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g-Natural metrics on unit tangent sphere bundles via a Musso-Tricerri process

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Dedicated to the memory of Professor F. Tricerri

Abstract: *E. Musso and F. Tricerri had given a process of construction of Riemannian metrics on tangent bundles and unit tangent bundles, over m-dimensional Riemannian manifolds (M, g), from some special quadratic forms an $OM \times \mathbb{R}^m$ and OM, respectively, where OM is the bundle of orthonormal frames [7]. We prove in this note that every Riemannian g-natural metric on the unit tangent sphere bundle over a Riemannian manifold can be constructed by the Musso-Tricerri's process. As a corollary, we show that every Riemannian g-natural metric on the unit tangent bundle, over a two-point homogeneous space, is homogeneous.*

Let (M, g) be a Riemannian manifold and TM its tangent bundle. Considering TM as a vector bundle associated with the bundle of orthonormal frames OM, E. Musso and F. Tricerri have constructed an interesting class of *Riemannian* metrics on TM [7]. This construction is not a classification *per se*, but it is a construction process of *Riemannian* metrics on TM from symmetric, positive semi-definite tensor fields Q of type $(2, 0)$ and rank $2m$ on $OM \times \mathbb{R}^m$, which are basic for the natural submersion $\Phi : OM \times \mathbb{R}^m \to TM$, $\Phi(v\varepsilon) = (x, \sum_i \varepsilon^e v_i)$, for $v = (x; v_1, \ldots, v_m) \in OM$ and $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^m) \in \mathbb{R}^m$. Recall that Q is *basic*

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means that Q is $O(m)$-invariant and $Q(X,Y)=0$, if X is tangent to a fiber of Φ. The construction can be presented as follows:

PROPOSITION 1 ([7]). *Let Q be a symmetric, positive semi-definite tensor field of type $(2,0)$ and rank $2m$ on $OM \times \mathbb{R}^m$, which is basic for the natural submersion $\Phi : OM \times \mathbb{R}^m \to TM$. Then there is a unique Riemannian metric G^Q on TM such that $\Phi^*(G^Q) = Q$. It is given by*

$$G^Q_{(x,u)}(X,Y) = Q_{(v,\varepsilon)}(X',Y'), \tag{1}$$

where (v,ε) belongs to the fiber $\Phi^{-1}(x,u), X, Y$ are elements of $(TM)_(x,u)$ and X', Y' are any tangent vectors to $OM \times \mathbb{R}^m$ at (v,ε) such that $\Phi_*(X') = X$ and $\Phi_*(Y') = Y$.

On the other hand, Musso and Tricerri proposed a similar process for constructing Riemannian metrics on the unit tangent sphere bundle T_1M from symmetric, positive semi-definite tensor fields $\tilde{Q}$ of type $(2,0)$ and rank $2m-1$ on OM, which are basic for the natural submersion $\psi_m : OM \to T_1M$, $\psi_m(v) = (x, v_m)$, for $v = (x; v_1, \ldots, v_m) \in OM$. Recall that $\tilde{Q}$ is *basic* means that $\tilde{Q}$ is $O(m-1)$-invariant and $\tilde{Q}(X,Y)=0$, if X is tangent to a fiber of ψ_m. Note that ψ_m is a submersion whose fibers can be identified with the subgroup $O(m-1)$ of $O(m)$ given by the matrices of the form $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, $A \in O(m-1)$. Then T_1M can be regarded as the quotient space $OM/O(m-1)$, and ψ_m is the natural projection. The construction can be stated as follows:

PROPOSITION 2 ([7]). *Let $\tilde{Q}$ be a symmetric, positive semi-definite tensor field of type $(2,0)$ and rank $2m$ on OM, which is basic for the natural submersion $\psi_m : OM \to T_1M$. Then there is a unique Riemannian metric $\tilde{G}^{\tilde{Q}}$ on T_1M such that $\psi_m^*(\tilde{G}^{\tilde{Q}}) = \tilde{Q}$. It is given by*

$$\tilde{G}^{\tilde{Q}}_{(x,u)}(X,Y) = \tilde{Q}_{(v)}(X',Y'), \tag{2}$$

where v belongs to the fiber $\psi_m^{-1}(x,u), X, Y$ are elements of $(T_1M)_(x,u)$ and X', Y' are any tangent vectors to OM at v such that $(\psi_m)_*(X') = X$ and $(\psi_m)_*(Y') = Y$.

The Musso-Tricerri processes described by Propositions 1 and 2, respectively, are compatible in the following sense:

PROPOSITION 3. *If a Riemannian metric G on TM is induced from a bilinear form Q on $OM \times \mathbb{R}^m$ by the Musso-Tricerri process described in Proposition 1, i.e., $\Phi^*(G) = Q$, then the induced metric $\tilde{G} := i^*(G)$ on T_1M, where i: $T_1M \to TM$ is the canonical injection, can be obtained from the bilinear form $\tilde{Q} := i^*(Q)$ on OM by the Musso-Tricerri process described in Proposition 2.*

Proof. Denote by i_m the map $OM \to OM\times\mathbb{R}^m, v \mapsto (v,0,\dots,0,1)$. Then the following diagram

$$\begin{array}{ccc} OM & \overset{i_m}{\to} & OM\times\mathbb{R}^m \\ \psi_m\downarrow & & \downarrow\Phi \\ T_1M & \overset{i}{\hookrightarrow} & TM \end{array} \tag{3}$$

commutes. If we consider $\tilde{Q} := i_m^* Q$, then $\tilde{Q}$ is a symmetric, semi-positive definite, tensor field of type (0, 2) on OM. We can prove by a bit longer routine computation that is basic for ψ_m and it is of rank $2m-1$. Furthermore, we have, by virtue of (3), that $\psi_m^*(\tilde{G}) = \psi_m^*(i^*(G)) = (i\circ\psi_m)^*(G) = i_m^*(\Phi^*(G)) = i_m^*(Q) = \tilde{Q}$. □

Now we shall prove that every Riemannian g-natural metric on the unit tangent sphere bundle T_1M of a Riemannian manifold (M,g) can be constructed by the Musso-Tricerri's scheme, given by Proposition 2. For this, let us recall some basic definition.

Let ∇ the Levi-Civita connection of g. Then the tangent space of TM at any point $(x,u)\in TM$ split into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

If $(x,u)\in TM$ is given then, for any vector $X\in M_x$, there exists a unique vector $X^h\in H_{(x,u)}$ such that $p_*X^h = X$, $p: TM\to M$ is the natural projection. We call X^h the *horizontal lift* of X to the point $(x,u)\in TM$. The *vertical lift* of a vector $X\in M_x$ to $(x,u)\in TM$ is a vector $X^v\in V_{(x,u)}$ such that $X^v(df) = Mf$, for all functions f on M. Here we consider 1-forms df on M as functions on TM (*i.e.*, $(df)(x,u) = uf$). Note that the map $X\to X^h$ is an isomorphism between the vector spaces M_x and $H_{(x,u)}$. Similarly, the map $X\to X^v$ is an isomorphism between the vector spaces M_x and $V_{(x,v)}$. Obviously, each tangent vector $\widetilde{Z}\in (TM)_{(x,u)}$ can be written in the form $\widetilde{Z} = X^h + Y^v$, where X, $Y\in M_x$ are uniquely determined vectors.

In an obvious way we can define horizontal and vertical lifts of vector fields on M.

If we fix an *F-metric* ξ on M, *i.e.*, a mapping $TM\oplus TM\oplus TM\to\mathbb{R}$ which is linear in the second and the third argument and smooth in the first argument, then there are three distinguished constructions of metrics on the tangent bundle TM, which are given as follows [5]:

(a) If we suppose that ξ is symmetric with respect to the last two arguments, then the *Sasaki lift* ξ^s of ξ is defined as follows:

$$\begin{cases} \xi^s_{(x,u)}(X^h,Y^h) = \xi(u;X,Y), \\ \xi^s_{(x,u)}(X^v,Y^h) = 0, \end{cases} \qquad \begin{cases} \xi^s_{(x,u)}(X^h,Y^v) = 0, \\ \xi^s_{(x,u)}(X^v,Y^v) = \xi(u;X,Y), \end{cases}$$

for all X, $Y \in M_x$. If ξ is non degenerate and positive definite with respect to the last two arguments for each fixed u, then ξ^s is a Riemannian metric on TM.

(b) The *horizontal lift* ξ^h of ξ is a pseudo-Riemannian metric on TM which is given by:

$$\begin{cases} \xi^h_{(x,u)}(X^h, Y^h) = 0, \\ \xi^h_{(x,u)}(X^v, Y^h) = \xi(u; X, Y), \end{cases} \quad \begin{cases} \xi^h_{(x,u)}(X^h, Y^v) = \xi(u; X, Y), \\ \xi^h_{(x,u)}(X^v, Y^v) = 0, \end{cases}$$

for all X, $Y \in M_x$. If ξ is positive definite with respect to the last two arguments, then ξ^s is of signature (m, m).

(c) The *vertical lift* ξ^v of ξ is a degenerate metric on TM which is given by:

$$\begin{cases} \xi^v_{(x,u)}(X^h, Y^h) = \xi(u; X, Y), \\ \xi^v_{(x,u)}(X^v, Y^h) = 0, \end{cases} \quad \begin{cases} \xi^v_{(x,u)}(X^h, Y^v) = 0, \\ \xi^v_{(x,u)}(X^v, Y^v) = 0, \end{cases}$$

for all X, $Y \in M_x$. For each fixed u, the rank of ξ^v is exactly that of ξ.

If $\xi = g$ is a Riemannian metric on M, then the three lifts of ξ just constructed coincide with the three well-known classical lifts of the metric g to TM.

Let (M, g) be non-oriented. Then it is known that all *natural* F-metrics are of the form

$$F(u; X, Y) = \alpha(\|u\|^2)g(X, Y) + \beta(\|u\|^2)g(X, u)g(Y, u),$$

where $\alpha(t)$, $\beta(t)$ are smooth functions on $[0, +\infty)$ and $\|u\| = \sqrt{g(u, u)}$ (see [4] and [2]). The three lifts above of *natural* F-metrics generate the class of g-natural metrics on TM (*cf.* [5] and [2] for the classification and the definition of g-natural metrics and [4] for the general definition of naturality).

More precisely, we have

Proposition 4. *Let (M, g) be a Riemannian manifold. Every g-natural metric G on TM is given by*

$$G = (\alpha_1 g + \beta_1 k)^s + (\alpha_2 g + \beta_2 k)^h + (\alpha_3 g + \beta_3 k)^v, \tag{4}$$

where α_i, β_i, $i = 1, 2, 3$, are smooth functions on $[0, +\infty)$, and k is the natural F-metric on M defined by

$$k(u; X, Y) = g(u, X)g(u, Y), \quad \textit{for all} \quad (u, x, Y) \in TM \oplus TM \oplus TM. \tag{5}$$

If we restrict an arbitrary g-natural metric (4) to a tangent sphere bundle $T_rM(r>0)$, then we obtain the metric $\widetilde{G}$ of the form

$$\widetilde{G} = a \cdot \widetilde{g^d} + b \cdot \widetilde{g^h} + c \cdot \widetilde{g^v} + d \cdot \widetilde{k^v}, \tag{6}$$

where $a = \alpha_1(r^2)$, $b = \alpha_2(r^2)$, $c = \alpha_3(r^2)$, $d = \beta_3(r^2)$ and $\widetilde{g^s}$, $\widetilde{g^h}$, $\widetilde{g^v}$ and $\widetilde{k^v}$ are the metrics on T_rM induced by g^s, g^h, g^v and k^v, respectively. We call such metrics on T_rM, induced by g-natural metrics, *g-natural metrics on T_rM.*

Riemannian g-natural metrics on tangent sphere bundles are characterized by

Proposition 5 ([1]). *Let $r>0$ and (M,g) be a Riemannian manifold. Then every Riemannian g-natural metric $\widetilde{G}$ on T_rm induced form a (possibly degenerate) g-natural G on TM, is of the form* (6)*, where a, b, c and d are constants satisfying the inequalities $a>0$, $a(a+c)-b^2>0$ and $a+c+dr^2>0$.*

Let $\theta = (\theta^1, \dots, \theta^m)$ denote the canonical 1-form on OM, and let π denote the natural projection $OM \xrightarrow{\pi} M$. Then

$$d\pi_v(X) = \sum_i \theta^i(X) v_i, \qquad v = (x; v_1, \dots, v_m).$$

If we denote with $\omega = (\omega^i_j)$ the connection form on OM, then we find that the forms

$$\pi_1^* \theta^i,\ i = 1, \dots, m;\ \pi_1^* \omega^i_j,\ 1 \le i \le j \le m;\ d\varepsilon^i,\ i = 1, \dots, m,$$

where $\pi_1 : OM \times \mathbb{R}^m \to OM$ denotes the first natural projection, determine an absolute parallelism on $OM \times \mathbb{R}^m$. We consider the 1-forms $\nabla \varepsilon^i$ on $OM \times \mathbb{R}^m$ defined by

$$\nabla \varepsilon^i = d\varepsilon^i + \sum_j \varepsilon^j \pi_1^* \omega^i_j. \tag{7}$$

The first author an M. Sarih have proved the following

Proposition 6 ([2]). *Every g-natural metric on the tangent bundle TM of a Riemannian manifold (M,g) can be constructed by the Musso-Tricerri's generalized scheme, given by Proposition* 1.

More precisely, and arbitrary g-natural metric G on TM, which is of the form (4) by Proposition 4, is induced by the symmetric tensor field Q of type $(2,0)$ on $OM \times \mathbb{R}^m$ given by

$$
\begin{aligned}
Q = (\alpha_1+\alpha_3)(r^2)\sum_i(\pi_1^*\theta^i)^2 + (\beta_1+\beta_3)(r^2)\left(\sum_i \varepsilon^i\pi_1^*\theta^i\right)^2 + \\
+\alpha_1(r^2)\sum_i(D\varepsilon^i)^2 + \beta_1(r^2)\left(\sum_i \varepsilon^i D\varepsilon^i\right)^2 + \\
+2\alpha_2(r^2)\sum_i \pi_i^*\theta^i D\varepsilon^i + 2\beta_2(r^2)\left(\sum_i \varepsilon^i\pi_1^*\theta^i\right)\left(\sum_i \varepsilon^i D\varepsilon^i\right),
\end{aligned} \tag{8}
$$

where $r^2 = \sum_i(\varepsilon^i)^2$.

Note that (8) is exactly the expression (3.4) of [2] with the abuse of notation $\theta = \pi_1^*\theta$ (*cf.* [2, p. 8, line 7 from below]).

Let us mention that in the proof of this result in [2], there occurred a little misprint which did not influence the correctness of the statement.

Combining this last proposition with Proposition 3, we obtain

Theorem 1. *Every Riemannian g-natural metric on the unit tangent sphere bundle T_1M of a Riemannian manifold (M,g) can be constructed by the Musso-Tricerri's scheme, given by Proposition* 2.

More precisely, if $\widetilde{G} = a\cdot\widetilde{g^s} + b\cdot\widetilde{g^h} + c\cdot\widetilde{g^v} + d\cdot\widetilde{k^v}$, is an arbitrary Riemannian g-natural metric on T_1M, then $\widetilde{G}$ is induced, via the Musso-Tricerri process, by the $(0,2)$-tensor field $\widetilde{Q} = (a+c)\sum_{i=1}^{m-1}(\theta^i)^2 + (a+c+d)(\theta^m)^2 + a\sum_{i=1}^{m-1}(\omega_m^i)^2 + 2b\sum_{i=1}^{m-1}\theta^i\omega_m^i$ on OM.

Proof. By Proposition 5, every Riemannian g-natural metric on T_1M is of the form $\widetilde{G} = a\cdot\widetilde{g^s} + b\cdot\widetilde{g^h} + c\cdot\widetilde{g^v} + d\cdot\widetilde{k^v}$,where a, b, c and d are constants such that $a>0$, $a(a+c)-b^2>0$ and $a+c+d>0$. Such a metric on T_1M is obviously induced by the g-natural metric $G = a\cdot g^s + b\cdot g^h + c\cdot g^v + d\cdot k^v$ on TM. If we consider, in Proposition 6, constant functions α_i, β_i; $i=1,2,3$, such that $\alpha_1 = a$, $\alpha_2 = b$, $\alpha_3 = c$, $\beta_3 = d$ and $\beta_1 = \beta_2 = 0$, then our G is induced by the symmetric tensor filed Q of type $(2,0)$ on $OM\times\mathbb{R}^m$ given by $Q = (a+c)\sum_{i=1}^m(\pi_1^*\theta^i)^2 + d(\sum_{i=1}^m\varepsilon^i\pi_1^*\theta^i)^2 + a\sum_{i=1}^m(\nabla\varepsilon^i)^2 + 2b\sum_{i=1}^m\pi_1^*\theta^i\nabla\varepsilon^i$, where $r^2 = \sum_{i=1}^m(\varepsilon^i)^2$. From Proposition 3, G is induced, via the Musso-Tricerri process, by the bilinear form $\widetilde{Q} = (i_m) * Q$ on OM, *i.e.*, by the form

$$\widetilde{Q} = (a+c)\sum_{i=1}^{m}((\pi_1 \circ i_m)^*\theta^i)^2 + d\left(\sum_{i=1}^{m}(\varepsilon^i \circ i_m\right)(\pi_1 \circ i_m)^*\theta^i)^2 + \\ + a\sum_{i=1}^{m}((i_m)^*\nabla\varepsilon^i)^2 + 2b\sum_{i=1}((\pi_1 \circ i_m)^*\theta^i)((i_m)^*\nabla\varepsilon^i). \tag{9}$$

But, it is easy to check that $\varepsilon^i \circ i_m = \delta^i_m$, where (δ^i_j) denote the Kronecker symbols. Then

$$r^2 \circ i_m = \sum_{i=1}^{m}(\varepsilon^i \circ i_m)^2 = 1 \quad \text{and} \quad (i_m)^*(d\varepsilon^i) = 0, \tag{10}$$

and it follows from (7) that

$$(i_m)^*(\nabla\varepsilon^i) = i_m^*(\pi_1^*\omega^i_m) = \omega^i_m, \tag{11}$$

and we have also

$$i_m^*(\pi_1^*\theta^i) = \theta^i, \tag{12}$$

since $\pi_1 \circ i_m = \mathrm{Id}_{OM}$. Hence, substituting from (10)-(12) into (9) and using the fact that $\omega^m_m = 0$ by the skew-symmetry of (ω^i_j), we obtain $\widetilde{Q} = (a+c)\sum_{i=1}^{m-1}(\theta^i)^2 + (a+c+d)(\theta^m)^2 + a\sum_{i=1}^{m-1}(\omega^i_m)^2 + 2b\sum_{i=1}^{m-1}\theta^i\omega^i_m$. □

REMARK 1. Theorem 1 is a kind of weak generalization of the Main theorem in [1], where the base manifold (M, g) was a round sphere S^m. In our weaker analogy, the base manifold (M, g) is arbitrary. (*Cf.* [1], Section 4 and the formulas (3.1), (3.2)).

Now, we prove that any Riemannian g-natural metric on the unit tangent bundle of a two-point homogeneous space is homogeneous. This will generalize a theorem proved in [7, p. 10] for the induced Sasaki metric.

THEOREM 2. *Let (M, g) be a two-point homogeneous space and let $\widetilde{G}$ be a Riemannian g-natural metric on T_1M. Then $(T_1M, \widetilde{G})$ is a homogeneous Riemannian space.*

Proof. Let $I(M,g)$ denote the group of isometries of (M,g). Then there is a natural left action of $I(M,g)$ on T_1M and OM, respectively, defined by the formulas

$$L_f(x,u) = (f(x), f_*u), \tag{13}$$

$$L_f(v) = (f(x), f_*u_1, \dots, f_*u_m), \tag{14}$$

where $f \in I(M,g)$, $(x,u) \in T_1M$ and $v := (x, u_1, \dots, u_m) \in OM$.

We claim that $\widetilde{G}$ is $I(M,g)$-invariant with respect to the action (13). It is well-known that the canonical 1-form *theta* and the Levi-Civita connection form ω are $I(M,g)$-invariant, *i.e.*,

$$L_f^*(\theta^i) = \theta^i, \tag{15}$$

$$L_f^*(\omega_j^i) = \omega_j^i. \tag{16}$$

Now, $\widetilde{G}$ is induced by the (0,2)-tensor filed $\widetilde{Q}$ from Theorem 1. By using (15) and (16) we obtain that $L_f^*(\widetilde{Q}) = \widetilde{Q}$. Moreover, $\widetilde{Q} = \psi_m^*(\widetilde{G})$ holds by the proof of Proposition 3.

We deduce that $\psi_m^*(\widetilde{G}) = L_f^*(\psi_m^*(\widetilde{G})) = (\psi_m \circ L_f)^*(\widetilde{G})$. But it is straightforward, form (13) and (14), that $\psi_m \circ L_f = L_f \circ \psi_m$. It follows then that $\psi_m^*(\widetilde{g}) = (L_f \circ \psi_m)^*(\widetilde{G}) = \psi_m^*(L_f^*(\widetilde{g}))$. Since ψ_m is a submersion, then $L_f^*(\widetilde{g}) = \widetilde{G}$, for all $f \in I(M,g)$. This proves our claim.

Next, it is classical that $I(M,g)$ is transitive on T_1M if and only if (M,g) is a two-point homogeneous space (*cf.* [9], p. 289). Hence if (M,g) is a two-point homogeneous space, then $I(M,g)$ acts transitively on T_1, M, as an isometry group. Consequently, $(T_1M, \widetilde{G})$ is a homogeneous Riemannian space.

For an alternative proof of Theorem 2 see [6]. ∎

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REFERENCES

[1] K. M. T. Abbassi – O. Kowalski: *Naturality of homogeneous metrics on Stiefel manifolds $SO(m+1)/SO(m-1)$*, Diff. Geometry and Appl., **28** (2010), 131–139

[2] K. M. T. Abbassi – M. Sarih: *On natural metrics on tangent bundles of Riemannian manifolds*, Archivum Math. (Brno), **41** (2005), 71–92.

[3] K. M. T. Abbassi – M. Sarih: *On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds*, Diff. Geometry and Appl., **22** (1) (2005), 19–47.

[4] I. Kolář – P. W. Michor – J. Slovák: *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993.

[5] O. Kowalski – M. Sekizawa: *Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles – a classification* –, Bull. Tokyo Gakugei Univ., **40** (4) (1988), 1–29.

[6] O. Kowalski – M. Sekizawa: *Invariance of g-natural metrics on tangent bundles, differential geometry and its applications*, in: Proc. Conf. in Honour of Leonhard Euler, Olomouc, August 2007, World Scientific Publishing Company, 2088, 171–181.

[7] E. Musso – F. Tricerri: *Riemannian metrics on tangent bundles*, Ann. Math. Pura Appl., **150** (4) (1988), 1–20.

[8] S. Sasaki: *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J., I: **10** (1958, 338–354; II: **14** (1962), 146–155.

[9] J. A. Wolf: *Spaces of constant curvature*, Publich or Perish, IV edition, Berkeley, 1977.

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