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# Trigonometric approach to convolution formulae of Bernoulli and Euler numbers

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ABSTRACT: *Summation formulae involving Bernoulli and Euler numbers as well as their convolutions are systematically reviewed by applying four classically elementary trigonometric identities.*

The Bernoulli and Euler numbers are important classical numbers and have wide applications in mathematics and physics. They can be defined, respectively, through the following trigonometric generating functions (see [12, Section 3.1.4], [14, Section 7.58] and [15, Section 2.5] for example)

$$x\cot x = \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!}B_{2n}, \tag{1}$$

$$\sec x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}E_{2n}. \tag{2}$$

According to the two elementary trigonometric relations

$$\tan x = \cot x - 2\cot(2x) \quad \text{and} \quad \csc x = \cot x + \tan\frac{x}{2}$$

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the following two power series expansions can easily be shown

$$x\tan x=\sum_{n=1}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}B'_{2n}, \tag{3}$$

$$x\csc x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}B''_{2n}; \tag{4}$$

where $B'_{2n}$ and $B''_{2n}$ denote, respectively, the two variants of Bernoulli numbers $B'_{2n}:=(1-4^n)B_{2n}$ and $B''_{2n}:=(2-4^n)B_{2n}$ in order to shorten lengthy expressions.

It is well-known that the sum of $n^{\text{th}}$ powers of the first $m$ natural numbers can be expressed in terms of Bernoulli numbers (*cf.* [5, Section 3.9] and [8, Section 6.5]):

$$\sum_{k=1}^{m}k^n=\frac{m+1}{n+1}\sum_{i=0}^{n}(m+1)^i\binom{n+1}{i+1}B_{n-i}.$$

Similar relations have recently been found by Liu and Luo [10] for the first $m$ odd positive integers, which motivated the authors [3] to work out four classes of arithmetic identities involving Bernoulli and Euler numbers.

Observe that the above mentioned arithmetic identities have been accomplished entirely by manipulating elementary trigonometric sums. This encourages the authors to explore thoroughly the trigonometric approach to the arithmetic sums involving Bernoulli and Euler numbers as well as their convolutions. Our investigation will be carried out by employing exclusively four basic trigonometric sum identities. In fact, the rest of the paper will be structured into four sections according to these trigonometric relations with each of them having five different reformulations, that result logically in further division of each section into five subsections. Each subsection will prove a general theorem of arithmetic convolution sum involving Bernoulli and/or Euler numbers, followed by several concrete identities.

Throughout the paper, we shall assume $\delta=0,1$ and $m,n\in\mathbb{N}_0$. In addition, the following Taylor series for sine and cosine functions

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\quad\text{and}\quad\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \tag{5}$$

will frequently be appealed without explanation.

Apart from the arithmetic formulae treated in this paper, there exist vast mathematical literature dealing with different approaches and identities for Bernoulli and Euler numbers as well as polynomials. The interested reader may consult, for instance, [1], [3], [4], [6] for convolution formulae, [7], [11] for Miki-type identities and [2], [13] for Bernoulli and Euler polynomials, as well as the handbook by Hansen [9, Section 50 and Section 51].

## 1 – Trigonometric sum concerning $\cos(2k+\gamma)x$

According to the following well-known formula

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) - \sin(\beta-\alpha)$$

we can evaluate via telescoping method the trigonometric sum

$$2\sin x\sum_{k=1}^{m}\cos(2k+\gamma)x = \sin(2m+\gamma+1)x - \sin(\gamma+1)x. \tag{6}$$

By means of five different reformulations of this identity, this section will investigate arithmetic sums involving Bernoulli and Euler numbers as well as their convolutions.

1.1 – Firstly, it is obvious that (6) is equivalent to the equation

$$2\sum_{k=1}^{m}\cos(2k+\gamma)x = \csc x\sin(2m+\gamma+1)x - \csc x\sin(\gamma+1)x.$$

Applying (4) and (5), we get the following power series expansion

$$2\sum_{n=0}^{\infty}\sum_{k=1}^{m}(-1)^n\frac{(2k+\gamma)^{2n}}{(2n)!}x^{2n} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{(2m+\gamma+1)^{2j+1}}{(2i)!(2j+1)!}B''_{2i}x^{2i+2j}+ \\ -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{(\gamma+1)^{2j+1}}{(2i)!(2j+1)!}B''_{2i}x^{2i+2j}.$$

Comparing the coefficients of $x^{2n}$, we find immediately the following identity.

Theorem 1 ($m \geq 0$ and $n \geq 0$).

$$\sum_{k=1}^{m}(2k+\gamma)^{2n} = \frac{(2m+\gamma+1)^{2n+1}}{2(2n+1)}\sum_{i=0}^{n}\binom{2n+1}{2i}\frac{B''_{2i}}{(2m+\gamma+1)^{2i}}+ \\ -\frac{(\gamma+1)^{2n+1}}{2(2n+1)}\sum_{i=0}^{n}\binom{2n+1}{2i}\frac{B''_{2i}}{(\gamma+1)^{2i}}.$$

This general theorem contains several interesting identities as special cases.

Corollary 2 ($m = 1$ and $\gamma = -1$ in Theorem 1: Liu and Luo [10, Equation 8]).

$$\sum_{k=0}^{n}\frac{4^n}{4^k}\binom{2n+1}{2k}B''_{2k} = 2n+1.$$

Corollary 3 ($m=1$ and $\gamma=-2$ in Theorem 1: Liu–Luo [10, Equation 5]).

$$\sum_{k=0}^{n}\binom{2n+1}{2k}B''_{2k}=0 \quad \textit{where} \quad n>0.$$

According to (2), (4) and (5), extracting the coefficients of $x^{2n}$ across the trigonometric equation $2\sin x\csc 2x=\sec x$, we recover another similar identity.

Lemma 4 (Chu–Wang [3, Equation 19a]).

$$\sum_{i=0}^{n}4^i\binom{2n+1}{2i}B''_{2i}=(2n+1)E_{2n}.$$

In Theorem 1, letting $\gamma=-\delta-1/2$ with $\delta=0,1$ and then simplifying the resulting equation through the last identity, we get the following transformation formula.

Proposition 5 ($\delta=0,1$ and $m,n\geq 0$).

$$\frac{(1-2\delta)(n+1/2)}{4^n(2m-\delta+1/2)^{2n+1}}E_{2n}=\sum_{i=0}^{n}\binom{2n+1}{2i}\frac{B''_{2i}}{(2m-\delta+1/2)^{2i}} -\frac{2(2n+1)}{(2m-\delta+1/2)^{2n+1}}\sum_{k=1}^{m}(2k-\delta-1/2)^{2n}.$$

When $\delta=0$ and $m=1,2$, this proposition yields the following two identities

$$\sum_{k=0}^{n}\binom{2n+1}{2k}\Big(\frac{2}{5}\Big)^{2k}B''_{2k}=\frac{2n+1}{5^{2n+1}}E_{2n}+4\frac{2n+1}{5^{2n+1}}3^{2n},$$
$$\sum_{k=0}^{n}\binom{2n+1}{2k}\Big(\frac{2}{9}\Big)^{2k}B''_{2k}=\frac{2n+1}{9^{2n+1}}\Big\{E_{2n}+4\cdot 3^{2n}+4\cdot 7^{2n}\Big\}.$$

Instead for $\delta=1$ and $m=1,2$, we get similarly two other identities

$$\sum_{k=0}^{n}\binom{2n+1}{2k}\Big(\frac{2}{3}\Big)^{2k}B''_{2k}=\frac{2n+1}{3^{2n+1}}(4-E_{2n}),$$
$$\sum_{k=0}^{n}\binom{2n+1}{2k}\Big(\frac{2}{7}\Big)^{2k}B''_{2k}=\frac{2n+1}{7^{2n+1}}\Big\{4(1+5^{2n})-E_{2n}\Big\}.$$

1.2 – Secondly, the identity (6) may equivalently be restated as

$$2\csc(2m+\gamma+1)x\sum_{k=1}^{m}\cos(2k+\gamma)x$$
$$=\csc x-\csc x\sin(\gamma+1)x\csc(2m+\gamma+1)x.$$

By means of (4) and (5), extracting the coefficients of $x^{2n-1}$ from the last equation leads us to the following transformation theorem.

Theorem 6 ($m \geq 0$ and $n \geq 0$).

$$\frac{B''_{2n}}{2(2m+\gamma+1)^{2n-1}}-\sum_{k=1}^{m}\sum_{i=0}^{n}\binom{2n}{2i}\frac{(2k+\gamma)^{2i}}{(2m+\gamma+1)^{2i}}B''_{2n-2i}$$
$$=\frac{(\gamma+1)^{2n+1}}{2(2n+1)(2m+\gamma+1)^{2n}}\sum_{0\leq i+j\leq n}\binom{2n+1}{2i,2j}\frac{(2m+\gamma+1)^{2j}}{(\gamma+1)^{2i+2j}}B''_{2i}B''_{2j}.$$

Several interesting identities follow immediately from this theorem.

Corollary 7 ($m = 1$ and $\gamma = -1$ in Theorem 6: Liu–Luo [10, Equation 13]).

$$\sum_{k=0}^{n}4^k\binom{2n}{2k}B''_{2k}=B''_{2n}.$$

Next letting $m = \gamma = 0$ in Theorem 6 gives directly the formula

$$\sum_{0\leq i+j\leq n}\binom{2n+1}{2i,2j}B''_{2i}B''_{2j}=(2n+1)B''_{2n}.$$

In fact, combining the series rearrangement with Corollary 3 we can show the following more general result.

Corollary 8 ($W \neq 0$).

$$\sum_{0\leq i+j\leq n}\binom{2n+1}{2i,2j}\frac{B''_{2i}B''_{2j}}{W^{2n-2i}}=(2n+1)B''_{2n}.$$

Finally, letting $\gamma = \delta - 1/2$ in Theorem 6 and then applying Lemma 4, we find the following transformation formula.

PROPOSITION 9 ($\delta = 0, 1$ and $m, n \geq 0$).

$$\frac{B''_{2n}}{(2m-\delta+1/2)^{2n-1}} = 2\sum_{i=0}^{n}\sum_{k=1}^{m}\binom{2n}{2i}\Big(\frac{2k-\delta-1/2}{2m-\delta+1/2}\Big)^{2i}B''_{2n-2i} - (\delta-1/2)\sum_{i=0}^{n}\binom{2n}{2i}\frac{E_{2i}B''_{2n-2i}}{(4m-2\delta+1)^{2i}}.$$

For $\delta = 0$ and $m = 0, 1, 2$, this proposition reduces to the following three identities

$$\sum_{k=0}^{n}\binom{2n}{2k}B''_{2k}E_{2n-2k} = 4^n B''_{2n},$$

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{B''_{2n-2k}}{5^{2k}}\Big\{3^{2k}+\frac{E_{2k}}{4}\Big\} = \frac{2^{2n-2}}{5^{2n-1}}B''_{2n},$$

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{B''_{2n-2k}}{9^{2k}}\Big\{3^{2k}+7^{2k}+\frac{E_{2k}}{4}\Big\} = \frac{2^{2n-2}}{9^{2n-1}}B''_{2n}.$$

Similarly, when $\delta = 1$ and $m = 1, 2$, we have two further formulae

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{B''_{2n-2k}}{3^{2k}}\Big\{1-\frac{E_{2k}}{4}\Big\} = \frac{2^{2n-2}}{3^{2n-1}}B''_{2n},$$

$$\sum_{k=0}^{n}\binom{2n}{2k}\frac{B''_{2n-2k}}{7^{2k}}\Big\{1+5^{2k}-\frac{E_{2k}}{4}\Big\} = \frac{2^{2n-2}}{7^{2n-1}}B''_{2n}.$$

**1.3 –** Thirdly, rewrite (6) equivalently in the following manner

$$2\csc(2m+\gamma+1)x\sum_{k=1}^{m}\cos x\cos(2k+\gamma)x = \cot x - \cot x\sin(\gamma+1)x\csc(2m+\gamma+1)x$$

and then recall the relation

$$2\cos x\cos(2k+\gamma)x = \cos(2k+\gamma+1)x + \cos(2k+\gamma-1)x. \tag{11}$$

In view of (1), (4) and (5), equating the coefficients of $x^{2n-1}$ across the penultimate equation gives rise to the following identity.

Theorem 10 ($m \geq 0$ and $n \geq 0$).

$$\sum_{k=1}^{m}\sum_{i=0}^{n}\binom{2n}{2i}\frac{(2m+\gamma+1)^{2i-1}}{(\gamma+1)^{2n+1}}B''_{2i}\left\{\begin{array}{c}(2k+\gamma+1)^{2n-2i}\\+(2k+\gamma-1)^{2n-2i}\end{array}\right\}$$
$$=\frac{4^n B_{2n}}{(\gamma+1)^{2n+1}}-\sum_{0\leq i+j\leq n}\binom{2n+1}{2i,2j}\frac{4^i(2m+\gamma+1)^{2j-1}}{(2n+1)(\gamma+1)^{2i+2j}}B_{2i}B''_{2j}.$$

As special cases of this theorem, three identities are displayed below. First letting $m=\gamma=0$ in Theorem 10, we have

$$\sum_{0\leq i+j\leq n}4^i\binom{2n+1}{2i,2j}B_{2i}B''_{2j}=(2n+1)4^n B_{2n}.$$

However, considering Corollary 3, we can show the following more general result.

Corollary 11 ($W \neq 0$).

$$\sum_{0\leq i+j\leq n}4^i\binom{2n+1}{2i,2j}\frac{B_{2i}B''_{2j}}{W^{2n-2i}}=(2n+1)4^n B_{2n}.$$

Corollary 12 ($m=1$ and $\gamma=-1$ in Theorem 10: Chu-Wang [3, Equation 9a]).

$$\sum_{k=0}^{n}\binom{2n}{2k}B''_{2k}=4^n B_{2n}.$$

Corollary 13 ($m=1$ and $\gamma=-3$ in Theorem 10: $n>0$).

$$\sum_{k=0}^{n}\binom{2n+1}{2k}B_{2k}=n+\frac{1}{2}.$$

**1.4 –** Fourthly, the identity (6) may equivalently be expressed as

$$2\cot(2m+\gamma+1)x\sum_{k=1}^{m}\cos(2k+\gamma)x=$$
$$=\csc x\cos(2m+\gamma+1)x-\csc x\sin(\gamma+1)x\cot(2m+\gamma+1)x.$$

On account of (1), (4) and (5), equating the coefficients of $x^{2n-1}$ across the last equation results in the following transformation theorem.

Theorem 14 ($m \geq 0$ and $n \geq 0$).

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{2n}{2i}(2m+\gamma+1)^{2n-2i}B''_{2i}\\
&=2\sum_{k=1}^{m}\sum_{i=0}^{n}4^i\binom{2n}{2i}(2m+\gamma+1)^{2i-1}(2k+\gamma)^{2n-2i}B_{2i}\\
&\quad+\sum_{0\leq i+j\leq n}4^j\binom{2n+1}{2i,2j}(2m+\gamma+1)^{2j-1}\frac{(\gamma+1)^{2n+1-2i-2j}}{2n+1}B''_{2i}B_{2j}.
\end{aligned}$$

When $m = 1$ and $\gamma = -1$, Theorem 14 reduces to the following equality

$$\sum_{k=0}^{n}\frac{4^n}{4^k}\binom{2n}{2k}B''_{2k}=\sum_{\ell=0}^{n}16^{\ell}\binom{2n}{2\ell}B_{2\ell}.$$

By extracting the coefficients of $x^{2n-1}$ across the following equation

$$\csc x\cos 2x = 2\cot 2x\cos x = \csc x - 2\sin x$$

we derive the two identities together.

Corollary 15 ($m = 1$ and $\gamma = -1$ in Theorem 14: Chu-Wang [3, Equation 10a]).

$$\sum_{k=0}^{n}\frac{4^n}{4^k}\binom{2n}{2k}B''_{2k}=\sum_{\ell=0}^{n}16^{\ell}\binom{2n}{2\ell}B_{2\ell}=B''_{2n}+4n.$$

1.5 – Finally, reformulate (6) equivalently as the following equality

$$\begin{aligned}
&2\cot(2m+\gamma+1)x\sum_{k=1}^{m}\cos x\cos(2k+\gamma)x\\
&=\cot x\cos(2m+\gamma+1)x-\cot x\sin(\gamma+1)x\cot(2m+\gamma+1)x.
\end{aligned}$$

With the help of (1), (5) and (11), extracting the coefficients of $x^{2n-1}$ across this equation, we get the following identity.

THEOREM 16 ($m \geq 0$ and $n \geq 0$).

$$\sum_{k=0}^{n} 4^k \binom{2n}{2k} (2m+\gamma+1)^{2n-2k} B_{2k}$$
$$= \sum_{k=1}^{m} \sum_{i=0}^{n} 4^i \binom{2n}{2i} (2m+\gamma+1)^{2i-1} B_{2i} \left\{ \begin{array}{c} (2k+\gamma+1)^{2n-2i} \\ +(2k+\gamma-1)^{2n-2i} \end{array} \right\}$$
$$+ \sum_{0 \leq i+j \leq n} 4^{i+j} \binom{2n+1}{2i, 2j} \frac{B_{2i} B_{2j}}{2n+1} (2m+\gamma+1)^{2j-1} (\gamma+1)^{2n-2i-2j+1}.$$

First letting $m = \gamma = 0$ in Theorem 16 results in the following relation

$$\sum_{0 \leq i+j \leq n} 4^{i+j} \binom{2n+1}{2i, 2j} \frac{B_{2i} B_{2j}}{2n+1} = \sum_{k=0}^{n} 4^k \binom{2n}{2k} B_{2k}$$

which can also be verified by applying the following lemma.

LAMMA 17.

$$\sum_{k=0}^{n} 4^k \binom{2n+1}{2k} B_{2k} = 2n+1.$$

This identity follows easily by equating the coefficients of $x^{2n}$ across the trigonometric equation $\sin x \cot x = \cos x$. Instead, by extracting the coefficients of $x^{2n-1}$ across the equalities

$$\cot^2 x \sin x = \cos x \cot x = \csc x - \sin x$$

we get the following closed formulae.

COROLLARY 18 ($m = \gamma = 0$ in Theorem 16).

$$\sum_{0 \leq i+j \leq n} 4^{i+j} \binom{2n+1}{2i, 2j} \frac{B_{2i} B_{2j}}{2n+1} = \sum_{k=0}^{n} 4^k \binom{2n}{2k} B_{2k} = B''_{2n} + 2n.$$

Finally we examine the case of Theorem 16 with $m = 1$ and $\gamma = -1$

$$2 \sum_{k=0}^{n} \binom{2n}{2k} B_{2k} = 4^n B_{2n} + \sum_{i=0}^{n} 4^i \binom{2n}{2i} B_{2i}.$$

Recalling Corollary 18, we find another similar closed formula.

COROLLARY 19 ($m = 1$ and $\gamma = -1$ in Theorem 16).

$$\sum_{k=0}^{n} \binom{2n}{2k} B_{2k} = n + B_{2n}.$$

**2 – Trigonometric sum concerning** $\sin(2k+\gamma)x$

By means of the trigonometric relation

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

it is not hard to compute the finite sum

$$2\sin x\sum_{k=1}^{m}\sin(2k+\gamma)x = \cos(\gamma+1)x - \cos(2m+\gamma+1)x. \tag{12}$$

According to five different reformulations of this identity, this section will investigate summation formulae involving Bernoulli and Euler numbers.

2.1 – Firstly, it is obvious that (12) is equivalent to the equation

$$2\sum_{k=1}^{m}\sin(2k+\gamma)x = \csc x\cos(\gamma+1)x - \csc x\cos(2m+\gamma+1)x.$$

Applying (4) and (5), we have the power series expansion

$$\begin{aligned}2\sum_{n=0}^{\infty}\sum_{k=1}^{m}(-1)^n\frac{(2k+\gamma)^{2n+1}}{(2n+1)!}x^{2n+1} = &\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{(\gamma+1)^{2j}}{(2i)!(2j)!}B''_{2i}x^{2i+2j-1}\\ &-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{(2m+\gamma+1)^{2j}}{(2i)!(2j)!}B''_{2i}x^{2i+2j-1}.\end{aligned}$$

Extracting the coefficients of $x^{2n-1}$ from both sides of the last equation and then simplifying the result, we derive the following formula.

Theorem 20 ($m \geq 0$ and $n \geq 1$).

$$\sum_{k=1}^{m}(2k+\gamma)^{2n-1} = \sum_{i=0}^{n}\frac{B''_{2i}}{4n}\binom{2n}{2i}\frac{(2m+\gamma+1)^{2n}}{(2m+\gamma+1)^{2i}} - \sum_{i=0}^{n}\frac{B''_{2i}}{4n}\binom{2n}{2i}\frac{(\gamma+1)^{2n}}{(\gamma+1)^{2i}}.$$

According to Corollary 7, letting $\gamma = -\delta-1/2$ in this theorem yields the formula.

Proposition 21 ($\delta = 0, 1$ and $m, n \geq 0$).

$$\sum_{i=0}^{n}\binom{2n}{2i}\frac{B''_{2i}}{(2m-\delta+1/2)^{2i}} = 4n\sum_{k=1}^{m}\frac{(2k-\delta-1/2)^{2n-1}}{(2m-\delta+1/2)^{2n}} + \frac{B''_{2n}}{(4m-2\delta+1)^{2n}}.$$

For $\delta = 0$ and $m = 1, 2$, the last theorem reduces to the following two identities

$$\sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{2}{5}\right)^{2k} B''_{2k} = \frac{1}{5^{2n}} \{8n \cdot 3^{2n-1} + B''_{2n}\},$$

$$\sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{2}{9}\right)^{2k} B''_{2k} = \frac{1}{9^{2n}} \{8n(3^{2n-1} + 7^{2n-1}) + B''_{2n}\}.$$

Similarly, when $\delta = 1$ and $m = 1, 2$, we get from Theorem 20 two further interesting identities

$$\sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{2}{3}\right)^{2k} B''_{2k} = \frac{1}{3^{2n}} \left\{8n + B''_{2n}\right\},$$

$$\sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{2}{7}\right)^{2k} B''_{2k} = \frac{1}{7^{2n}} \left\{8n(1 + 5^{2n-1}) + B''_{2n}\right\}.$$

2.2 – Secondly, the identity (12) may equivalently be restated as

$$\begin{aligned} &2 \sec(2m + \gamma + 1)x \sum_{k=1}^{m} \sin(2k + \gamma)x = \\ &= \csc x \cos(\gamma + 1)x \sec(2m + \gamma + 1)x - \csc x. \end{aligned}$$

By means of (2), (4) and (5), extracting the coefficients of $x^{2n-1}$ across this equation yields the following identity.

Theorem 22 ($m \geq 0$ and $n \geq 1$).

$$\begin{aligned} &\frac{B''_{2n}}{4n(2m + \gamma + 1)^{2n}} - \sum_{k=1}^{m} \sum_{i=1}^{n} \binom{2n-1}{2i-1} \frac{(2k + \gamma)^{2i-1}}{(2m + \gamma + 1)^{2i}} E_{2n-2i} \\ &= \frac{(\gamma + 1)^{2n}}{4n(2m + \gamma + 1)^{2n}} \sum_{0 \leq i+j \leq n} \binom{2n}{2i, 2j} \frac{(2m + \gamma + 1)^{2j}}{(\gamma + 1)^{2i+2j}} B''_{2i} E_{2j}. \end{aligned}$$

Letting $m = \gamma = 0$, Theorem 22 gives directly the formula

$$\sum_{0 \leq i+j \leq n} \binom{2n}{2i, 2j} B''_{2i} E_{2j} = B''_{2n}.$$

In fact, applying Corollary 45, we can show the following more general result.

Corollary 23 ($W \neq 0$).

$$\sum_{0 \leq i+j \leq n} \binom{2n}{2i, 2j} \frac{B''_{2i} E_{2j}}{W^{2n-2i}} = B''_{2n}.$$

Taking $\gamma = \delta - 1/2$ in Theorem 22 and then appealing to Corollary 7, we derive the following transformation.

PROPOSITION 24 ($\delta = 0$, and $m \geq 0$, $n \geq 1$).

$$\sum_{k=1}^{m}\sum_{i=1}^{n}\binom{2n-1}{2i-1}\frac{(2k-\delta-1/2)^{2i-1}}{(2m-\delta+1/2)^{2i}}E_{2n-2i}$$
$$=\frac{B''_{2n}}{4n(2m-\delta+1/2)^{2n}}-\frac{1}{4n}\sum_{i=0}^{n}\binom{2n}{2i}\frac{B''_{2i}E_{2n-2i}}{4^i(2m-\delta+1/2)^{2i}}.$$

2.3 – Thirdly, rewrite (12) equivalently in the following manner

$$2\sec(2m+\gamma+1)x\sum_{k=1}^{m}\cos x\sin(2k+\gamma)x$$
$$=\cot x\cos(\gamma+1)x\sec(2m+\gamma+1)x-\cot x$$

and then recall the trigonometric relation

$$(15)\qquad 2\cos x\sin(2k+\gamma)x=\sin(2k+\gamma+1)x+\sin(2k+\gamma-1)x.$$

In view of (1), (2) and (5), extracting the coefficients of $x^{2n+1}$ from the penultimate equation and then simplifying the result, we derive the following arithmetic formula.

THEOREM 25 ($m \geq 0$ and $n \geq 0$).

$$\sum_{0\leq i+j\leq n+1}4^i\binom{2n+2}{2i,2j}(2m+\gamma+1)^{2j}\frac{(\gamma+1)^{2n+2-2i-2j}}{2n+2}B_{2i}E_{2j}$$
$$=4^{n+1}\frac{B_{2n+2}}{2n+2}-\sum_{k=1}^{m}\sum_{i=0}^{n}\binom{2n+1}{2i}(2m+\gamma+1)^{2i}E_{2i}\left\{\begin{matrix}(2k+\gamma+1)^{2n+1-2i}\\+(2k+\gamma-1)^{2n+1-2i}\end{matrix}\right\}.$$

Two examples of this theorem are given below as applications.

Taking $m = \gamma = 0$ in Theorem 25, we have directly the formula

$$\sum_{0\leq i+j\leq n}4^i\binom{2n}{2i,2j}B_{2i}E_{2j}=4^nB_{2n}.$$

In fact, by means of Corollary 45, we can show the following more general result.

COROLLARY 26 ($W \neq 0$).

$$\sum_{0\leq i+j\leq n}4^i\binom{2n}{2i,2j}\frac{B_{2i}E_{2j}}{W^{2n-2i}}=4^nB_{2n}.$$

Letting $m=1$ and $\gamma=-1$ in Theorem 10, we have the following transformation

$$\sum_{i+j=n+1}\binom{2+2n}{2i,2j}B_{2i}E_{2j}=B_{2n+2}-(n+1)\sum_{i=0}^{n}\binom{2n+1}{2i}E_{2i}.$$

Evaluating the last sum by Corollary 34 and then replacing $n$ by $n-1$, we get the following convolution formula between Bernoulli and Euler numbers.

Corollary 27.

$$\sum_{k=0}^{n}\binom{2n}{2k}B_{2k}E_{2n-2k}=B_{2n}\Big\{1+2^{2n-1}-2^{4n-1}\Big\}.$$

This can also be verified by equating the coefficients $x^{2n-1}$ across the following trigonometric equation

$$\sec x\cot\tfrac{x}{2}=\tan\tfrac{x}{2}+\cot\tfrac{x}{2}.$$

**2.4 –** Fourthly, the identity (12) may equivalently be expressed as

$$\begin{aligned}&2\tan(2m+\gamma+1)x\sum_{k=1}^{m}\sin(2k+\gamma)x\\&=-\csc x\sin(2m+\gamma+1)x+\csc x\cos(\gamma+1)x\tan(2m+\gamma+1)x.\end{aligned}$$

On account of (3), (4) and (5), we can equate the coefficients of $x^{2n}$ across the last equation and obtain the following arithmetic formula.

Theorem 28 ($m\geq 0$ and $n\geq 0$).

$$\begin{aligned}&2\sum_{k=1}^{m}\sum_{i=0}^{n}4^i\binom{2n+1}{2i}(2m+\gamma+1)^{2i-1}(2k+\gamma)^{2n+1-2i}B'_{2i}\\&=-\sum_{0\leq i+j\leq n+1}4^i\binom{2n+2}{2i,2j}(2m+\gamma+1)^{2i-1}\frac{(\gamma+1)^{2n+2-2i-2j}}{2n+2}B'_{2i}B''_{2j}\\&\quad-\sum_{i=0}^{n}\binom{2n+1}{2i}(2m+\gamma+1)^{2n+1-2i}B''_{2i}.\end{aligned}$$

Letting $m = \gamma = 0$ in this theorem and then keeping in mind of Corollary 3, we find the following strange-looking identity.

Corollary 29 ($n > 1$).

$$\sum_{0 \le i+j \le n} 4^i \binom{2n}{2i, 2j} B'_{2i} B''_{2j} = 0.$$

2.5 – Finally, reformulate (6) equivalently as the following equality

$$\begin{aligned} &2\tan(2m+\gamma+1)x \sum_{k=1}^{m} \cos x \sin(2k+\gamma)x = \\ &= \cot x \cos(\gamma+1)x \tan(2m+\gamma+1)x - \cot x \sin(2m+\gamma+1)x. \end{aligned}$$

With the help of the trigonometric relation (15), we can extract, according to (1), (3) and (5), the coefficients of $x^{2n}$ across the last equation and establish the following formula.

Theorem 30 ($m \ge 0$ and $n \ge 0$).

$$\begin{aligned} &\sum_{k=1}^{m} \sum_{i=0}^{n} 4^i \binom{2n+1}{2i} (2m+\gamma+1)^{2i-1} B'_{2i} \left\{ \begin{array}{c} (2k+\gamma+1)^{2n+1-2i} \\ +(2k+\gamma-1)^{2n+1-2i} \end{array} \right\} \\ &= -\sum_{i=0}^{n} 4^i \binom{2n+1}{2i} (2m+\gamma+1)^{2n-2i+1} B_{2i} \\ &\quad - \sum_{0 \le i+j \le n+1} 4^{i+j} \binom{2n+2}{2i, 2j} \frac{B_{2i} B'_{2j}}{2n+2} (2m+\gamma+1)^{2j-1} (\gamma+1)^{2n+2-2i-2j}. \end{aligned}$$

When $m = \gamma = 0$, the last expression yields the following transformation

$$\sum_{0 \le i+j \le n+1} 4^{i+j} \binom{2n+2}{2i, 2j} B_{2i} B'_{2j} = -(2n+2) \sum_{i=0}^{n} 4^i \binom{2n+1}{2i} B_{2i}.$$

Evaluating the last sum by Lemma 17 and then replacing $n$ by $n-1$, we get the following convolution formula for Bernoulli numbers.

Corollary 31.

$$\sum_{0 \le i+j \le n} 4^{i+j} \binom{2n}{2i, 2j} B_{2i} B'_{2j} = 2n(1-2n).$$

This can also be proved by equating the coefficients $x^{2n-2}$ across the following trigonometric equation
$$\cos x \tan x \cot x = \cos x.$$

## 3 – Alternating sum concerning $\sin(2k+\gamma)x$

Recalling the trigonometric formula

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

we have the finite trigonometric sum

$$2\cos x\sum_{k=1}^{m}(-1)^k \sin(2k+\gamma)x = (-1)^m \sin(2m+\gamma+1)x - \sin(\gamma+1)x. \tag{16}$$

By means of five different reformulations of this identity, this section will investigate convolution formulae involving Bernoulli and Euler numbers.

3.1 – Firstly, it is obvious that (16) is equivalent to the equation

$$2\sum_{k=1}^{m}(-1)^k \sin(2k+\gamma)x = (-1)^m \sec x \sin(2m+\gamma+1)x - \sec x \sin(\gamma+1)x.$$

According to (2) and (5), equating the coefficients of $x^{2n+1}$ across the last equation, we find the following formula.

Theorem 32 ($m \geq 0$ and $n \geq 0$).

$$2\sum_{k=1}^{m}(-1)^k(2k+\gamma)^{2n+1} = (-1)^m\sum_{i=0}^{n}\binom{2n+1}{2i+1}(2m+\gamma+1)^{2i+1}E_{2n-2i} - \sum_{i=0}^{n}\binom{2n+1}{2i+1}(\gamma+1)^{2i+1}E_{2n-2i}.$$

Two known identities can be recovered directly from this theorem.

Corollary 33 ($m = 1$ and $\gamma = -1$ in Theorem 32: Chu-Wang [3, Equation 17a]).

$$\sum_{i=0}^{n}\frac{4^n}{4^i}\binom{2n+1}{2i}E_{2i} = 1.$$

Comparing the case $\gamma = -\delta$ of this theorem with the identity due to CHU and WANG [3, Theorem 7], we recover another formula.

COROLLARY 34 (HANSEN [9, Equation 51.1.2] and CHU-WANG [3, Equation 16a: $n > 0$]).

$$\sum_{i=0}^{n} \binom{2n-1}{2i} E_{2i} = -4^n \frac{B'_{2n}}{2n}.$$

Letting $m = 1, 2$ in this theorem, we get respectively the following two identities

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \Big\{ (\gamma+1)^{2k+1} + (\gamma+3)^{2k+1} \Big\} E_{2n-2k} = 2(\gamma+2)^{2n+1},$$

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \Big\{ (\gamma+5)^{2k+1} - (\gamma+1)^{2k+1} \Big\} E_{2n-2k} = 2\Big\{ (\gamma+4)^{2n+1} - (\gamma+2)^{2n+1} \Big\}.$$

3.2 – Secondly, the identity (16) may equivalently be restated as

$$2\csc(2m+\gamma+1)x \sum_{k=1}^{m} (-1)^k \sin(2k+\gamma)x$$
$$= (-1)^m \sec x - \sec x \sin(\gamma+1)x \csc(2m+\gamma+1)x.$$

By means of (2), (4) and (5), equating the coefficients of $x^{2n}$ across this equation and then simplifying the result, we get the following identity.

THEOREM 35 ($m \geq 0$ and $n \geq 0$).

$$\frac{(-1)^m(2n+1)}{(2m+\gamma+1)^{2n-1}} E_{2n} - 2\sum_{k=1}^{m}\sum_{i=0}^{n} (-1)^k \binom{2n+1}{2i+1} \frac{(2k+\gamma)^{2i+1}}{(2m+\gamma+1)^{2i}} B''_{2n-2i}$$
$$= \frac{(\gamma+1)^{2n+1}}{(2m+\gamma+1)^{2n}} \sum_{0\leq i+j\leq n} \binom{2n+1}{2i,2j} \frac{(2m+\gamma+1)^{2i}}{(\gamma+1)^{2i+2j}} B''_{2i} E_{2j}.$$

Letting $m = \gamma = 0$ in Theorem 35 gives directly the formula

$$\sum_{0\leq i+j\leq n} \binom{2n+1}{2i,2j} B''_{2i} E_{2j} = (2n+1)E_{2n}.$$

In fact, applying Corollary 3, we can prove the following more general result.

Corollary 36 ($W \neq 0$).

$$\sum_{0\le i+j\le n}\binom{2n+1}{2i,2j}\frac{B''_{2i}E_{2j}}{W^{2n-2j}}=(2n+1)E_{2n}.$$

3.3 – Thirdly, rewrite (16) equivalently in the following manner

$$\begin{aligned}&2\csc(2m+\gamma+1)x\sum_{k=1}^{m}(-1)^k\sin x\sin(2k+\gamma)x\\&=(-1)^m\tan x-\tan x\sin(\gamma+1)x\csc(2m+\gamma+1)x\end{aligned}$$

and recall the trigonometric relation

$$2\sin x\sin(2k+\gamma)x=\cos(2k+\gamma-1)x-\cos(2k+\gamma+1)x.$$

In view of (3), (4) and (5), extracting the coefficient of $x^{2n-1}$ across the penultimate equation, we get the identity.

Theorem 37 ($m \geq 0$ and $n \geq 0$).

$$\begin{aligned}&\sum_{k=1}^{m}\sum_{i=0}^{n}(-1)^k\binom{2n}{2i}\frac{(2m+\gamma+1)^{2i-1}}{(\gamma+1)^{2n+1}}B''_{2i}\left\{\begin{matrix}(2k+\gamma-1)^{2n-2i}\\-(2k+\gamma+1)^{2n-2i}\end{matrix}\right\}\\&=(-1)^m\frac{4^nB'_{2n}}{(\gamma+1)^{2n+1}}-\sum_{0\le i+j\le n}\binom{2n+1}{2i,2j}\frac{4^i(2m+\gamma+1)^{2j-1}}{(2n+1)(\gamma+1)^{2i+2j}}B'_{2i}B''_{2j}.\end{aligned}$$

When $m=\gamma=0$, Theorem 37 yields the following identity

$$\sum_{0\le i+j\le n}4^i\binom{2n+1}{2i,2j}B'_{2i}B''_{2j}=(2n+1)4^nB'_{2n}.$$

This identity can also be shown by equating the coefficients $x^{2n-1}$ across the following trigonometric equation

$$\tan x\csc x\sin x=\tan x.$$

Furthermore, we can verify through Corollary 3, the following more general result.

corollary 38 ($W \neq 0$).

$$\sum_{0\le i+j\le n}4^i\binom{2n+1}{2i,2j}\frac{B'_{2i}B''_{2j}}{W^{2n-2i}}=(2n+1)4^nB'_{2n}.$$

Corollary 39 ($m = 1$ and $m = -3$ in Theorem 37: $n > 0$).

$$\sum_{i=0}^{n} \binom{2n+1}{2i} B'_{2i} = -n - 1/2.$$

We remark that this identity is also the linear combination of Corollary 13 and Lemma 17.

3.4 – Fourthly, the identity (16) may equivalently be expressed as

$$\begin{aligned} &2\cot(2m+\gamma+1)x \sum_{k=1}^{m} (-1)^k \sin(2k+\gamma)x = \\ &= (-1)^m \sec x \cos(2m+\gamma+1)x - \sec x \sin(\gamma+1)x \cot(2m+\gamma+1)x. \end{aligned}$$

On account of (1), (2) and (5), equating the coefficients of $x^{2n}$ across this equation and then simplifying the result, we get the following identity.

Theorem 40 ($m \geq 0$ and $n \geq 0$).

$$\begin{aligned} &\sum_{k=1}^{m} \sum_{i=0}^{n} (-1)^k \binom{2n+1}{2i} 2^{2i+1} (2m+\gamma+1)^{2i-1} (2k+\gamma)^{2n+1-2i} B_{2i} \\ &= (2n+1) \sum_{i=0}^{n} (-1)^m \binom{2n}{2i} (2m+\gamma+1)^{2n-2i} E_{2i} \\ &\quad - \sum_{0 \leq i+j \leq n} 4^j \binom{2n+1}{2i, 2j} (2m+\gamma+1)^{2j-1} (\gamma+1)^{2n+1-2i-2j} E_{2i} B_{2j}. \end{aligned}$$

When $m = \gamma = 0$, it yields the following expression

$$\sum_{0 \leq i+j \leq n} 4^j \binom{2n+1}{2i, 2j} E_{2i} B_{2j} = (2n+1) \sum_{i=0}^{n} \binom{2n}{2i} E_{2i}.$$

According to Corollary 45, this gives rise to the following formula.

Corollary 41 ($n > 0$).

$$\sum_{0 \leq i+j \leq n} 4^i \binom{2n+1}{2i, 2j} B_{2i} E_{2j} = 0.$$

**3.5 –** Finally, reformulate (16) equivalently as the following equality

$$\begin{aligned}&2\cot(2m+\gamma+1)x\sum_{k=1}^{m}(-1)^k\sin x\sin(2k+\gamma)x\\&=(-1)^m\tan x\cos(2m+\gamma+1)x+\tan x\sin(\gamma+1)x\cot(2m+\gamma+1)x.\end{aligned}$$

With the help of (1), (3) and (5), extracting the coefficient of $x^{2n-1}$ across the last equation leads us to the following identity.

Theorem 42 ($m \geq 0$ and $n \geq 0$).

$$\begin{aligned}&\sum_{k=1}^{m}\sum_{i=0}^{n}(-1)^k\binom{2n}{2i}\frac{4^i(2m+\gamma+1)^{2i-1}}{(\gamma+1)^{2n+1}}B_{2i}\left\{\begin{array}{c}(2k+\gamma-1)^{2n-2i}\\-(2k+\gamma+1)^{2n-2i}\end{array}\right\}\\&=\sum_{i=0}^{n}(-1)^m\binom{2n}{2i}\frac{4^i(2m+\gamma+1)^{2n-2i}}{(\gamma+1)^{2n+1}}B'_{2i}\\&\quad-\sum_{0\leq i+j\leq n}\binom{2n+1}{2i,2j}\frac{4^{i+j}(2m+\gamma+1)^{2j-1}}{(2n+1)(\gamma+1)^{2i+2j}}B'_{2i}B_{2j}.\end{aligned}$$

When $m=\gamma=0$, it yields the following transformation

$$\sum_{0\leq i+j\leq n}4^{i+j}\binom{2n+1}{2i,2j}\frac{B'_{2i}B_{2j}}{2n+1}=\sum_{i=0}^{n}4^i\binom{2n}{2i}B'_{2i}.$$

By extracting the coefficients of $x^{2n-1}$ across the expansion of the trigonometric relation

$$\tan x\cot x\sin x=\cos x\tan x=\sin x,$$

we can show further the following two closed formulae.

Corollary 43.

$$\sum_{0\leq i+j\leq n}4^{i+j}\binom{2n+1}{2i,2j}\frac{B'_{2i}B_{2j}}{2n+1}=\sum_{i=0}^{n}4^i\binom{2n}{2i}B'_{2i}=-2n.$$

## 4 – Alternating sum concerning $\cos(2k+\gamma)x$

In view of the following trigonometric relation

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

it is almost trivial to derive that

$$2\cos x \sum_{k=1}^{m} (-1)^k \cos(2k+\gamma)x = (-1)^m \cos(2m+\gamma+1)x - \cos(\gamma+1)x. \tag{18}$$

According to five different reformulations of this identity, this section will investigate convolution identities involving Bernoulli and Euler numbers.

4.1 – Firstly, it is obvious that (18) is equivalent to the equation

$$2\sum_{k=1}^{m} (-1)^k \cos(2k+\gamma)x = (-1)^m \sec x \cos(2m+\gamma+1)x - \sec x \cos(\gamma+1)x.$$

According to (2) and (5), we have the following power series expansions

$$2\sum_{k=1}^{m}\sum_{n=0}^{\infty} (-1)^{n+k} \frac{(2k+\gamma)^{2n}}{(2n)!} x^{2n} + \sum_{i,j\geq 0} (-1)^{i+j} \frac{(\gamma+1)^{2j}}{(2i)!(2j)!} E_{2i} x^{2i+2j}$$
$$= \sum_{i,j\geq 0} (-1)^{m+i+j} \frac{(2m+\gamma+1)^{2j}}{(2i)!(2j)!} E_{2i} x^{2i+2j}.$$

Equating the coefficients of $x^{2n}$ across this equation, we find the transformation.

THEOREM 44 ($m \geq 0$ and $n \geq 0$).

$$2\sum_{k=1}^{m} (-1)^k (2k+\gamma)^{2n} = (-1)^m \sum_{i=0}^{n} \binom{2n}{2i} (2m+\gamma+1)^{2n-2i} E_{2i}$$
$$- \sum_{i=0}^{n} \binom{2n}{2i} (\gamma+1)^{2n-2i} E_{2i}.$$

Comparing the case $\gamma = -\delta$ of this theorem with the identity due to Chu and Wang [3, Theorem 10], we recover the following well-known identity.

Corollary 45 (Stromberg [14, Section 7.58]: $n > 0$).

$$\sum_{i=0}^{n} \binom{2n}{2i} E_{2i} = 0.$$

For $m = 1$ and $\gamma = -3$, the last theorem recovers another interesting identity.

Corollary 46 ($m = 1$ and $\gamma = -3$ in Theorem 44: Chu-Wang [3, Equation 23a]).

$$\sum_{i=0}^{n} \frac{4^n}{4^i} \binom{2n}{2i} E_{2i} = 2 - E_{2n}.$$

4.2 – Secondly, the identity (18) may equivalently be restated as

$$\begin{aligned} &2 \sec(2m+\gamma+1)x \sum_{k=1}^{m} (-1)^k \cos(2k+\gamma)x \\ &= (-1)^m \sec x - \sec x \sec(2m+\gamma+1)x \cos(\gamma+1)x. \end{aligned}$$

By means of (2), (4) and (5), equating the coefficients of $x^{2n}$ across this equation yields the following identity.

Theorem 47 ($m \geq 1$ and $n \geq 0$).

$$\begin{aligned} &\frac{(-1)^m E_{2n}}{(2m+\gamma+1)^{2n}} - 2 \sum_{k=1}^{m} \sum_{i=0}^{n} (-1)^k \binom{2n}{2i} \frac{(2k+\gamma)^{2i}}{(2m+\gamma+1)^{2i}} E_{2n-2i} \\ &= \frac{(\gamma+1)^{2n}}{(2m+\gamma+1)^{2n}} \sum_{0 \leq i+j \leq n} \binom{2n}{2i, 2j} \frac{(2m+\gamma+1)^{2i}}{(\gamma+1)^{2i+2j}} E_{2i} E_{2j}. \end{aligned}$$

When $m = \gamma = 0$, it reduces to the following identity

$$\sum_{0 \leq i+j \leq n} \binom{2n}{2i, 2j} E_{2i} E_{2j} = E_{2n}.$$

By exchanging the summation order and then applying Corollary 45, we can show the following more general result.

Corollary 48 ($W \neq 0$).

$$\sum_{0 \leq i+j \leq n} \binom{2n}{2i, 2j} \frac{E_{2i} E_{2j}}{W^{2n-2i}} = E_{2n}.$$

**4.3 –** Thirdly, rewrite (18) equivalently in the following manner

$$\begin{aligned}&2\sec(2m+\gamma+1)x\sum_{k=1}^{m}(-1)^k\sin x\cos(2k+\gamma)x\\&=(-1)^m\tan x-\tan x\sec(2m+\gamma+1)x\cos(\gamma+1)x\end{aligned}$$

and recall to the trigonometric relation

$$2\sin x\cos(2k+\gamma)x=\sin(2k+\gamma+1)x-\sin(2k+\gamma-1)x. \tag{19}$$

In view of (2), (3) and (5), extracting the coefficients of $x^{2n+1}$ across the penultimate equation results in the following general transformation.

Theorem 49 ($m\geq 0$ and $n\geq 0$).

$$\begin{aligned}&\sum_{k=1}^{m}\sum_{i=0}^{n}(-1)^k\binom{2n+1}{2i}\frac{(2m+\gamma+1)^{2i}}{(\gamma+1)^{2n+2}}E_{2i}\left\{\begin{matrix}(2k+\gamma+1)^{2n-2i+1}\\-(2k+\gamma-1)^{2n-2i+1}\end{matrix}\right\}\\&=\frac{(-1)^{m+1}2^{2n+1}}{(n+1)(\gamma+1)^{2n+2}}B'_{2n+2}+\sum_{0\leq i+j\leq n+1}\binom{2n+2}{2i,2j}\frac{4^i(2m+\gamma+1)^{2j}}{(2n+2)(\gamma+1)^{2i+2j}}B'_{2i}E_{2j}.\end{aligned}$$

Three identities can be derived from this theorem as consequences.

Firstly, when $m=\gamma=0$, the theorem yields the following identity

$$\sum_{0\leq i+j\leq n}4^i\binom{2n}{2i,2j}B'_{2i}E_{2j}=4^nB'_{2n}.$$

Applying Corollary 45, this can be generalized to the following general formula.

Corollary 50 ($W\neq 0$).

$$\sum_{0\leq i+j\leq n}4^i\binom{2n}{2i,2j}\frac{B'_{2i}E_{2j}}{W^{2n-2i}}=4^nB'_{2n}.$$

Secondly, taking $m=1$ and $\gamma=-1$, we have the transformation expression

$$\sum_{i=0}^{n+1}\binom{2n+2}{2i}B'_{2i}E_{2n+2-2i}=-B'_{2n+2}-(n+1)\sum_{i=0}^{n}\binom{2n+1}{2i}E_{2i}.$$

Evaluating the last sum by Corollary 34 and then replacing $n$ by $n-1$, we get the following interesting convolution formula.

Corollary 51.

$$\sum_{i=0}^{n}\binom{2n}{2i}B'_{2i}E_{2n-2i}=(2^{2n-1}-1)B'_{2n}.$$

This can also be verified by equating the coefficients of $x^{2n-1}$ across the following trigonometric equation

$$\tan x \sec x = \tan x - \tan \frac{x}{2}.$$

Finally, letting $m = 1$ and $\gamma = -3$, we find another closed formula, which is , in fact, also a linear combination of Corollary 18 and Corollary 19.

Corollary 52 ($m = 1$ and $\gamma = -3$ in Theorem 49).

$$\sum_{i=0}^{n} \binom{2n}{2i} B'_{2i} = -n - B'_{2n}.$$

**4.4 –** Fourthly, the identity (18) may equivalently be expressed as

$$\begin{aligned} &2\tan(2m+\gamma+1)x \sum_{k=1}^{m} (-1)^k \cos(2k+\gamma)x \\ &= (-1)^m \sec x \sin(2m+\gamma+1)x - \sec x \tan(2m+\gamma+1)x \cos(\gamma+1)x. \end{aligned}$$

On account of (19), we can extract, via (2), (3) and (5), the coefficients of $x^{2n-1}$ across this equation. Simplifying the result gives the following identity.

Theorem 53 ($m \geq 0$ and $n \geq 0$).

$$\begin{aligned} &2\sum_{k=1}^{m}\sum_{i=0}^{n+1} (-1)^k \binom{2n+2}{2i} 4^i (2m+\gamma+1)^{2i-1} (2k+\gamma)^{2n+2-2i} B'_{2i} \\ &= (2n+2)\sum_{i=0}^{n} (-1)^{m+1} \binom{2n+1}{2i} (2m+\gamma+1)^{2n+1-2i} E_{2i} \\ &\quad - \sum_{0\leq i+j\leq n+1} 4^i \binom{2n+2}{2i, 2j} (2m+\gamma+1)^{2i-1} (\gamma+1)^{2n+2-2i-2j} B'_{2i} E_{2j}. \end{aligned}$$

When $m = -\gamma = 1$, this theorem reduced a simplified transformation.

Corollary 54 ($m = 1$ and $\gamma = -1$ in Theorem 53).

$$\sum_{i=0}^{n} 4^{2i} \binom{2n}{2i} B'_{2i} E_{2n-2i} = 8n + 2\sum_{i=0}^{n} 4^{2i} \binom{2n}{2i} B'_{2i}.$$

4.5 – Finally, reformulate (18) equivalently as the following equality

$$\begin{aligned}&2\tan(2m+\gamma+1)x\sum_{k=1}^{m}(-1)^k\sin x\cos(2k+\gamma)x\\&=(-1)^m\tan x\sin(2m+\gamma+1)x+\tan x\tan(2m+\gamma+1)x\cos(\gamma+1)x.\end{aligned}$$

Similarly with the help of (3) and (5), extracting the coefficient of $x^{2n}$ across the last equation, we establish the following identity.

Theorem 55 ($m \geq 0$ and $n \geq 0$).

$$\begin{aligned}&\sum_{k=1}^{m}\sum_{i=0}^{n}(-1)^k\binom{2n+1}{2i}4^i\frac{(2m+\gamma+1)^{2i-1}}{(\gamma+1)^{2n+2}}B'_{2i}\left\{\begin{matrix}(2k+\gamma+1)^{2n-2i+1}\\-(2k+\gamma-1)^{2n-2i+1}\end{matrix}\right\}\\&=\sum_{i=0}^{n}(-1)^m\binom{2n+1}{2i}4^i\frac{(2m+\gamma+1)^{2n-2i+1}}{(\gamma+1)^{2n+2}}B'_{2i}\\&\quad+\sum_{0\leq i+j\leq n+1}4^{i+j}\binom{2n+2}{2i,2j}\frac{(2m+\gamma+1)^{2j-1}}{(2n+2)(\gamma+1)^{2i+2j}}B'_{2i}B'_{2j}.\end{aligned}$$

When $m=\gamma=0$, it yields the following strange identity

$$\sum_{0\leq i+j\leq n+1}4^{i+j}\binom{2n+2}{2i,2j}B'_{2i}B'_{2j}=-(2n+2)\sum_{i=0}^{n}4^i\binom{2n+1}{2i}B'_{2i}.$$

By extracting the coefficients of $x^{2n}$ across the expansion of the trigonometric relation

$$\tan^2 x\cos x=\tan x\sin x=\sec x-\cos x,$$

we have the following two convolution formulae.

Corollary 56.

$$\sum_{i=0}^{n}4^i\binom{2n+1}{2i}B'_{2i}=(2n+1)(E_{2n}-1).$$

Corollary 57 ($n>0$).

$$\sum_{0\leq i+j\leq n}4^{i+j}\binom{2n}{2i,2j}B'_{2i}B'_{2j}=2n(2n-1)(1-E_{2n-2}).$$

Taking $m = -\gamma = 1$ in Theorem 55 and then replacing $n$ by $n-1$, we have the transformation formula

$$\sum_{i=0}^{n} \frac{4^n}{4^i}\binom{2n}{2i} B'_{2i}B'_{2n-2i} = n\sum_{i=0}^{n}\binom{2n-1}{2i}(2-4^i)B'_{2i}.$$

Evaluating the last sum by Corollary 39 and Corollary 56, we derive further the following convolution identity.

Corollary 58 ($m = 1$ and $m = -1$ in Theorem 55: $n > 1$).

$$\sum_{i=0}^{n} \frac{4^n}{4^i}\binom{2n}{2i} B'_{2i}B'_{2n-2i} = n(1-2n)E_{2n-2}.$$

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