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A note on the topology and geometry of $F_4 I$

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Abstract: *We determine the intersection numbers and the structure of the rational cohomology ring of the symmetric space $F_4/(Sp(3)Sp(1))$ by using index theory and its quaternion-Kähler structure.*

1 – Introduction

An oriented connected irreducible Riemannian $4n$-manifold M is called a quaternion-Kähler manifold, $n \geq 2$, if its linear holonomy is contained in the group $Sp(n)Sp(1)$. Examples of such manifolds were given in [7], where Wolf showed that each compact centerless Lie group G is the isometry group of a quaternion-Kähler symmetric space given as the conjugacy class of a copy of $Sp(1)$ in G determined by a highest root of G. Thus, the symmetric space

$$F_4 I = \frac{F_4}{Sp(3)Sp(1)}$$

is a 28-dimensional quaternion-Kähler manifold. Although the cohomology of homogeneous spaces has been extensively studied, and the integral cohomology of $F_4 I$ was determined in [3], here we give a description of the rational cohomology ring $H^*(F_4 I; \mathbb{Q})$ in terms of classes determined by its quaternion-Kähler structure. The motivation for this work is the need to understand the topological structure of general quaternion-Kähler manifolds, whose rational cohomology we

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expect to be generated by a small number of cohomology classes. This is indeed the case for the space F_4I, as its Poincaré polynomial shows

$$\begin{aligned} P_{F_4I}(t) &= (1+t^4+t^8+t^{12}+t^{16}+t^{20})(1+t^8) \\ &= 1+t^4+2t^8+2t^{12}+2t^{16}+2t^{20}+t^{24}+t^{28}. \end{aligned}$$

The note is organized as follows. In Section 2 we compute the intersection pairings of the relevant characteristic classes arising from the quaternion-Kähler structure of F_4I (see Theorem 2.1). In Section 3 we determine the ring structure of $H^*(F_4I;Q)$ by using the intersection numbers (see Theorem 3.1). In Section 4, as a corollary of our calculations, we compute explicitly the Pontrjagin classes and numbers of F_4I, which may be of use in other geometrical contexts. In Section 5, we revisit Ishitoya and Toda's result [3] on the torsion free part of the integral cohomology of F_4I in terms of our characteristic classes.

2 – Intersection numbers

The holonomy group $Sp(7)Sp(1) \subset SO(28)$ of a 28-dimensional quaternion-Kähler manifold M determines the following factorization of the complexified tangent bundle [6]

$$TM_c = E \otimes H \tag{1}$$

where the fibres of the (locally defined) bundles E and H are isomorphic to the standard representations $\mathbb{C}^{14}$ and $\mathbb{C}^2$ of $Sp(7)$ and $Sp(1)$ respectively. Furthermore, for F_4I, the representation E decomposes further under $Sp(3) \subset Sp(7)$

$$E = \bigwedge\nolimits_0^3 \tilde{E} \tag{2}$$

where $\tilde{E} \cong \mathbb{C}^6$ is the standard representation of $Sp(3)$, and $\bigwedge_0^p \tilde{E}$ denotes the irreducible representation of $Sp(3)$ obtained as the primitive subspace of $\bigwedge^p \tilde{E}$ with respect to wedging by a symplectic form. Furthermore, the faithful 26-dimensional representation of F_4 also decomposes under $Sp(3)Sp(1)$

$$26 = \bigwedge\nolimits_0^2 \tilde{E} \oplus \tilde{E} \otimes H, \tag{3}$$

where the left hand side now denotes a rank 26 trivial vector bundle on F_4I (cf. [1]). Note that (2) implies that the characteristic classes of E are given in terms of those of the rank 6 bundle $\tilde{E}$, and (3) implies relations between the characteristic classes of $\tilde{E}$ and H. More precisely, by computing the first three components of the Chern character of $\bigwedge_0^2 \tilde{E} \oplus \tilde{E} \otimes H$ and equating them to zero we find that

$$c_2(\tilde{E}) = u,$$
$$c_6(\tilde{E}) = c_4(\tilde{E})u;$$

where $u = -c_2(H)$. This provides us with two candidates for the generators of $H^*(F_4I)$: u in dimension 4 and $c_4(\tilde{E})$ in dimension 8. From now on, we shall denote

$$c_4 = c_4(\tilde{E}).$$

Thus, our first task is to compute the pairings

$$u^7, \quad c_4u^5, \quad c_4^2u^3, \quad c_4^3u, \tag{4}$$

where the notation really means the evaluation of representatives of these 28-dimensional cohomology classes on the fundamental cycle of F_4I. In order to compute such pairings, we will make use of a Hilbert polynomial given by the index of certain twisted Dirac operators [6, 5]. More precisely, we will use the polynomial in q given by

$$f(q) = \operatorname{ind}(\partial\!\!\!/ \otimes S^qH) = \left\langle \widehat{A} \cdot \operatorname{ch}(S^qH), [F_4I] \right\rangle,$$

where $\widehat{A}$ denotes the $\widehat{A}$-genus of the manifold, ch denotes the Chern character and S^qH denotes the q-th symmetric power of H.

On the one hand, due to (1), (2) and (3), the coefficients of $f(q)$ are linear combinations of the intersection pairings in (4). Namely,

$$\begin{aligned}
f(q) = {} & \frac{u^7q^{15}}{1307674368000} + \frac{u^7q^{14}}{87178291200} + \frac{u^7q^{13}}{37362124800} - \frac{u^7q^{12}}{2874009600} \\
& + \left(\frac{u^5c_4}{4105728000} - \frac{u^7}{522547200}\right)q^{11} + \left(\frac{u^7}{2612736000} + \frac{u^5c_4}{373248000}\right)q^{10} \\
& + \left(\frac{229u^7}{10973491200} + \frac{59u^5c_4}{10973491200}\right)q^9 + \left(\frac{13u^7}{406425600} - \frac{13u^5c_4}{406425600}\right)q^8 \\
& + \left(-\frac{151u^7}{3657830400} - \frac{149u^5c_4}{457228800} + \frac{221u^3c_4^2}{18289152000}\right)q^7 \\
& + \left(-\frac{113u^5c_4}{81648000} + \frac{221u^3c_4^2}{2612736000} - \frac{31u^7}{522547200}\right)q^6 \\
& + \left(-\frac{17u^5c_4}{18711000} + \frac{1037u^3c_4^2}{9580032000} + \frac{107u^7}{1368576000}\right)q^5
\end{aligned}$$

$$+\left(-\frac{1751u^3c_4^2}{5748019200}+\frac{2603u^5c_4}{359251200}-\frac{1751u^7}{5748019200}\right)q^4$$
$$+\left(\frac{739163u^5c_4}{52306974720}+\frac{402959uc_4^3}{7846046208000}-\frac{3201281u^3c_4^2}{784604620800}-\frac{385673u^7}{523069747200}\right)q^3$$
$$+\left(-\frac{13528111u^3c_4^2}{1307674368000}+\frac{1237813u^5c_4}{261534873600}+\frac{3721u^7}{20922789888}+\frac{402959uc_4^3}{2615348736000}\right)q^2$$
$$+\left(\frac{2713u^7}{4828336128}-\frac{3383123u^3c_4^2}{980755776000}+\frac{535039uc_4^3}{7846046208000}-\frac{769633u^5c_4}{140107968000}\right)q$$
$$+\left(\frac{12899u^7}{373621248000}+\frac{294779u^3c_4^2}{93405312000}-\frac{12899uc_4^3}{373621248000}-\frac{294779u^5c_4}{93405312000}\right).$$

On the other hand, these indices can be seen as holomorphic Euler characteristics of the twistor space

$$Z = Z(F_4I) = \frac{F_4}{Sp(3)U(1)}$$

of F_4I by twistor transform [6, 5]. Namely,

$$\begin{aligned}\operatorname{ind}(\partial\!\!\!/ \otimes S^qH) &= \chi(Z, \mathcal{O}(L^{(q-7)/2})),\\ &= \sum_{i=0}^{15}(-1)^i \dim H^i(Z, \mathcal{O}(L^{(q-7)/2})),\end{aligned}$$

where L is the positive line bundle over Z which restricted to the $\mathbb{C}P^1$-fibres is $\mathcal{O}(2)$. These holomorphic Euler characteristics can be computed by means of the Bott-Borel-Weil theorem and the Weyl dimension formula as follows [4]. Let $R(\mathfrak{sp}(3)\oplus\mathfrak{u}(1))$ be the set of roots of $Sp(3)U(1)\subset F_4$, R^+ be the set of positive roots of F_4 with $R(\mathfrak{sp}(3)\oplus\mathfrak{u}(1))$ generated by simple roots, $\delta = \frac{1}{2}\sum_{\alpha\in R+}\alpha$. Let $V(\lambda)$ be an irreducible representation for $Sp(3)U(1)$ with highest weight $\lambda\in R(\mathfrak{sp}(3)\oplus\mathfrak{u}(1))$ and $\mathbf{V}(\lambda)$ the corresponding homogeneous vector bundle on F_4I. By the Bott-Borel-Weil theorem and the Weyl dimension formula [4] we have

$$\chi(Z, \mathcal{O}(\mathbf{V}(\lambda))) = (-\mathbf{1})^{\mathbf{s}}\prod_{\alpha\in\mathbf{R}+}\frac{\langle\alpha,\delta+\lambda\rangle}{\langle\alpha,\delta\rangle},$$

where

$$s = \sharp\{\alpha\in R+ \mid \langle\alpha,\delta+\lambda\rangle < 0\}.$$

Let $\mathfrak{H}$ be the Cartan subalgebra of $(\mathfrak{f}_4)_c$ spanned by the following basic roots

$$\{\alpha_1 = (1,-1,0,0), \alpha_2 = (0,1,-1,0), \alpha_3 = (0,0,2,0), \alpha_4 = (-1,-1,-1,1)\}.$$

The coordinates have been chosen so that $\mathfrak{sp}(3)$ has the Cartan subalgebra spanned by $\{\alpha_1,\alpha_2,\alpha_3\}$ which is orthogonal to the maximal root $\rho = (0,0,0,2)$.

In this case $\delta = (3,2,1,8)$. The roots coming from $Sp(3)$ are thus embedded canonically in the first three coordinates and the one coming from $U(1)$ corresponds to the last coordinate. The bundle $L^{(q-7)/2}$ corresponds to $\frac{(q-7)}{2}(0,0,0,2)$. When adding δ we get $(3,2,1,q+1)$. Therefore

$$\begin{aligned} f(q) = \chi(Z(F_4I), \mathcal{O}(L^{(q-7)/2})) &= \frac{1}{8583708672000}q^{15} + \frac{1}{572247244800}q^{14} \\ &+ \frac{1}{245248819200}q^{13} - \frac{13}{245248819200}q^{12} - \frac{59}{204374016000}q^{11} \\ &+ \frac{1}{11147673600}q^{10} + \frac{253}{78033715200}q^{9} + \frac{13}{2890137600}q^{8} - \frac{1111}{111476736000}q^{7} \\ &- \frac{541}{22295347200}q^{6} + \frac{23}{9083289600}q^{5} + \frac{8567}{245248819200}q^{4} + \frac{4751}{357654528000}q^{3} \\ &- \frac{29}{1907490816}q^{2} - \frac{1}{113541120}q. \end{aligned}$$

Equating the coefficients of the two expressions of the polynomial $f(q)$ we get the intersection pairings which, by the way, show a remarkable symmetry.

THEOREM 2.1. *Let $u = -c_2(H)$ and $c_4 = c_4(\tilde{E})$ where H and $\tilde{E}$ are the locally defined bundles by the isotropy factors of F_4I. The intersection numbers are the following*

$$u^7 = \frac{39}{256}, \quad c_4u^5 = \frac{3}{256}, \quad c_4^2u^3 = \frac{3}{256}, \quad c_4^3u = \frac{39}{256}.$$

3 – Cohomology ring

Armed with the intersection numbers of Theorem 2.1 and the Poincaré polynomial of F_4I, we can now compute the generators of $H^*(F_4I)$ and their relations.

- In dimension 4: u is non-degenerate, so it is non-zero in $H^4(F_4I)$.
- In dimension 8: We have two classes u^2 and c_4. Suppose

$$au^2 + bc_4 = 0.$$

Then

$$\begin{aligned} au^7 + bc_4u^5 &= 0, \\ ac_4u^5 + bc4^2u^3 &= 0, \\ ac_4^2u^3 + bc4^3u &= 0, \end{aligned}$$

which has no non-trivial solutions for a and b when we substitute the intersection numbers. Therefore, u^2 and c_4 generate $H^8(F_4I)$.

- In dimension 12: We have two classes u_3 and c_4u. Suppose

$$au^3 + bc_4u = 0.$$

Then we get the same system of equations as above

$$\begin{aligned} au^7 + bc_4u^5 &= 0, \\ ac_4u^5 + bc4^2u^3 &= 0, \\ ac_4^2u^3 + bc4^3u &= 0, \end{aligned}$$

which has no non-trivial solutions for a and b. Therefore, u^3 and c_4u generate $H^{12}(F_4I)$.

- In dimension 16: We have three classes: u^4, c_4u^2 and c_4^2 . Since $H^{16}(F_4I)$ is 2-dimensional, we must find the relation between these classes. Suppose

$$au^4 + bc_4u^2 + c_4^2 = 0.$$

Then we get

$$\begin{aligned} au^7 + bc_4u^5 + c_4^2u^3 &= 0, \\ ac_4u^5 + bc_4^2u^3 + c_4^3u &= 0, \end{aligned}$$

which have a unique solution

$$a = 1, \quad b = -14,$$

so that

$$c_4^2 = -u^4 + 14c_4u^2.$$

Moreover, u^4 and c_4u^2 are linearly independent since

$$au^4 + bc_4u^2 = 0$$

implies

$$\begin{aligned} au^7 + bc_4u^5 &= 0, \\ ac_4u^5 + bc4^2u^3 &= 0, \\ ac_4^2u^3 + bc4^3u &= 0, \end{aligned}$$

whose only solution is the trivial one. Therefore, u^4 and c_4u^2 generate $H^{16}(F_4I)$.

- In dimension 20: We have three classes u^5, c_4u^3 and c_4^2u. Suppose

$$au^5 + bc_4u^3 + c_4^2u = 0.$$

Then

$$au^7 + bc_4u^5 + c_4^2u^3 = 0,$$
$$ac_4u^5 + bc_4^2u^3 + c_4^3u = 0,$$

which have a unique solution

$$a = 1, \quad b = -14.$$

Thus,

$$c_4^2u = -u^5 + 14c_4u^3,$$

which comes from the relation found in dimension 16. Moreover, u^5 and c_4u^3 are linearly independent since

$$au^5 + bc_4u^3 = 0$$

implies

$$au^7 + bc_4u^5 = 0,$$
$$ac_4u^5 + bc_4^2u^3 = 0,$$

whose only solution is the trivial one. Therefore, u^5 and c_4u^3 generate $H^{20}(F_4I)$.

- In dimension 24: We have four classes u^6, c_4u^4, $c_4^2u^2$ and c_4^3. In this case, $H^{24}(F_4I)$ is 1-dimensional and we see that if

$$au^6 + c_4u^4 = 0,$$

then

$$a = -\frac{1}{13},$$

and the other classes can all be put in terms of u^6

$$13c_4u^4 = u^6,$$
$$13c_4^2u^2 = u^6,$$
$$c_4^3 = u^6.$$

Hence, we have proved the following.

Theorem 3.1. *Let $u = -c_2(H)$ and $c_4 = c_4(\tilde{E})$ where H and $\tilde{E}$ are the locally defined bundles by the isotropy factors of F_4I. The rational comohomology ring of F_4I is*

$$H^*(F_4I;\mathbb{Q}) = \mathbb{Q}[u, c_4]/(c_4^2 + u^4 - 14c_4u^2, u^6 - 13c_4u^4).$$

4 – Pontrjagin classes and numbers

As a corollary of the intersection numbers and relations we obtain the Pontrjagin numbers of F_4I.

Corollary 4.1. *The Pontrjagin numbers of F_4I are given as follows:*

$$\begin{aligned}
p_7 &= 348,\\
p_1^7 &= 2496,\\
p_2^3p_1 &= 8424,\\
p_2p_3p_1^2 &= 4932,\\
p_2^2p_3 &= 5904,\\
p_3^2p_1 &= 3972,\\
p_2^2p_1^3 &= 6192,\\
p_4p_2p_1 &= 4842,\\
p_3p_1^4 &= 3048,\\
p_2p_1^5 &= 3888,\\
p_6p_1 &= 2091,\\
p_4p_3 &= 2832,\\
p_5p_2 &= 2718,\\
p_4p_1^3 &= 4188,\\
p_5p_1^2 &= 3246,
\end{aligned}$$

where p_i denotes the i^{th} Pontrjagin class of F_4I.

Proof. This follows from the relations described in the previous sections and

$$\begin{aligned}
p_1 &= 4u,\\
p_2 &= 26u^2 - 14c_4,\\
p_3 &= 84u^3 - 76c_4u,\\
p_4 &= 281u^4 + 1866c_4u^2 + 65c_4^2,\\
&= 216u^4 + 2776c_4u^2,\\
p_5 &= 720u^5 + 7376c_4u^3 + 576c_4^2u,\\
&= 144u^5 + 15440c_4u^3,\\
p_6 &= 1620u^6 + 11864c_4u^4 + 12724c_4^2u^2 - 80c_4^3,\\
&= 44608c_4u^4,\\
p_7 &= 3200u^7 + 10624c_4^2u^3 + 5760c_4u^5 - 2176c_4^3u,\\
&= 348.
\end{aligned}$$

□

5 – Torsion-free part of the integral cohomology of F_4I

We can go a little further by revisiting the following result of Ishitoya and Toda [3] about the torsion-free part of the integral cohomology of F_4I.

THEOREM 5.1. [3] *The torsion-free part of the integral cohomology of F_4I can be described as follows*

$$H^*(F_4I;\mathbb{Z})_{tf} = \frac{\mathbb{Z}[f_4, f_8, f_{12}]}{(f_4^3 - 12f_4f_8 + 8f_{12}, f_4f_{12} - 3f_8^2, f_8^3 - f_{12}^2)},$$

where $f_i \in H^{4i}(F_qI, \mathbb{Z})$, $i = 4, 8, 12$.

First, let us observe that $4u = p_1(F_4I)$ is integral and indivisible. If $4u = m\xi$ with $\xi \in H^4(F_4I;\mathbb{Z})$ an indivisible class and m an non-zero integer, then

$$\left(\frac{4u}{m}\right)^7 = \frac{4^3 39}{m^7}$$

should be an integer, which can only happen if $m = \pm 1$. Thus, let us set

$$f_4 = 4u.$$

Taking the relations in Theorem 5.1 we see that

$$f_{12} = -\frac{1}{8}f_4^3 + \frac{3}{2}f_4f_8,$$
$$f_8^2 = -\frac{1}{24}f_4^4 + \frac{1}{2}f_4^2f_8,$$
$$f_4^6 = \frac{104}{11}f_4^4f_8,$$

so that

$$u^5f_8 = \frac{33}{128},$$
$$u^3f_8^2 = \frac{7}{16},$$
$$uf_8^3 = \frac{3}{4}.$$

By setting $f_8 = au^2 + bc_4$ we get three equations

$$a^2u^7 + 2abc_4u^5 + b^2c_4^2u^3 = \frac{7}{16},$$
$$au^7 + bc_4u^5 = \frac{33}{128},$$
$$a^3u^7 + 3a^2bc_4u^5 + 3ab^2c_4^2u^3 + b^3c_4^3u = \frac{3}{4},$$

i.e.

$$\frac{39}{256}a + \frac{3}{256}b = \frac{33}{128},$$
$$\frac{39}{256}a^2 + \frac{3}{128}ab + \frac{3}{256}b^2 = \frac{7}{16},$$
$$\frac{39}{256}a^3 + \frac{9}{256}a^2b + \frac{9}{256}ab^2 + \frac{39}{256}b^3 = \frac{3}{4},$$

with unique solution

$$a = \frac{5}{3}, \quad b = \frac{1}{3},$$

i.e.

$$f_8 = \frac{5}{3}u^2 + \frac{1}{3}c_4, \quad \text{and} \quad f_{12} = 2u^3 + 2c_4u.$$

It is interesting to notice that

$$6f_8 = 10u^2 + 2c_4 = c_4(\tilde{E} \otimes H) \quad \text{and} \quad f_{12} = e([\tilde{E} \otimes H]_{\mathbb{R}})$$

where $[\tilde{E} \otimes H]_{\mathbb{R}}$ denotes the underlying real vector bundle of $\tilde{E} \otimes H$, so that these classes have a geometrical interpretation.

This results can be used to reinterpret the integral cohomology ring of the twistor space $Z(F_4I)$, which is torsion free. In [2], they calculated such a cohomology ring using a Schubert calculus approach. It may be interesting to investigate the geometry arising from that description in combination with the geometry encoded in the Chern classes u and c_4.

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