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Deformations of algebraic subvarieties

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Dedicated to Professor Marialuisa de Resmini

Abstract: *In this paper, we use (bi)semicosimplicial language to study the classical problem of infinitesimal deformations of a closed subscheme in a fixed smooth variety, defined over an algebraically closed field of characteristic 0. In particular, we give an explicit description of the differential graded Lie algebra controlling this problem.*

– Introduction

In the last fifty years, deformation theory has played an important role in algebraic and complex geometry. The main goal is the classification of families of geometric objects in such a way that the classifying space (the so called moduli space) is a reasonable geometric space. In particular, each point of our moduli space corresponds to one geometric object (class of isomorphism). The study of small deformations of the complex structures of complex manifolds started with the works of K. Kodaira and D.C. Spencer [KoSp58] and M. Kuranishi [Ku71]. Then, A. Grothendieck [Gr59], M. Schlessinger [Schl68] and M. Artin [Ar76] formalized this theory translating it into a functorial language. The idea is that, with a infinitesimal deformations of a geometric object, we can associate a deformation functor of Artin rings $F : \mathbf{Art} \to \mathbf{Set}$. For example, we can study the functor Def_X of infinitesimal deformations of a variety X or the functor Hilb_X^Z of infinitesimal deformations of a subvariety Z in a fixed variety X. The fundamental fact is that, using these functors, we are able to study the formal neighborhood of the points in the moduli space. In particular, we can

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determine the tangent space or analyze the obstructions (smoothness) problem [Ma07, Ia07, IM09].

A modern approach to the study of deformation functors, associated with geometric objects, is via differential graded Lie algebras or, in general, via L_∞-algebras. At this stage, we can think about these structures as a generalization of differential graded vector spaces in which we also have a bracket, plus some compatibility conditions between the differential and the bracket. Once we have a differential graded Lie algebra L, we can define the associated deformation functor $\mathrm{Def}_L : \mathbf{Art} \to \mathbf{Set}$, using the solutions of the Maurer-Cartan equation up to gauge equivalence.

The guiding principle is the idea due, at least, to P. Deligne, V. Drinfeld, D. Quillen and M. Kontsevich [Kon03] that "in characteristic zero every deformation problem is controlled by a differential graded Lie algebra". In other words, if F is the deformation functor associated with a geometric problem, then there exists a differential graded Lie algebra L (up to quasi-isomorphism) such that $\mathrm{Def}_L \cong F$. We point out that it is easier to study a deformation functor associated with a differential graded Lie algebra but, in general, it is not an easy task to find the right differential graded Lie algebra (up to quasi-isomorphism) associated with the problem [Kon94]. A first example, in which the associated differential graded Lie algebra is well understood, is the case of deformations of complex manifolds. If X is a complex compact manifold, then the infinitesimal deformations of X are controlled by its Kodaira-Spencer algebra KS_X, see [GM90, Ma04b, Ma09] and [Ia06, Theorem II.7.3]. We recall that $KS_X = \oplus_i \Gamma(X, \mathcal{A}_X^{0,i}(\Theta_X))$, where $\mathcal{A}_X^{0,i}(\Theta_X)$ is the sheaf of the $(0,i)$-forms on X, with values in the holomorphic tangent bundle Θ_X.

In general, if we work over an algebraically closed field of characteristic zero, different from the complex numbers, then we can not use the Kodaira-Spencer algebra.

A strategy to solve this problem and "produce" differential graded Lie algebras, is via semicosimplicial objects [Hin97, Pr03, FMM08, FIM09]. Actually, the fundamental idea goes back to K. Kodaira and D.C. Spencer: "a deformation of X is regarded as the gluing of the same polydisks via different identifications" [Kod86, pag. 182]. In other words, a deformation of a geometric object consists in deforming the object locally and then glue back together these local deformations. Then, from the algebraic point of view, we have to find the algebraic objects that control locally the deformations and then glue them together. Thus, we can think at a semicosimplcial object as a sequence of objects, that controls locally the deformations, and a sequence of maps, that controls the gluing. For example, let X be a smooth projective variety, over an algebraically closed field $\mathbb{K}$ of characteristic 0, with tangent sheaf Θ_X. Given an affine open cover $\mathcal{U} = \{U_i\}$ of X, we can define the Čech semicosimplicial Lie algebra $\Theta_X(\mathcal{U})$, *i.e.*, we have a sequence of Lie algebras $\{\mathfrak{g}_k = \prod_{i_0<\cdots<i_k} \Theta_X(U_{i_0\cdots i_k})\}$ and a "lot" of maps among them, that are the

restrictions to open subsets. In particular, $\mathfrak{g}_0 = \prod_i \Theta_X(U_i)$ and each $\Theta_X(U_i)$ controls the infinitesimal deformations of U_i; moreover, the maps controls the gluing of deformations, see [FMM08] and [IM09, Section 5].

In general, we will have a semicosimplicial differential graded Lie algebra, $\mathfrak{g}^\Delta = \{\mathfrak{g}_k\}_k$, with $\mathfrak{g}_0$ that controls the deformations of each open of the cover, as in the case of deformations of varieties or of coherent sheaves [FIM09, FIM].

Next, once we have a semicosimplicial differential graded Lie algebra $\mathfrak{g}^\Delta$, we need to find out just one differential graded Lie algebra. Following [NaA87, FMM08], there is a canonical way to define a differential graded Lie algebra $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$, using the Thom-Whitney construction. In conclusion, given a geometric deformation problem, if we are able to associate with it a semicosimplicial differential graded Lie algebra, then we can find out just one differential graded Lie algebra controlling our problem.

Inspired by these ideas, in this paper we use semicosimplicial language to study infinitesimal deformations of closed subschemes. More precisely, let X be a smooth variety, defined over an algebraically closed field $\mathbb{K}$ of characteristic 0, and $Z \subset X$ a closed subscheme. Denote by Hilb_X^Z the functor of infinitesimal deformations of Z in X and by Hilb'^Z_X the subfunctor of locally trivial infinitesimal deformations. We recall that $\mathrm{Hilb}_X^Z = \mathrm{Hilb}'^Z_X$, whenever Z is smooth. For $\mathbb{K} = \mathbb{C}$ and Z smooth, the analysis of this problem via differential graded Lie algebra is due to M. Manetti [Ma07]. Here, we extend his work to all algebraically closed fields $\mathbb{K}$ of characteristic 0, using semicosimplicial language; more precisely, it is convenient to use bisemicosimplial Lie algebras. Indeed, let Θ_X be the tangent sheaf of X and $\Theta_X(-\log Z)$ the sheaf of tangent vectors to X which are tangent to Z. Denote by $\chi : \Theta_X(-\log Z) \hookrightarrow \Theta_X$ the inclusion of sheaves of Lie algebras. We can associate with $\Theta_X(-\log Z)$ and Θ_X the Čech semicosimplicial Lie algebra $\Theta_X(-\log Z)(\mathcal{U})$ and $\Theta_X(\mathcal{U})$, respectively; and so we can consider the bisemicosimplicial Lie algebra $\chi^\blacktriangle : \Theta_X(-\log Z)(\mathcal{U}) \to \Theta_X(\mathcal{U})$. Once again, using the Thom-Whitney construction, we can define a differential graded Lie algebra $\mathrm{Tot}^\blacktriangle_{TW}(\chi^\blacktriangle)$. This algebra controls the deformations of the closed subscheme Z; more precisely, we prove the following theorem.

Theorem (A). *Let X be a smooth variety, defined over an algebraically closed field $\mathbb{K}$ of characteristic 0, and $Z \subset X$ a closed subscheme. Then, there exists an isomorphism of functors* $\mathrm{Def}_{\mathrm{Tot}^\blacktriangle_{TW}(\chi^\blacktriangle)} \cong \mathrm{Hilb}'^Z_X$. *In particular, if $Z \subset X$ is smooth, then* $\mathrm{Def}_{\mathrm{Tot}^\blacktriangle_{TW}(\chi^\blacktriangle)} \cong \mathrm{Hilb}^Z_X$.

In a forthcoming paper, we will use this theorem to study the obstruction to deformations of Z in X, via the semiregularity map.

The paper goes as follows: the first section is intended for the nonexpert reader and is devoted to recall the basic notions of differential graded Lie algebras and their role in deformation theory.

In Section 2, we introduce semicosimplicial objects and total constructions. In particular, we review semicosimplicial differential graded Lie algebras, the corresponding Thom-Whitney DGLA and the associated deformation functors.
Sections 3 is devoted to bisemicosimplicial objects and, again, to the total constructions and the associated deformation functors. In particular, we describe the bisemicosimplicial Lie algebra $\chi^{\blacktriangle} : \Theta_X(-\log Z)(\mathcal{U}) \to \Theta_X(\mathcal{U})$, associated with the inclusion $\chi : \Theta_X(-\log Z) \hookrightarrow \Theta_X$.
In Section 4, we go back to geometric applications and we prove Theorem A.

NOTATION. Throughout the paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. All vector spaces, linear maps, tensor products etc. are intended over $\mathbb{K}$. We denote by **Set** the category of sets (in a fixed universe) and by **Art** the category of local Artinian $\mathbb{K}$-algebras (with residue field $\mathbb{K}$). If A is an object in **Art**, then $\mathfrak{m}_A$ denotes its maximal ideal.

1 – Review of differential graded Lie algebras

A differential graded vector space is a pair (V, d), where $V = \oplus_{i\in\mathbb{Z}} V^i$ is a $\mathbb{Z}$-graded vector space and d is a differential of degree $+1$, *i.e.*, $d : V^i \to V^{i+1}$ and $d \circ d = 0$. For every integer n, we define a new differential graded vector space $V[n]$, by setting

$$V[n]^i = V^{n+i} \qquad \text{and} \qquad d_{V[n]} = (-1)^n d_V.$$

DEFINITION 1.1. *A differential graded Lie algebra (DGLA for short) is a triple $(L, [\ ,\ ], d)$, where $(L = \oplus_{i\in\mathbb{Z}} L^i, d)$ is a differential graded vector space and $[\ ,\ ] : L \times L \to L$ is a bilinear map of degree zero, called bracket, satisfying the following conditions:*

(1) (*graded skewsymmetry*) $[a, b] = -(-1)^{\deg(a)\deg(b)}[b, a]$;
(2) (*graded Jacobi identity*) $[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)}[b, [a, c]]$;
(3) (*graded Leibniz rule*) $d[a, b] = [da, b] + (-1)^{\deg(a)}[a, db]$.

EXAMPLE 1.2. If $L = \oplus L^i$ is a DGLA, then L^0 is a Lie algebra in the usual sense; vice-versa, every Lie algebra is a differential graded Lie algebra concentrated in degree 0 (and differential zero).

EXAMPLE 1.3. If L is a DGLA and B is a commutative $\mathbb{K}$-algebra, then $L \otimes B$ has a natural structure of DGLA, given by

$$[l \otimes a, m \otimes b] = [l, m] \otimes ab;$$

$$d(l \otimes a) = dl \otimes a.$$

A morphism of differential graded Lie algebras $\phi\colon L \to M$ is a linear map that preserves degrees and commutes with brackets and differentials. A quasi-isomorphism of DGLAs is a morphism that induces an isomorphism in cohomology. Two DGLAs L and M are said to be quasi-isomorphic if they are equivalent under the equivalence relation generated by: $L \sim M$ if there exists a quasi-isomorphism $\phi : L \to M$.

1.1 – Deformation functor associated with a DGLA

Definition 1.4. *Let L be a DGLA; then, the Maurer-Cartan functor associated with L is the functor*

$$MC_L : \mathbf{Art} \to \mathbf{Set},$$

$$MC_L(A) = \left\{ x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x,x] = 0 \right\}.$$

Note that in the previous equation we use the DGLA structure on $L \otimes m_A$ induced by the one on L (see Example 1.3).

Definition 1.5. *Two elements x and $y \in L^1 \otimes m_A$ are gauge equivalent if there exists $a \in L^0 \otimes m_A$ such that*

$$y = e^a * x := x + \sum_{n \geq 0} \frac{[a, -]^n}{(n+1)!}([a,x] - da).$$

The operator $*$ is called the gauge action of the group $\exp(L^0 \otimes m_A)$ on $L \otimes m_A$; indeed, $e^a * e^b * x = e^{a \bullet b} * x$, where $\bullet$ is the Baker-Campbell-Hausdorff product in the nilpotent DGLA $L \otimes m_A$, *i.e.*, $a \bullet b = a + b + \frac{1}{2}[a,b] + \frac{1}{12}[a,[a,b]] - \frac{1}{12}[b,[b,a]] + \cdots$.

Definition 1.6. *The deformation functor associated with a differential graded Lie algebra L is:*

$$\mathrm{Def}_L : \mathbf{Art} \to \mathbf{Set},$$

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}_L(A)}{\text{gauge}} = \frac{\{x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0\}}{\exp(L^0 \otimes \mathfrak{m}_A)}.$$

Remark 1.7. Every morphism of DGLAs induces a natural transformation of the associated deformation functors. If L and M are quasi-isomorphic DGLAs, then the associated functor Def_L and Def_M are isomorphic [SS79, GM88, GM90], [Ma99, Corollary 3.2], or [Ma04b, Corollary 5.52].

2 – Semicosimplicial objects

Let Δ_{mon} be the category whose objects are the finite ordinal sets $[n] = \{0, 1, \ldots, n\}$, $n = 0, 1, \ldots$, and whose morphisms are order-preserving injective maps among them. Every morphism in Δ_{mon}, different from the identity, is a finite composition of coface morphisms:

$$\partial_k \colon [i-1] \to [i], \qquad \partial_k(p) = \begin{cases} p & \text{if } p < k \\ p+1 & \text{if } k \leq p \end{cases}, \qquad k = 0, \ldots, i.$$

The relations about compositions of them are generated by

$$\partial_l \partial_k = \partial_{k+1} \partial_l \,, \qquad \text{for every } l \leq k.$$

Definition 2.1. *According to [$EZ50, We94$], a semicosimplicial object in a category* $\mathbf{C}$ *is a covariant functor* $A^{\Delta} \colon \Delta_{\mathrm{mon}} \to \mathbf{C}$. *Equivalently, a semicosimplicial object* A^{Δ} *is a diagram in* $\mathbf{C}$*:*

$$A_0 \rightrightarrows A_1 \; \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \; A_2 \; \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \; \cdots,$$

where each A_i *is in* $\mathbf{C}$*, and, for each* $i > 0$*, there are* $i + 1$ *morphisms*

$$\partial_k \colon A_{i-1} \to A_i, \qquad k = 0, \ldots, i,$$

such that $\partial_l \partial_k = \partial_{k+1} \partial_l$*, for any* $l \leq k$.

Example 2.2. Let $\chi : L \to M$ be a morphism in a category $\mathbf{C}$. Then, we can consider it as a semicosimplicial object in $\mathbf{C}$, by extension with zero, *i.e.*,

$$\chi^{\Delta} : \quad L \rightrightarrows M \; \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \; 0 \cdots, \quad \partial_0 = \chi, \quad \partial_1 = 0,$$

Example 2.3. Let X be a smooth variety, defined over an algebraically closed field of characteristic 0. Let $\mathcal{U} = \{U_i\}$ be an affine open cover and $\mathcal{F}$ a sheaf of Lie algebras on X. Then, we can define the Čech semicosimplicial Lie algebra $\mathcal{F}(\mathcal{U})$ as the semicosimplicial Lie algebra

$$\mathcal{F}(\mathcal{U}) : \quad \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i<j} \mathcal{F}(U_{ij}) \; \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \; \prod_{i<j<k} \mathcal{F}(U_{ijk}) \; \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \; \cdots,$$

where the coface maps $\partial_h\colon \prod_{i_0<\cdots<i_{k-1}} \mathcal{F}(U_{i_0\cdots i_{k-1}}) \to \prod_{i_0<\cdots<i_k} \mathcal{F}(U_{i_0\cdots i_k})$ are given by

$$\partial_h(x)_{i_0\ldots i_k} = x_{i_0\ldots\widehat{i_h}\ldots i_k|U_{i_0\cdots i_k}}, \qquad \text{for } h=0,\ldots,k.$$

2.1 – The total construction

Given a semicosimplicial differential graded vector space

$$V^{\Delta}: V_0 \rightrightarrows V_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} V_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots,$$

the graded vector space $\bigoplus_{n\geq 0} V_n[-n]$ has two differentials, *i.e.*,

$$d = \sum_n (-1)^n d_n, \qquad \text{where} \quad d_n \text{ is the differential of } V_n,$$

and

$$\partial = \sum_i (-1)^i \partial_i, \qquad \text{where} \quad \partial_i \text{ are the coface maps.}$$

More explicitly, if $v \in V_n^i$, then the degree of v is $i+n$ and

$$d(v) = (-1)^n d_n(v) \in V_n^{i+1},\ \partial(v) = \partial_0(v) - \partial_1(v) + \cdots + (-1)^{n+1}\partial_{n+1}(v) \in V_{n+1}^i.$$

Since $d\partial + \partial d = 0$, we define $\operatorname{Tot}(V^{\Delta})$ as the graded vector space $\bigoplus_{n\geq 0} V_n[-n]$, endowed with the differential $D = d + \partial$.

Remark 2.4. In Example 2.3, the total complex $\operatorname{Tot}(\mathcal{F}(\mathcal{U}))$, associated with the Čech semicosimplicial Lie algebra $\mathcal{F}(\mathcal{U})$, is nothing else that the Čech complex $\check{C}(\mathcal{U},\mathcal{F})$ of the sheaf $\mathcal{F}$.

There is also another way to associate with a semicosimplicial differential graded vector space V^{Δ} a differential graded vector space. Namely, let $(A_{PL})_n$ be the differential graded commutative algebra of polynomial differential forms on the standard n-simplex $\{(t_0,\ldots,t_n)\in\mathbb{K}^{n+1} \mid \sum t_i = 1\}$ [FHT01]:

$$(A_{PL})_n = \frac{\mathbb{K}[t_0,\ldots,t_n,dt_0,\ldots,dt_n]}{(1-\sum t_i, \sum dt_i)}.$$

For every n,m the tensor product $V_n \otimes (A_{PL})_m$ is a differential graded vector space and then also $\prod_n V_n \otimes (A_{PL})_n$ is a differential graded vector space. Denote by

$$\delta^k\colon (A_{PL})_n \to (A_{PL})_{n-1}, \quad \delta^k(t_i) = \begin{cases} t_i & \text{if } 0\leq i<k \\ 0 & \text{if } i=k \\ t_{i-1} & \text{if } k<i \end{cases}, \qquad k=0,\ldots,n,$$

the face maps, for every $0 \leq k \leq n$; then, there are well-defined morphisms of differential graded vector spaces

$$Id \otimes \delta^k \colon V_n \otimes (A_{PL})_n \to V_n \otimes (A_{PL})_{n-1},$$
$$\partial_k \otimes Id \colon V_{n-1} \otimes (A_{PL})_{n-1} \to V_n \otimes (A_{PL})_{n-1}.$$

The Thom-Whitney differential graded vector space $\mathrm{Tot}_{TW}(V^\Delta)$ of V^Δ is the differential graded subvector space of $\prod_n V_n \otimes (A_{PL})_n$, whose elements are the sequences $(x_n)_{n\in\mathbb{N}}$ satisfying the equations

$$(Id \otimes \delta^k)x_n = (\partial_k \otimes Id)x_{n-1}, \quad \text{for every} \;\; 0 \leq k \leq n.$$

LEMMA 2.5. *The differential graded vector spaces* $\mathrm{Tot}(V^\Delta)$ *and* $\mathrm{Tot}_{TW}(V^\Delta)$ *are quasi-isomorphic.*

PROOF. See [Whi57, Dup76, Dup 78, NaA87, Get04, FMM08, CG08] for explicit description of the quasi-isomorphism. □

Let

$$\mathfrak{g}^\Delta : \quad \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \;\substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow}\; \mathfrak{g}_2 \;\substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow}\; \cdots,$$

be a semicosimplicial differential graded Lie algebra. Since, every DGLA is, in particular, a differential graded vector space, we can consider the associated total complex $\mathrm{Tot}(\mathfrak{g}^\Delta)$. Even if all $\mathfrak{g}_i$ are DGLAs, there is no natural DGLA structure on $\mathrm{Tot}(\mathfrak{g}^\Delta)$ [FiMa07, IM09]

EXAMPLE 2.6. Let $\chi : L \to M$ be a morphism of DGLAs, then, following Example 2.2, we can associate with it a semicosimplicial DGLA. Its total complex $\mathrm{Tot}(\chi^\Delta)$ is nothing else than the (suspension of the) mapping cone complex associated with χ. Even in this simple case, it is not possible to define a canonical DGLA structure on $\mathrm{Tot}(\chi^\Delta)$, such that the projection $\mathrm{Tot}(\chi^\Delta) \to L$ is a morphism of DGLAs [IM09, Example 3.1].

However, in the case of semicosimplicial DGLAs, we can apply the Thom-Whitney construction to g^Δ: it turns out that $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$ has a structure of DGLA [NaA87, FMM08] .

REMARK 2.7. Using the homotopy transfer, the DGLA structure of $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$ induces an L_∞-algebra structure $\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)$ on the differential graded vector space $\mathrm{Tot}(\mathfrak{g}^\Delta)$, such that $\widetilde{\mathrm{Tot}}(\mathfrak{g}^\Delta)$ and $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$ are quasi-isomorphic; see [FiMa07, FMM08] or [IM09, Corollary 3.3].

2.2 – Deformation functor associated with semicosimplicial DGLAs

Let $\mathfrak{g}^\Delta$ be a semicosimplicial DGLA. Applying the Thom-Whitney construction of the previous section, we can consider the DGLA $\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)$ and so the associated deformation functor $\mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)}$. Beyond this way, there is another natural, and more geometric, way to define a deformation functor associated with $\mathfrak{g}^\Delta$, see [Pr03, Definitions 1.4 and 1.6], [FMM08, Section 3] or [FIM09, Definition 2.1 and 2.2].

More precisely, if $\mathfrak{g}^\Delta$ is a semicosimplicial DGLA, we can define the functor

$$Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)\colon \mathbf{Art} \to \mathbf{Set},$$

such that, for all $A \in \mathbf{Art}$, $Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)(A)$ is the set of the pairs $(l,m) \in (\mathfrak{g}^1_0 \otimes \mathfrak{m}_A) \oplus (\mathfrak{g}^0_1 \otimes \mathfrak{m}_A)$, satisfying the following conditions:

(1) $dl + \frac{1}{2}[l,l] = 0$;
(2) $\partial_1 l = e^m * \partial_0 l$;
(3) $\partial_0 m \bullet -\partial_1 m \bullet \partial_2 m = dn + [\partial_2\partial_0 l, n]$, for some $n \in \mathfrak{g}^{-1}_2 \otimes \mathfrak{m}_A$.

Moreover, we define the functor

$$H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)\colon \mathbf{Art} \to \mathbf{Set},$$

such that

$$H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)(A) = \frac{Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)(A)}{\sim},$$

where (l_0, m_0) and $(l_1, m_1) \in Z^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)(A)$ are equivalent under the relation $\sim$ if and only if there exist elements $a \in \mathfrak{g}^0_0 \otimes \mathfrak{m}_A$ and $b \in \mathfrak{g}^{-1}_1 \otimes \mathfrak{m}_A$, such that

(1) $e^a * l_0 = l_1$;
(2) $-m_0 \bullet -\partial_1 a \bullet m_1 \bullet \partial_0 a = db + [\partial_0 l_0, b]$.

Example 2.8. Let L be a differential graded Lie algebra, then it can be considered as a semicosimplicial DGLA $\mathfrak{L}^\Delta$ by zero extension, *i.e.*, $\mathfrak{L}^\Delta_0 = L$ and $\mathfrak{L}^\Delta_i = 0$, for all $i > 0$. In this case, the above functors $Z^1_{\mathrm{sc}}(\exp \mathfrak{L}^\Delta)$ and $H^1_{\mathrm{sc}}(\exp \mathfrak{L}^\Delta)$ reduce to MC_L and Def_L, respectively.

Example 2.9. If $\chi : L \to M$ is a morphism of DGLAs, then we can consider it as a simple case of semicosimplicial DGLA χ^Δ, extending χ by zero (see Example 2.2).

In this case, the functors $Z^1_{\mathrm{sc}}(\exp \chi^\Delta)$ and $H^1_{\mathrm{sc}}(\exp \chi^\Delta)$ coincide with the functors MC_χ and Def_χ defined in [Ma07, Section 2]. More precisely, we have

$$\mathrm{Def}_\chi(A) = \frac{\mathrm{MC}_\chi(A)}{\exp(L^0 \otimes \mathfrak{m}_A) \times \exp(dM^{-1} \otimes \mathfrak{m}_A)},$$

where

$$\mathrm{MC}_\chi(A)=\left\{(x,e^a)\in(L^1\otimes\mathfrak{m}_A)\times\exp(M^0\otimes\mathfrak{m}_A)\,|\,dx+\frac{1}{2}[x,x]=0,\ e^a*\chi(x)=0\right\},$$

and the gauge action of $\exp(L^0\otimes\mathfrak{m}_A)\times\exp(dM^{-1}\otimes\mathfrak{m}_A)$ is given by the formula

$$(e^l,e^{dm})*(x,e^a)=(e^l*x,e^{dm}e^ae^{-\chi(l)})=(e^l*x,e^{dm\bullet a\bullet-\chi(l)}).$$

In particular, if $\chi: L\to M$ is an injective morphism of DGLAs, then for every $A\in\mathbf{Art}$, we have

$$\mathrm{MC}_\chi(A)=\left\{e^a\in\exp(M^0\otimes\mathfrak{m}_A)\mid\ e^{-a}*0\in L^1\otimes\mathfrak{m}_A\right\}.$$

Under this identification, the gauge action becomes

$$\exp(L^0\otimes\mathfrak{m}_A)\times\mathrm{MC}_\chi(A)\to\mathrm{MC}_\chi(A),\qquad (e^m,e^a)\mapsto e^ae^{-m},$$

and then

$$\mathrm{Def}_\chi(A)=\frac{\mathrm{MC}_\chi(A)}{\exp(L^0\otimes\mathfrak{m}_A).}$$

EXAMPLE 2.10. If all $\mathfrak{g}_i=0$, for all $i>1$, then the functors $Z^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)$ and $H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)$ reduce to the functors $\mathrm{MC}_{(\partial_0,\partial_1)}$ and $\mathrm{Def}_{(\partial_0,\partial_1)}$, respectively, associated with the pair of morphisms of DGLAs $\partial_0,\partial_1:\mathfrak{g}_0\to\mathfrak{g}_1$, introduced in [Ia08].

EXAMPLE 2.11. If each $\mathfrak{g}_i$ is concentrated in degree zero, *i.e.*, $\mathfrak{g}^\Delta$ is a semicosimplicial Lie algebra, then the functors $Z^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)$ and $H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)$ reduce to the one defined in [FMM08, Section 3]. More explicitly, in this case, we have

$$Z^1_{sc}(\exp\mathfrak{g}^\Delta)(A)=\{x\in\mathfrak{g}_1\otimes\mathfrak{m}_A\mid e^{\partial_0x}e^{-\partial_1x}e^{\partial_2x}=1\},$$

and $x\sim y$ if and only if there exists $a\in\mathfrak{g}_0\otimes\mathfrak{m}_A$, such that $e^{-\partial_1a}e^xe^{\partial_0a}=e^y$.

Therefore, given a semicosimplicial DGLA $\mathfrak{g}^\Delta$, we can define two deformations functor, $\mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)}$ and $H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta)$. The relation between these functors is given by the following theorem.

THEOREM 2.12. *Let $\mathfrak{g}^\Delta$ be a semicosimplicial DGLA such that $H^k(\mathfrak{g}_i)=0$, for all i and for all $k<0$. Then, there exists a natural isomorphism of deformation functors*

$$\mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)}\simeq H^1_{\mathrm{sc}}(\exp\mathfrak{g}^\Delta).$$

Proof. In the case of semicosimplicial Lie algebra, this theorem was proved in [FMM08, Theorem 6.8]. For the general case, see [FIMM09, Theorem 7.6]. ∎

3 – Bisemicosimplicial objects

In this section, we generalize the notion of semicosimplicial objects, defining bisemicosimplicial objects.

Definition 3.1. *According to [GJ99, Chapter IV], a bisemicosimplicial object* $A^{\blacktriangle}$ *in a category* $\mathbf{C}$ *is a covariant functor* $A^{\blacktriangle}\colon \Delta_{\mathrm{mon}} \times \Delta_{\mathrm{mon}} \to \mathbf{C}$*; equivalently, a bisemicosimplicial object in* $\mathbf{C}$ *is a semicosimplicial object in the category of semicosimplicial object in* $\mathbf{C}$*. More explicitly, it consists of objects* $A_{i,j}$*, for all* $i, j \geq 0$*, and morphisms* $\partial_k^{V_i}$ *and* $\partial_s^{H_j}$ *in* $\mathbf{C}$*, for each* $i, j > 0$*, such that*

$$\partial_k^{V_i}\colon A_{i,j-1} \to A_{i,j}, \qquad k = 0, \dots, j,$$

$$\partial_s^{H_j}\colon A_{i-1,j} \to A_{i,j}, \qquad s = 0, \dots, i,$$

and the following compatibility conditions are satisfied

$$\begin{aligned} \partial_l^{V_i} \circ \partial_k^{V_i} &= \partial_{k+1}^{V_i} \circ \partial_l^{V_i}, & \forall l \leq k, \\ \partial_s^{H_j} \circ \partial_t^{H_j} &= \partial_{t+1}^{H_j} \circ \partial_s^{H_j}, & \forall s \leq t, \\ \partial_s^{H_{j+1}} \circ \partial_k^{V_i} &= \partial_k^{V_{i+1}} \circ \partial_s^{H_j}, & \forall s \leq i+1,\ k \leq j+1. \end{aligned}$$

We shall say that the object $A_{i,j}$ has bidegree (i, j) or precisely horizontal degree i and vertical degree j, and that $\partial_{\bullet}^{H_j}$ and $\partial_{\bullet}^{V_i}$ are horizontal (height j) and vertical (column i) morphisms, respectively. In particular, for all fixed j, $(A_{\bullet,j}, \partial_{\bullet}^{H_j})$ is a (horizontal) semicosimplicial object in $\mathbf{C}$; analogously, for all fixed i, $(A_{i,\bullet}, \partial_{\bullet}^{V_i})$ is a (vertical) semicosimplicial object in $\mathbf{C}$. To sum up, a bisemicosimplicial object $A^{\blacktriangle}$ looks like a diagram

$$\begin{array}{ccccccc}
\vdots & \rightrightarrows & \vdots & \substack{\to\\\to\\\to} & \vdots & \substack{\to\\\to\\\to\\\to} & \cdots \\
\upuparrows\upuparrows & & \upuparrows\upuparrows & & \upuparrows\upuparrows & & \\
A_{0,2} & \rightrightarrows & A_{1,2} & \substack{\to\\\to\\\to} & A_{2,2} & \substack{\to\\\to\\\to\\\to} & \cdots \\
\uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \\
A_{0,1} & \rightrightarrows & A_{1,1} & \substack{\to\\\to\\\to} & A_{2,1} & \substack{\to\\\to\\\to\\\to} & \cdots \\
\upuparrows & & \upuparrows & & \upuparrows & & \\
A_{0,0} & \rightrightarrows & A_{1,0} & \substack{\to\\\to\\\to} & A_{2,0} & \substack{\to\\\to\\\to\\\to} & \cdots,
\end{array}$$

where each line and each column is a semicosimplicial object and each square commutes in a simplicial sense, *i.e.*, for all s, k, i and j, the following diagram commutes

$$\begin{tikzcd}
A_{i,j+1} \arrow[r, "\partial_s^{H_{j+1}}"'] & A_{i+1,j+1} \\
A_{i,j} \arrow[u, "\partial_k^{V_i}"] \arrow[r, "\partial_s^{H_j}"'] & A_{i+1,j} \arrow[u, "\partial_k^{V_{i+1}}"']
\end{tikzcd}$$

Example 3.2. Every semicosimplicial object in a category $\mathbf{C}$ can be considered as a bisemicosimplcial object concentrated in zero (vertical or horizontal) degree.

Bisemicosimplicial objects naturally arise in simple situation. Indeed, let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a variety X, with value in a category $\mathbf{C}$. Fix an affine open cover $\mathcal{U} = \{U_i\}$. Then, as in Example 2.3, we denote by $\mathcal{F}(\mathcal{U})$ and $\mathcal{G}(\mathcal{U})$ the associated Čech semicosimplicial objects in $\mathbf{C}$. Next, let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Since φ commutes with restrictions of every open subsets, it induces a morphism $\varphi^{\Delta} : \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})$ of semicosimplcial objects. Finally, as in Example 2.2, we can consider the semicosimplicial extension of φ^{Δ} (by zero) to get a bisemicosimplcial object $\varphi^{\blacktriangle} : \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})$ in $\mathbf{C}$. This construction is commutative, *i.e.*, we can firstly extend φ (by zero) to get a semcosimplicial sheaf of object in $\mathbf{C}$, and then apply the Čech semicosimplicial construction to all sheaves.

Example 3.3. Let X be a smooth variety, defined over an algebraically closed field of characteristic 0, and $\mathcal{U} = \{U_i\}$ be an affine open cover. Let $Z \subset X$ be a closed subscheme of X. We denote by $\Theta_X(-\log Z)$ the sheaf of germs of tangent vectors to X which are tangent to Z [Se06, Section 3.4.4]. We recall that, if $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf of Z in X, then $\Theta(-\log Z) = \{f \in Der(\mathcal{O}_X, \mathcal{O}_X) \mid f(\mathcal{I}) \subset \mathcal{I}\}$. Let $\chi : \Theta_X(-\log Z) \hookrightarrow \Theta_X$ be the inclusion of sheaves of Lie algebras. Then, we can associate with $\Theta_X(-\log Z)$ and Θ_X the Čech semicosimplicial Lie algebra $\Theta_X(-\log Z)(\mathcal{U})$ and $\Theta_X(\mathcal{U})$, respectively. Finally, extending the morphism χ by zero, we get a a bisemicosimplicial Lie algebra $\chi^{\blacktriangle} : \Theta_X(-\log Z)(\mathcal{U}) \to \Theta_X(\mathcal{U})$.

More explicitly, we have the following diagram

$$
\begin{array}{ccccc}
\vdots & \rightrightarrows & \vdots & \Rrightarrow & 0 \\
\upuparrows\upuparrows & & \upuparrows\upuparrows & & \upuparrows\upuparrows \\
\prod_{i<j<k} \Theta_X(-\log Z)(U_{ijk}) & \rightrightarrows & \prod_{i<j<k} \Theta_X(U_{ijk}) & \Rrightarrow & 0 \\
\upuparrows\uparrow & & \upuparrows\uparrow & & \upuparrows\uparrow \\
\prod_{i<j} \Theta_X(-\log Z)(U_{ij}) & \rightrightarrows & \prod_{i<j} \Theta_X(U_{ij}) & \Rrightarrow & 0 \\
\upuparrows & & \upuparrows & & \upuparrows \\
\prod_{i} \Theta_X(-\log Z)(U_{i}) & \rightrightarrows & \prod_{i} \Theta_X(U_{i}) & \Rrightarrow & 0.
\end{array}
$$

3.1 – The total construction

Let $V^{\blacktriangle} = (V^*_{n,m}, d_{n,m})_{n,m}$ be a bisemicosimplicial differential graded vector space; in particular, we recall that each line and each column is a semicosimplicial differential graded vector space. Then, as in Section 2.1, with each horizontal semicosimplicial differential graded vector space $(V^{\Delta}_{\bullet,m}, \partial^{H_m}_{\bullet})$, we can associate the total complex $\mathrm{Tot}(V^{\Delta}_{\bullet,m})$. We recall that $\mathrm{Tot}(V^{\Delta}_{\bullet,m}) = \bigoplus_{n\geq 0} V^*_{n,m}[-n]$ and its differential is $D_m = \sum_n (-1)^n d_{n,m} + \sum_j (-1)^j \partial^{H_m}_j$. In this way, we construct a semicosimplicial differential graded vector space $\mathrm{Tot}^{H,\Delta}(V^{\blacktriangle})$

$$
\begin{array}{c}
\vdots \\
\upuparrows\upuparrows \\
\mathrm{Tot}(V^{\triangle}_{\bullet,2}) \\
\upuparrows\uparrow \\
\mathrm{Tot}(V^{\triangle}_{\bullet,1}) \\
\upuparrows \\
\mathrm{Tot}(V^{\triangle}_{\bullet,0}).
\end{array}
$$

In particular, we can still apply the total construction to $\mathrm{Tot}^{H,\Delta}(V^{\blacktriangle})$ to obtain the differential graded vector space $\mathrm{Tot}(\mathrm{Tot}^{H,\Delta}(V^{\blacktriangle}))$. More explicitly, $\mathrm{Tot}(\mathrm{Tot}^{H,\Delta}(V^{\blacktriangle})) = \bigoplus_m \mathrm{Tot}(V^{\Delta}_{\bullet,m})[-m] = \bigoplus_{n,m} V^*_{n,m}[-n-m]$ and the differential is $D = \sum_m (-1)^m D_m + \sum_k (-1)^k \partial^{V_\bullet}_k = \sum_{m,n} (-1)^{m+n} d_{n,m} + \sum_{j,m} (-1)^{j+m} \partial^{H_m}_j + \sum_k (-1)^k \partial^{V_\bullet}_k$.

Analogously, given $V^{\blacktriangle} = (V^*_{n,m}, d_{n,m})_{n,m}$, we can firstly focus our attention on each vertical semicosimplicial differential graded vector space $(V^{\Delta}_{n,\bullet}, \partial^{V_n}_{\bullet})$. As before, we can associate with each column its total complex, to get a semicosimplicial differential graded vector space $\mathrm{Tot}^{V,\Delta}(V^{\blacktriangle})$

$$\mathrm{Tot}(V^{\Delta}_{0,\bullet}) \rightrightarrows \mathrm{Tot}(V^{\Delta}_{1,\bullet}) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathrm{Tot}(V^{\Delta}_{2,\bullet}) \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \cdots,$$

In this case, $\mathrm{Tot}(V^{\Delta}_{n,\bullet}) = \bigoplus_m V^*_{n,m}[-m]$ and its differential is given by $D'_n = \sum_m (-1)^m d_{n,m} + \sum_j (-1)^j \partial^{V_n}_j$. Then, applying again the total construction to $\mathrm{Tot}^{V,\Delta}(V^{\blacktriangle})$, we get the differential graded vector space $\mathrm{Tot}(\mathrm{Tot}^{V,\Delta}(V^{\blacktriangle}))$. In this case, we have $\mathrm{Tot}(\mathrm{Tot}^{V,\Delta}(V^{\blacktriangle})) = \bigoplus_n \mathrm{Tot}(V^{\Delta}_{n,\bullet})[-n] = \bigoplus_{n,m} V^*_{n,m}[-n-m]$ and the differential is $D' = \sum_n (-1)^n D'_n + \sum_k (-1)^k \partial^{H_\bullet}_k = \sum_{m,n} (-1)^{n+m} d_{n,m} + \sum_{j,n} (-1)^{j+n} \partial^{V_n}_j + \sum_k (-1)^k \partial^{H_\bullet}_k$.

Moreover, we can also consider the total complex $(\mathrm{Tot}^{\blacktriangle}(V^{\blacktriangle}), D)$ associated with the triple complex $(V^*_{n,m}, d_{n,m}, \partial^V, \partial^H)$. More explicitly, $\mathrm{Tot}^{\blacktriangle}(V^{\blacktriangle})^i = \bigoplus_{n,m} V_{n,m}[-n-m]^{i-n-m}$ and the differential is given by $D = d + \partial_1 + \partial_2$, where $d = \sum_{m,n} (-1)^{m+n} d_{n,m}$, $\partial_1 = \sum_{j,m} (-1)^{j+m} \partial^{H_m}_j$ and $\partial_2 = \sum_k (-1)^k \partial^{V_\bullet}_k$.

LEMMA 3.4. *Let $V^{\blacktriangle} = (V^*_{n,m}, d_{n,m})_{n,m}$ be a bisemicosimplicial differential graded vector space. Then, the associated differential graded vector spaces $(\mathrm{Tot}^{\blacktriangle}(V^{\blacktriangle}), D)$, $\mathrm{Tot}(\mathrm{Tot}^{H,\Delta}(V^{\blacktriangle}))$ and $\mathrm{Tot}(\mathrm{Tot}^{V,\Delta}(V^{\blacktriangle}))$ are quasi isomorphic.*

PROOF. It follows from a standard computation, using spectral sequence. □

As in the previous section, we can also apply the Thom-Whitney construction instead of the total complex construction. Also in this case, we get two differential graded vector spaces $\mathrm{Tot}_{TW}(\mathrm{Tot}^{H,\Delta}_{TW})$ and $\mathrm{Tot}_{TW}(\mathrm{Tot}^{V,\Delta}_{TW})$ depending, a priori, on the order of the construction. There is also a more direct way, based on the Thom-Whitney construction, to associate a differential graded vector space with a bisemicosimplicial differential graded vector space.

DEFINITION 3.5. *Let $V^{\blacktriangle} = (V_{n,m})$ be a bisemicosimplicial DGLA. The Thom-Whitney DGLA $\mathrm{Tot}^{\blacktriangle}_{TW}(V^{\blacktriangle})$ is defined as the sub-differential graded vector space of $\prod_{n,m} V_{n,m} \otimes (A_{PL})_n \otimes (A_{PL})_m$, whose elements are sequences $(x_{n,m})_{n,m}$ satisfying the relations:*

$$(\partial^{H_m}_k \otimes Id \otimes Id) x_{n,m} = (Id \otimes \delta^k \otimes Id) x_{n+1,m}, \quad \textit{for every } 0 \leq k \leq n,$$

and

$$(\partial^{V_n}_k \otimes Id \otimes Id) x_{n,m} = (Id \otimes Id \otimes \delta^k) x_{n,m+1}, \quad \textit{for every } 0 \leq k \leq m.$$

More explicitly, we are considering sequence of elements $(x_{n,m})_{n,m} = x_{n,m} \otimes \alpha_n \otimes \beta_m \in V_{n,m} \otimes (A_{PL})_n \otimes (A_{PL})_m$ *such that*

$$\partial_k^{H_m} x_{n,m} \otimes \alpha_n \otimes \beta_m = x_{n+1,m} \otimes \delta^k \alpha_{n+1} \otimes \beta_m$$

and

$$\partial_k^{V_n} x_{n,m} \otimes \alpha_n \otimes \beta_m = x_{n,m+1} \otimes \alpha_n \otimes \delta^k \beta_{m+1}.$$

Lemma 3.6. *Let* $V^\blacktriangle = (V_{n,m})$ *be a bisemicosimplicial differential graded vector space; then, the Thom-Withney construction does not depend on the order, i.e.,* $\mathrm{Tot}_{TW}(\mathrm{Tot}_{TW}^{H,\Delta}) \cong \mathrm{Tot}_{TW}(\mathrm{Tot}_{TW}^{V,\Delta}) \cong \mathrm{Tot}_{TW}^\blacktriangle(V^\blacktriangle)$

Proof. It follows from the explicit description of the Thom-Withney construction. ∎

If $\mathfrak{g}^\blacktriangle$ is a bisemicosimplicial DGLAs, then, as in the semicosimplicial case, the differential graded vector space $\mathrm{Tot}_{TW}^\blacktriangle(\mathfrak{g}^\blacktriangle)$ inherits a structure of DGLA.

Remark 3.7. As for the semicosimplicial case, the differential graded vector spaces $\mathrm{Tot}_{TW}^\blacktriangle(\mathfrak{g}^\blacktriangle)$ and $\mathrm{Tot}^\blacktriangle(\mathfrak{g}^\blacktriangle)$ are quasi-isomorphic. In a forthcoming paper, we will use the DGLA structure of $\mathrm{Tot}_{TW}^\blacktriangle(\mathfrak{g}^\blacktriangle)$ and the homotopy transfer to define a canonical L_∞-algebra structure $\widetilde{\mathrm{Tot}}^\blacktriangle(\mathfrak{g}^\blacktriangle)$ on $\mathrm{Tot}^\blacktriangle(\mathfrak{g}^\blacktriangle)$, such that $\widetilde{\mathrm{Tot}}^\blacktriangle(\mathfrak{g}^\blacktriangle)$ and $\mathrm{Tot}_{TW}^\blacktriangle(\mathfrak{g}^\blacktriangle)$ are quasi-isomorphic L_∞-algebra.

3.2 – Deformation functors associated with a bisemicosimplicial DGLA

In this section, we will describe how we can associate a deformation functor with a bisemicosimplicial DGLA. In Section 2.2, we introduced the deformation functor $H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta)$ associated with a semicosimplicial DGLA $\mathfrak{g}^\Delta$. Moreover, Theorem 2.12 states that $H^1_{\mathrm{sc}}(\exp \mathfrak{g}^\Delta) \simeq \mathrm{Def}_{\mathrm{Tot}_{TW}(\mathfrak{g}^\Delta)}$, whenever $H^k(\mathfrak{g}_i) = 0$, for all i and for all $k < 0$.

Next, let $\mathfrak{g}^\blacktriangle$ be a bisemicosimplicial DGLA. In the previous section, we associate with $\mathfrak{g}^\blacktriangle$, the semicosimplicial DGLA $\mathrm{Tot}_{TW}^{H,\Delta}$ and $\mathrm{Tot}_{TW}^{V,\Delta}$. Therefore, we can naturally associate with $\mathfrak{g}^\blacktriangle$ the two deformations functors $H^1_{\mathrm{sc}}(\exp \mathrm{Tot}_{TW}^{H,\Delta})$ and $H^1_{\mathrm{sc}}(\exp \mathrm{Tot}_{TW}^{V,\Delta})$. Moreover, we associate with $\mathfrak{g}^\blacktriangle$ the Thom-Whitney DGLA $\mathrm{Tot}_{TW}^\blacktriangle(\mathfrak{g}^\blacktriangle)$ and its deformation functor $\mathrm{Def}_{\mathrm{Tot}_{TW}^\blacktriangle(\mathfrak{g}^\blacktriangle)}$. The following theorem explains the relation between all these functors.

Thorem 3.8. *Let* $\mathfrak{g}^\blacktriangle$ *be a bisemicosimplicial DGLA such that* $H^k(\mathfrak{g}_{i,j}) = 0$ *for all* i, j *and* $k < 0$. *Then, there exist natural isomorphisms of deformation functors*

$$H^1_{\mathrm{sc}}(\exp \mathrm{Tot}_{TW}^{H,\Delta}) \cong \mathrm{Def}_{\mathrm{Tot}_{TW}(\mathrm{Tot}_{TW}^{H,\Delta}(V^\blacktriangle))} \cong$$

$$\cong \mathrm{Def}_{\mathrm{Tot}_{TW}^\blacktriangle(\mathfrak{g}^\blacktriangle)} \cong \mathrm{Def}_{\mathrm{Tot}_{TW}(\mathrm{Tot}_{TW}^{V,\Delta}(\mathfrak{g}^\blacktriangle))} \cong H^1_{\mathrm{sc}}(\exp \mathrm{Tot}_{TW}^{V,\Delta}).$$

Proof. The cohomological constraint of the hypothesis implies that $H^k(\mathrm{Tot}^{H,\Delta}_{TW}(\mathfrak{g}^\blacktriangle)_m) = H^k(\mathrm{Tot}^{V,\Delta}_{TW}(\mathfrak{g}^\blacktriangle)_n) = 0$, for all n, m and for all $k < 0$. Therefore, the first and last isomorphisms follow from Theorem 2.12. The remaining isomorphisms follow from Lemma 3.6. ∎

Example 3.9. (Example 3.3 revisited) Let X be a smooth variety, $Z \subset X$ a closed subscheme and $\mathcal{U} = \{U_i\}_i$ an affine open cover of X. Denote by $\chi : \Theta_X(-\log Z) \hookrightarrow \Theta_X$ the inclusion of sheaves of Lie algebras. Following Example 3.3, we have the bisemicosimplicial Lie algebra $\chi^\blacktriangle : \Theta_X(-\log Z)(\mathcal{U}) \to \Theta_X(\mathcal{U})$ and so we can consider the associated DGLA $\mathrm{Tot}^\blacktriangle_{TW}(\chi^\blacktriangle)$. Moreover, as in the the previous construction, we can associate with χ two semicosimplicial DGLAs. he easiest way is to consider the induced morphism of DGLA $\chi_{TW} : \mathrm{Tot}_{TW}(\Theta_X(-\log Z)(\mathcal{U})) \to \mathrm{Tot}_{TW}(\Theta_X(\mathcal{U}))$, and view it as a semicosimplcial DGLA by zero extension (see Example 2.2), *i.e.*,

$$\chi^\Delta_{TW} : \quad \mathrm{Tot}_{TW}(\Theta_X(-\log Z)(\mathcal{U})) \underset{\chi_{TW}}{\overset{0}{\rightrightarrows}} \mathrm{Tot}_{TW}(\Theta_X(\mathcal{U})) \rightrightarrows\!\!\!\rightarrow 0 \cdots .$$

Analogously, if we apply the Thom-Whitney construction firstly on the rows, then we get the semicosimplcial DGLA T^Δ

$$\begin{array}{c}
\vdots \\
\upuparrows\upuparrows \\
T_2 = \mathrm{Tot}_{TW}(\prod_{i<j<k} \Theta_X(-\log Z)(U_{ijk}) \to \prod_{i<j<k} \Theta_X(U_{ijk})) \\
\upuparrows\uparrow \\
T_1 = \mathrm{Tot}_{TW}(\prod_{i<j} \Theta_X(-\log Z)(U_{ij}) \to \prod_{i<j} \Theta_X(U_{ij})) \\
\upuparrows \\
T_0 = \mathrm{Tot}_{TW}(\prod_{i} \Theta_X(-\log Z)(U_{i}) \to \prod_{i} \Theta_X(-\log Z)(U_{i})).
\end{array}$$

In this second case, the vertical maps are the restrictions to open subsets (see Example 2.3). The previous Theorem 3.8 implies that there exist isomorphisms of deformation functors

$$\mathrm{Def}_{\chi_{TW}} \cong \mathrm{Def}_{\mathrm{Tot}^\blacktriangle_{TW}(\chi^\blacktriangle)} \cong H^1_{\mathrm{sc}}(\exp T^\Delta).$$

We recall that the functor $\mathrm{Def}_{\chi_{TW}}$ is isomorphic to $H^1_{\mathrm{sc}}(\exp \chi^\Delta_{TW})$ (see Example 2.9). More explicitly, since χ is injective, for all $A \in \mathbf{Art}$, the set $\mathrm{Def}_{\chi_{TW}}(A)$ is given by

$$\mathrm{Def}_{\chi_{TW}}(A) = \frac{\mathrm{MC}_{\chi_{TW}}(A)}{\sim},$$

where

$$\begin{aligned}\mathrm{MC}_{\mathcal{X}_{TW}}(A) = \{a \in \mathrm{Tot}_{TW}(\Theta_X(\mathcal{U}))^0 \otimes \mathfrak{m}_A)|\\ e^{-a} * 0 \in \mathrm{Tot}_{TW}(\Theta_X(-\log Z)(\mathcal{U}))^1 \otimes \mathfrak{m}_A\},\end{aligned}$$

and $e^a \sim e^{a'}$ if and only if there exist $b \in \mathrm{Tot}_{TW}(\Theta_X(-\log Z)(\mathcal{U}))^0 \otimes \mathfrak{m}_A$, such that $e^{a'} = e^a e^{-b}$.

4 – Application: Deformations of subvarieties in a fixed smooth variety

Let X be a smooth variety, defined over an algebraically closed field $\mathbb{K}$ of characteristic 0, and $Z \subset X$ a closed subscheme of X. We recall the definition of infinitesimal deformations of Z in X fixed, full details can be found in [Se06].

Definition 4.1. *Let $A \in$ **Art**. An infinitesimal deformation of Z in X over* $\mathrm{Spec}(A)$ *is a cartisian diagram*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, dash] & Z_A \subset X \times \mathrm{Spec}(A) \arrow[d, dash, "\pi"] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r, "a"] & \mathrm{Spec}(A), \end{tikzcd}$$

where π is a flat map induced by the projection from $X \times \mathrm{Spec}(A)$ to $\mathrm{Spec}(A)$. *The associated infinitesimal deformation functor is*

$$\mathrm{Hilb}^Z_X : \mathbf{Art} \to \mathbf{Set},$$

such that

$$\mathrm{Hilb}^Z_X(A) = \{\textit{infinitesimal deformations of } Z \textit{ in } X \textit{ over } \mathrm{Spec}(A)\}.$$

Moreover, we can define the sub-functor

$$\mathrm{Hilb}'^Z_X : \mathbf{Art} \to \mathbf{Set},$$

where

$$\mathrm{Hilb}'^Z_X(A) = \{\text{locally trivial infinitesimal deformations of } Z \text{ in } X \text{ over } \mathrm{Spec}(A)\}.$$

We recall that, an infinitesimal deformation Z_A of Z in X over $\mathrm{Spec}(A)$ is called locally trivial if, for every point $z \in Z$, there exists an open neighbourhood $U_z \subset Z$ such that

$$\begin{tikzcd} U_z \arrow[r, "i"] \arrow[d, no head] & Z_{A|U_z} \arrow[d, no head] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r, "a"] & \mathrm{Spec}(A), \end{tikzcd}$$

is a trivial deformation of U_z. Whenever $Z \subset X$ is smooth, then every deformation of Z in X is locally trivial and so $\mathrm{Hilb}_X^Z = \mathrm{Hilb'}_X^Z$.

Next, following Examples 3.3 and 3.9, denote by $\chi : \Theta_X(-\log Z) \hookrightarrow \Theta_X$, the inclusion of sheaves of Lie algebras, and by $\chi^\blacktriangle : \Theta_X(-\log Z)(\mathcal{U}) \to \Theta_X(\mathcal{U})$, the associated bisemicosimplicial Lie algebra.

THEOREM 4.2. *Let X be a smooth variety, defined over an algebraically closed field $\mathbb{K}$ of characteristic 0, and $Z \subset X$ a closed subscheme. Then, there exists an isomorphism of functors* $\mathrm{Def}_{\mathrm{Tot}_{TW}^\blacktriangle(\chi^\blacktriangle)} \cong \mathrm{Hilb'}_X^Z$. *In particular, if $Z \subset X$ is smooth, then* $\mathrm{Def}_{\mathrm{Tot}_{TW}^\blacktriangle(\chi^\blacktriangle)} \cong \mathrm{Hilb}_X^Z$.

PROOF. For $\mathbb{K} = \mathbb{C}$ and Z smooth, this theorem was already proved in [Ma07, Theorem 5.2]ManettiSemireg, without the use of semicosimplicial language.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an affine open cover of X and $\mathcal{V} = \{V_i = U_i \cap Z\}_{i\in I}$ the induced one on Z. Let Z_A be a locally trivial deformation of Z in X over $\mathrm{Spec}(A)$. Then, Z_A is obtained by gluing the trivial deformations $V_i \otimes A$ in $U_i \otimes A$ along the double intersections $V_{ij} \otimes A$, such that the induced deformation of X is trivial. Therefore, it is determined by a sequence $\{\theta_{ij}\}_{i<j}$ of automorphisms of the sheaves of A-algebras

$$\begin{tikzcd} & \mathcal{O}_Z(V_{ij}) & \\ \mathcal{O}_Z(V_{ij}) \otimes A \arrow[ur] \arrow[rr, "\theta_{ij}", "\simeq"'] & & \mathcal{O}_Z(V_{ij}) \otimes A \arrow[ul] \\ & A, \arrow[ul] \arrow[ur] & \end{tikzcd}$$

satisfying the cocycle condition

$$\theta_{jk}\theta_{ik}^{-1}\theta_{ij} = \mathrm{Id}_{\mathcal{O}_Z(V_{ijk})\otimes A}, \qquad \forall\, i < j < k \in I, \tag{1}$$

and such that there exist automorphisms α_i of $\mathcal{O}_X(U_i) \otimes A$ satisfying

$$\theta_{ij} = \alpha_i{}^{-1}\alpha_j, \qquad \forall i < j. \tag{2}$$

Note that Equation (2) implies (1). Since we are in characteristic zero, we can take the logarithms and write $\theta_{ij} = e^{d_{ij}}$, for some $d_{ij} \in \Theta_X(-\log Z)(U_{ij}) \otimes \mathfrak{m}_A$, and $\alpha_i = e^{a_i}$, with $a_i \in \Theta_X(U_i) \otimes \mathfrak{m}_A$.

Therefore, a locally trivial deformation of Z in X over Spec(A) is equivalent to the datum of a sequence $\{a_i\}_i \in \prod_i \Theta_X(U_i) \otimes \mathfrak{m}_A$, such that

$$e^{-a_i}e^{a_j} \in \exp(\Theta_X(-\log Z)(U_{ij}) \otimes \mathfrak{m}_A), \qquad \forall\, i < j \in I.$$

As regards the equivalence relation, let Z_A and Z'_A be two deformations of Z in X over Spec(A). Denote by $\theta_{ij} = e^{d_{ij}} = e^{-a_i}e^{a_j}$ and $\theta'_{ij} = e^{d'_{ij}} = e^{-a'_i}e^{a'_j}$ the data associated with Z_A and Z'_A, respectively. The deformations Z_A and Z'_A are isomorphic if, for every i, there exists an automorphism β_i of $\mathcal{O}_Z(V_i) \otimes A$, such that $\theta_{ij} = \beta_i{}^{-1}\theta'_{ij}\beta_j$, for every $i < j$, and satisfying the compatibility relation $\alpha'_i\beta_i = \alpha_i$. Taking again logarithms, an isomorphism between Z_A and Z'_A is equivalent to the existence of a sequence $\{b_i\}_i \in \prod_i \Theta_X(-\log Z)(U_i) \otimes \mathfrak{m}_A$, such that $e^{a_i} = e^{a'_i}e^{b_i}$.

Next, from the DGLA point of view, we showed in Example 3.9, that $\mathrm{Def}_{\mathrm{Tot}^{\blacktriangle}_{TW}(\chi^{\blacktriangle})} \cong H^1_{\mathrm{sc}}(\exp \chi^{\blacktriangle}) \cong \mathrm{Def}_{\chi_{TW}}$. Therefore, it is enough to prove that $\mathrm{Hilb}'^{Z}_{X} \cong \mathrm{Def}_{\chi_{TW}}$, with $\chi_{TW} : \mathrm{Tot}_{TW}(\Theta_X(-\log Z)(\mathcal{U})) \hookrightarrow \mathrm{Tot}_{TW}(\Theta_X(\mathcal{U}))$; and it follows from the explicit description of $\mathrm{Def}_{\chi_{TW}}$. Indeed, $\mathrm{MC}_{\chi_{TW}}(A)$ is the set of all $a \in \mathrm{Tot}_{TW}(\Theta_X(\mathcal{U}))^0 \otimes \mathfrak{m}_A$, such that $e^{-a} * 0 \in \mathrm{Tot}_{TW}(\Theta_X(-\log Z)(\mathcal{U}))^1 \otimes \mathfrak{m}_A$, *i.e.*, $a = \{a_i\}_i \in \prod_i \Theta_X(U_i) \otimes \mathfrak{m}_A$, such that $e^{-a_i}e^{a_j} \in \exp(\Theta_X(-\log Z)(U_{ij}) \otimes \mathfrak{m}_A)$. Moreover, $a \sim a'$ if and only is there exist $b \in \mathrm{Tot}_{TW}(\Theta_X(-\log Z)(\mathcal{U}))^0 \otimes \mathfrak{m}_A$, such that $e^{a'} = e^a e^{-b}$, *i.e.*, $b = \{b_i\}_i \in \prod_i \Theta_X(-\log Z)(U_i) \otimes \mathfrak{m}_A$ such that $e^{a_i} = e^{a'_i}e^{b_i}$. ∎

Remark 4.3. In a forthcoming paper, we will use this theorem to study the obstructions to the deformations of Z in X, via the semiregularity map.

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