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Some models of geometries after Hilbert's Grundlagen

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Dedicated to Professor Marialuisa de Resmini

ABSTRACT: *We investigate the contribution of Max Dehn to the development of non-Archimedean geometries and the contribution of his student Ruth Moufang to the development of non-Desarguesian geometries.*

1 – Introduction

In 1899, David Hilbert published the *Grundlagen der Geometrie*, a book that opened up research in the foundations of geometry. In fact, the Grundlagen took the axiomatic method both as a culmination of geometry and as the beginning of a new phase of research. In that new phase, the links between the postulates were not seen as the cold expression of their logical relations or interdependence, but as the creation of new geometries having equal importance at the research level. In particular, the starting point of research on non-Archimedean geometries was the investigation of the independence of Archimedes'axiom(1) from other axioms and the starting point of research on non-desarguesian geometries was the investigation of the independence of Desargues'theorem(2) from the axioms of plane geometry.

KEY WORDS AND PHRASES: *David Hilbert – Max Dehn – Roberto Bonola – Ruth Moufang – Non-Desarguesian geometry – Non-Archimedean geometry – Octonions*
A.M.S. CLASSIFICATION: 01A70, 01A60, 51A35, 05B35, 17D05.

(1) As is well known, Archimedes'axiom states that if A and B are two segments, with A smaller than B ($A < B$), then there exists a positive integer n such that n times A is greater than B ($nA > B$).

(2) As is well known, Desargues's theorem states that if two triangles $a_1b_1c_1$, $a_2b_2c_2$ are in perspective from a point V, then the lines containing the opposite edges intersect in three collinear points, d_1, d_2, d_3.

The present paper aims to describe some models of non-Archimedean and non-desarguesian geometries that was born after the *Grundlagen*[3].

2 – Max Dehn and non-archimedean geometries

As is well known, Hilbert devoted Chapter II of his *Grundlagen der Geometrie* to proving the independence and non-contradictoriness of axioms. In particular, he proved the independence of Archimedes' axiom from the other ones. More precisely, he showed that Archimedes' axiom is not a consequence of axioms I (of incidence (connection)), II (of order), III (of parallelism), and IV (of congruence)[4] by exhibiting a geometry where Archimedes'axiom fails to be valid [Hilbert 1899].

On Hilbert's suggestion, Dehn studied the relationship between Legendre's theorems[5] and Archimedes'axiom. This last analysis is agreed with the point of view of Hilbert's Grundlagen. In fact, in the proofs of Legendre's theorems, that we can find in the literature (*i.e.* those of Euclid, those of Saccheri and those of Legendre himself) Archimedes'axiom is used, in a more or less explicit way. In the optic of Hilbert, and consequently of Dehn, it is remarkable to study whether these theorems really depend on this axiom.

Max Dehn was one of Hilbert's most prominent students. He was Born in Hamburg in 1878 and he received his doctorate in Göttingen at age twenty-one, under Hilbert's supervision, with the dissertation *Die Legendre'schen Satze über die Winkelsumme im Dreieck* on the foundations of geometry [Dehn 1900a]. He got his Habilitation in Munich in 1901, with a thesis in which he solved the third of the twenty-three problems Hilbert posed at the International Congress of Mathematicians in Paris in 1900 [Dehn 1900b],[Dehn 1901]; he was the first to solve one of Hilbert's problems. His solution showed that Archimede's axiom was needed to prove that two tetrahedra have the same volume, if they have the same altitudes as well as bases of the same area.

M. Dehn was Privatdozent in Munich from 1901 until 1911 and became Ordinarius in Breslau in 1913. He moved to the University of Frankfurt in 1921 where he lectured until 1935. Moreover, he published several valuable essays on the relationship between Greek philosophy and mathematics. In 1922 the seminar of history of Mathematics was founded, in Frankfurt, and Dehn was the

(3) This work is an elaboration of the two papers [Cerroni 2004], [Cerroni 2007], with some integrations.

(4) Usually, axioms III are on congruence and axioms IV are on parallelism,like in the more recent edition of the Grundlagen.

(5) These theorems are already in [Saccheri 1733], and in Italy they are called Saccheri's theorems. We recall that Legendre's theorems state that:

1. The sum of the angles of a triangle is equal to or less than two right angles.
2. If in a triangle the sum of the angles is equal to two right angles, it is so in every triangle.

driving force of this institution. The seminar on the history of Mathematics was held each semester until 1935. The rule of the seminar was to study the most important mathematical discoveries from all epochs in the original version.

In 1939, since he was a Jew, he emigrated from Germany to Copenhagen and later to Trondheim in Norway, where he took over the post of a vacationing colleague at the Technical University until 1940. Since German soldiers occupied Trondheim, Dehn and his wife, in October of 1941, emigrated to the United States via the trans-Siberian railway (from Moscov the tracks extended some 9,200 km to Vladivostok).

Dehn related their journey in a talk he gave at Idaho Southern not long after his arrival there [Dehn 1941]. According to that narrative, at the frontier between Norway and Sweden their luggage was "*ransacked*" and they were treated "*extremely unkindly and roughly*" by the border guards. They were delayed three weeks in Stockholm, apparently because of an outbreak of plague in Manchukuo and Vladivostok, but actually Dehn thought, for "*obscure political*" reasons. In the end they took the Amur River route and so did not pass through Manchukuo. At last the necessary tickets and travel document were issued, the Dehns were vaccinated and they flew on to Moscow.

During the several days they spent crossing the "*endless Russian plain*", the temperature at times fell low and Dehn developed a life-threatening combination of influenza and pneumonia, for which he was treated in Irkutsk. When the Dehns finally reached Vladivostok, they were forced to remain six more days while waiting for a ship to Kobe. The crossing to Japan proved to be very rough and cramped. He said nothing about the subsequent voyage to san Francisco, where he and his wife arrived on New Year's Day.

In United States, Dehn led a rather itinerant life until he found a position where he felt more or less comfortable. At the beginning Dehn spent one and a half year as a Professor of Mathematics and Philosophy at the State University of Idaho at Pocatello. The next year Dehn worked at the Illinois Institute of Technology in Chicago and after at St John's College in Annapolis, Maryland, where he was specially unhappy. Finally, in 1945 Dehn arrived at the final station in his life. This was Black Mountain College in North Carolina. He stayed there for the last 7 years of his life, leaving only for short periods as a guest lecturer in Madison, Wisconsin. He died in 1952 in Black Mountain, North Carolina [Dawson Jr. 2002], [Gillispie 1970-1990], [Siegel 1965].

M. Dehn, in his dissertation *Die Legendre'schen Sätze über die Winkelsumme im Dreieck*, analysed the relationship between Legendre's theorems and Archimedes' axiom. In particular, he asked:

"*Can one prove Legendre's theorems without an axiom of continuity, i.e. without making use of the Archimedian axiom?*"(6) [Dehn 1900a, p. 405].

(6) *The original texts in the following are faithfully translated by the author.*

To answer this question, Dehn first showed that Legendre's second theorem is only a consequence of the incidence, order and congruence axioms by proving, in a geometry (named *pseudogeometry*) [Dehn 1900a, pp. 406-411], where such axioms hold and Archimedes' axiom does not hold, the following more general theorem:

> "*If the angle sum of one triangle is less than two right angles then this is true for every triangle.*
> *If the angle sum of one triangle is equal to two right angles then it is so for every triangle.*
> *If the angle sum of one triangle is greater than two right angles then the same holds for every triangle.*" [Dehn 1900a, pp. 430-431].

Note that, the second statement is Legendre's second theorem.

Subsequently, Dehn showed that it is impossible to prove Legendre's first theorem with the incidence, the order, the congruence axioms and without Archimedes' axiom, by constructing a *Non-Legendrian Geometry* in which there are infinitely many lines parallel to a fixed line through a point, Archimede's axiom does not hold and the sum of the inner angles of a triangle is greater than two right-angles and constructing a *Semi-Euclidean Geometry* in which there are infinitely many lines parallel to a fixed line through a point, Archimede's axiom does not hold but the sum of the inner angles of every triangle is still equal to two right angles [Dehn 1900a, pp. 431-438].

2.1 – Hilbert analytic model of non-Archimedean geometry

Hilbert, in Chapter II of his *Grundlagen der Geometrie*, constructed a non-archimedean number system on which he based an analytic geometry. In particular, he considered the set $\Omega(t)$ consisting of the algebraic functions of t obtained from the set of polynomials with rational coefficients in t by the five operations of addition, subtraction, multiplication, division and the operation $\sqrt{1+\omega^2}$, where ω is a function derived from the previous five operations.

The set $\Omega(t)$ is countable and it can be regarded as the set of real- valued functions of a real variable defined at all but a finite number of points. Moreover, if c is a function in $\Omega(t)$, *i.e.* c is an algebraic function of t, it will vanish only on a finitely many values of t. Therefore, c, for positive large values of t, is either always positive or always negative. The usual operations are valid in $\Omega(t)$, and if a and b are two functions in $\Omega(t)$, a will be greater than b ($a > b$) or a less than b ($a < b$) if $a - b$ is always positive or always negative, respectively.

Let n be a positive integer, then n is less than t ($n < t$) since $n - t$ is always negative for large positive values of t. Consider the numbers 1 and t in $\Omega(t)$. Then, every multiple of 1 is always less than t, so $\Omega(t)$ is a non-archimedean number system.

Hilbert constructed an analytic geometry on this number system as follows: (x, y, z), where x, y, $z \in \Omega(t)$, is a point; $ux + vy + wz + r = 0$, where u, v, w, $r \in \Omega(t)$, is a plane; a line is the intersection of two planes [Hilbert 1899, pp. 24-26].

It easy to see that such a geometry is non-archimedean; indeed, on the basis of the above, a line segment the length of which is n times that of the unit segment will never exceed a segment of length t on the same line.

2.2 – The non-Legendrian Geometry

As it is well known, Legendre's first theorem affirms that the sum of the inner angles of a triangle is less or equal than two right-angles. Dehn showed that the Archimede axiom is needed to prove the previous theorem and analyzed the relationships between Archimede's axiom, the number of lines through a point parallel to a fixed line and the sum of inner angles of a triangle [Dehn 1900a, pp. 431-436].

It is known that if the Archimede axiom holds, there exists the following relationships between the hypothesis on the existence and the number of parallel lines through a point and the sum of the inner angles of a triangle: if the sum of the inner angles of a triangle is greater, equal or less than two right-angles then, no line passes through a point parallel to a fixed line, there exists exactly one line through a point parallel to a fixed line, there exist infinite lines through a point parallel to a fixed line, respectively.

Dehn constructed a non-archimedean geometry where through a point there exist infinite parallel lines and where the sum of inner angles of a triangle is greater than two right-angles, therefore as he wrote:

"*For all that is shown the non-validity of Legendre's first theorem and the hypothesis on the sum of the inner angles of a triangle, as Saccheri said, is not related with the hyphotesis on the existence and number of parallel lines through a point*" [Dehn 1900a, p. 432].

Dehn considered the non-archimedean number system introduced by Hilbert, that is the set $\Omega(t)$ of the algebraic functions of t obtained from t with the five operations of addition, subtraction, multiplication, division and the operation $\sqrt{1+\omega^2}$, where ω is a function obtained by the previous five operations and constructed an analytic geometry over the set $\Omega(t)$ as follows: the points are the pairs (x, y), with x, y in $\Omega(t)$, and the lines have the equations $ux + vy + w = 0$, with u, v, w in $\Omega(t)$ [Dehn 1900a, p. 432]. In the previous geometry all the axioms hold with the exclusion of Archimede's axiom.

Dehn constructed, over this non-archimedean plane, an *elliptic* or *Riemaniann* geometry as follows. He took the conic

$$x^2 + y^2 + 1 = 0,$$

and considered as points and lines of the elliptic geometry, all the points and the lines of the non-archimedean plane together with the line at infinity with its points and, as congruences of the elliptic geometry, the real transformations that fix the conic[7]. Then, he considered, as points of the new geometry, the points of the *elliptic* geometry (x, y) satisfyting the following conditions:

$$\frac{-n}{t} < x < \frac{n}{t}$$
$$\frac{-n}{t} < y < \frac{n}{t}$$

where n is an integer, and as a lines, the lines whose points satisfied the previous condition [Dehn 1900a, pp. 433]. Dehn showed that all the axioms are valid except Euclid's axioms and Archimede's axiom and that the sum of the inner angles of a triangle is greater that two right-angles [Dehn 1900a, pp.433-436]. Thus, as he wrote:

> "*We have constructed a geometry where axioms* I, II *and* IV *hold, where through one point there exist infinite lines parallel to a fixed line and where the sum of the inner angles of a triangle is greater than two right-angles. The Archimede axiom does not hold. Then it is shown that Legendre's first theorem does not hold without the help of the Archimede axiom. We call the constructed geometry "Non-Legendrian "geometry [...]*" [Dehn 1900a, p. 436].

2.3 – The Semi-Euclidean Geometry

Dehn continued his analysis on the relationship between the sum of inner angles of a triangle and the hypothesis on the existence and the number of parallel lines through a point by constructing another kind of geometry. He considered the above non-archimedean plane and constructed over it a new geometry as follows: the points of the new geometry are the points (x, y) of the non-archimedean plane satisfying the following condition,

$$-n < x < n$$
$$-n < y < n,$$

where n is a positive integer and the lines are the lines of the non-archimedean plane whose points satisfying the condition above [Dehn 1900a, p. 436]. Dehn showed that in this geometry axioms I, II and IV hold and moreover, since the segments and the angles are defined as in the euclidean geometry, Legendre's first theorem is valid:

[7] This construction was done by Klein [Klein 1871].

"*[...] Moreover, the theorems of the classic euclidean geometry are valid in the limited zone. The sum of inner angles is equal to two right-angles in every triangles.*" [Dehn 1900a, p. 437].

However, it is easy to see that through a point there exist infinite lines parallel to a fixed line. To show this Dehn considered the line through the points $(t, 0)$ and $(0, 1)$; this is a line of the new geometry, since it passes through the points $(0, 1)$ and $(1, \frac{t-1}{t})$ which are points of the new geometry, but intersects the x axis in a point that is not a point of the new geometry. Then, he considered the line through the points $(-t, 0)$ and $(0, 1)$; this line is a line of the new geometry which intersects the x axis in a point that is not a point of the new geometry. The two previous lines pass through the point $(1, 0)$ and are parallel to the x axis. So, Dehn showed that:

"*This is a non-archimedean geometry in which the parallel axiom is not valid but where the sum of the inner angles of a triangle is equal to two right-angles.*" [Dehn 1900a, p. 438].

Thus, Dehn constructed a geometry where the theorems of the Euclidean geometry hold, but where the parallel axiom does not; Dehn called this geometry *Semi-Euclidean* geometry.

Hilbert was struck by this kind of geometry which he called a *remarkable geometry* so that he constructed, in his lectures on foundations of geometry of 1902, another model of *semi-Euclidean* geometry, inspired by Dehn's result [Hallet and Ulrich 2004].

Dehn summed up the previous results in the following diagram:

The sum of the inner angles of a triangle is:	Lines through a fixed point and parallel to a given line:		
	No parallel lines	One parallel line	Infinite parallel lines
$> 2R$	Elliptic Geometry	(impossible)	Non-Legendre geometry
= 2R	(impossible)	Euclidean geometry	Semi-Euclidean geometry
$< 2R$	(impossible)	(impossible)	Hyperbolic geometry

Hilbert also summarized the results in detail in his conclusion to the French and English translations of the *Festschrift* [Hilbert 1899], and from the second edition of the *Grundlagen*, there are short remarks on Dehn's work at the end of Chapter III.

3 – Italian school and Bonola research on Saccheri's theorem

A pioneer in the study of non-archimedean geometry was Giuseppe Veronese (1854-1917). In the work *Fondamenti di Geometria* [Veronese 1891] he constructed, in abstract manner, a geometry in which he postulated the existence of a segment which is infinitesimal with respect to another, and where the straight line of geometry is not equated with the continuous straight line of Dedekind.

This work brought about many critics and it caused a national and international discussion(8) which involved, from the others, Rodolfo Bettazzi (1861-1941), George Cantor (1845-1918), Wilhelm Killing (1847-1923), Tullio Levi Civita (1873-1941), Giuseppe Peano (1858-1932), Arthur Moritz Schönflies (1853, 1928), Otto Stolz (1842-1905), and Giulio Vivanti (1859-1949).

In 1893 there is a turning-point in the discussion; Tullio Levi Civita published the work *Sugli infiniti ed infinitesimi attuali quali elementi analitici* [Levi Civita 1893] in which he constructed from the reals number, in an analitically way, a number system whose numbers (the *monosemii*) are the marks of the infinite and infinitesimal segments of Veronese.

The discussion about the possibility of the existence of infinite and infinitesimal segments ended with the publication of Hilbert's *Grundlagen*, as we have seen in the previous sections and it is possible that the international discussion influenced Hilbert, who knew the work of Veronese and referred to it as *deep work* [Hilbert 1899, p. 48].

Strangely enough, the approach of Veronese, who anticipated from many points of view Hilbert's work, did not eventually produce an Italian school in this kinds of studies(9). So mach so that, R. Bonola was more influenced by the works of M. Dehn than the ones of G. Veronese.

Roberto Bonola was born in 1874 in Bologna and there died prematurely in 1911. He graduated in Mathematics in 1898 under the supervision of F. Enriques, who choose him as his assistant. In 1900 he became a teacher of mathematics in schools for girls, first in Petralia Sottana, then in Pavia, where he spent the best years of his short life. In 1902 he became assistant to the course of Calculus at the University of Pavia and in 1904 he gave lectures on the Foundations of

(8)See for examples the works [Bettazzi 1891], [Bettazzi 1892], [Cantor 1895], [Cantor 1897], [Killing 1895-1897], [Killing 1897], [Levi Civita 1893], [Levi Civita 1898],[Peano 1892], [Peano 1892a], [Schönflies 1897], [Schönflies 1897a], [Stolz 1883], [Stolz 1891], [Veronese 1892], [Veronese 1896], [Veronese 1897], [Veronese 1898], [Vivanti 1891], [Vivanti 1891a].

(9)For a study in depht of the Italian question see [Avellone et al. 2002] and [Bottazzini 2001].

Geometry. Moreover, from 1904 to 1907, he taught a mathematics course for Chemistry and Natural Science students. In 1909 he obtained the *Libera Docenza* of Projective Geometry and in 1910 he became Ordinary Professor on the Regio Istituto Superiore di Magistero femminile in Rome. He was seriously sick since 1900 and he died while he established in Rome. [Amaldi 1911].

Bonola was among the very few Italians who were deeply interested in Dehn's work. At the time, he was working under the supervision of Enriques on non-Euclidean geometry from an historical point of view [Bonola 1906], which can be considered his main work.

He was thus deeply interested in understanding the role of Archimedes' axiom in the proof of Saccheri's theorem(10). In his work [Bonola 1905], he demonstrated, in a direct way without the use of Archimedes' axiom, Saccheri's theorem on the sum of the inner angles of a triangle:

"*This note aims at giving a direct and elementary proof of the result by Dehn, that is of the proof of Saccheri's theorem, without the use of Archimedes' axiom.*" [Bonola 1905, p. 652].

In fact, Bonola shared his master's (F. Enriques) vision of geometry, *i.e.* as a deeply intuitive discipline. Thus, only a direct proof could really satisfy our intuitive vision:

"*The way followed by Dehn to prove, without Archimedes' axiom, Saccheri's theorem is very elegant and logically complete. Geometrical intuition, however, needed a direct proof, that is a proof without formal systems, constructed over abstract concepts, that only formally satisfies the geometrical properties.*" [Bonola 1905, p. 652].

He started from the research of Father Saccheri [Saccheri 1733] on Euclid's V axiom. He considered the birectangular isosceles quadrilateral ABCD (= 1 right angle and AB=CD), that is now called the Saccheri Quadrilateral and distinguished the three Hypothesis: the one of the right angle, the one of the acute angle and the one of the obtuse angle.

He then demonstrated Saccheri's theorem: "*If one of the three previous Hypotheses is valid in a Saccheri Quadrilateral, this hypothesis is valid in every Saccheri Quadrilateral*" without using Archimedes' axiom and since Saccheri's theorem on the sum of the inner angles of a triangle is a consequence of this theorem, the aim is achieved [Bonola 1905]. To prove Saccheri's theorem, he considered a plane in which the axioms of connection, order, and congruence are satisfied, distinguishing two cases: *the closed line* and *the open line.*

(10) Bonola called Saccheri's theorem the second theorem of Legendre.

4 – M. Dehn research program, Moufang planes and their coordinatizating algebra

As we have seen in the previous sections, the foundations of geometry represent a type of mathematical inquiry that highly suited some characteristic features of Dehn's mind. However, he was not primarily interested in finding minimal sets of axioms or in separating the postulates of a given discipline into sets of weaker ones and then proving their independence and completeness. That kind of axiomatic approach is well represented by Moritz Pasch (1843-1930), whose book *Vorlesungen über neuere Geometrie* first appeared in 1882. The second edition of this book appeared in 1926 with a supplementary part by Dehn, entitled *Die Grundlegung der geometrie in historischer Entwicklung* [Dehn 1926]. Whereas Pasch emphasized precision and detail, Dehn focused on insight and ideas. He was interested in finding solid and simple foundations for a theory, in particular for projective geometry:

"*[t]he aim of the foundations of projective geometry, as well as of metric geometry, is to transform the projective relations (collineations) into algebraic relations.*" [Dehn 1926, pp. 213-214].

As this makes clear, Dehn was interested in the relationships between algebra and geometry. In particular, he was deeply impressed by the part of Hilbert's *Grudlagen* in which it is showed that the incidence axioms of (projective) geometry together with a single incidence theorem, namely, Pappus's theorem, are equivalent to the definition of a field, and that the same axioms together with Desargues's theorem define a skew field [Magnus and Maufang 1954], [Magnus 1978-1979].

This kind of approach was to become a real *Research Program* for Dehn and he inspired many students. One of Dehn's students was Ruth Moufang. In a series of difficult papers, she proved that there exists a third theorem of the type of Desargues and Pappus, *the theorem of the complete quadrilateral.*

Ruth Moufang was born in Darmstadt, Germany on 10 January, 1905. Her interest in mathematics was first stimulated at the Realgymnasium in Bad Kreuznach, which she attended from 1921 to 1924. She then studied mathematics at the University of Frankfurt from 1925 until 1930 and she passed the teacher's examination in 1929.

She took her Ph.D. in 1931 under Dehn's supervision with a dissertation on projective geometry. After, she spent a year in Rome with a research fellowship. She lectured from 1932 to 1933 at the University of Königsberg, where she took a course with Emmy Noether, was encouraged in the study of mathematics by Kurt Reidemeister (1893-1971), and met Richard Brauer (1901-1977).

She returned to Frankfurt in 1934 and held a Lehrauftrag there while writing her Habilitationsschrift. On 9 February, 1937, Ruth Moufang became the third woman in Germany to receive the Habilitation in mathematics.

The logical course of events would have been for her to become a Privatdozent, but in March 1937, she received a letter from the Ministry of Education informing her that the policies of the Third Reich required a professor to be a *leader* of students in more than just the academic sphere. Since the student population was mostly male, they did not think it feasible to appoint women professors.

Although they had no objection if she devoted herself solely to research, since there was no permanent position at the university to do only research, she left to work for the Krupps Research Institute in the autumn of 1937 and stayed there until 1946. She was the first German woman with a doctorate to be employed as an industrial mathematician. In this period, she published several papers in theoretical physics, in particular elasticity theory [Moufang 1941-1942], [Moufang 1946-1947], [Moufang 1948].

After the war, the University of Frankfurt was looking for first-rate mathematicians who had not joined any Nazi organization under Hitler. In 1946, Moufang moved there and was given the venia legendi. She served as Privatdozent until her appointment as associate professor in December 1947. In February 1957, she became the first woman in Germany to be appointed full professor and remained at the University of Frankfurt until her retirement in 1970. In the postwar years, she published almost nothing, although she was a successful teacher and had many Ph.D. students. She died in Frankfurt on 26 November, 1977.

4.1 – Moufang planes

Moufang's works from 1931 and 1937(11), marked the starting point of a new mathematical specialty in the algebraic analysis of projective planes that drew upon a mixture of geometry and algebra. She studied what are known today as *Moufang planes* and *Moufang loops*. Her studies became part of the foundations of geometry and were inspired by Max Dehn's work.

Her dissertation of 1931 inaugurated the systematic study of non-Desarguesian planes. In her first works, written between 1931 and 1932 and suggested by M. Dehn, she studied problems about *incidence theorems* in a projective plane. In particular, she investigated, in general, when one incidence configuration follows from another and examined what the consequences of this are on the introduction of coordinates:

"*[t]he following study which has arisen from a suggestion of Herr Dehn, is at the foundations of a very general problem in projective planes: let there be two incidence configurations, the problem is to decide whether or not, under*

(11)See [Moufang 1931], [Moufang 1931a], [Moufang 1932], [Moufang 1932a], [Moufang 1933], [Moufang 1933a], [Moufang 1934], [Moufang 1937].

the axioms of order and incidence, one follows from the other." [Moufang 1931, p. 536].

In her main work, *Alternative Körper und der Satz von Vollstadingen Vierseit* (D_9) [Moufang 1933a], she constructed the non-Desarguesian planes coordinatizited by an alternative division ring[12] (of characteristic $\neq 2$), that now are named *Moufang planes*, exhibiting a delicate interplay between geometry and algebra.

Of particular interest here, however, is how Moufang hit upon the idea of relating alternative division rings to the geometric construction of Moufang planes. As noted, Moufang's early work concerned the problem of considering the relationship between various *theorems on configurations* in a projective plane. In the paper *Die Schnittpunktsätze des projektiven speziellen Fünfecknetzes in ihrer Abhängigkeit voneinander* [Moufang 1932], proposed by Max Dehn, she investigated some special cases of Desargues's theorem, analyzing whether from one of these configurations the other ones follow and whether they are equivalent to the complete quadrilateral theorem.

In particular, she considered a special case of Desargues's theorem, which she called D_9. Here, the triangles $1'2'3'$ and 123 are in perspective with respect to the point a, one side of each triangle goes through one vertex of the other, two vertices of one of the perspective triangles lie on two sides of the other one.

At this point in her argument, Moufang followed Hilbert's method. As she explained,

"*[w]e now investigate, as in Hilbert's Grundlagen der Geometrie (Chapter V, 7th Edition), to what extent the calculus of a skew field (which obtains in the presence of Desargues's theorem) obtains under the configuration* D_9". [Moufang 1932, p. 766].

Therefore, she introduced the operations of addition and multiplication, as in the *Grundlagen*, by using D_9 and obtained the following rules:

$$(*)\qquad \begin{aligned} \alpha+\beta &= \beta+\alpha \\ (\alpha+\beta)+\gamma &= \alpha+(\beta+\gamma) \\ \alpha(\beta+\gamma) &= \alpha\beta+\alpha\gamma \\ (\beta+\gamma)\alpha &= \beta\alpha+\gamma\alpha \end{aligned}$$

if $\alpha \neq 0$, then there exists α^{-1}, such that $\alpha^{-1}\alpha = 1 = \alpha\alpha^{-1}$, and $\alpha^{-1}(\alpha\beta) = \beta = (\beta\alpha)\alpha^{-1}$. [Moufang 1932, pp. 767-771].

[12] Recall that an alternative division ring is a triple $(A, +, \cdot)$, where $(A, +)$ is an Abelian group and where $(A, \cdot)$ is a quasigroup with identity (loop), in which the distributive laws and the *alternative laws* are satisfied: $\forall a, b \in, A$ $a(ab) = (aa)b$, $a(ba) = (ab)a$, $(ba)a = b(aa)$. The multiplication is thus, in general, non-associative.

Thus, she showed that, in general, the multiplication is non associative.

In [Moufang 1932], she did not recognize the equivalence between a structure with properties (*) and an alternative division ring. She also neither constructed planes coordinatized by such a structure nor showed that D_9 is weaker than Desargues's theorem. In fact, her paper was completely geometric and ended with the following question: can Desargues's theorem follow from the configuration D_9? She conjectured an answer in the negative.

After completing this work, Moufang went, to Königsberg in 1932, where she was influenced by both Reidemeister and Brauer. Her stay there was decisive for her growth as an algebraist and for her subsequent research.

In fact, her main work, *Alternative Korper und der Satz von Vollstadingen Vierseit* (D_9) [Moufang 1933a] was written while she was in Königsberg. There, she thanked Brauer explicitly "*for the hint that, according to the introduction of the paper of Herr Zorn*", the number system she had constructed was a "*generalization of Cayley's number system, as presented in* 133*, Dickson's Algebren und ihre Zahlentheorie* (*p.* 264)", namely, a Cayley-Dickson system [Moufang 1933a, p. 222].

This paper is more algebraic than the others, she exhibited a delicate interplay between geometry and algebra. Her approach was systematic and followed Hilbert's method:

"*Hilbert had shown that a subset of his axioms for plane geometry (essentially the incidence axioms) together with the incidence theorem of Desargues allows for the introduction of coordinates on a straight line that are elements of a skew field.*

He proceeded as follows: he had defined the operations of addition and multiplication and their inverses, using the incidence theorems. So, Desargues's theorem and the incidence axioms are sufficient to prove the calculus rules except for the commutative rule of multiplication. Conversely, he had constructed a geometry in which Desargues's theorem is valid, by using the elements of a skew field.

We investigate in the same way, by using a particular case of Desargues's theorem, which is equivalent to the theorem of the complete quadrilateral (we call this theorem D_9*).*" [Moufang 1933a, p. 207].

Therefore, she first considered a plane coordinatized by a structure that satisfies (*) and proved, following Hilbert's method, that it is a projective plane and that the configuration D_9 is valid [Moufang 1933a, p. 211-215]. She also established the equivalence of D_9 and an alternative division ring in a purely algebraic way [Moufang 1933a, pp. 216-219], by showing that a structure that satisfies (*) satisfies the *alternative rules*, and conversely. She closed, as Hilbert did, by exhibiting a non-Desarguesian number system and by constructing what are now called *Moufang planes.*

Thus, whereas Hilbert had shown that Desargues's theorem together with the incidence axioms of planes allows one to introduce coordinates in a projective plane which are elements of a skew field, and conversely, Moufang proved that the configuration D_9 (or, equivalently, the complete quadrilateral theorem) holds in a projective plane if and only if it can be coordinatized by an alternative division ring (of characteristic $\neq 2$). The non-Desarguesian Moufang planes are of this type.

While Moufang planes are not the first examples of non-Desarguian planes, they are very important since they gave rise to the systematic study of such planes. Ruth Moufang recognized the connections between the geometric properties of planes and the algebraic properties of the coordinatizing structure. In 1943, Marshall Hall introduced a general way to coordinatize every projective plane by planary ternary rings and provided a classification that exploited the relationship between the algebraic and geometric properties [Hall 1943]. Moufang's results and techniques thus led in a crucial way to a *modern* method of classification of algebraic and geometric structures.

4.2 – The Alternative division ring

The general structure of alternative division ring was risen with the discovery of the Octonions, who are an example of such structure. Here, we sketch their history. The Octonions were discovered in 1843 by John Thomas Graves (1806-1870), who called them octaves. He was an Irish jurist, a mathematician and he was a friend of W.R. Hamilton. He is credited both with inspiring Hamilton to discover the Quaternions [Baez 2001].

The Octonions were discovered independently by Arthur Cayley (1821-1895) who published the first paper on them in 1845 [Cayley 1845]. Subsequently, in 1847 Cayley showed that they are not associative [Cayley 1847] and in 1881 he found their moltiplicative table [Cayley 1881]. Therefore, they are sometimes referred to as Cayley numbers or the Cayley algebra.

In 1912, after 31 years, Leonard Eugene Dickson (1874-1954) showed that the Octonions are a division ring [Dickson 1912] and in 1914 he found a new description of the Octonions as an ordered pairs of quaternions, with multiplication and conjugation defined exactly as for the quaternions [Dickson 1914]. This description is called Cayley-Dickson construction.

Max Zorn first introduced abstract alternative division rings in his paper, *Theorie der alternativen Ringe* [Zorn 1930], after he has noted that Cayley numbers satisfies the "Alternative laws"; he credited with Emil Artin (1898-1962) some results. Zorn also showed that a finite alternative division ring is a skew field. It followed that a finite Moufang plane is Desarguesian.

R. Moufang also opened up new avenues in the field of abstract algebra. In the work [Moufang 1934], she studied the multiplicative structure of alternative division rings, constructing the objects now called *Moufang loops.* The main theorem in this area was then showed by L.A. Skornjakov [Stornjakov 1950, pp. 74-84] and Richard Bruck and Erwin Kleinfeld [Bruk and Kleinfeld 1951, p. 887], namely, that any alternative division ring of characteristic $\neq 2$ is either associative or a Cayley-Dickson algebra over its center.

The intrinsic link between the Moufang planes and the Octonions was further confirmed by studies of Ernst Pasqual Jordan (1902-1980). In 1949, he showed that the idempotent elements of the exceptional algebra $h_3(\mathcal{O})$[13] are a Moufang plane [Jordan 1949].

In the first half of '900 was found a link between the Octonions and the theory of Lie algebras. Between 1887 and 1890 Wilhelm Killing classified the simple Lie algebras and he recognized the five exceptional algebras, and then the corresponding five exceptional Lie groups: G_2, F_4, E_6, E_7, E_8. The Octonions intervened in this context, because, it is possible to describe four of these exceptional groups through to them.

In 1914, Elie Cartan (1869-1951) showed that the group of automorphisms of the Octonions is G_2 [Cartan 1914]. In 1950, Armand Borel (1923-2003) observed that F_4 is the group of isometries of the Moufang plane [Borel 1950]. In the same year Claude Chevalley (1909-1984) and Richard Schafer (1918 -) showed that F_4 is the automorphism group of the exceptional Jordan algebra $h_3(\mathcal{O})$ and they described E_6 as the group of linear transformations of $h_3(\mathcal{O})$ that preserves the determinant [Chevalley and Schafer 1950]. Finally, in 1954 Hans Freudenthal (1905-1990) described the group E_7 as the automorphisms group of a 56-dimensional structure of Octonions [Freudenthal 1954].

5 – Conclusions

It follows from the previous considerations that the approach of Hilbert was at least partially realized by his students, in particular by Max Dehn. Dehn's approach was "modern", and he was interested, moreover, precisely in the mutual interrelation between algebraic structure and geometric relations. This point of view marked the starting point of a "new mathematics", one whose principal author was Ruth Moufang. In her works, she embraced Dehn's "modern" approach as well as ideas on "structural" algebra that were increasingly defining a school of algebraic research around Emmy Noether.

We may, also, note that Veronese's pioneering work did not give rise to a real mathematical school, but to a lasting debate on the subject of non-Archimedean geometry also within Italian geometers. Infact, what it is happened in German and in America did not happen in Italy. Before Hilbert, the contributions of the Italian geometrical school, with Riccardo De Paolis (1854-1892), Federico Enriques (1871-1946), Gino Fano (1871-1952), Giuseppe Peano (1858-1932), Mario

[13] Which is the algebra of Hermitian matrices over Octonions with product $ab = \frac{1}{2}(ab + ba)$.

Pieri (1860-1913), Corrado Segre (1863-1924) and Giuseppe Veronese (1854-1917) to the Foundantions of Projective Geometry were considerable.

After Hilbert, the foundations of geometry in Italy were totally replaced by the works of the German and the American schools. One of the reasons was that the Italian school considered the foundations as the end of a process of building a geometric theory [Avellone et al. 2002] contrary to the Grundlagen that inspired a new phase of geometry researches, as we have seen.

Furthermore, we find interesting to analyze the "case" of the Octonions. In fact their history shows how an abstract theory can find unexpected applications long after its introduction. It seemed that the role of the Octonions in Mathematics was simply to be an example of non-associative structure and even within the same Cayley after their introduction, forgot to them for about 30 years. As we have seen, it was due to arrive in the first half of the twentieth century to find their first important applications both the projective geometry and the theory of Lie.

Surprisingly, in the eighties it was assumed a link between the Octonions and the String theory. The physicists have found that in the spaces of dimensions 3, 4, 6 and 10 each spinor can be represented as a pair of elements of the same algebra. This is satisfied only if the space has dimension 2 plus the dimensions of a normed division algebra. So that, from the dimensions 1, 2, 4, 8 are obtained just 3, 4, 6 and 10. Therefore, the String theories candidate to be defined are, respectively, the Real, the Complex, the Quaternionic and the Octonionic and it seems that the most authoritative to be a model of physical reality is just that Octonionic.

This assumption made them return the focus of scientific research today and made them the protagonists of many books both informative and technical as [Stewart 2008], [Conway and Smith 2003].

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