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Some landmarks in the history of the tangential Cauchy Riemann equations

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Abstract: *We discuss the origins of the tangential Cauchy Riemann equation beginning with W. Wirtinger in 1926, and trace the largely unknown early developments until the emergence of the $\overline{\partial}_b-$ Neumann complex in the 1960s.*

Vienna is a most appropriate venue for a program centered on the $\overline{\partial}-$ Neumann Problem. Not only did the calculus of the differential operators $\partial/\partial z_j$ and $\partial/\partial \overline{z}_j$ originate in the work of Wilhelm Wirtinger, Professor at the University of Vienna, but to my knowledge Wirtinger also was the first person to have thought of what today we call the *tangential Cauchy Riemann equations* and the corresponding notion of *(tangential) Cauchy-Riemann* ($= CR$) *functions*. Since much of the modern literature seems to be unaware of this work and of other early work on "tangential analytic functions", it may be useful to trace the path from these origins to the modern theory of the tangential $\overline{\partial}-$Neumann Complex as developed by J. J. Kohn and H. Rossi in the 1960s.

Key Words and Phrases: *Tangential Cauchy-Riemann functions – CR Extension – Global CR Extension theorem – Local CR embedding problem*
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1 – The Beginning

Wilhelm Wirtinger (1865 - 1945) was born in Ybbs on the Danube and studied mathematics at the Universität Wien. He earned his doctorate in 1887 with Emil Weyr and Gustav Ritter von Escherich, working on triple evolutions in the plane. For the next three years he expanded his mathematical horizons in Berlin and Göttingen, where he was strongly influenced by F. Klein. In 1890 he earned the Habilitation in Vienna, and after a few years as assistant he was appointed to a chair at the University of Innsbruck in 1895. He returned to Vienna in 1905 to assume a chair at his alma mater, where he stayed until his retirement in 1935. Wirtinger was productive across a broad spectrum of mathematics and mathematical physics, ranging from complex analysis and number theory to relativity theory and capillary waves. He was well recognized internationally as one of the leading mathematicians of his days. Among his nine doctoral students are W. Blaschke (1908, Wien) and L. Vietoris (1920, Wien). Other well known mathematicians such as Schreier, Gödel, Radon, and Tausky-Todd studied with him.

Most relevant for the present discussion is Wirtinger's 1926 paper *Zur formalen Theorie der Funktionen von mehr komplexen Veränderlichen* [Wir]. Starting with a (smooth) function $F(x_1, ..., x_{2n})$ of the real variables x_β, $\beta = 1, ..., 2n$, Wirtinger introduces the complex functions $z_j = x_{2j-1} + ix_{2j}$ and their conjugates $\overline{z_j}$, $j = 1, ..., n$ and thinks of F as a function of the z_j and $\overline{z_j}$ via $x_{2j-1} = \frac{1}{2}(z_j + \overline{z_j})$ and $x_{2j} = \frac{1}{2i}(z_j - \overline{z_j})$. Formal application of the chain rule leads to

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} + \frac{1}{i}\frac{\partial}{\partial x_{2j}}\right) \text{ and } \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} - \frac{1}{i}\frac{\partial}{\partial x_{2j}}\right).$$

F is then an *analytic* function of $z_1, ..., z_n$ precisely when F satisfies the Cauchy-Riemann equations

$$\frac{\partial F}{\partial \overline{z_j}} = 0, \text{ or, equivalently, } \frac{\partial \overline{F}}{\partial z_j} = 0,\ j = 1, ..., n.$$

As a first elegant application of this point of view, Wirtinger notes that if W is the real part of an analytic function, or more generally, a linear combination $aF + b\overline{G}$, where F and G are analytic in $z_1, ..., z_n$, then obviously

$$\frac{\partial^2 W}{\partial z_j \partial \overline{z_k}} = 0 \text{ for all } j, k = 1, ..., n.$$

Conversely, if W satisfies these equations, W must be such a linear combination, at least locally. In fact, the $1-$form $\omega_1 = \sum \frac{\partial W}{\partial z_j} dz_j$ has analytic coefficients and is clearly closed, hence it is (locally) the differential dF of a function $F = \int \omega_1$

which is analytic in $z_1, ..., z_n$. Similarly, $\omega_2 = \sum \frac{\partial W}{\partial \overline{z_j}} d\overline{z_j}$ is the differential $d\overline{G}$ of a function $\overline{G} = \int \omega_2$ which depends analytically on $\overline{z_1}, ..., \overline{z_n}$, i.e., G depends analytically on $z_1, ..., z_n$. Since $dW = \omega_1 + \omega_2 = d(F + \overline{G})$, W and $F + \overline{G}$ differ by a constant. In particular, if W is *real* valued, then $\omega_2 = \overline{\omega_1}$, and hence $G = F$, so W is the real part of an analytic function.

Influenced by Riemann's point of view, who considered functions $f(x, y)$ on a $2-$ dimensional manifold which are analytic in $z = x + iy$, in the sense that their differential df is just a multiple of dz, Wirtinger generalizes this idea to the setting of an $m-$ dimensional manifold M_m, with (real) coordinates $t = (t_1, ..., t_m)$. Given a positive integer n, with $n < m \leq 2n$, he introduces $2n$ real functions $x_\gamma(t)$, $y_\gamma(t)$, $1 \leq \gamma \leq n$ on M_m, and the corresponding complex valued functions $z_\gamma = x_\gamma + iy_\gamma$, subject to the nondegeneracy condition

$$rank \left[\frac{\partial z_\gamma}{\partial t_\lambda} \quad \frac{\partial \overline{z_\gamma}}{\partial t_\lambda} \right]_{m \times 2n} = m,$$

so that the points on M_m are uniquely determined by the values of the z_j and $\overline{z_j}$. Furthermore, the functions $z_1(t), ..., z_n(t)$ are assumed to be independent, i.e., $dz_1 \wedge .... \wedge dz_n \neq 0$. Wirtinger then introduces the concept of a *complex valued function* $\Phi(t)$ *on* M_m *which depends on* $z_1, ..., z_n$ (" ... *eine Funktion* $\Phi(t)$ [welche] *als Funktion der* z_γ ... *dargestellt werden kann*" [Wir, p. 364]) by the condition that

$$rank \begin{bmatrix} \dfrac{\partial \Phi}{t_\lambda} \\ \dfrac{\partial z_\gamma}{\partial t_\lambda} \end{bmatrix} < n + 1.$$

In the language of differential forms, this means that $d\Phi \wedge dz_1 \wedge .... \wedge dz_n \equiv 0$ on M_m, or $d\Phi = \sum a_\gamma(t)\, dz_\gamma$. Equivalently, the partial derivatives $\frac{\partial \Phi}{\partial t_\lambda}$, $\lambda = 1, ..., m$, must satisfy a system of $m - n > 0$ linear equations, i.e., there exist linear differential operators $X_k = \sum X_k^\lambda \frac{\partial}{\partial t_\lambda}$, $k = 1, ..., m - n$, with complex valued coefficients X_k^λ on M_m, such that Φ is "analytic in $z_1, ..., z_n$" if and only if

$$X_k(\Phi) = \sum_{\lambda=1}^{m} X_k^\lambda \frac{\partial \Phi}{\partial t_\lambda} = 0, \;\; k = 1, ..., m - n.$$

This system so generated has two basic properties:

a) The only *real* valued solutions are constants.
b) *span* $\{X_1, ..., X_{m-n}\}$ is closed under Lie brackets.

Conversely, starting with such a system which satisfies a) and b), Wirtinger notes that if there exist n independent solutions $z_1, ..., z_n$, then all solutions of $X_k(\Phi) = 0$, $k = 1, ..., m - n$, on M_m can be thought of as analytic functions of these independent solutions. The key problem thus involves proving the existence of such independent solutions. Motivated by the classical case $m = 2$, $n = 1$, which Riemann studied by means of extremal properties, i.e, via the Dirichlet problem, Wirtinger first attempts to introduce appropriate variational problems and integral invariants in the higher dimensional case $n > 1$, $m = 2n$. In modern language, Wirtinger was considering an integrable almost complex structure on M_{2n}, and he was trying to extend Riemann's methods to prove that the given data defined a complex manifold M_{2n}. But he quickly realized that *"bis zu bestimmten Existenzsätzen noch ein weiter Weg ist.*(1) ([Wir], p. 372). He surely was right: it took over 30 years until the problem was eventually solved by A. Newlander and L. Nirenberg ([NeNi]).

Wirtinger then turned to the case $m < 2n$ and set up some explicit computations in the case $m = 3$, $n = 2$, thereby attempting to outline a strategy to solve what eventually became known as the *(local) embedding problem for abstract CR -structures.* He seemed prescient, as he stated that such investigations, *if they can be carried out at all*, would be much more difficult and complicated (op. cit, p. 375). In fact, moving ahead half a century, L. Nirenberg showed in 1974 that there is in general no solution in this particular dimension, even assuming a definite Levi form [Nir]. On the other hand, in a remarkable tour de force, M. Kuranishi [Kur] proved in 1982 that the answer is positive in the hypersurface case $m = 2n - 1$ with definite Levi form, provided $n \geq 5$. Subsequently, T. Akahori was able to extend Kuranishi's work to the case $n = 4$ [Aka]. The case $n = 3$ (i.e. $m = 5$) remains open to this date. Wirtinger's intuition thus was remarkably accurate. Realizing these difficulties with continuing along the path initiated by Riemann, Wirtinger ended his paper with the statement *"Vielleicht hätte Riemann auch Ideen zur Überwindung dieser Schwierigkeiten gehabt."*(2)

2 – Early CR Extension Results

As noted above, there was no progress for a long time regarding the deep question about *existence* of solutions to the system of partial differential equations introduced by Wirtinger. However, Wirtinger's idea of "analytic functions of the complex variables $z_1, ..., z_n$" on a real manifold led to other important developments. Remarkable results were obtained just a few years after Wirtinger's paper in the concrete setting in which the real manifold M_{2n-1} is a submanifold of $\mathbb{C}^n$, where the complex coordinates $z_1, ..., z_n$ trivially provide n independent solutions of Wirtinger's system. Here a most natural question is to examine the relationship between functions analytic on M_{2n-1} in Wirtinger's sense, and the functions analytic in $z_1, ..., z_n$ in the ambient space in the classical sense.

(1) *The path to specific existence theorems is still long.*
(2) *Perhaps Riemann would also have had ideas to overcome these difficulties.*

Clearly restrictions to M_{2n-1} of such classical analytic functions, as well as suitable boundary values of such functions defined on only one side of M_{2n-1} , are solutions of the corresponding Wirtinger system. The obvious question then is whether all solutions are, essentially, of this type.

Making reference to Wirtinger's 1926 paper, Francesco Severi gave an affirmative answer in 1931 in the real analytic category [Sev].

THEOREM (SEVERI 1931). *If $M_{2n-1} \subset \mathbb{C}^n$ is real analytic, and if f is a real analytic function on M_{2n-1} which satisfies the Wirtinger condition $df \wedge dz_1 \wedge \ldots \wedge dz_n = 0$ in a neighborhood of a point $P \in M_{2n-1}$, then there exists a function $F \in \mathcal{O}(U)$ on an open neighborhood U of P in $\mathbb{C}^n$, such that $F|_{M_{2n-1} \cap U} = f$.*

The proof, which is essentially trivial in case $n = 1$, involves an elegant application of Severi's method to pass from real to complex variables in appropriate power series. Severi proved the theorem in case $n = 2$, but his proof works in higher dimensions as well with the obvious modifications. Via the identity theorem, the result is easily globalized. By applying the classical Hartogs extension theorem, Severi thus obtains the following generalization of the Hartogs theorem to the case of Wirtinger's tangential analytic functions.

GLOBAL *CR* EXTENSION THEOREM, REAL ANALYTIC CASE .*If $n > 1$ and the bounded region $D \subset \mathbb{C}^n$ has connected real analytic boundary bD , then any real analytic function f which satisfies $df \wedge dz_1 \wedge \ldots \wedge dz_n = 0$ on bD has a holomorphic extension to D.*

The *local* extension theory in the *differentiable* case is considerably more complicated. Apparently unaware of Severi's work, in 1936 Helmuth Kneser studied the problem on M_3 in $\mathbb{C}^2$ and produced examples to show that differentiable functions satisfying the Wirtinger condition are not necessarily the boundary values of classical holomorphic functions [Kne]. In fact, Kneser considered a generalization of Wirtinger's differential condition on $M_3 = M$ to a Morera type condition (A) for *continuous* functions f, as follows. *A continuous function f satisfies condition (A) on the 3-dimensional manifold M if $\int_{bG} f\, dz_1 \wedge dz_2 = 0$ for every subregion $G \subset M$ with C^1 boundary bG.* Kneser showed that for f of class C^1, condition (A) is equivalent to Wirtinger's differential condition. More significantly, in analogy to the E. E. Levi extension phenomenon for holomorphic functions, Kneser proved a deep *local* one-sided extension result for such continuous *CR* functions near a strictly Levi pseudoconvex boundary point, as follows.

THEOREM (KNESER 1936). *Assume that $P \in bD$ and that D is strictly Levi pseudoconvex at P.*[3] *Then there exist neighborhoods $V \subset\subset U$ of P, such*

[3] This implies in particular that bD is of class C^2 near P.

that every continuous function f on $U \cap bD$ which satisfies condition (A) can be extended continuously to a function holomorphic on $V \cap D$.

In the proof, Kneser first showed that the geometric hypothesis implied that after a local holomorphic change of coordinates one could assume that the boundary was strictly Euclidean convex.(4) In this geometrically simple setting Kneser then produced the holomorphic extension via an explicit integral formula which was a suitably adapted variant of the Cauchy integral formula for polydiscs. Condition (A) is the critical ingredient that makes the proof work.

Incidentally, just as Severi had done in the real analytic case, Kneser also proved the corresponding *global* version.

Global CR Extension Theorem, Strictly Pseudoconvex Case. *If the bounded region $D \subset \mathbb{C}^2$ has connected strictly pseudoconvex boundary bD then any continuous (weakly) CR function f on bD has a holomorphic extension to D.*

To my knowledge, Kneser's result is the the first global CR extension theorem in the differentiable category, albeit under some restrictive geometric conditions.

Unfortunately, the phenomenal progress in global complex function theory in higher dimensions achieved by K. Oka and H. Cartan beginning in the mid 1930s, as well as the political climate in Germany and the disruptions of the second world war, relegated the investigations begun by Wirtinger, Severi, and Kneser to the sidelines, to the extent that for all practical purposes they were forgotten and did not get proper recognition for a long time.

3 – Results of Lewy and Fichera in the 1950s

In the early 1950s there was renewed interest in fundamental investigations in the theory of partial differential equations. One major result of this period was the proof of existence of fundamental solutions for every linear partial differential operator with *constant* coefficients, obtained independently by L. Ehrenpreis and B. Malgrange. Furthermore, the multivariable classical Cauchy-Riemann equations presented a central example of an *overdetermined* system which required new methods for its study. Lastly, the system of linear partial differential equations introduced by Wirtinger in 1926 provided natural important classes of examples which were not covered by the Ehrenpreis - Malgrange theory. Although it is not clear how much Wirtinger's ideas were known in those days,

(4) This seems to be the earliest explicit occurence of what has become a well known standard tool.

times were certainly ripe for studying such more general equations. In particular, Hans Lewy, who had earned his doctorate in Göttingen with R. Courant in 1925 and had moved to the University of California in Berkeley in 1935 after he was forced to emigrate from Germany, began to investigate a linear differential equation with non-constant coefficients which is equivalent to Wirtinger's equation for "analytic" functions on submanifolds in the case of a 3-dimensional submanifold in $\mathbb{C}^2$ [Lew 1]. While Lewy did not mention Wirtinger's name in his paper, he made explicit reference to Severi's 1931 extension theorem in the real analytic case, and thus it is most likely that he also knew Wirtinger's work, which is prominently cited in Severi's paper. Unaware of Hellmuth Kneser's 1936 work, Lewy proved Kneser's extension theorem for continously differentiable CR functions in the strictly pseudoconvex case. Shortly thereafter, Lewy used an explicit example of the equation studied earlier to produce the first example - and at that time quite unexpected - of a smooth first order complex linear partial differential equation in 3 real variables without any solutions [Lew 2]. Lewy's results generated much interest and became widely known.

These developments probably contributed to overshadowing the remarkable extension result for CR functions obtained in Italy by G. Fichera [Fic] around the same time. Motivated by Severi, Fichera showed in 1957 that Severi's *"global CR extension theorem"* (i.e., the CR version of Hartogs' Theorem) remained true *without* assuming real analyticity and without any geometric restrictions. Fichera's proof, based on the solution of the Dirichlet problem, required the given data to be of class $C^{1+\varepsilon}$. His work subsequently inspired E. Martinelli to modify his 1942 integral formula proof of the classical Hartogs Theorem to produce a simple proof of the Severi-Fichera global extension result in the C^1 category [Mar].(5) However, these results about the *global* CR extension problem remained virtually unrecognized outside of Italy for a long time.

4 – The modern theory

The global CR extension theorem came to the forefront in 1965, when J.J. Kohn and H. Rossi, inspired by Lewy's local extension theorem, introduced tangential (p, q) forms and the $\overline{\partial}_b-$complex on smooth boundaries of domains in complex manifolds [KoRo]. By using Kohn's then new deep regularity results for the $\overline{\partial}-$Neumann problem, Kohn and Rossi proved the holomorphic extension of C^∞ global CR functions from the connected boundary of domains in Stein manifolds, assuming that the Levi form has at least one positive eigenvalue at each point on the boundary. They also proved corresponding extension results for $\overline{\partial}_b-$ closed forms in higher degree. Their work marks the beginning of the *modern* theory of tangential CR functions and forms, either incorporated in the

(5) More recently, this author used the Bochner-Martinelli kernel to give a simple proof of the *local* Kneser-Lewy extension theorem for continuous (weak) CR functions [Ran 2].

$\overline{\partial}_b-$complex and in the theory of the $\overline{\partial}_b-$Neumann problem, or as the principal object of study in numerous settings. The reader may consult the recent monographs by M. S. Baouendi, P. Ebenfelt, and L. Rothschild [BER] and So-Chin Chen and Mei-Chi Shaw [ChSh] for an overview of many developments since then.

Unfortunately, Kohn and Rossi were apparently unaware of the earlier work on the global CR extension theorem by Severi, Fichera, and Martinelli. Furthermore, a remark in the introduction of their 1965 paper connected the global CR extension theorem to S. Bochner's 1943 proof of the classical Hartogs extension theorem. Shortly thereafter this linkage led L. Hörmander to crediting Bochner with the proof of the global CR extension theorem in his well known 1966 monograph. This erroneous attribution became widely accepted since then, even though there is no evidence in the published record that Bochner stated and proved such a theorem, nor that he had even been thinking about tangential CR functions. The historical record was eventually corrected beginning in 1999, when this author learned of the long forgotten 1936 paper of H. Kneser. The interested reader should consult [Ran 1, 3] for more details.

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