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A Dunkl-classical d-symmetric d-orthogonal polynomial set

Y. BEN CHEIKH – M. GAIED

Abstract: *In this paper, we apply a d-orthogonality preserving operator to the Humbert polynomials to derive a new Dunkl-classical d-orthogonal polynomials generalizing the Humbert ones. For the resulting polynomials, we state a $(d+1)$-order recurrence relation and a $(d+1)$-order differential-difference equation. We also obtain an explicit expression of the d-dimensional functional vector for which the d-orthogonality holds. We show that the components of this d-symmetric Dunkl-classical d-orthogonal polynomial set are classical d-orthogonal.*

1 – Introduction

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients. A linear operator L acting on $\mathcal{P}$ is said to be a lowering operator if it fulfills:

$$L(\mathcal{P}) = \mathcal{P}, \quad L(1) = 0 \quad \text{and} \quad \deg(L(x^n)) = n-1, \quad n \in \mathbb{N}^* := \{1, 2, \dots\}.$$

Some lowering operators were used, in the orthogonal polynomials theory, to classify orthogonal polynomials according to the following definition:

Definition 1.1. An orthogonal polynomial set $\{P_n\}_{n\geq 0}$ is said to be L-classical whenever the polynomial set $\{Q_n = LP_{n+1}\}_{n\geq 0}$ is also orthogonal.

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Among such lowering operators, we mention the derivative operator D, the difference operator Δ and the Hahn operator H_q. In the sequel, for shorter, we write "classical" instead of "D-classical".

The literature on these topics is extremely vast. We quote, for instance, [1], [2], [14], [16], [18]. Notice also that Definition 1.1 was extended by replacing the "orthogonal" property by "d-orthogonal". The notion of d-orthogonal polynomials is a generalization of orthogonal polynomials in the sense that the polynomials satisfy orthogonality conditions with respect to d functionals. That was introduced in [20], [25].

Recently, we consider a further lowering operator to treat analogue questions. That is $T_\mu := D + 2\mu H_{-1}$, $\mu > -1/2$, the Dunkl operator to introduce the so called T_μ-classical (or Dunkl-classical) polynomials. In [7], we characterize the T_μ-classical symmetric orthogonal polynomials. In [8], we introduce a T_μ-classical d-symmetric d-orthogonal polynomial family generalizing the Gould-Hopper ones by solving a characterization problem. In this work, we introduce a second example of T_μ-classical d-symmetric d-orthogonal polynomial set as the range of the Humbert polynomials by a suitable d-orthogonality preserving operator. Notice by the way that Humbert polynomials include many special cases considered in the literature (see Subsection 2.4). The outline of the paper is as follows. In Section 2, we recall some definitions and results to be used in the sequel. In Section 3, we introduce a new d-symmetric polynomial set generalizing the Humbert polynomials. For a restricted condition on d, we show that these polynomials are Dunkl-classical d-orthogonal and we explicitly express the d-dimensional functional for which the d-orthogonality holds. In Section 4, for positive integer d, we derive a $(d+1)$-order differential-difference equation satisfied by the generalized Humbert polynomials. For the components of these polynomials we state an hypergeometric representation. From which, we deduce that these components are classical d-orthogonal. Finally, in Section 5, we discuss the significance of the generalized Humbert polynomials, the method how these polynomials were introduced and some questions arising in the d-orthogonal polynomial theory.

2 – Preliminaries and notations

Throughout this paper, we shall use the following notations, definitions and formulas.

2.1 – Dunkl-operator

Let μ be a real number with $\mu > -1/2$. The Dunkl operator T_μ is defined in the linear space of entire functions as follows [13]

$$T_\mu \phi(x) = \phi'(x) + \mu \frac{\phi(x) - \phi(-x)}{x}. \tag{2.1}$$

The operator T_0 is reduced to the derivative operator D.

One easily verifies that

$$T_\mu x^n = \frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} x^{n-1}, \; n \in \mathbb{N}^*, T_\mu(1) = 0 \tag{2.2}$$

where γ_μ is defined by

$$\gamma_\mu(2n) := \frac{2^{2n} n! \, \Gamma(n+\mu+1/2)}{\Gamma(\mu+1/2)} = (2n)! \, \frac{\Gamma(n+\mu+1/2)\Gamma(1/2)}{\Gamma(n+1/2)\Gamma(\mu+1/2)} \tag{2.3}$$

and

$$\gamma_\mu(2n+1) := \frac{2^{2n+1} n! \, \Gamma(n+\mu+3/2)}{\Gamma(\mu+1/2)} = (2n+1)! \, \frac{\Gamma(n+\mu+3/2)\Gamma(1/2)}{\Gamma(n+3/2)\Gamma(\mu+1/2)}. \tag{2.4}$$

γ_μ plays the role of generalized factorial [23], since

$$\gamma_\mu(n+1) = (n+1+2\mu\theta_{n+1})\gamma_\mu(n), \quad n \in \mathbb{N} := \{0, 1, 2, \dots\},$$

where θ_n is defined to be 0 if $n \in 2\mathbb{N}$ and 1 if $n \in (2\mathbb{N}+1)$.

The associated commutative algebra of Dunkl operator T_μ is intertwined with the algebra of the standard derivative operator D by a unique linear and homogeneous isomorphism V_μ in the linear space $\mathcal{P}$ of polynomials with complex coefficients in the sens that:

(i) $V_\mu(1) = 1$;
(ii) $V_\mu(\mathcal{P}_n) = \mathcal{P}_n$, where $\mathcal{P}_n$ denotes the linear subspace of polynomials of degree at most n;
(iii) $T_\mu V_\mu = V_\mu D$.

The expression of V_μ in terms of its action on the canonical basis of $\mathcal{P}$ is given by

$$V_\mu(x^n) = \frac{\left(\frac{1}{2}\right)_{[\frac{n+1}{2}]}}{\left(\mu+\frac{1}{2}\right)_{[\frac{n+1}{2}]}} x^n = \frac{n!}{\gamma_\mu(n)} x^n, \quad n \in \mathbb{N}, \tag{2.6}$$

where, and in what follows, $[x]$ denotes the greatest integer in x and $(a)_p$ the Pochhammer symbol given by $(a)_p = \frac{\Gamma(a+p)}{\Gamma(a)}$, $p \in \mathbb{N}$.

This amounts to the following integral representation [13]

$$V_\mu(f(x)) = \frac{\Gamma(\mu+1/2)}{\Gamma(1/2)\Gamma(\mu)} \int_{-1}^{1} f(xt)(1-t)^{\mu-1}(1+t)^\mu dt.$$

2.2 – d-orthogonal polynomials

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the effect of the functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. A polynomial sequence $\{P_n\}_{n\geq 0}$ is called a *polynomial set* (PS, for shorter) if and only if $\deg P_n = n$ for all non-negative integer n. The PS $\{P_n\}_{n\geq 0}$ is called monic if $P_n(x) = x^n + \pi_{n-1}(x)$ with deg $\pi_{n-1} \leq n-1$.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials. The dual sequence associated with $\{P_n\}_{n\geq 0}$ is a sequence of forms $\{u_k\}_{k\geq 0}$ such that

$$\langle u_k, P_n\rangle = \delta_{k,n}, \quad n, k \geq 0.$$

Throughout this paper, d denotes a positive integer.

Definition 2.1 ([25]). A PS $\{P_n\}_{n\geq 0}$ is called *d-orthogonal* (d-OPS, for shorter) with respect to the d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$ if it satisfies the following orthogonality relations:

$$\begin{cases} \langle \Gamma_k, P_r P_n\rangle = 0, & r > nd + k, \quad n \in \mathbb{N}, \\ \langle \Gamma_k, P_n P_{nd+k}\rangle \neq 0, & n \in \mathbb{N}, \end{cases} \tag{2.7}$$

for each integer k belonging to $N_d := \{0, 1, \ldots, d-1\}$.

For $d = 1$, the d-orthogonality is reduced to the orthogonality.

The d-dimensional functional $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$ given in this definition is not unique according to the following result.

Lemma 2.2 ([11]). *If a PS $\{P_n\}_{n\geq 0}$ is d-orthogonal with respect to a d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$, then this PS is also d-orthogonal with respect to the d-dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \ldots, u_{d-1})$, where the forms $u_0, u_1, \ldots, u_{d-1}$ are the d first elements of the dual sequence $\{u_n\}_{n\geq 0}$ associated with $\{P_n\}_{n\geq 0}$.*

2.3 – d-Symmetric polynomials

Definition 2.3 ([10]). A PS $\{P_n\}_{n\geq 0}$ is called d-symmetric if it fulfills for all $n \in \mathbb{N}$,

$$P_n(w_{d+1}x) = w_{d+1}^n P_n(x) \tag{2.8}$$

where $w_{d+1} = \exp(2i\frac{\pi}{d+1})$.

For $d = 1$, $w_2 = -1$, the PS $\{P_n\}_{n\geq 0}$ is symmetric *i.e.* $P_n(-x) = (-1)^n P_n(x)$.

Lemma 2.4 ([10]). *A PS $\{P_n\}_{n\geq 0}$ is d-symmetric if and only if there exist $(d+1)$ sequences $\{P_n^k\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, such that $P_{(d+1)n+k}(x) = x^k P_n^k(x^{d+1})$, $n \geq 0$.*

The $(d+1)$ PSs $\{P_n^k\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, are called the components of the d-symmetric PS $\{P_n\}_{n\geq 0}$.

Lemma 2.5 ([10]). *Let $\{P_n\}_{n\geq 0}$ be a monic d-OPS. Then $\{P_n\}_{n\geq 0}$ is d-symmetric if and only if $\{P_n\}_{n\geq 0}$ satisfies the $(d+1)$-order recurrence relation:*

$$\begin{cases} P_n(x) = x^n, & n \in \mathbb{N}_{d+1} \\ P_{n+1}(x) = xP_n(x) - \gamma_n P_{n-d}(x), & n \geq d \end{cases} \tag{2.9}$$

where $\gamma_n \neq 0$, $n \geq d$.

2.4 – Humbert polynomials

The Humbert polynomials are generated by [15]

$$\left(1-(d+1)xt+t^{d+1}\right)^{-\nu} = \sum_{n\geq 0} h_{n,d+1}^{\nu}(x)\ t^n \tag{2.10}$$

where $\nu > -\frac{1}{2}$ and $\nu \neq 0$.

The explicit representation of the Humbert polynomials is given by [5]

$$h_{n,d+1}^{\nu}(x) = \sum_{k=0}^{[\frac{n}{d+1}]} \frac{(-1)^k(\nu)_{n-dk}}{k!(n-(d+1)k)!}\left((d+1)x\right)^{n-(d+1)k}. \tag{2.11}$$

Let us give an overview of some special cases that were investigated in the literature.

- *Gegenbauer polynomials*: by letting $d=1$ in (2.10), we meet the Gegenbauer polynomials $\{C_n^{\nu}(x)\}_{n\geq 0}$.
- *Pincherle type polynomials*: for $d=2$, the Humbert polynomials are reduced to the Pincherle type polynomials [22], which, in the limiting case $\nu = -\frac{1}{2}$, are reduced to the Pincherle polynomials.
- *Chebyshev type d-OPS*: by letting $\nu = 1$ in (2.10), we meet the Chebyshev type d-OPS of the second kind $\{U_n(.;d)\}_{n\geq 0}$ studied by Douak and Maroni [12] and generated by:

$$\left(1+bt^{d+1}-xt\right)^{-1} = \sum_{n\geq 0} U_n(x;d)\ t^n,\ b\neq 0. \tag{2.12}$$

  For $d=b=1$, these polynomials are reduced to the *Chebyshev polynomials of the second kind* $\{U_n(.)\}_{n\geq 0}$.
- *Legendre type d-OPS*: by letting $\nu = \frac{1}{2}$ in (2.10), we meet the d-OPS of Legendre type $\{L_n(.\,;d)\}_{n\geq 0}$ considered by Lamiri and Ouni [17]. This PS is a natural extension of the Legendre ones.

- *Kinney polynomials*: for $\nu = \frac{1}{d+1}$, the Humbert PS $\{h^{\nu}_{n,d+1}(x)\}_{n\geq 0}$ is reduced to the Kinney PS $\{K_n(.;d)\}_{n\geq 0}$. That includes as particular cases, the Legendre polynomials $\{L_n(x)\}_{n\geq 0}$ ($d=1$), and the Pincherle type polynomials $\{P_n^{\frac{1}{3}}(x)\}_{n\geq 0}$ ($d=2$).

Lemma 2.6 ([5]). *The Humbert polynomials $\{h^{\nu}_n\}_{n\geq 0}$ are d-symmetric classical d-orthogonal.*

3 – A Dunkl-classical d-symmetric d-orthogonal polynomial set

Replacing the derivative operator in the definition of classical d-OPS, introduced by Douak and Maroni, by the Dunkl operator T_μ, one obtains the following.

Definition 3.1. A PS $\{P_n\}_{n\geq 0}$ is called T_μ-classical (or Dunkl-classical) d-orthogonal if and only if both $\{P_n\}_{n\geq 0}$ and $\{T_\mu P_{n+1}\}_{n\geq 0}$ are d-orthogonal.

Next, we introduce and study a T_μ-classical d-OPS.

3.1 – Generalized Humbert polynomials

The generalized Gegenbauer polynomials $S_n^{(\alpha,\beta)}(x)$ are orthogonal with respect to the weight function:

$$|x|^{2\beta+1}(1-x^2)^{\alpha}; \qquad -1\leq x\leq 1.$$

For $\beta = -1/2$, these polynomials are reduced to Gegenbauer polynomials. In [7], we gave the relation linking the generalized Gegenbauer polynomials $\{S_n^{(\alpha,\mu-1/2)}\}_{n\geq 0}$ and the Gegenbauer polynomials $\{C_n^{\alpha+\mu+1/2}\}_{n\geq 0}$. That is

$$V_\mu(C_n^{\alpha+\mu+1/2}) = S_n^{(\alpha,\mu-1/2)} \tag{3.1}$$

where V_μ is the isomorphism defined by (2.6).

Starting from this identity, we remark that, for this case, the operator V_μ preserves two main properties of the Gegenbauer polynomials, the symmetry and the orthogonality, while the "classical" property is replaced by the T_μ-classical ones. On the other hand, from Lemma 2.6, we notice that these three properties of the Gegenbauer polynomials have corresponding ones satisfied by the Humbert polynomials, another generalization of Gegenbauer polynomials. That suggests us to consider the polynomials:

$$\mathcal{H}^{(\nu,\mu-1/2)}_{n,d+1}(x) = \frac{\gamma_\mu(n)}{n!}V_\mu(h^{\nu}_{n,d+1}(x)), \tag{3.2}$$

in order to introduce a further example of T_μ-classical d-orthogonal polynomial set. We refer to these polynomials as *generalized Humbert polynomials.*

In the sequel, for the sake of simplicity, we will adopt the notation: $\mathcal{H}_n^{\nu}(.,d+1) := \mathcal{H}_{n,d+1}^{(\nu,\mu-1/2)}(.)$, $n \in \mathbb{N}$.

Theorem 3.2. *The PS* $\{\mathcal{H}_n^{\nu+\gamma}(.,d+1)\}_{n\in\mathbb{N}}$ *is a Dunkl-classical d-symmetric d-OPS if*

$$\text{(3.3)}\qquad \begin{cases} \gamma = \dfrac{d}{d+1}(2\mu+1), \\ (\mu,d) \in \{0\}\times\mathbb{N}^* \ \textit{or}\ (\mu,d) \in (]-1/2,+\infty[)\setminus\{0\}\times(2\mathbb{N}+1). \end{cases}$$

To prove this result, we need the following.

Lemma 3.3. *The generalized Humbert Polynomials* $\{\mathcal{H}_n^{\nu+\gamma}(.,d+1)\}_{n\in\mathbb{N}}$ *satisfy the* $(d+1)$*-order recurrence relation:*

$$\text{(3.4)}\qquad \begin{cases} \mathcal{H}_{n+1}^{\nu+\gamma}(x,d+1) = \dfrac{(d+1)(\nu+\gamma+n)}{n+1}x\mathcal{H}_n^{\nu+\gamma}(x,d+1) + \\ \quad -\dfrac{\gamma_\mu(n)(n-d)!(n+(d+1)\nu+2\mu d\theta_n)}{(n+1)!\gamma_\mu(n-d)}\mathcal{H}_{n-d}^{\nu+\gamma}(x,d+1),\ n\geq d, \\ \mathcal{H}_n^{\nu+\gamma}(x,d+1) = x^n, \quad n\in\mathbb{N}_{d+1}, \end{cases}$$

where γ, μ *and* d *satisfies* (3.3).

Proof. In order to prove (3.4), we put

$$\mathcal{H}_n^{\nu+\gamma}(x,d+1) = \sum_{k=0}^{[\frac{n}{d+1}]} C_{n,k}x^{n-(d+1)k}.$$

Taking account of (2.11), (3.2) and (2.6), ones obtains

$$C_{n,k} = \frac{(-1)^k\gamma_\mu(n)(\nu+\gamma)_{n-dk}(d+1)^{n-(d+1)k}}{k!n!\gamma_\mu(n-(d+1)k)}.$$

The coefficient of $x^{n+1-(d+1)k}$ in $\frac{(d+1)(\nu+\gamma+n)}{n+1}x\mathcal{H}_n^{\nu+\gamma}(x,d+1) - \mathcal{H}_{n+1}^{\nu+\gamma}(x,d+1)$, $k \in \left\{1,2,\dots,\left[\frac{n}{d+1}\right]\right\}$, is given by

$$\frac{(d+1)(\nu+\gamma+n)}{n+1}C_{n,k} - C_{n+1,k} = \\ = \frac{(-1)^{k+1}\gamma_\mu(n)(\nu+\gamma)_{n-dk}(d+1)^{n+1-(d+1)k}}{k!(n+1)!\gamma_\mu(n+1-(d+1)k)} \times A$$

where

$$A = k(n + (\nu+\gamma)(d+1) - d - 2\mu d\theta_{n+1}) + \mu(n+\nu+\gamma)(-1)^n \left(1 - (-1)^{(d+1)k}\right).$$

Next, we consider the case (3.3). Then

$$A = k(n + \nu(d+1) + 2\mu d\theta_n)$$

and

$$\frac{(d+1)(\nu+\gamma+n)}{n+1} C_{n,k} - C_{n+1,k} = \frac{\gamma_\mu(n)(n-d)!(n+(d+1)\nu+2\mu d\theta_n)}{(n+1)!\gamma_\mu(n-d)} C_{n-d,k-1}.$$

That leads to (3.4). □

Proof of Theorem 3.1. From Lemma 2.5 and Lemma 3.4, we deduce that $\{\mathcal{H}_n^{\nu+\gamma}(x, d+1)\}_{n\geq 0}$ is a d-symmetric d-OPS.

Now, let T_μ operate on both sides of (3.2). Taking account of Equation (2.11), using (2.2) and the following transformation

$$(a)_{i+j} = (a)_i (a+i)_j, \quad i, j \in \mathbb{N} \tag{3.5}$$

one obtains

$$T_\mu \mathcal{H}_n^{\nu+\gamma}(x, d+1) = \frac{(d+1)(\nu+\gamma)\gamma_\mu(n)}{n\gamma_\mu(n-1)} \mathcal{H}_{n-1}^{\nu+\gamma+1}(x, d+1).$$

It follows that the PS $\{H_n^{\nu+\gamma}(., d+1)\}_{n\geq 0}$ is Dunkl-classical. □

3.2 – d-dimensional functionals

In this subsection, we express explicitly the d-dimensional functional for which we have the d-orthogonality of the generalized Humbert polynomials. Then, we consider some special cases.

According to Lemma 2.2, we will determinate the d first elements of the corresponding dual sequence to construct the d forms ensuring the d-orthogonality of these polynomials. To this end, we follow an approach used by Lamiri and Ouni [17] based on the inversion formula. We state the following.

Theorem 3.4. *With the conditions*:

$$(\mu, d) \in \{0\} \times \mathbb{N}^* \ \text{or} \ (\mu, d) \in (]-1/2, +\infty[) \setminus \{0\} \times (2\mathbb{N}+1),$$

the generalized Humbert PS $\{\mathcal{H}_n^{\nu}(.,d+1)\}_{n\geq 0}$, $\nu-\mu>-\frac{1}{2}$, *defined by* (3.2) *is a d-OPS with respect to the d-dimensional functional vector* $\mathcal{U}={}^t(u_0,u_1,\ldots,u_{d-1})$ *given by*:

$$\langle u_r, x^n\rangle = \delta_{r,i}\int_0^{d^{-\frac{d}{d+1}}} \xi^n\ \varphi_{r,d}(\xi)d\xi, \tag{3.6}$$

where $n=i+(d+1)k$, $k\in\mathbb{N}$, $i=0,1,\ldots,d$, $r=0,1,\ldots,d-1$ *and*

$$\varphi_{r,d}(\xi)=\frac{2^r r!\left[\frac{r}{2}\right]!(\mu+1/2)_{[\frac{r+1}{2}]}\prod_{j=1}^{d}\Gamma\left(\frac{\nu+r+j}{d}\right)}{\gamma_\mu(r)(d+1)^{r-1}(\nu)_r\prod_{j=1}^{q}\Gamma\left(\frac{\left[\frac{r}{2}\right]+j}{q}\right)\prod_{j=1}^{q}\Gamma\left(\frac{\mu-1/2+\left[\frac{r+1}{2}\right]+j}{q}\right)} \times\xi^{-(r+1)}G^{d+1,0}_{d+1,\ d+1}\left(d^d\xi^{d+1}\,\middle|\,\begin{matrix}\alpha_1,\ldots,\alpha_{d+1}\\ \beta_1,\ldots,\beta_{d+1}\end{matrix}\right) \tag{3.7}$$

where

$$\alpha_j=\begin{cases}\dfrac{\nu+r+j}{d}, & 1\leq j\leq d,\\ 1, & j=d+1\end{cases}$$

and

$$\beta_j=\begin{cases}\dfrac{\left[\frac{r}{2}\right]+j}{q}, & 1\leq j\leq \dfrac{d+1}{2},\\ \dfrac{\mu-1/2+\left[\frac{r+1}{2}\right]+j}{q}-1, & \dfrac{d+1}{2}<j\leq d+1\end{cases}$$

with $q=\frac{d+1}{2}$.

Here, $G^{m,\ n}_{p,\ q}$, *designates the Meijer's G-function defined by* [19, p. 143]*:*

$$G^{m,\ n}_{p,\ q}\left(z\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right)=(2\pi i)^{-1}\int_{\mathcal{L}} z^{\xi}\frac{\prod_{j=1}^{m}\Gamma(b_j-\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\xi)}d\xi,$$

where (a_p) *abbreviates the set* $\{a_1,a_2,\ldots,a_p\}$. *We refer to* [19, p. 144] *for the details regarding the type of the contour* $\mathcal{L}$.

Proof. Recall first that the inversion formula related to the Humbert polynomial set $\left\{h^{\nu}_{n,d+1}\right\}_{n\geq 0}$ is given by [3]:

$$x^n = \sum_{j=0}^{[\frac{n}{d+1}]} \frac{(\nu+n-(d+1)j)}{(\nu)_{n+1-j}} \frac{n!}{(d+1)^n j!}\ h^{\nu}_{n-(d+1)j,\ d+1}(x). \tag{3.8}$$

Letting V_μ operate on both sides of (3.8) and using (2.6) and (3.2), we deduce that the inversion formula related to the generalized Humbert polynomial set $\{\mathcal{H}^{\nu+\gamma}_n(.,d+1)\}_{n\geq 0}$ is given by

$$x^n = \sum_{j=0}^{[\frac{n}{d+1}]} \frac{(\nu+n-(d+1)j)}{(\nu)_{n+1-j}} \frac{\gamma_\mu(n)}{(d+1)^n j!} \frac{(n-(d+1)j)!}{\gamma_\mu(n-(d+1)j)} \mathcal{H}^{\nu}_{n-(d+1)j}(x,d+1). \tag{3.9}$$

According to the definition of a linear functional vector, we have from (3.9):

$$\langle u_r, x^n\rangle = \delta_{r,i} \frac{(\nu+r)}{(d+1)^{r+(d+1)k}} \frac{\gamma_\mu(r+(d+1)k)}{k!(\nu)_{r+1+dk}} \frac{r!}{\gamma_\mu(r)} \tag{3.10}$$

where $n = i+(d+1)k$, $k \in \mathbb{N}$, $i = 0,1,\dots,d$, $r = 0,1,\dots,d-1$.

Taking account of (2.3) and (2.4), one obtains, for all $n \in \mathbb{N}$,

$$\gamma_\mu(n) = 2^n[\frac{n}{2}]!(\mu+1/2)_{[\frac{n+1}{2}]}.$$

The use of the identities (3.10), (3.5) and the following transformation:

$$(a)_{m\,k} = m^{m\,k} \prod_{j=0}^{m-1} \left(\frac{a+j}{m}\right)_k, \qquad k = 0,1,2,\dots, \tag{3.11}$$

leads, with $q = \frac{d+1}{2}$, to

$$\gamma_\mu(r+(d+1)k) = 2^{r+(d+1)k} \left(\left[\frac{r}{2}\right]+qk\right)!(\mu+1/2)_{[\frac{r+1}{2}]+qk}$$

$$= 2^{r+(d+1)k} \left[\frac{r}{2}\right]!q^{(d+1)k} \prod_{j=1}^{q} \left(\frac{\left[\frac{r}{2}\right]+j}{q}\right)_k (\mu+1/2)_{[\frac{r+1}{2}]} \prod_{j=0}^{q-1} \left(\frac{\mu+1/2+\left[\frac{r+1}{2}\right]+j}{q}\right)_k$$

and

$$\langle u_r, x^n\rangle = \delta_{r,i}\frac{2^r r!\left[\frac{r}{2}\right]!\prod_{j=1}^{q}\left(\frac{\left[\frac{r}{2}\right]+j}{q}\right)_k(\mu+1/2)_{\left[\frac{r+1}{2}\right]}\prod_{j=0}^{q-1}\left(\frac{\mu+1/2+\left[\frac{r+1}{2}\right]+j}{q}\right)_k}{\gamma_\mu(r)(d+1)^r(\nu)_r d^{dk}k!\prod_{j=1}^{d}\left(\frac{\nu+r+j}{d}\right)_k}$$

$$=\delta_{r,i}\frac{2^r r!\left[\frac{r}{2}\right]!(\mu+1/2)_{\left[\frac{r+1}{2}\right]}\prod_{j=1}^{d}\Gamma\left(\frac{\nu+r+j}{d}\right)}{\gamma_\mu(r)(d+1)^r(\nu)_r\prod_{j=1}^{q}\Gamma\left(\frac{\left[\frac{r}{2}\right]+j}{q}\right)\prod_{j=1}^{q}\Gamma\left(\frac{\mu-1/2+\left[\frac{r+1}{2}\right]+j}{q}\right)}\cdot A_{k,r}$$

with

$$A_{k,r}=\frac{\prod_{j=1}^{q}\Gamma\left(\frac{\left[\frac{r}{2}\right]+j}{q}+k\right)\prod_{j=1}^{q}\Gamma\left(\frac{\mu-1/2+\left[\frac{r+1}{2}\right]+j}{q}+k\right)}{d^{dk}k!\prod_{j=1}^{d}\Gamma\left(\frac{\nu+r+j}{d}+k\right)}.$$

Setting

$$\gamma_j=\begin{cases}\dfrac{\left[\frac{r}{2}\right]+j}{q}+k-1, & 1\le j\le \dfrac{d+1}{2},\\[2ex] \dfrac{\mu-1/2+\left[\frac{r+1}{2}\right]+j}{q}+k-2, & \dfrac{d+1}{2}<j\le d+1,\end{cases}$$

$$l_j=\begin{cases}\dfrac{\nu(d+1)+(d+1)r-2d\left[\frac{r}{2}\right]+(1-d)j}{d(d+1)}, & 1\le j\le\dfrac{d+1}{2},\\[2ex] \dfrac{\nu(d+1)+(d+1)r-2d\left[\frac{r+1}{2}\right]+(1-d)j-(2\mu-1)d}{d(d+1)}+1, & \dfrac{d+1}{2}<j\le d,\\[2ex] -1-\dfrac{\mu-1/2+\left[\frac{r+1}{2}\right]}{q}, & j=d+1,\end{cases}$$

we obtain

$$A_{k,r}(d) = \frac{1}{d^{dk}} \prod_{j=1}^{d+1} \left(\frac{\Gamma(\gamma_j + 1)}{\Gamma(\gamma_j + l_j + 1)} \right) \tag{3.13}$$

and $\sum_{j=1}^{d+1} l_j = \nu - \mu + 1/2$.

On the other hand, if $\sum_{j=1}^{d+1} l_j > 0$, the first author and Douak [6] showed that

$$\begin{aligned} {}_pF_q\left(\begin{matrix}(a_p)\\(\gamma_q + l_q + 1)\end{matrix}; x\right) = & \prod_{i=1}^{q}\left(\frac{\Gamma(\gamma_i + 1 + l_i)}{\Gamma(\gamma_i + 1)}\right) \\ & \times \int_0^1 G_{q,\ q}^{q,\ 0}\left(t \,\middle|\, \begin{matrix}(\gamma_q + l_q)\\(\gamma_q)\end{matrix}\right) {}_pF_q\left(\begin{matrix}(a_p)\\(\gamma_q + 1)\end{matrix}; xt\right) dt, \end{aligned} \tag{3.14}$$

where the ${}_pF_q$, as usual, denotes the generalized hypergeometric functions defined by:

$${}_pF_q\left(\begin{matrix}\alpha_1, & \ldots, & \alpha_p & & \\ & & & ; & z\\ \beta_1, & \ldots, & \beta_q & & \end{matrix}\right) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \frac{z^m}{m!}, \tag{3.15}$$

- p and q are positive integers or zero (interpreting an empty product as 1);
- z is a complex variable;
- the numerator parameters $\alpha_1, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \ldots, \beta_q$ take in complex values. β_j; $j \in \mathbb{N}^*_{q+1}$: being non-negative integers.

The identity (3.14), for $x = 0$ and $q = d + 1$, is reduced to

$$\prod_{j=1}^{d+1} \left(\frac{\Gamma(\gamma_j + 1)}{\Gamma(\gamma_j + 1 + l_j)} \right) = \int_0^1 G_{d+1,\ d+1}^{d+1,\ 0}\left(t \,\middle|\, \begin{matrix}(\gamma_{d+1} + l_{d+1})\\(\gamma_{d+1})\end{matrix}\right) dt. \tag{3.16}$$

Thus, for $\nu > -\frac{1}{2}$, the identity (3.13) can be rewritten under the form

$$A_{k,r}(d) = \frac{1}{d^{dk}}$$

$$\int_0^1 G_{d+1,d+1}^{d+1,0}\left(t \,\middle|\, \begin{matrix} \frac{\nu + r + 1}{d} - 1 + k, \ldots, \frac{\nu + r + d}{d} - 1 + k, k \\ \frac{\left[\frac{r}{2}\right] + 1}{q} - 1 + k, \ldots, \frac{\left[\frac{r}{2}\right] + q}{q} - 1 + k, \frac{\mu + 1/2 + \left[\frac{r+1}{2}\right]}{q} + \\ +k - 1, \ldots, \frac{\mu + 1/2 + \left[\frac{r+1}{2}\right] + q - 1}{q} + k - 1 \end{matrix}\right) dt.$$

Then, according to the transformation [[24], p. 46]

$$z^k G^{m,n}_{p,\,q}\left(z\,\middle|\,\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{matrix}\right)=G^{m,n}_{p,\,q}\left(z\,\middle|\,\begin{matrix}\alpha_1+k,\dots,\alpha_p+k\\ \beta_1+k,\dots,\beta_q+k\end{matrix}\right), \tag{3.17}$$

we get

$$A_{k,r}(d)=$$

$$\int_0^1\left(\frac{t}{d^d}\right)^k G^{d+1,0}_{d+1,\,d+1}\left(t\,\middle|\,\begin{matrix}\frac{\nu+r+1}{d}-1,\dots,\frac{\nu+r+d}{d}-1,0\\ \frac{\left[\frac{r}{2}\right]+1}{q}-1,\dots,\frac{\left[\frac{r}{2}\right]+q}{q}-1,\frac{\mu+1/2+\left[\frac{r+1}{2}\right]}{q}+\\ -1,\dots,\frac{\mu+1/2+\left[\frac{r+1}{2}\right]+q-1}{q}-1\end{matrix}\right)dt.$$

That, upon the change of variables $t=d^d\xi^{(d+1)}$, leads to

$$\begin{aligned}A_{k,r}(d)=&\int_0^{d^{-\frac{d}{d+1}}}\xi^{k(d+1)}G^{d+1,\,0}_{d+1,\,d+1}\cdot\\ &\cdot\left(d^d\xi^{d+1}\,\middle|\,\begin{matrix}\frac{\nu+r+1}{d}-1,\dots,\frac{\nu+r+d}{d}-1,0\\ \gamma_1-k,\dots,\gamma_{d+1}-k\end{matrix}\right)(d+1)d^d\xi^d d\xi.\end{aligned}\tag{3.18}$$

Substituting (3.18) in (3.12), we obtain

$$\langle u_r,x^n\rangle=\delta_{r,i}\frac{2^r r!\left[\frac{r}{2}\right]!(\mu+1/2)_{\left[\frac{r+1}{2}\right]}\prod_{j=1}^{d}\Gamma\left(\frac{\nu+r+j}{d}\right)}{\gamma_\mu(r)(d+1)^{r-1}(\nu)_r\prod_{j=1}^{q}\Gamma\left(\frac{\left[\frac{r}{2}\right]+j}{q}\right)\prod_{j=0}^{q-1}\Gamma\left(\frac{\mu+1/2+\left[\frac{r+1}{2}\right]+j}{q}\right)}$$

$$\begin{aligned}&\times\int_0^{d^{-\frac{d}{d+1}}}\xi^{r+k(d+1)}\xi^{-(r+1)}\left(d^d\xi^{d+1}\right)\;G^{d+1,\,0}_{d+1,\,d+1}\times\\ &\times\left(d^d\xi^{d+1}\,\middle|\,\begin{matrix}\frac{\nu+r+1}{d}-1,\dots,\frac{\nu+r+d}{d}-1,\;0\\ \gamma_1-k,\dots,\gamma_{d+1}-k\end{matrix}\right)d\xi.\end{aligned}$$

That, by virtue of (3.17), leads to (3.6). □

3.2.1 – Special cases

In this subsection, we consider some particular cases of generalized Humbert polynomials by specializing the parameters d and μ.

Case 1 (Generalized Gegenbauer polynomials). Letting $d = 1$ in (3.2), we meet the generalized Gegenbauer polynomials $\left\{S_n^{(\nu-\mu-1/2,\mu-1/2)}(x)\right\}_{n\geq 0}$. Indeed, from (3.7) with $d = 1$ and the transformation 24, p. 46]

$$G^{m,n}_{p,\,q}\left(z\,\middle|\,\begin{matrix}\alpha_1,\dots,\alpha_{p-1},\beta_1\\ \beta_1,\dots,\beta_q\end{matrix}\right)= \tag{3.19}$$
$$=G^{m-1,n}_{p-1,q-1}\left(z\,\middle|\,\begin{matrix}\alpha_1,\dots,\alpha_{p-1}\\ \beta_2,\dots,\beta_q\end{matrix}\right);\ m,p,q\geq 1;$$

we have

$$\varphi_{0,1}(\xi)=\frac{2\Gamma(\nu+1)}{\Gamma\left(\mu+\frac{1}{2}\right)}\,\xi^{-1}\,G^{1,\,0}_{1,\,1}\left(\xi^2\,\middle|\,\begin{matrix}\nu+1\\ \mu+\frac{1}{2}\end{matrix}\right).$$

Taking account of the following identity [6]

$$G^{1,\,0}_{1,\,1}\left(x\,\middle|\,\begin{matrix}\alpha+\beta\\ \alpha\end{matrix}\right)=\frac{1}{\Gamma(\beta)}\,(1-x)^{\beta-1}\,x^{\alpha}, \tag{3.20}$$

we obtain

$$\varphi_{0,1}(\xi)=\frac{2\Gamma(\nu+1)}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}\xi^{2\mu}\left(1-\xi^2\right)^{\nu-\mu-\frac{1}{2}}. \tag{3.21}$$

Consequently, the linear functional u_0 of the generalized Gegenbauer polynomials is given by

$$\langle u_0,x^n\rangle=\frac{\Gamma(\nu+1)}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}\int_{-1}^{1}\xi^n|\xi|^{2\mu}\left(1-\xi^2\right)^{\nu-\mu-\frac{1}{2}}d\xi.$$

If moreover $\mu = 0$, we meet the Gegenbauer polynomials $\{C_n^\nu\}_{n\geq 0}$. These polynomials are orthogonal with respect to the well known weight function

$$\varphi_{0,1}(\xi) = \frac{\nu(\Gamma(\nu))^2}{\pi\Gamma(2\nu)2^{1-2\nu}}\left(1-\xi^2\right)^{\nu-\frac{1}{2}}, \quad -1 \leq \xi \leq 1.$$

Case 2 (Humbert polynomials). In this case, $\mu = 0$, $d \in \mathbb{N}^*$ and

$$\begin{aligned}\{\beta_1, \beta_2, \dots, \beta_{d+1}\} &= \left\{\frac{\left[\frac{r}{2}\right]+j}{q},\ 1 \leq j \leq \frac{d+1}{2}\right\} \bigcup \\ &\bigcup \left\{\frac{-1/2+\left[\frac{r+1}{2}\right]+j}{q}-1,\ \frac{d+1}{2} < j \leq d+1\right\} \\ &= \left\{\frac{r+1}{d+1}, \frac{r+2}{d+1}, \dots, \frac{r+(d+1)}{d+1}\right\}.\end{aligned}$$

Since $r = [\frac{r}{2}] + [\frac{r+1}{2}]$, then $2^r[\frac{r}{2}]!(1/2)_{[\frac{r+1}{2}]} = r!$ and (3.7) is reduced to

$$\begin{aligned}\varphi_{r,d}(\xi) &= \frac{r!\displaystyle\prod_{j=1}^{d}\Gamma\left(\frac{\nu+r+j}{d}\right)}{(d+1)^{r-1}(\nu)_r\displaystyle\prod_{j=1}^{d+1}\Gamma\left(\frac{r+j}{d+1}\right)}\times \\ &\times \xi^{-(r+1)} G^{d+1,0}_{d+1,\ d+1}\left(d^d\xi^{d+1}\,\middle|\, \begin{matrix}\frac{\nu+r+1}{d}, \dots, \frac{\nu+r+d}{d}, 1 \\ \frac{r+1}{d+1}, \dots, \frac{r+(d+1)}{d+1}\end{matrix}\right).\end{aligned} \tag{3.22}$$

That was obtained by Lamiri and Ouni [17].

4 – Properties of the generalized Humbert polynomials

4.1 – A T_μ-equation

In this subsection, we state a $(d+1)$-order differential-difference equation satisfied by the generalized Humbert polynomials. To this end, we need the linear operator U_μ defined on polynomials by means of

$$U_\mu(x^n) := \frac{(n+1)\gamma_\mu(n)}{\gamma_\mu(n+1)} x^{n+1}, \tag{4.1}$$

where $\gamma_\mu(n)$ is defined by (2.3) and (2.4).

We have the following.

THEOREM 4.1. *The generalized Humbert polynomial $\mathcal{H}_n^\nu(.,d+1)$, $n = 0,1,\dots,$ satisfy the following $(d+1)$-order differential-difference equation*:

$$\left(T_\mu^{d+1} - (U_\mu T_\mu - n)\prod_{j=0}^{d-1}\left(d\left(U_\mu T_\mu - n + d + 1\right) + (d+1)(\nu + n - j - 1)\right)\right) y = 0, \quad n > d. \tag{4.2}$$

PROOF. The Humbert polynomials $h_{n,d+1}^\nu$, $n = 0,1,\dots,$ satisfy the following $(d+1)$-order differential equation [5]:

$$L_n(y) := \left(D^{d+1} - (xD - n)\prod_{j=0}^{d-1}\left(d\left(xD - n + d + 1\right) + (d+1)(\nu + n - j - 1)\right)\right) y = 0, \quad n > d.$$

From the identities (2.5) and (4.1), we deduce that $D = V_\mu^{-1} T_\mu V_\mu$ and $U_\mu V_\mu = V_\mu X$.

Then

$$V_\mu(XD) = U_\mu T_\mu V_\mu$$

and

$$V_\mu(XD)^k = (U_\mu T_\mu)^k V_\mu. \tag{4.3}$$

Put

$$L_n = D^{d+1} + \sum_{k=0}^{d} \alpha_{n,k}(XD)^k. \tag{4.4}$$

Letting V_μ operate on both sides of (4.4) and using (4.3), we deduce that

$$V_\mu L_n(h_{n,d+1}^\nu) = \widetilde{L_n}(V_\mu(h_{n,d+1}^\nu)) = \widetilde{L_n}(\mathcal{H}_n^\nu(.,d+1)) = 0$$

where

$$\widetilde{L_n} := T_\mu^{d+1} + \sum_{k=0}^{d} \alpha_{n,k} (U_\mu T_\mu)^k. \qquad \square$$

Put $d = 1$ in (4.2), we meet the second-order differential-difference equation satisfied by the generalized Gegenbauer polynomials $\{S_n^{(\alpha,\mu-1/2)}\}_{n\geq 0}$. Indeed, from (4.2), with $d = 1$, we have

$$\left(T_\mu^2 - (U_\mu T_\mu - n)\,(U_\mu T_\mu + n + 2\nu)\right) y = 0, \quad n > 1. \tag{4.5}$$

Since, for $k = 0, 1, \ldots, n$ and $n - k \in 2\mathbb{N}$,

$$(n - U_\mu T_\mu)\,(U_\mu T_\mu + n + 2\nu)\, x^k =$$

$$= \left(-x^2 T_\mu^2 - 2(\alpha+1) x T_\mu + \frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} \left(\frac{\gamma_\mu(n-1)}{\gamma_\mu(n-2)} + 2(\alpha+1)\right)\right) x^k,$$

with $\alpha = \nu - \mu - 1/2$, (4.5) becomes [7]

$$\left((1-x^2)T_\mu^2 - 2(\alpha+1)xT_\mu + \frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} \left(\frac{\gamma_\mu(n-1)}{\gamma_\mu(n-2)} + 2(\alpha+1)\right)\right) S_n^{(\alpha,\mu-1/2)}(x) = 0.$$

4.2 – Components of generalized Humbert polynomials

According to Lemma 2.4, the components $\{\mathcal{H}_n^{\nu,k}\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, of the generalized Humbert PS $\{\mathcal{H}_n^{\nu}\}_{n\geq 0}$ are defined by

$$\mathcal{H}^{\nu}_{(d+1)m+k}(x, d+1) = \frac{\gamma_\mu(n(d+1)+k)}{(n(d+1)+k)!} x^k \mathcal{H}_m^{\nu,k}(x^{d+1}, d+1), \quad k \in \mathbb{N}_{d+1}. \tag{4.6}$$

With $d = 1$, the identity (4.6) is reduced to the classical relation between generalized Gegenbauer polynomials and Jacobi polynomials.

Next we give some properties of these components.

Theorem 4.2. *The components $\{\mathcal{H}_n^{\nu,k}(., d+1)\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, are classical d-orthogonal.*

The proof of this theorem, follows from the following three lemmas.

Lemma 4.3. *The components* $\{\mathcal{H}_n^{\nu,k}(.,d+1)\}_{n\geq 0}$, $k\in\mathbb{N}_{d+1}$, *are generated by*

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{H}_n^{\nu,k}(x,d+1)\ t^n=\\ &=\frac{(\nu)_k}{\gamma_\mu(k)}(d+1)^k(1+t)^{-k-\nu}{}_{d+1}F_d\left(\begin{matrix}\Delta\,(d+1,k+\nu)\\ \Delta_\mu^*\,(d+1,k+1)\end{matrix};\left(\frac{d+1}{1+t}\right)^{d+1}xt\right)\end{aligned}\tag{4.7}$$

where $\Delta(n,a)$ *abbreviates the array of* n *parameters* $\frac{a+j-1}{n}$, $j=1,\dots,n$ *and*

$$\Delta_\mu^*(n,l):=\left\{\frac{l+j+2\mu\theta_{l+j}}{n};\ j=0,1,\dots,n-1\right\}\Big\backslash\left\{\frac{n}{n}\right\}.$$

Proof. The Humbert polynomials are generated by (2.10), which can be rewritten in the form

$$(1+t^{d+1})^{-\nu}\sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}\left(\frac{(d+1)xt}{1+t^{d+1}}\right)^n=\sum_{n\geq 0}h_{n,d+1}^{\nu}(x)\ t^n.\tag{4.8}$$

Let $\Pi_{[d+1,k]}$ be the linear operator on formel power series defined by

$$\Pi_{[d+1,k]}f(z)=\frac{1}{d+1}\sum_{l=0}^{d}w_{d+1}^{-kl}f(w_{d+1}^{l}z)\ ,\quad k\in\mathbb{N}_{d+1}.\tag{4.9}$$

Applying the operator $\Pi_{[d+1,k]}, k\in\mathbb{N}_{d+1}$, to the two members of the identity (4.8) considered as functions of the variable x, and using the fact that the Humbert polynomials $h_{n,d+1}^{\nu}(x)$ are d-symmetric, we obtain

$$\begin{aligned}&\sum_{n\geq 0}h_{n(d+1)+k,d+1}^{\nu}(x)\ t^{n(d+1)+k}=\\ &=(1+t^{d+1})^{-\nu}\sum_{n=0}^{\infty}\frac{(\nu)_{n(d+1)+k}}{(n(d+1)+k)!}\left(\frac{(d+1)xt}{1+t^{d+1}}\right)^{n(d+1)+k}.\end{aligned}$$

Letting V_μ operate on both sides of the last identity considered as functions of the variable x and using (2.6) and (3.2), we deduce that the generalized Humbert polynomial set $\{\mathcal{H}_n^{\nu}(x,d+1)\}_{n\geq 0}$ is generated by

$$\begin{aligned}&\sum_{n\geq 0}\frac{(n(d+1)+k)!}{\gamma_\mu(n(d+1)+k)}\mathcal{H}_{n(d+1)+k}^{\nu}(x,d+1)\ t^{n(d+1)+k}=\\ &=(1+t^{d+1})^{-\nu}\sum_{n=0}^{\infty}\frac{(\nu)_{n(d+1)+k}}{\gamma_\mu(n(d+1)+k)}\left(\frac{(d+1)xt}{1+t^{d+1}}\right)^{n(d+1)+k}.\end{aligned}$$

Notice that

$$\begin{aligned}\gamma_\mu(n(d+1)+k) &= \prod_{j=1}^{n(d+1)+k} (j+2\mu\theta_j)\\ &= (d+1)^{n(d+1)}\gamma_\mu(k) \prod_{j=1}^{n(d+1)} \left(\frac{j+k+2\mu\theta_{j+k}}{d+1}\right)\\ &= (d+1)^{n(d+1)}\gamma_\mu(k) \prod_{l=1}^{d+1}\prod_{j=0}^{n-1}\left(\frac{k+l+j(d+1)+2\mu\theta_{k+l+j(d+1)}}{d+1}\right).\end{aligned}$$

Since $j(d+1)$ is even, for $j=0,1,\ldots,d$, we have

$$\gamma_\mu(n(d+1)+k) = $$

$$= (d+1)^{n(d+1)}\gamma_\mu(k)\; n! \prod_{l=1, l+k\neq d+1}^{d+1} \left(\frac{k+l+2\mu\theta_{k+l}}{d+1}\right)_n .$$

Using (3.5) and (3.11), we write

$$\sum_{n=0}^{\infty} \frac{(n(d+1)+k)!}{\gamma_\mu(n(d+1)+k)} \mathcal{H}^{\nu}_{n(d+1)+k}(x,d+1)\; t^{n(d+1)+k} =$$

$$= (1+t^{d+1})^{-\nu} \sum_{n=0}^{\infty} \frac{(\nu)_k \prod_{j=0}^{d}\left(\frac{\nu+k+j}{d+1}\right)_n}{\gamma_\mu(k) n! \prod_{l=1,l+k\neq d+1}^{d+1}\left(\frac{k+l+2\mu\theta_{k+l}}{d+1}\right)_n} \left(\frac{(d+1)xt}{1+t^{d+1}}\right)^{n(d+1)+k} =$$

$$= \frac{(\nu)_k}{\gamma_\mu(k)}((d+1)xt)^k(1+t^{d+1})^{-k-\nu}\,{}_{d+1}F_d\left(\begin{matrix}\Delta\,(d+1,k+\nu)\\ \Delta^*_\mu\,(d+1,k+1)\end{matrix};\left(\frac{(d+1)xt}{1+t^{d+1}}\right)^{d+1}\right).$$

Which, by virtue of (4.6) leads to (4.7). □

LEMMA 4.4. *The components $\mathcal{H}^{\nu,k}_n(.,d+1)$, $k\in\mathbb{N}_{d+1}$, have the following hypergeometric representation.*

$$\mathcal{H}^{\nu,k}_n(x,d+1) =$$

$$= \frac{(\nu)_k(d+1)^k}{\gamma_\mu(k)} \frac{(-1)^n(\nu+k)_n}{n!}\;{}_{d+1}F_d\left(\begin{matrix}-n,\Delta\,(d,\nu+k+n)\\ \\ \Delta^*_\mu\,(d+1,k+1)\end{matrix};d^d x\right). \tag{4.10}$$

Proof. From (4.7), we have

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{\nu,k}(x,d+1)\, t^n =$$

$$= \sum_{l=0}^{\infty} \frac{(\nu)_k \prod_{j=0}^{d} \left(\frac{\nu+k+j}{d+1}\right)_l (d+1)^{l(d+1)+k}}{\gamma_\mu(k) l! \prod_{j=1, j+k\neq d+1}^{d+1} \left(\frac{k+j+2\mu\theta_{k+j}}{d+1}\right)_l} (1+t)^{-(k+\nu)-l(d+1)}\, x^l t^l$$

$$= \sum_{l=0}^{\infty} \frac{(\nu)_k \prod_{j=0}^{d} \left(\frac{\nu+k+j}{d+1}\right)_l (d+1)^{l(d+1)+k}}{\gamma_\mu(k) l! \prod_{j=1, j+k\neq d+1}^{d+1} \left(\frac{k+j+2\mu\theta_{k+l}}{d+1}\right)_l} \sum_{n=0}^{\infty} \frac{(-1)^n((k+\nu)+l(d+1))_n}{n!} t^n x^l t^l$$

$$= \sum_{n=0}^{\infty}\sum_{l=0}^{n} \frac{(\nu)_k \prod_{j=0}^{d} \left(\frac{\nu+k+j}{d+1}\right)_l (d+1)^{l(d+1)+k}}{\gamma_\mu(k) l! \prod_{j=1, j+k\neq d+1}^{d+1} \left(\frac{k+j+2\mu\theta_{k+l}}{d+1}\right)_l} \frac{(-1)^{n-l}(k+\nu+l(d+1))_{n-l}}{(n-l)!} x^l t^n$$

$$= \frac{(\nu)_k (d+1)^k}{\gamma_\mu(k)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{l=0}^{n} \frac{(-n)_l (\nu+k)_{n+dl}}{l! \prod_{j=1, j+k\neq d+1}^{d+1} \left(\frac{k+j+2\mu\theta_{k+l}}{d+1}\right)_l} x^l t^n$$

$$= \frac{(\nu)_k (d+1)^k}{\gamma_\mu(k)} \sum_{n=0}^{\infty} \frac{(-1)^n (\nu+k)_n}{n!} \sum_{l=0}^{n} \frac{(-n)_l d^{dl} \prod_{j=0}^{d-1} \left(\frac{\nu+k+n+j}{d}\right)_l}{l! \prod_{j=1, j+k\neq d+1}^{d+1} \left(\frac{k+j+2\mu\theta_{k+l}}{d+1}\right)_l} x^l t^n.$$

Equaling the coeffitients of t^n and using (3.15), we obtain (4.10). ∎

Lemma 4.5 ([17]). *The polynomials defined by*

$${}_{d+1}F_d \left(\begin{matrix} -n, \Delta\left(d, \nu+k+n\right) \\ \beta_1, \ \ldots, \ \beta_d \end{matrix} ; x \right)$$

are classical d-orthogonal.

5 – Concluding remarks

In this section, we discuss the significance of the polynomials given by (3.2) and the method how these polynomials were introduced. As example of a special function, we show that these polynomials are a generalization of Gegenbauer polynomials having a property related to an orthogonality notion and as example of a T_μ-classical d-OPS, we show that these polynomials give negative answers to two questions arising in the d-orthogonal polynomial theory and suggest an open one.

5.1 – Generalized Gegenbauer polynomials and some orthogonality notions

The literature on generalizations of Gegenbauer polynomials contains several references. But only a few ones have a property related to an orthogonality notion. Let us give an overview of some different generalizations that were investigated in the literature.

- The Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$ are orthogonal with respect to the weight function:

$$(1-x)^\alpha(1+x)^\beta; \qquad -1\leq x\leq 1.$$

  For $\alpha=\beta$, the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$ becomes the Gegenbauer polynomial.
- The generalized Gegenbauer polynomials $\{S_n^{(\alpha,\beta)}\}_{n\geq 0}$ are orthogonal with respect to the weight function:

$$|x|^{2\beta+1}(1-x^2)^\alpha; \qquad -1\leq x\leq 1.$$

  For $\beta=-1/2$, these polynomials are reduced to Gegenbauer polynomials.
- The Humbert polynomials $\{h_{n,d+1}^\nu\}_{n\geq 0}$ generated by (2.10). They are d-orthogonal with respect to the weights functions $\varphi_{r,d}$, $r=0,1,\dots,d$, given by (3.22).
- The generalization given by Milovanovic [21]:

$$\pi_N(z)=2^{-n}z^\nu \widehat{P}_n^{(\alpha,\beta_\nu)}(2z^{2m}-1), \quad N=2mn+\nu, \quad n=\left[\frac{N}{2m}\right],$$

  where $\nu\in\mathbb{N}_{2m}$, $\beta_\nu=\gamma+(2\nu+1-2m)/(2m)$, and $\widehat{P}_n^{(\alpha,\beta)}(x)$ denotes the monic Jacobi polynomial.
  They are orthogonal relative to the inner product

$$(f,g)=\int_0^1\left(\sum_{s=0}^{2m-1} f(x\epsilon_s)\overline{g(x\epsilon_s)}\right)w(x)dx,$$

where $m \in \mathbb{N}$, $\epsilon_s = \exp(i\pi s/m)$, $s \in \mathbb{N}_{2m}$, and

$$w(x) = (1 - x^{2m})^{\alpha} x^{2m\gamma}, \quad \alpha > -1, \quad \gamma > -\frac{1}{2m}.$$

That corresponds to an orthogonal polynomial set over the star (OPS/$\star$, for shorter).

The link between the aforementioned polynomial sets and the obtained polynomials in this paper and defined by (3.2) may be summarized by the following scheme:

5.2 – The "L-classical" d-symmetric d-OPS and its components

Two general interesting questions may be discussed about a d-symmetric d-OPS $\{P_n\}_{n\geq 0}$, its components $\{P_n^k\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, and L-classical property:

Question 1. If $\{P_n\}_{n\geq 0}$ is L-classical, what about its components?

Question 2. If all the components $\{P_n^k\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, are L-classical, what about $\{P_n\}_{n\geq 0}$?

As far as we know, only a particular case of Question 1, where L is the derivative operator D, was considered in the literature. Indeed, if $\{P_n\}_{n\geq 0}$ is classical, Douak and Maroni [10] showed that the first component $\{P_n^0\}_{n\geq 0}$ is classical and, recently, Blel [9] showed that all the components $\{P_n^k\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, are classical. For the converse of this result, one can formulate Question 2 as follows.

Question 2.1. If all the components $\{P_n^k\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, are classical, is $\{P_n\}_{n\geq 0}$ too?

To generalize Blel result, one can formulate Question 1 as follows.

Question 1.1. If $\{P_n\}_{n\geq 0}$ is L-classical, are all the components $\{P_n^k\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, too?

The example (3.2), introduced and studied in this paper, gives negative answers to the two last questions. Indeed, from Theorem (3.2) and Theorem (4.2), we deduce that the PS given by (3.2) is T_μ-classical but its components are classical. Another example of T_μ-classical d-symmetric d-OPS having classical d-orthogonal components was treated in [8] where we showed that the components of the Gould-Hopper type polynomials are of Laguerre type which are classical d-orthogonal according to Theorem 1 in [4]. These two examples suggest us the following particular case of Question 1.

Question 1.2. If $\{P_n\}_{n\geq 0}$ is T_μ-classical, are the components $\{P_n^k\}_{n\geq 0}$, $k \in \mathbb{N}_{d+1}$, classical?

This question remains open.

5.3 – About the introduction of a new d-OPS

In this work, we introduce and study a Dunkl-classical d-symmetric d-OPS. As far as we know, the method used here to introduce new d-OPSs is original. In fact, most of the known d-OPSs were introduced as solutions of characterization problems or as components of d-symmetric d-OPSs while this polynomial set was introduced as a range of another one, the Humbert PS, by a suitable d-orthogonality preserving operator V_μ.

In a forthcoming investigation we will benefit from the present method to derive new Dunkl-classical d-symmetric d-OPSs.

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Indirizzo degli Autori:

Y. Ben Cheikh – Département de Mathématiques – Faculté des Sciences de Monastir – 5019 Monastir, Tunisia
E-mail: youssef.bencheikh@planet.tn

M. Gaied – Institut Supérieure de Mathématiques Appliquées et d'Informatique de Kairouan, Tunisia