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Some remarks on Calabi-Yau manifolds

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Dedicated to Professor Marialuisa de Resmini

Abstract: *Here we focus on the geometry of the "mirror quintic" Y and its generalizations. In particular, we illustrate how to obtain new birational models of Y.*

1 – Introduction

Let X be a complex, compact, connected Kähler manifold. X is said to be a Calabi-Yau variety if i) the canonical bundle is trivial and ii) there are no p-holomorphic forms for $p \neq 0, n$, where n is the complex dimension of X. i) implies that there is a unique (up to scalars) global top degree holomorphic form and ii) can be rephrased in terms of Hodge numbers, that is to say, $h^{p,0} \neq 0$ for p in the range above. We remark that $h^{0,0} = 1$ because X is connected and $h^{n,0} = 1$ because the canonical bundle $K_X = \Omega^n_X$ is trivial.

For applications in Mathematics and Physics it is important to give a definition of singular Calabi-Yau varieties. These are normal compact manifolds with Gorenstein canonical singularities such that the dualizing sheaf is trivial and the Hodge numbers $h^{p,0} \neq 0$ for $p \neq 0, n$. In most of the applications we shall deal with, X will be a global quotient, i.e., a smooth variety with an action of a finite group $G \subset SL(n, \mathbb{C})$.

It is easy to give examples of smooth Calabi-Yau manifold in low dimension. Elliptic curves and K3 surfaces are the only examples of Calabi-Yau manifolds

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in dimension one and two, respectively. Noticeably, in these cases the condition of being Calabi-Yau uniquely determines the structure of the Hodge diamond. This is no longer true for higher dimensional examples.

We start our talk by going over an intriguing example: a family of quintic threefolds in $\mathbb{P}^4$. This family was introduced by Dwork in the sixties, and has been extensively studied in connection with Number Theory [10] and Physics (see, for instance, [5]). Clearly, a smooth quintic in $\mathbb{P}^4$ is Calabi-Yau by adjunction and the Lefschetz Theorem. Hence, the generic member of the Dwork pencil is a Calabi-Yau manifold. Further, the five singular members are singular Calabi-Yau manifolds according to the definition recalled above.

A group $G \cong (\mathbb{Z}/125\mathbb{Z})^3$ acts on the Dwork pencil X_t. Generically, the quotient has a smooth resolution Y_t, which is a Calabi-Yau manifold. There is a strange duality - first pointed out in [7] - among the Hodge numbers of X_t and those of Y_t for generic t. More specifically, X_t and Y_t are said to be *mirror symmetric.*

Given a family of Calabi-Yau manifolds $\mathcal{F}_t$, it is natural to ask whether $\mathcal{F}_t$ is birational to Y_t or not. In [2] we answer this question for six families. Some of them are birational to Y_t modulo a finite group. One of them is exactly the family investigated in [8].

We finally remark that the Dwork pencil X_t^{n+1} can be generalized to any degree. We investigate its properties in [3]. Here we show how the geometry of X_t^{n+1} can be intricate by describing a special subvariety that exists in even dimensional projective space.

2 – The mirror quintic

Let $X_t \to \mathbb{P}^1$ be the Dwork pencil, where

$$X_t := \left\{x_1^5 + \ldots + x_5^5 - 5tx_1 \ldots x_5 = 0\right\}. \tag{1}$$

It is easy to check that for $t^5 \neq 1$, the fiber of the Dwork pencil is a smooth Calabi-Yau manifold. For $t = \infty$ the fiber is a union of hyperplanes.

Proposition 2.1. *For $t^5 = 1$ X_t is a singular Calabi-Yau.*

Proof. First, notice that the singularities are normal because the singular set has codimension more than one: see [15], p. 76. Moreover, they are Gorenstein by [13], p. 314. Furthermore, an ordinary double point is canonical: see, for instance, [11]. Finally, it is an exercise to show that $h^{i,0}(X_t) = 0$. □

Let us now compute the Hodge numbers of the general fiber of the Dwork pencil. By definition of Calabi-Yau manifold, it suffices to compute $h^{1,1}$ and $h^{2,1}$. The former equals the dimension of $H^2(X_t, \mathbb{C})$ by Lefschetz's Theorem, which is 1. The Euler characteristic of X_t is given by $c_3(X_t)$, which can be computed by the Euler exact sequence and the exact sequence, which defines the tangent space to X. More precisely, we have

$$c(X_t) = \frac{(1+u)^5}{(1+5u)},$$

where $c(X_t)$ is the total Chern polynomial. Hence we get $c_3(X_t) = -200$. This yields $h^{2,1} = 101$.

There is an abelian group that acts on X_t for all t. Set

$$G := \left\{ (a_1, \dots, a_5) \in (\mathbb{Z}/5\mathbb{Z})^5 : \sum_i a_i \equiv 0 \mod 5 \right\} / < (a, a, a, a, a) > .$$

The group G acts on the projective space $\mathbb{P}^4$ in the following way:

$$(a_1, \dots, a_5) \cdot (x_1 : \dots : x_5) = (\zeta^{a_1} x_1 : \dots : \zeta^{a_5} x_5),\ \zeta^5 = 1, \zeta \neq 1,$$

where ζ is a primitive fifth root of unity. If the a_i's are equal to each other, the action becomes trivial; hence we mod out by the subgroup of diagonal elements. The condition $\sum_i a_i \equiv 0 \mod 5$ preserves the term $x_1 \dots x_5$; so the group G acts on X_t for any t. Modding out by the subgroup of diagonal elements allows one to set one of the coordinates equal to zero. Since the sum of the remaining coordinates has to be congruent to zero mod 5, the group G depends on three coordinates. Hence it is isomorphic to $(\mathbb{Z}/5\mathbb{Z})^3$, whose order is 125. As proved in [17], the set of 125 nodes is transitive with respect to the action of G for $t^5 = 1$.

The group G acts on X_t with nontrivial stabilizers. Suppose $x_j = x_k = 0$ for $j, k \in \{1, \dots, 5\}$. Then $\{x_j = x_k = 0\} \cap X_t$ is a plane quintic curve with generic stabilizer isomorphic to $\mathbb{Z}/5\mathbb{Z}$. If three coordinates are equal to zero, then the stabilizer is isomorphic to $(\mathbb{Z}/5\mathbb{Z})^2$.

A monomial $x_1^{k_1} \dots x_4^{k_4}$ is invariant under G if and only if $k_1 \equiv k_2 \equiv k_3 \equiv k_4 \mod 5$. Thus the quotient map $p : X_t \to X_t/G$ is given by

$$(x_1 : \dots : x_5) \to (x_1 \dots x_5 : x_1^5 : \dots : x_5^5).$$

The quotient is thus a threefold in $\mathbb{P}^5$ which satisfies the following equations:

$$z_1 + z_2 + \dots + z_5 - 5t z_0 = 0, \qquad z_0^5 = z_1 z_2 z_3 z_4 z_5, \tag{2}$$

where z_i are a system of homogeneous coordinates in $\mathbb{P}^5$.

The image of the curves $\{x_j = x_k = 0\} \cap X_t$ is given by $z_0 = z_j = z_k = 0$ and $z_1 + \ldots + z_5 = 0$, which is isomorphic to $\mathbb{P}^1$. The points with stabilizer $(\mathbb{Z}/5\mathbb{Z})^2$ satisfy the condition $x_i = x_j = x_k = 0$ for distinct $i, j, k \in \{1, 2, 3, 4, 5\}$. For each triple i, j, k they give a point in X_t/G.

The Calabi-Yau manifold X_t has a unique (up to scalars) top degree differential form. It can be written down explicitely as follows:

$$\omega := \mathrm{Res}_{X_t}\left(\frac{\sum_{i=1}^{n}(-1)^i x_i \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_i} \wedge \ldots \wedge \mathrm{d}x_5}{F_t}\right),$$

where $F_t := x_1^5 + \ldots + x_5^5 - 5tx_1 \ldots x_5$.

This form is clearly invariant under the action of G. This means that $G \subset SL(3, \mathbb{C})$; hence the quotient has Gorenstein singularities. For these orbifolds there exists a desingularization, which is a smooth Calabi-Yau threefold Y_t. Moreover, by [19] the Hodge structure of the cohomology of Y_t is the same as the Hodge structure of the orbifold cohomology of X_t/G. Let us briefly recall the definition of these groups.

2.1 – The orbifold cohomology groups

We briefly summarize some facts on orbifold cohomology: for more details the reader is referred to [4]. Let X be an n-dimensional complex orbifold. Define $\widetilde{X}$ to be the set of pairs $(p, ((g))_{G_p})$ for $p \in X$ and (g) is the conjugacy class of g in the local isotropy group G_p. It is known that $\widetilde{X}$ is an orbifold called the *inertia orbifold*. This orbifold admits a decomposition in connected components, the nontwisted sector X and the twisted sectors $X_{(g)}$ for $g \neq 1$.

Any $g \in G_p$ acts on the tangent space T_pX via a diagonal matrix

$$D = \mathrm{diag}(e^{2\pi i r_1}, \ldots, e^{2\pi i r_n}),$$

where $r_i \in [0, 1)$. The degree shifting number $i_{(g)}$ is defined to be $\sum_i r_i$. If $g \in SL(n, \mathbb{C})$, then $i_{(g)}$ is an integer. Moreover, we have

$$i_{(g)} + i_{(g^{-1})} = n - dim_{\mathbb{C}} X_{(g)}. \tag{3}$$

The d-th orbifold group is defined to be

$$H^d_{orb}(X) := \bigoplus_{(g)} H^{d-2i_{(g)}}(X).$$

In particular, if $X = Y/G$ is a global quotient of a smooth variety Y by a finite group G, then

$$H^d_{orb}(X) := \bigoplus_{(g) \in G_*} H^{d-2i_{(g)}}(Y^g/C(g)),$$

where Y^g is the fixed locus of g, $C(g)$ is the centralizer in G, and G_* is a set of representatives of conjugacy classes in G.

Now, let us compute the Hodge numbers of Y_t. As before, it suffices to compute $h^{1,1}$ and $h^{2,1}$. The whole cohomology ring of the mirror quintic has been computed in [12]. Here we obtain the numbers mentioned above via direct methods.

Since X_t/G is Gorenstein, the degree shifting number is always an integer. The twisted sectors coincide with $Y^g/C(g)$ for $g \neq 1$. They are points or isomorphic to $\mathbb{P}^1$. By (3), the degree shifting number of g is 1 or 2, respectively. Clearly, the degree shifting number of the identity is zero.

A direct computation of the elements of G shows that there are 24 elements that do not fix anything, namely the S_4-orbit of $(1,2,3,4,0) \in G$. If three of the components of $g := (a_1, a_2, a_3, a_4, 0) \in G$ are equal, then g fixes a quintic curve whose image in X_t/G is a $\mathbb{P}^1$. If there are two pairs of the components of g that are equal, then g fixes ten points, which become two points under the quotient map $p : X_t \to X_t/G$.

Lemma 2.2.

i) *There are* 40 *elements g in G such that $i_{(g)} = 1$ and $i_{(g^{-1})} = 1$.*

ii) *There are* 30 *elements g in G such that $i_{(g)} = 1$ and $i_{(g^{-1})} = 2$.*

Proof.

i) We need to count all elements g such that $Y^g/C(g)$ is isomorphic to $\mathbb{P}^1$. As mentioned before, three components in $g = (a_1, a_2, a_3, a_4, 0)$ must be equal. This proves the claim.

ii) Since 24 elements do not move anything, we are left with $125-1-24-40 = 60$ elements. These come in pairs (g, g^{-1}). Therefore, ii) is completely proved. □

Proposition 2.3. *The Hodge numbers $h^{1,1}(Y_t)$ and $h^{2,1}(Y_t)$ are equal to* 101 *and* 1, *respectively.*

Proof. It suffices to compute $h^2_{orb}(X_t/G)$ and $h^3_{orb}(X_t/G)$. By definition, we have

$$h^2_{orb}(X_t/G) = h^2(X_t)^G \bigoplus_{g\neq 1} h^0(X_t^g/G).$$

We have $h^2(X_t)^G = 1$ since $h^2(X_t)$ is one-dimensional. By Lemma 2.2, we have $h^0(X_t^g/G) = 100$, since the elements in ii) yield two connected components in X_t^g/G. Note that $C(g) = G$ since the group is abelian.

As for $h^3_{orb}(X_t/G)$, we have

$$h^3_{orb}(X_t/G) = h^3(X_t)^G \bigoplus_{g\neq 1} h^1(X_t^g/G).$$

For $g \neq 1$ we have no contribution because X_t^g/G is either a point or a projective line. This leaves us with the computation of $h^3(X_t)^G$. The dimension of the space of invariants can be expressed in terms of the Euler characteristics of the fixed loci (Holomorphic Lefschetz Formula). In particular, we have

$$h^3(X_t)^G = \frac{1}{|G|}\sum_g tr\left(g^*|H^3(X_t)\right),$$

where g^* is the transformation induced by g on $H^3(X_t)$. Further, we have

$$\chi(X_t^g) = \sum_i (-1)^i tr\left(g^*|H^i(X_t)\right) = 4 - tr\left(g^*|H^3(X_t)\right).$$

Hence we have

$$h^3(X_t)^G = 4 - \frac{1}{|G|}\sum_g \chi(X_t^g).$$

On the other hand, X_t^g can be a plane quintic or 10 points. Therefore, we have

$$h^3(X_t)^G = 4 - \frac{1}{|G|}\{-200 + 40(-10) + 60(10)\} = 4.$$

Since $h^{3,0}(Y_t) = 1$, we have

$$h^{2,1}(Y_t) = h^{2,1}_{orb}(X_t/G) = \frac{1}{2}(4-2) = 1.$$

□

2.2 – Generalizations

The Dwork pencil can be generalized to any degree n. More precisely, we can consider the pencil $X_t^{n+1} \to \mathbb{P}^1$, where $X_t^{n+1} = Z(F_t^{n+1}) \subset \mathbb{P}^n$ and

$$F_t^{n+1} := \sum_i^{n+1} x_i^{n+1} - nt\prod_i^{n+1} x_i.$$

In [3] we investigate the geometry of this generalized pencil and its quotients by various automorphism groups. As n varies, the geometry might be rather intricate as the following proposition shows.

Let us consider the following subvariety Z of $\mathbb{P}^n$ for $n \equiv 0 \bmod 2$, namely:

$$\begin{cases} x_1 + \ldots + x_{n+1} = 0 \\ x_1^2 + \ldots + x_{n+1}^2 = 0 \\ \ldots\ldots\ldots \\ x_1^{n/2} + \ldots + x_{n+1}^{n/2} = 0 \end{cases}$$

Lemma 2.4. *Let $\mathbb{Q}(\lambda)$ be an extension of the rational field. Choose $n-1$ distinct non-zero rational numbers $c_1, \dots, c_{n-1}$. The determinant V of the Vandermonde matrix $V(\lambda, c_1, \dots, c_{n-1})$ is not rational.*

Proof. Suppose, on the contrary, that V is a rational number. If we expand with respect to the column of the powers of λ, it is easy to see that λ satisfies a polynomial with rational coefficients. Hence, the extension $\mathbb{Q}(\lambda)$ is algebraic and the Galois group is finite. If V is rational, it is fixed by any element σ of the Galois group. We thus have

$$\det\begin{pmatrix} 0 & 1 & \dots & 1 \\ \lambda - \sigma(\lambda) & c_1 & \dots & c_{n-1} \\ \dots & \dots & \dots & \dots \\ \lambda^{n-1} - \sigma(\lambda^{n-1}) & c_1 & \dots & c_{n-1}^{n-1} \end{pmatrix} = 0.$$

The determinant of the matrix

$$\begin{pmatrix} c_1 & \dots & c_{n-1} \\ \dots & \dots & \dots \\ c_1^{n-1} & \dots & c_{n-1}^{n-1} \end{pmatrix}.$$

is given by

$$c_1 c_2 \dots c_{n-1} \prod_{r<s} (c_r - c_s),$$

which is different from zero. This means that the first column of the matrix in (4) is a linear combination with rational coefficients of the other columns, which are rational numbers. Thus, we have $(\sigma - I)(\lambda) = d \in \mathbb{Q}$. Suppose $\sigma^m = I$. If we apply $\sigma^{m-1} + \dots + I$ to both members, we get $0 = md$; hence $\lambda = \sigma(\lambda)$ for any σ in the Galois group. This would mean that λ is rational against the assumptions. □

Theorem 2.5. *The subvariety Z is smooth and is contained in X_1^{n+1}.*

Proof. First of all, we notice that Z is defined by the equations $p_1 = p_2 = \dots = p_{n/2} = 0$, where the p_j's are the Newton symmetric functions. The elementary symmetric functions e_j can be written in terms of the p_j. It is easy to check that the subvariety Z can be defined via the equations $e_1 = e_2 = \dots = e_{n/2} = 0$. This said, we recall that X_1^{n+1} is given by $p_{n+1} - (n+1)e_{n+1} = 0$. Since n is even, this equation is equivalent to

$$\sum_{j=1}^{n} (-1)^{n+1-j} p_j e_{n+1-j} = 0. \tag{5}$$

If $e_1 = \ldots = e_{n/2} = 0$, then equation (5) is satisfied.

Second, the jacobian J of the system of equations defining Z is given by

$$\begin{pmatrix} 1 & \ldots & 1 \\ 2x_1 & \ldots & 2x_{n+1} \\ \ldots & \ldots & \ldots \\ \frac{n}{2}x_1^{\frac{n}{2}-1} & \ldots & \frac{n}{2}x_{n+1}^{\frac{n}{2}-1} \end{pmatrix}$$

If we choose any $n/2$ columns, we get a Vandermonde matrix. If a point of Z has at least $n/2$ different coordinates, there exists a minor of J different from zero. We need to show that a point with at most $n/2$ different coordinates does not belong to Z. This implies that Z is smooth. Suppose, on the contrary, that a point $P := [\lambda_0 : \ldots : \lambda_0 : \ldots : \lambda_{\frac{n}{2}-2} : \ldots : \lambda_{\frac{n}{2}-2}]$ belongs to Z. We can assume $\lambda_i \neq \lambda_j$. Let k_i be the number of times λ_i appears as a coordinate of P. Notice that $\sum_i k_i = n + 1$. The λ_i's and the k_i's satisfy the following system of equations:

$$(6) \qquad \begin{cases} k_0 + \ldots + k_{\frac{n}{2}-2} = n+1 \\ k_0\lambda_0 + \ldots + k_{\frac{n}{2}-2}\lambda_{\frac{n}{2}-2} = 0 \\ k_0\lambda_0^2 + \ldots + k_{\frac{n}{2}-2}\lambda_{\frac{n}{2}-2}^2 = 0 \\ \ldots\ldots\ldots\ldots\ldots \\ k_0\lambda_0^{n/2} + \ldots + k_{\frac{n}{2}-2}\lambda_{\frac{n}{2}-2}^{n/2} = 0 \end{cases}$$

Let us consider the linear system $\Lambda X = N$, where Λ is the $(n/2+1) \times (n/2-1)$ matrix

$$\begin{pmatrix} 1 & \ldots & 1 \\ \lambda_0 & \ldots & \lambda_{\frac{n}{2}-2} \\ \ldots & \ldots & \ldots \\ \lambda_0^{n/2} & \ldots & \lambda_{\frac{n}{2}-2}^{n/2} \end{pmatrix},$$

X is the column of unknowns and N is the column vector $(n+1, 0, \ldots, 0)^t$. Since $\lambda_i \neq \lambda_j$, the matrix Λ contains a minor V of size $(n/2-1) \times (n/2-1)$ different from zero, so the system has a unique solution, which is given by the integers k_i for any given P. By standard linear algebra, we have

$$(7) \qquad k_l = (n+1)(-1)^{l+1}\frac{V_l}{\det(V)},$$

where V_l is the determinant of the matrix obtained from V by removing the l-th column.

Notice that if some of the λ_i's coincide, the k_i would be zero, so we would get a smaller system and we could proceed as in the case where $\lambda_i \neq \lambda_j$.

Third, we can assume that λ_i is in $\mathbb{Z}$ for any i. To do this, it suffices to show that under our assumptions all λ_i's are in the rational field. Suppose there exist $\lambda_{i_1}, \ldots, \lambda_{i_f}$ not in $\mathbb{Q}$. If $f \geq 2$, there exist λ_{i_r} and λ_{i_s} not in $\mathbb{Q}$. Then, there exists an element of the Galois group of the extension $\mathbb{Q}(\lambda_{i_1}, \ldots, \lambda_{i_f})$ over $\mathbb{Q}$ which exchanges λ_{i_r} and λ_{i_s}. It is easy to check that under this element k_{i_r} is mapped onto $-k_{i_s}$. Since k_{i_r} is an integer, we must have $k_{i_r} + k_{i_s} = 0$. This means that $k_{i_r} = k_{i_s} = 0$. In other words, we can disregard λ_{i_r} and λ_{i_s}. If f is even, we can disregard all the λ_i's not in $\mathbb{Q}$. If f is odd, we are left with the extension $\mathbb{Q}(\lambda_l)$ over $\mathbb{Q}$. In other words, there is only one λ_l not rational and the other ones are rational numbers. If we take into account k_l, then V_l is rational. Recall that the λ_j are all distinct. If none of them is zero, we reach a contradiction by Lemma 2.4. If one of them is zero (this is the only possible case because the λ's are all distinct), we can cancel a column from the matrix Λ and apply the result of Lemma 2.4.

Let us recap what we have proved so far. If P is a point in Z with at most $(n/2)-1$ different entries, the coordinates of P are integer numbers given by the formula (7). More explicitly, the solutions are given by

$$k_l = (n+1)(-1)^{l+1} \frac{\lambda_0 \ldots \widehat{\lambda_l} \ldots \lambda_t}{\prod_{r<l}(\lambda_r - \lambda_l) \prod_{s>l}(\lambda_l - \lambda_s)},$$

where $t = (n/2) - 2$ and $l \in \{0, \ldots, (n/2) - 2\}$. Since the subvariety Z is defined by symmetric equations, we can assume that the λ_i's are ordered so that $\prod_{r<l}(\lambda_r - \lambda_l) \prod_{s>l}(\lambda_l - \lambda_s)$ is positive.

Since $0 \leq k_l \leq n+1$, we should have

$$(-1)^{l+1} \lambda_0 \ldots \widehat{\lambda_l} \ldots \lambda_t \geq 0$$

for any l. If all the λ_i's were positive, k_0 would be negative against the assumptions. If the number of negative λ_i's is odd, k_0 would be negative. If the number of positive λ_i's is even, k_1 would be negative. If all the λ_i's are negative and t is odd, k_0 would be negative. If all the λ_i's are negative and t is even, k_1 would be negative. In any case, there exists a k_i which is negattive, whereas all the k_i's are positive by assumption. □

3 – Birational Models of the Mirror Quintic

It is important to understand whether a given Calabi-Yau is indeed new or birational to an existing one. Let us consider the following families:

	F_t
1	$x_1^5 + x_2^5 + \ldots + x_5^5 - 5tx_1x_2\cdots x_5$
2	$x_1^4x_2 + x_2^4x_3 + x_3^4x_4 + x_4^4x_5 + x_5^4x_1 - 5tx_1x_2\cdots x_5$
3	$x_1^4x_2 + x_2^4x_3 + x_3^4x_4 + x_4^4x_1 + x_5^5 - 5tx_1x_2\cdots x_5$
4	$x_1^4x_2 + x_2^4x_3 + x_3^4x_1 + x_4^5 + x_5^5 - 5tx_1x_2\cdots x_5$
5	$x_1^4x_2 + x_2^4x_3 + x_3^4x_1 + x_4^4x_5 + x_5^4x_4 - 5tx_1x_2\cdots x_5$
6	$x_1^4x_2 + x_2^4x_1 + x_3^5 + x_4^5 + x_5^5 - 5tx_1x_2\cdots x_5$

Each of them can be rewritten in the form

$$F_{A,t} := \sum_{i=1}^{5}\prod_{j=1}^{5} x_j^{a_{ij}} - 5tx_1x_2\cdots x_5,$$

where

$$a_{i1} + a_{i2} + \ldots + a_{i5} = 5, \quad a_{1j} + a_{2j} + \ldots + a_{5j} = 5.$$

If we set

$$z_i := \prod_{j=1}^{5} x_j^{a_{ij}}, \qquad z_1z_2\cdots z_5 = (x_1x_2\cdots x_5)^5,$$

we get the equations (2). This means that there exists a non-constant rational map

$$q_{A,t} : X_{A,t} \longrightarrow X_t/G, \qquad (x_1 : \ldots : x_5) \longmapsto (z_0 : z_1 \ldots : z_5),$$

where $z_0 := x_1x_2\cdots x_5$.

If we show that $q_{A,t}$ is birationally equivalent to a quotient map $X_{A,t} \to X_{A,t}/H_A$ for some group H_A, then Y_t is birational equivalent to $X_{A,t}/H_A$, thereby yielding a birational model of Y_t. In some cases, H_A is the identity group. We have shown that $q_{A,t}$ is birationally equivalent to a quotient map in [2]. To state the theorem, we need to define the group H_A.

Let d be the smallest positive integer such that $B := dA^{-1}$ has integer entries. Set

$$X_{dI,t} := Z(F_{dI,t}) \subset \mathbb{P}^{n-1}, \qquad F_{dI,t} = \sum_{j=1}^{n} y_j^d - nt\left(\prod_{j=1}^{n} y_j\right)^m,$$

$$d = mn.$$

We introduce a map

$$\phi_A : X_{dI,t} \longrightarrow X_{A,t}, \qquad (y_1 : \ldots : y_n) \longmapsto (x_1 : \ldots : x_n),$$

$$x_j = \prod_{k=1}^{n} y_k^{b_{jk}}.$$

For $a = (a_1, \dots, a_n) \in (\mathbb{Z}/d\mathbb{Z})^n$ define the automorphism g_a on $\mathbb{P}^{n-1}$ in the following way:

$$g_a(y_1 : \dots : y_n) := (\zeta^{a_1} y_1 : \dots : \zeta^{a_n} y_n).$$

Set

$$\Gamma_d := \{g_a : a = (a_1, \dots, a_n),\ a_1 + \dots + a_n \equiv 0 \mod n\}/\langle g_{(1,1,\dots,1)}\rangle.$$

It is an easy exercise to show that

$$\Gamma_d \cong \mathbb{Z}/m\mathbb{Z} \times (\mathbb{Z}/d\mathbb{Z})^{n-2}.$$

Γ_d induces an action on $X_{A,t}$. Indeed, we have:

$$\phi_A(g_a(y)) = (\zeta^{a'_1} x_1 : \dots : \zeta^{a'_n} x_n),\ a'_j = \sum_{k=1}^{n} a_k b_{jk};$$

so

$$\Gamma_d \longrightarrow Aut(X_{A,t}), \qquad g_a \longmapsto g_{Ba} = g_{a'}. \tag{8}$$

Let Γ_A and H_A be the kernel and the image of the homomorphism (8). Then the following holds ([2])

Theorem 3.1. *Let A be an $n \times n$ matrix with non-negative integer entries such that the sum of the entries in any row and column is equal to n and such that $X_{A,t}$ is irreducible. Then:*

$\phi_{A,t} : X_{dI,t} \longrightarrow X_{A,t}$, is birational to the quotient map

$$X_{dI,t} \longrightarrow X_{dI,t}/\Gamma_A,$$

$q_{A,t} : X_{A,t} \longrightarrow \overline{M}_t$, is birational to the quotient map

$$X_{A,t} \longrightarrow X_{A,t}/H_A,$$

and thus $q_{A,t} \circ \phi_{A,t} : X_{dI,t} \longrightarrow \overline{M}_t$, is birational to the quotient map

$$X_{dI,t} \longrightarrow X_{dI,t}/\Gamma_d.$$

Remark 3.2. If we consider the second family

$$S_t := \left\{ x_1^4 x_2 + x_2^4 x_3 + \ldots + x_5^4 x_1 - 5t x_1 x_2 \ldots x_5 = 0 \right\},$$

the Theorem above and direct computation (with MAGMA) yield that S_t/H_t is birational to Y_t, where H_t is isomorphic to $\mathbb{Z}/41\mathbb{Z}$. This answers positively a conjecture posed by Greene, Plesser and Roan [8].

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