Assume* (1), (2). *Then there exists a distributional solution* (u,z) *of the system* (3); *the function* u *belongs to* $W_0^{1,\frac{rN}{(r+1)(N-2)}}(\Omega)$, *the function* z *belongs to* $W_0^{1,\frac{sN}{(s+1)(N-2)}}(\Omega)$.

Proof. We use

$$\frac{r}{a}\,\frac{[\frac{1}{n}+|T_n(u_n)|]^{r-2}[\frac{1}{n}+T_n(u_n)]-(\frac{1}{n})^{r-1}}{r-1},$$

$$\frac{s}{b}\,\frac{[\frac{1}{n}+|T_n(z_n)|]^{s-2}[\frac{1}{n}+T_n(z_n)]-(\frac{1}{n})^{s-1}}{s-1}$$

as test functions in (5). We have

$$\left|\begin{array}{l}\displaystyle \frac{r}{a}\int_\Omega |\nabla T_n(u_n)|^2\left[\frac{1}{n}+|T_n(u_n)|\right]^{r-2}\le\\[2ex] \displaystyle c_r\|f\|_{L^p(\Omega)}\left[\int_\Omega\left[\frac{1}{n}+|T_n(u_n)|][\frac{1}{n}+|T_n(u_n)|\right]^{(r-1)p'}\right]^{\frac{1}{p'}}\\[2ex] \displaystyle +r\sum_{i=1}^N\int_\Omega\left[\frac{1}{n}+|T_n(u_n)|\right]^{r-2}\left[\frac{1}{n}+T_n(u_n)\right]\left[\frac{1}{n}+|T_n(z_n)|\right]^s\frac{\partial T_n(u_n)}{\partial x_i}.\end{array}\right.$$

and

$$\left|\begin{array}{l}\displaystyle \frac{s}{b}\int_\Omega |\nabla T_n(z_n)|^2\left[\frac{1}{n}+|T_n(z_n)|\right]^{s-2}\le\\[2ex] \displaystyle c_s\|g\|_{L^q(\Omega)}\left[\int_\Omega\left[\frac{1}{n}+|T_n(z_n)|\right]^{(s-1)q'}\right]^{\frac{1}{q'}}\\[2ex] \displaystyle +s\sum_{i=1}^N\int_\Omega\left[\frac{1}{n}+|T_n(z_n)|\right]^{s-2}\left[\frac{1}{n}+T_n(z_n)\right]\left[\frac{1}{n}+|T_n(u_n)|\right]^r\frac{\partial T_n(z_n)}{\partial x_i}.\end{array}\right.$$

We add the two inequalities. The Divergence Theorem implies that the sum of the last two terms is zero, since

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\frac{1}{n}+|T_n(u_n)|\right]^r \left[\frac{1}{n}+|T_n(z_n)|\right]^s$$
$$= r\sum_{i=1}^{N}\int_\Omega \left[\frac{1}{n}+|T_n(u_n)|\right]^{r-2}\left[\frac{1}{n}+T_n(u_n)\right]\left[\frac{1}{n}+|T_n(z_n)|\right]^s \frac{\partial T_n(u_n)}{\partial x_i}$$
$$+ s\sum_{i=1}^{N}\int_\Omega \left[\frac{1}{n}+|T_n(z_n)|\right]^{s-2}\left[\frac{1}{n}+T_n(z_n)\right]\left[\frac{1}{n}+|T_n(u_n)|\right]^r \frac{\partial T_n(z_n)}{\partial x_i}.$$

Then we have

$$\frac{r}{a}\int_\Omega |\nabla T_n(u_n)|^2\left[\frac{1}{n}+|T_n(u_n)|\right]^{r-2} + \frac{s}{b}\int_\Omega |\nabla T_n(z_n)|^2\left[\frac{1}{n}+|T_n(z_n)|\right]^{s-2}$$
$$\leq c_r\|f\|_{L^p(\Omega)}\left[\int_\Omega\left[\frac{1}{n}+|T_n(u_n)|\right]^{(r-1)p'}\right]^{\frac{1}{p'}}$$
$$+c_s\|g\|_{L^q(\Omega)}\left[\int_\Omega\left[\frac{1}{n}+|T_n(z_n)|\right]^{(s-1)q'}\right]^{\frac{1}{q'}}$$

Thanks to Sobolev inequality and to an elementary inequality on the left hand side and to Hölder inequality on right hand side, since $\frac{\frac{rN}{N-2}}{(r-1)p'} > 1$ and $\frac{\frac{sN}{N-2}}{(s-1)q'} > 1$ (as a consequence of (1)), we can deduce again that the sequences $\{T_n(u_n)\}$, $\{T_n(z_n)\}$ are bounded in $L^{\frac{rN}{N-2}}(\Omega)$, $L^{\frac{sN}{N-2}}(\Omega)$. Then in the two right hand sides of (5), $T_n(u_n)|T_n(z_n)|^s$ is bounded in $L^\gamma(\Omega)$ with $\gamma = \frac{rN}{(r+1)(N-2)}$ ($\gamma > 1$, if $r > \frac{N-2}{2}$) and $T_n(z_n)|T_n(u_n)|^r$ is bounded in $L^\mu(\Omega)$ with $\mu = \frac{sN}{(s+1)(N-2)}$ ($\mu > 1$, if $s > \frac{N-2}{2}$). Thus, with the Calderon-Zygmund theory, we conclude that the sequences

$$\{u_n\},\ \{z_n\}\ \text{ are bounded in } W_0^{1,\frac{rN}{(r+1)(N-2)}}(\Omega),\ W_0^{1,\frac{sN}{(s+1)(N-2)}}(\Omega).$$

Then there exist two functions u, z and two subsequences $\{u_{n_k}\}$, $\{z_{n_k}\}$ such that u_{n_k} converges weakly in $W_0^{1,\frac{rN}{(r+1)(N-2)}}(\Omega)$ and a.e. to u, z_{n_k} converges weakly in $W_0^{1,\frac{sN}{(s+1)(N-2)}}(\Omega)$ and a.e. to z. Of course $T_{n_k}(u_{n_k})(x)$ converges a.e. to $u(x)$ and $T_{n_k}(z_{n_k})(x)$ converges a.e. to $z(x)$, which implies, thanks to the estimates in $L^\gamma(\Omega)$ and $L^\mu(\Omega)$, that $T_{n_k}(u_{n_k})(x)$ converges weakly to $u(x)$ in $L^\gamma(\Omega)$ and $T_{n_k}(z_{n_k})(x)$ converges weakly to $z(x)$ in $L^\mu(\Omega)$.

Then it is possible to pass to the limit in (5) and we prove the existence of $u \in W_0^{1,\frac{rN}{(r+1)(N-2)}}(\Omega)$ and $z \in W_0^{1,\frac{sN}{(s+1)(N-2)}}(\Omega)$ solutions of (3). ∎

Remark 2.2 Note that the solutions belong to $W_0^{1,2}(\Omega)$ only if $N = 3$.

Remark 2.3 If in (3) we consider differential operators whose principal part is $-\mathrm{div}(M(x)\nabla v)$ and $-\mathrm{div}(N(x)\nabla w)$ instead of $-\Delta v$ and $-\Delta w$, it is possible to repeat the proof of Theorem 2.1 if the matrices M and N are elliptic and sufficiently regular so that it is possible to use either the classical Calderon-Zygmund theory or the results recently proved by Haïm Brezis (see [7]).

Moreover if the matrices M and N are elliptic and bounded, we can repeat the proof of Theorem 2.1, but we cannot use the Calderon-Zygmund theory. Then we can prove the existence of solutions u, z in $W_0^{1,2}(\Omega)$, if γ, $\mu \geq 2$, which is possible only under the assumptions r, $s \geq 2$ and $N = 3$.

3 – More summability

Lemma 2.1 *Assume that the sequence $\{w_n\}$ converges weakly to w in $W_0^{1,\gamma}(\Omega)$ and that $\|T_n(w_n)\|_{W_0^{1,\lambda}(\Omega)} \leq R$ with $1 < \gamma < \lambda$. Then w belongs to $W_0^{1,\lambda}(\Omega)$.*

Proof. Thanks to the properties of the truncation, for every $k > 0$, $T_k(w_n)$ converges weakly to $T_k(w)$ in $W_0^{1,\gamma}(\Omega)$. Moreover, for every $k > 0$, we have

$$\int_\Omega |\nabla T_k(w_n)|^\lambda \leq \int_\Omega |\nabla T_n(w_n)|^\lambda \leq R, \quad n > k.$$

Thus $T_k(w_n)$ converges weakly to $T_k(w)$ also in $W_0^{1,\lambda}(\Omega)$ and we have

$$\int_\Omega |\nabla T_k(w)|^\lambda \leq R.$$

Then the Fatou Lemma (as $k \to \infty$) implies that

$$\int_\Omega |\nabla w|^\lambda \leq R.$$

∎

Theorem 2.5. *If $2(N-2) > r > \max\left(\frac{N}{2} - 1, 2\right)$, the solution u belongs not only to $W_0^{1,\frac{rN}{(r+1)(N-2)}}(\Omega)$, but also to $W_0^{1,\frac{rN}{r(N-1)-(N-2)}}(\Omega)$. If $2(N-2) > s > \max\left(\frac{N}{2} - 1, 2\right)$, the solution z belongs not only to $W_0^{1,\frac{sN}{(s+1)(N-2)}}(\Omega)$, but also to $W_0^{1,\frac{sN}{s(N-1)-(N-2)}}(\Omega)$.*

Proof. Let $r \geq 2$. Define $\frac{rN}{r(N-1)-(N-2)} = \alpha$ and note that $1 < \alpha < 2$, if $r \geq 2$. Inequality (6), Hölder inequality and $2/\alpha > 1$ imply that

$$\begin{aligned}
&\int_\Omega |\nabla T_n(u_n)|^\alpha \\
&= \int_\Omega \frac{|\nabla T_n(u_n)|^\alpha}{\left[\frac{1}{n} + |T_n(u_n)|\right]^{\frac{\alpha(r-2)}{2}}} \left[\frac{1}{n} + |T_n(u_n)|\right]^{\frac{\alpha(r-2)}{2}} \\
&\leq C_1 \left[\int_\Omega \left(\frac{1}{n} + |T_n(u_n)|\right)^{\frac{\alpha(r-2)}{2-\alpha}}\right]^{\frac{2-\alpha}{\alpha}}
\end{aligned}$$

Since $\frac{rN}{N-2} = \frac{\alpha(r-2)}{2-\alpha}$, the right hand side is bounded, as we proved in Theorem 2.1. Remark that $\alpha > \frac{rN}{(r+1)(N-2)}$, if $2(N-2) > r$, so that the proof is complete using Lemma 2.4. ∎

Theorem 2.6. *If* $2 > r > \max\left(\frac{N}{2} - 1, 1\right)$, *then* $u \in W_0^{1,\frac{Nr}{N-2+r}}(\Omega)$. *If* $2 > s > \max\left(\frac{N}{2} - 1, 1\right)$, *then* $z \in W_0^{1,\frac{Ns}{N-2+s}}(\Omega)$.

Proof. If $1 < r < 2$, then (6) implies that

$$\int_\Omega \frac{|\nabla T_n(u_n)|^2}{\left(\frac{1}{n} + |T_n(u_n)|\right)^{2-r}} \leq C_{r,s,f,g}.$$

Now we follow [4]. Let $\mu = \frac{Nr}{N-2+r}$ (note that $1 < \mu < 2$), then $\mu^* = \frac{Nr}{N-2} = \frac{(2-r)\mu}{2-\mu}$. We have

$$\int_\Omega \frac{|\nabla T_n(u_n)|^\mu}{(\frac{1}{n} + |T_n(u_n)|)^{\frac{(2-r)\mu}{2}}} \left(\frac{1}{n} + |T_n(u_n)|\right)^{\frac{(2-r)\mu}{2}}$$

$$\leq C_{r,s,f,g}^{\frac{\mu}{2}} \left[\int_\Omega \left(\frac{1}{n} + |T_n(u_n)|\right)^{\frac{(2-r)\mu}{2-\mu}}\right]^{1-\frac{\mu}{2}} = C_{r,s,f,g}^{\frac{\mu}{2}} \left[\int_\Omega \left(\frac{1}{n} + |T_n(u_n)|\right)^{\mu^*}\right]^{1-\frac{\mu}{2}}.$$

So that

$$\int_\Omega |\nabla T_n(u_n)|^{\frac{Nr}{N-2+r}} \leq C_{r,s,f,g}.$$

Note that $\frac{rN}{(r+1)(N-2)} \leq \frac{Nr}{N-2+r}$ if $N \geq 3$, so that the proof is complete using Lemma 2.4. ∎

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