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Characterizing Geometric Designs

To Marialuisa J. de Resmini on the occasion of her retirement

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Abstract: *We conjecture that the classical geometric 2-designs $PG_d(n, q)$, where $2 \leq d \leq n-1$, are characterized among all designs with the same parameters as those having line size $q+1$. The conjecture is known to hold for the case $d = n-1$ (the Dembowski-Wagner theorem) and also for $d = 2$ (a recent result established by Tonchev and the present author). Here we extend this result to the cases $d = 3$ and $d = 4$. The general case remains open and seems to be difficult.*

1 – Introduction

In this note, we are concerned with the problem of characterizing the classical geometric designs $PG_d(n, q)$, where d is in the range $2 \leq d \leq n-2$, among all designs with the same parameters. For the convenience of the reader, we first recall basic facts about these designs. Let Π denote $PG(n, q)$, the n-dimensional projective space over the field $GF(q)$ with q elements. Then the points and d-spaces of Π form a 2-(v, k, λ) design $\mathcal{D} = PG_d(n, q)$ with parameters

$$v = \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q = (q^{n+1}-1)/(q-1),$$

$$k = \begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q = (q^{d+1}-1)/(q-1),$$

$$r = \begin{bmatrix} n \\ d \end{bmatrix}_q, \lambda = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q \text{ and } b = \begin{bmatrix} n+1 \\ d+1 \end{bmatrix}_q,$$

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where $\begin{bmatrix} n \\ i \end{bmatrix}_q$ denotes the number of i-dimensional subspaces of an n-dimensional vector space over $GF(q)$. These so-called Gaussian coefficients are given explicitly as

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-i+1}-1)}{(q^i-1)(q^{i-1}-1)\cdots(q-1)}.$$

Furthermore, the lines of the design $\mathcal{D}$ are just the lines of Π; in particular, all lines of $\mathcal{D}$ have cardinality $q+1$.(1) All these facts are well-known.

The classical designs are far from being characterized by their parameters. This is well-known for the case $d = n-1$: the number of symmetric 2-designs with the parameters of a classical point-hyperplane design $PG_{n-1}(n,q)$ grows exponentially, and a similar result also holds for affine 2-designs with the parameters of a classical point-hyperplane design $AG_{n-1}(n,q)$. These results were originally established by the author [5] in 1984, whose bounds were subsequently improved in several papers [7, 8, 9]. In a recent paper, the author and Tonchev established the corresponding result for the number of 2-designs with the parameters of $PG_d(n,q)$, where d is in the range $2 \le d \le n-2$.

This naturally poses the problem of characterizing the classical geometric designs $PG_d(n,q)$ among all designs with the same parameters. Again, the case $d = n-1$ has been settled for a long time: Dembowski and Wagner obtained several elegant characterizations in a celebrated paper [2] which appeared in 1960; see also [1] for a proof. One of their results characterizes the designs $PG_{n-1}(n,q)$ by their line size, namely $q+1$, and an analogous result was recently established by Tonchev and the present author [6] for the case $d = 2$.

In contrast, not that much is known for the cases $3 \le d \le n-2$. The only result I am aware of is due to Lefèvre-Percsy [10], who proved that a smooth(2) design with the parameters of $PG_d(n,q)$, where $d \ge 2$ and $q \ge 4$, but q not necessarily a prime power, is classical if and only if all lines have size at least $q+1$.

Unfortunately, Lefèvre-Percsy's hypothesis that the design should be smooth is a very severe restriction; moreover, the assumption $q \ge 4$ seems somewhat unnatural. Therefore the problem of finding a nicer characterization remains open. In this direction, I offer the following

(1) Recall that the *line* determined by two points of a design is defined as the intersection of all blocks containing these two points. See [1] for background on designs, and [3, 4] for background on finite projective spaces.

(2) Recall that the *plane* determined by three non-collinear points of a design is defined as the intersection of all blocks containing the three given points. In general, planes may be properly contained in other planes. This undesirable phenomenon is excluded if one requires the design to be *smooth*, that is, if one assumes that any three non-collinear points are contained in a constant number of blocks, which is then usually denoted by ϱ. See [1] for details.

Conjecture 1.1. A design with the parameters of $PG_d(n,q)$, where $2 \leq d \leq n-1$ and where $q \geq 2$ is not necessarily a prime power, is classical (so that q is actually a prime power) if and only if all lines have size $q+1$.

As mentioned before, this conjecture is already known to hold for the case $d = n-1$ (the Dembowski-Wagner theorem) and also for $d = 2$ (by the recent result of [6]). In the present note, I will establish the validity of Conjecture 1.1 for the cases $d = 3$ and $d = 4$. For the convenience of the reader, I shall also repeat the simple proof for the case $d = 2$ as a warm-up. It will become apparent that the problem gets more and more involved as d grows, so that a general proof will most likely require some major new idea. Even the next open case $d = 5$ seems to be rather challenging.

My proofs will repeatedly appeal to a simple, but extremely useful result concerning subspaces of linear spaces. Recall that a *linear space* Σ is just a pairwise balanced design with joining number $\lambda = 1$; therefore one speaks of *lines* instead of *blocks* in this context. A *subspace* of Σ is a subset S of the point set with the property that each line intersecting S in at least two points is entirely contained in S; thus the lines of Σ induce a linear space on S. The result alluded to gives bounds on the cardinality of a proper subspace, see [1, I.8.4]. As we shall only require the case where Σ has constant line size k (so that Σ is actually a 2-design), we merely state this special case:

Lemma 1.2. *Let S be a proper subspace of a $2-(v,k,1)$-design Σ. Then the cardinality of S satisfies the bound $|S| \leq (v-1)/(k-1)$.*

Finally, the subspace *spanned* by a subset U of the point set of a linear space Σ is, of course, just the smallest subspace S of Σ containing U.

2 – The cases $d = 2$ and $d = 3$

We begin by recalling the case $d = 2$ from [6]:

Theorem 2.1. *Let $\mathcal{D}'$ be a 2-design with the same parameters as the classical design $\mathcal{D} = PG_2(n,q)$, where $n \geq 3$ and where $q \geq 2$ is not necessarily a prime power. Then $\mathcal{D}'$ is isomorphic to the classical design if and only if all lines of $\mathcal{D}'$ have size $q+1$.*

Proof. The condition that all lines have size $q+1$ is trivially necessary. Thus assume that this condition is satisfied. Note that all blocks of $\mathcal{D}'$ have cardinality $k = q^2+q+1$ in this case. Consider an arbitrary block B. Any two points of B define a unique line of $\mathcal{D}'$ which, by our hypothesis, has size $q+1$. Thus the lines of B induce a $(q^2+q+1, q+1, 1)$-design on B, and hence every block of $\mathcal{D}'$ carries the structure of a projective plane of order q.

We next claim that an arbitrary line ℓ of $\mathcal{D}'$ and an arbitrary point $p \notin \ell$ determine a unique block of $\mathcal{D}'$; in other words, the blocks of $\mathcal{D}'$ containing ℓ partition the points not in ℓ. Note first that no two such blocks can intersect outside ℓ, since each block is a projective plane and since a line of a plane together with a point outside spans the entire plane. Now it suffices to count: there are $q^n + ... + q^2$ points outside ℓ and there are $q^{n-2} + \ldots + q + 1$ blocks containing ℓ, each of which has q^2 points not in ℓ.

It is now easily seen that the points and lines of $\mathcal{D}'$ satisfy the Veblen-Young axioms and therefore define a projective space Π; see, for instance, [1, Section XII.1]. In view of the parameters of $\mathcal{D}'$, we have $\Pi \cong PG(n, q)$, and thus $\mathcal{D}' \cong \mathcal{D}$. ∎

As we shall see, the case $d = 3$ is already more involved:

THEOREM 2.2. *Let $\mathcal{D}'$ be a 2-design with the same parameters as the classical design $\mathcal{D} = PG_3(n, q)$, where $n \geq 4$ and where $q \geq 2$ is not necessarily a prime power. Then $\mathcal{D}'$ is isomorphic to the classical design if and only if all lines of $\mathcal{D}'$ have size $q + 1$.*

PROOF. The condition that all lines have size $q + 1$ is trivially necessary. Thus assume that this condition is satisfied. Note that all blocks of $\mathcal{D}'$ have cardinality $k = q^3 + q^2 + q + 1$ in this case. Consider an arbitrary block B. Any two points of B define a unique line of $\mathcal{D}'$ which, by our hypothesis, has size $q+1$. Thus the lines of B induce a linear space Σ_B with constant line size $q+1$ on B. We want to show that an arbitrary line ℓ of $\mathcal{D}'$ and an arbitrary point $p \notin \ell$ again span a projective plane of order q, as in the case $d = 2$; this will require more work than before.

Step 1. Let ℓ be a line, B a block through ℓ, and $p \notin \ell$ a point of B. Then the subspace S of Σ_B spanned by p and ℓ is either a projective plane of order q or equal to B. In the latter case, B is the only block containing ℓ and p.

As each of the $q + 1$ points p' of ℓ determines together with p a line pp' of size $q+1$, it is clear that S has at least $q(q+1)+1$ points; in case of equality, S is obviously a projective plane of order q. But by Lemma 1.2, a proper subspace of Σ_B contains at most

$$\frac{v-1}{k-1} = \frac{q^3 + q^2 + q}{q} = q^2 + q + 1$$

points, and therefore either S is a proper subspace of cardinality $q^2 + q + 1$ of Σ_B, or $S = B$. As any two blocks intersect in a proper subspace (of either of these blocks), the assertion follows.

Step 2. Let ℓ be a line and $p \notin \ell$ a point of $\mathcal{D}'$. Then there are exactly

$$\varrho = q^{n-3} + \ldots + q + 1$$

blocks of $\mathcal{D}'$ containing both p and ℓ.[3]

We first fix the line ℓ and determine the average number of blocks containing ℓ and a point $p \notin \ell$. Now ℓ is on

$$\lambda = \frac{(q^{n-1}-1)(q^{n-2}-1)}{(q^2-1)(q-1)}$$

blocks, each of which contains q^3+q^2 further points. As there are altogether $q^n+q^{n-1}+\ldots+q^2$ points not in ℓ, a short computation shows that the desired average number is precisely the quantity ϱ defined in the assertion. Hence it suffices to check that ϱ is also an upper bound for the number ϱ_p of blocks containing ℓ and some given point $p \notin \ell$. Obviously, we may assume $\varrho_p \geq 2$ for this purpose. Then the ϱ_p blocks through p and ℓ intersect in a common subspace S of cardinality q^2+q+1, by Step 1. Moreover, no two of these blocks can share a point not in S, by Lemma 1.2. As there are exactly $q^n+\ldots+q^4+q^3$ points $p' \notin S$, we obtain indeed

$$\varrho_p \leq \frac{q^n+\ldots+q^4+q^3}{q^3} = \varrho.$$

Step 3. Let Σ denote the linear space induced by the lines of $\mathcal{D}'$. Then the subspace spanned by any three non-collinear points of $\mathcal{D}'$ is a projective plane of order q.

This follows immediately by combining Steps 1 and 2.

Step 4. The linear space Σ is isomorphic to $PG_1(n,q)$.

Using Step 3, one easily checks that the points and lines of $\mathcal{D}'$ satisfy the Veblen-Young axioms and therefore define a projective space Π; see, for instance, [1, section XII.1]. In view of the parameters of $\mathcal{D}'$, we have $\Sigma \cong PG_1(n,q)$.

Step 5. $\mathcal{D}'$ is isomorphic to $PG_3(n,q)$.

After Step 4, it still remains to show that the blocks of $\mathcal{D}'$ are actually the 3-subspaces of Π. Let us first note that the subspaces of cardinality q^2+q+1 of Σ are just the planes of Π. This is clear, as any given line ℓ and any point $p \notin \ell$ determine the same projective plane of order q in both structures, namely the union of the $q+1$ lines pp' where p' runs over the points of ℓ; see Step 3. By Step 2 and by the counting argument used there, any given plane S is in exactly ϱ blocks, which give rise to a partition of the points not in S. Hence S and any such point p determine a unique block, namely the union of the q^2+q+1 lines pp' where p' runs over the points of S. But this is also the point set of the 3-subspace of Π determined by S and p. □

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(3) This will establish that $\mathcal{D}'$ is smooth, and hence we could then, for $q \geq 4$, appeal to the result of [10]. Of course, this would leave the cases $q=2$ and $q=3$ unresolved.

3 – The case $d = 4$

We now settle the case $d = 4$ of Conjecture 1.1 , using similar arguments as for the case $d = 3$; as already mentioned, this turns out to be quite involved.

THEOREM 3.1. *Let $\mathcal{D}'$ be a 2-design with the same parameters as the classical design $\mathcal{D} = PG_4(n, q)$, where $n \geq 5$ and where $q \geq 2$ is not necessarily a prime power. Then $\mathcal{D}'$ is isomorphic to the classical design if and only if all lines of $\mathcal{D}'$ have size $q + 1$.*

PROOF. The condition that all lines have size $q + 1$ is trivially necessary. Thus assume that this condition is satisfied. Note that all blocks of $\mathcal{D}'$ have cardinality $k = q^4 + q^3 + q^2 + q + 1$ in this case. Consider an arbitrary block B. Any two points of B define a unique line of $\mathcal{D}'$ which, by our hypothesis, has size $q + 1$. Thus the lines of B induce a linear space Σ_B with constant line size $q + 1$ on B. Again, we need to show that an arbitrary line ℓ of $\mathcal{D}'$ and an arbitrary point $p \notin \ell$ span a projective plane of order q, as in the cases $d = 2$ and $d = 3$; this will require considerably more work than before.

As before, let Σ denote the linear space induced by the lines of $\mathcal{D}'$ on the point set V of $\mathcal{D}'$. For any subset X of V, we shall denote the subspace of Σ spanned by X as $S(X)$. For any line ℓ and any point $p \notin \ell$, we put $S(p, \ell) := S(\{p\} \cup \ell)$ and write $\varrho(p, L)$ for the number of blocks containing both p and ℓ, and hence all of $S(p, \ell)$. As in the proof of Theorem 2.2, one easily obtains

Step 1. $S(p, \ell) \geq q^2 + q + 1$ for each line ℓ and each point $p \notin \ell$.

But now we get a further possibility for the precise structure of $S(p, \ell)$:

Step 2. Let ℓ be a line, B a block through ℓ, and $p \notin \ell$ a point of B. Then $S(p, \ell)$ is either a projective plane of order q, or a proper maximal subspace of Σ_B, or equal to B. In the latter case, B is the only block containing both ℓ and p.

To see this, note that a proper subspace of Σ_B contains at most $q^3 + q^2 + q + 1$ points, by Lemma 1.2. If $S(p, \ell)$ is not a projective plane of order q, it has at least $q^2 + q + 2$ points, by Step 1.(4) Using Lemma 1.2 again, any subspace properly containg S then has to have cardinality at least $q^3 + q^2 + 2q + 1$, and hence has to be equal to B.

Step 3. Let ℓ be a line. Then the average number of blocks containing ℓ and a point $p \notin \ell$ is given by

$$\varrho = \frac{(q^{n-2} - 1)(q^{n-3} - 1)}{(q^2 - 1)(q - 1)}.$$

(4) Actually it would be easy to obtain a stronger bound, namely $2q^2 + 2q + 2$, but the weak version given above will already suffice. However, it seems not possible to obtain a precise cardinality for this – as we shall see, anyway entirely hypothetical – case.

The line ℓ is on

$$\lambda = \frac{(q^{n-1}-1)(q^{n-2}-1)(q^{n-3}-1)}{(q^3-1)(q^2-1)(q-1)}$$

blocks, each of which contains $q^4+q^3+q^2$ further points. As there are altogether $q^n+q^{n-1}+\ldots+q^2$ points not in ℓ, we get

$$\varrho = \frac{(q^{n-1}-1)(q^{n-2}-1)(q^{n-3}-1)q^2(q^2+q+1)}{(q^3-1)(q^2-1)(q-1)q^2(q^{n-2}+\ldots+q+1)},$$

and a short computation gives the desired result for ϱ.

Step 4. Let ℓ be a line, and p a point not on ℓ. Then $S(p,\ell)$ is a projective plane of order q, provided that $\varrho(p,\ell) \geq q^{n-4}+\ldots+q^2+q+2$.

Assume otherwise. Then Step 2 shows that $S := S(p,\ell)$ is a proper maximal subspace of Σ_B for every block B containing both p and ℓ, so that any two distinct blocks B and B' containing both p and ℓ intersect precisely in S. Moreover, as we have seen in the proof of Step 2,

$$q^2+q+2 \leq |S(p,\ell)| \leq q^3+q^2+q+1.$$

Hence there are at most $q^n+\ldots+q^4+q^3-1$ points not in S, and each of these points is on at most one block containing S. Also, each such block contains at least q^4 such points. Hence we get

$$\varrho(p,\ell) \leq \frac{q^n+\ldots+q^4+q^3-1}{q^4} < q^{n-4}+\ldots+q^2+q+2,$$

contradicting the hypothesis.

Step 5. Let ℓ be a line, p a point not on ℓ, and assume that $S := S(p,\ell)$ is a projective plane of order q. Given any point $p' \notin S$, let us put $S' = S'(p') := S(\ell \cup \{p,p'\})$. Then either S' is contained in at most one block, or $|S'| = q^3+q^2+q+1$. Moreover, there are at most $\tau := q^{n-3}+\ldots+q+1$ subspaces of cardinality q^3+q^2+q+1 containing S, and equality holds if and only if S and any point $p' \notin S$ always span a subspace of cardinality q^3+q^2+q+1.

Note that S is a proper subspace of S', and therefore, by Lemma 1.2, $|S'| \geq q^3+q^2+q+1$. If S' is contained in two distinct blocks, it is a proper subspace of both of them, and another application of Lemma 1.2 gives the desired equality. Now the second assertion is easily seen, as there are exactly $q^n+\ldots+q^4+q^3$ choices for p'.

Step 6. Let S' be any subspace of Σ of cardinality q^3+q^2+q+1. Then there are at most $\sigma := q^{n-4}+\ldots+q+1$ blocks containing S', and equality holds if and only if all such blocks give rise to a partition of the set V' of all points of Σ not contained in S'.

We may assume that there are two distinct blocks containing S', so that S' is a proper maximal subspace of every block containing it. Then no two such blocks can intersect in a point of V'. Hence we indeed get at most

$$\frac{q^n+\ldots+q^5+q^4}{q^4} = \sigma$$

blocks through S', and clearly equality holds iff these blocks partition V'.

Step 7. Let ℓ be a line, p a point not on ℓ, and assume $\varrho(p,\ell) \geq \varrho$. Then $S := S(p,\ell)$ is a projective plane of order q, and one actually has $\varrho(p,\ell) = \varrho$. Moreover, there are exactly τ subspaces S' of cardinality q^3+q^2+q+1 containing S, each of which lies in precisely σ blocks, and the blocks containing a given S' give rise to a partition of the set of points of Σ not contained in S'.

By Step 4, S is a projective plane of order q. Let us write $\varrho(p,\ell) = \varrho+\epsilon$, where $\epsilon \geq 0$, and let us denote the cardinality of the set X of all points $p' \notin S$ which are on at most one block B containing S by x. We now count the number f of all flags (p', B) with $p' \notin S \subset B$ in two ways. As each block B through S contains exactly q^4+q^3 points $p' \notin S$, we get

$$(1)\quad f = (\varrho+\epsilon)(q^4+q^3) = (q^{n-3}+\ldots+q+1)(q^{n-4}+\ldots+q+1)q^3+\epsilon(q^4+q^3).$$

On the other hand, counting via the points $p' \notin S$ first, we also obtain

$$(2)\qquad f \leq x+(q^n+\ldots+q^4+q^3-x)\sigma,$$

as each point in X is on at most one block B containing S, whereas each of the $q^n+\ldots+q^4+q^3-x$ points $p' \notin S \cup X$ determines a subspace $S' = S'(p')$ of cardinality q^3+q^2+q+1 by Step 5, which is then on at most σ blocks B through S by Step 6. Therefore

$$\begin{aligned}(q^{n-4}+\ldots+q+1)((q^n+\ldots+q^4+q^3)-(q^n+\ldots+q^4+q^3-x))\\ \leq x-\epsilon(q^4+q^3) \leq x,\end{aligned}$$

forcing $x = \epsilon = 0$. Thus indeed $\varrho(p,\ell) = \varrho$, and we have also proved that each point $p' \notin S$ is on at least two blocks B through S. Now Step 5 shows that there are exactly τ subspaces of cardinality q^3+q^2+q+1 containing S, and that these subspaces give rise to a partition of the set of all points $p' \notin S$. As $\epsilon = 0$, equation (1) above becomes

$$f = (q^n+\ldots+q^4+q^3)\sigma,$$

and therefore (using $x = 0$) the inequality (2) has to hold with equality, which is only possible if each of the subspaces $S' = S'(p')$ of cardinality $q^3 + q^2 + q + 1$ determined by the points not in S is actually on exactly σ blocks B containing S. Now the final assertion follows from Step 6.

Step 8. Let Σ denote the linear space induced by the lines of $\mathcal{D}'$. Then the subspace spanned by any three non-collinear points of $\mathcal{D}'$ is a projective plane of order q. Moreover, any four non-planar points determine a subspace of cardinality $q^3 + q^2 + q + 1$.

By Step 3, the average number of blocks containing three non-collinear points is ϱ. By Step 7, this is also an upper bound for the number of blocks containing any three given non-collinear points, and hence any three such points always lie in exactly ϱ common blocks. Hence the conclusions of Step 7 hold for any three non-collinear points, and then an appeal to Step 5 establishes also the second assertion.

Step 9. The linear space Σ is isomorphic to $PG_1(n, q)$.

Using the first assertion of Step 8, one easily checks that the points and lines of $\mathcal{D}'$ satisfy the Veblen-Young axioms and therefore define a projective space Π; see, for instance, [1, Section XII.1]. In view of the parameters of $\mathcal{D}'$, we have $\Sigma \cong PG_1(n, q)$.

Step 10. $\mathcal{D}'$ is isomorphic to $PG_4(n, q)$.

It still remains to show that the blocks of $\mathcal{D}'$ are actually the 4-subspaces of Π. Again, we first note that the subspaces of cardinality $q^2 + q + 1$ of Σ are simply the planes of Π; this follows as in the proof of Theorem 2.2.

By Step 8, any given plane S together with any point $p' \notin S$ determines a unique subspace S' of cardinality $q^3 + q^2 + q + 1$, which has to be the union of the $q^2 + q + 1$ lines $p's$ where s runs over the points of S. But this is also the point set of the 3-subspace of Π determined by S and p'. Therefore the subspaces of cardinality $q^3 + q^2 + q + 1$ of Σ are precisely the 3-spaces of Π.

Finally, by Steps 6 and 7, the blocks containing any given 3-subspace S' partition the points not in S'. Hence S' and any such point p'' determine a unique block of $\mathcal{D}'$, namely the union of the $q^3 + q^2 + q + 1$ lines $s'p''$ where s' runs over the points of S'. But this is also the point set of the 4-subspace of Π determined by S' and p''. $\square$

4 – Conclusion

We have seen that the correct line size $q + 1$ characterizes any design with the parameters of $PG_d(n, q)$ as this classical geometric design, provided that $d \in \{2, 3, 4, n-1\}$. These results provide considerable evidence for the validity of

Conjecture 1.1 . The key step in our proofs was establishing that any three non-collinear points always determine a projective plane of order q, or, in other words, that the given design is smooth (which was a hypothesis in the characterization result of Lefèvre-Percsy [10]). Unfortunately, this key step becomes more and more involved as the dimension d grows, the reason being that the number of a priori possibilities for the subspace spanned by a line ℓ and a point p not in this line grows with d. For the first open case, namely $d = 5$, we would for the first time have to consider the possibility that $S(p, \ell)$ is neither a projective plane of order q, nor a proper maximal subspace of a block B, nor B itself, but some proper, non-maximal subspace of Σ_B. This suggests that even this case will be a lot more complex than the case $d = 4$, which was already rather involved. Thus, settling Conjecture 1.1 in general seems a challenging problem which will most likely require some additional ideas.

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