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Regularity results for planar quasilinear equations

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ABSTRACT: *We study the Dirichlet problem for the quasilinear elliptic equation*

$$-\text{div } A(x, \nabla v) = f$$

in a planar domain Ω*, when* f *belongs to the Zygmund space* $L(\log L)^{\frac{1}{2}}(\log\log L)^{\frac{1}{2}}(\Omega)$*. We prove that the gradient of the variational solution* $v \in W_0^{1,2}(\Omega)$ *belongs to the Zygmund space* $L^2 \log\log L(\Omega)$*.*

1 – Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with C^1-boundary. We consider the following Dirichlet problem

$$\begin{cases} -\text{div } A(x, \nabla v) = f & \text{in } \Omega \\ v \in W_0^{1,2}(\Omega), \end{cases} \tag{1.1}$$

where $A : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a mapping such that:

$$x \to A(x, \xi) \quad \text{is measurable for any } \xi \in \mathbb{R}^2; \tag{1,2}$$

$$\xi \to A(x, \xi) \quad \text{is continuous for almost every } x \in \Omega. \tag{1.3}$$

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Moreover we assume that there exists $K \geqslant 1$ such that for almost every $x \in \Omega$ we have

(1.4) $\quad |A(x,\xi) - A(x,\eta)| \leqslant K|\xi - \eta| \qquad$ (Lipschitz continuity)

(1.5) $\quad |\xi - \eta|^2 \leqslant K\langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle \qquad$ (strong monotonicity)

(1.6) $\quad A(x,0) = 0$

for any vectors ξ and η in $\mathbb{R}^2$ (see [18]).

In [9] an existence and uniqueness theorem for the Dirichlet problem for the equation div $A(x,\nabla v) = f$ is proved where $f \in L^1(\Omega)$ and the solution v belongs to the so called *grand Sobolev space* $W_0^{1,2)}(\Omega)$ i.e. the space of function $v \in W_0^{1,1}(\Omega)$ whose gradient $|\nabla v|$ satisfies

$$\sup_{1<s<2}\left[(2-s)\int_\Omega |\nabla v|^s dx\right]^{\frac{1}{s}} = \|v\|_{W_0^{1,2)}} < \infty.$$

Note that the space of such functions $W_0^{1,2)}(\Omega)$ is slightly larger than $W_0^{1,2}(\Omega)$ and this is the appropriate space when the right-hand side f is assumed to be only L^1−integrable (see [9], [11] for more details).

In this paper we study cases where the solution v is the variational $W_0^{1,2}(\Omega)$-solution, under the assumption

$$f \in L(\log L)^{\frac{1}{2}}(\log\log L)^{\frac{1}{2}}(\Omega) \subset L(\log L)^{\frac{1}{2}}(\Omega). \tag{1.7}$$

Let us observe that by the Sobolev-Trudinger imbedding in the plane

$$W_0^{1,2}(\Omega) \hookrightarrow EXP_2(\Omega), \tag{1.8}$$

hypothesis (1.7) guarantees that f belongs to the dual space of $W_0^{1,2}(\Omega)$ and then, at least in the linear case $A(x,\xi) = \mathcal{A}(x)\xi$ the Lax-Milgram Theorem ensure that there exists a unique solution $v \in W_0^{1,2}(\Omega)$.

The case where f belongs to the Zygmund space

$$f \in L(\log L)^\delta(\Omega) \subset L^1(\Omega), \qquad \text{for } \frac{1}{2} \leqslant \delta \leqslant 1 \tag{1.9}$$

is treated in [3] (see also [2], [21] for the case $\delta = 1$) where e.g. the authors prove that under the assumption (1.9), there is a unique solution $v \in W_0^{1,2}(\Omega)$ to the Dirichlet problem (1.1) with $\nabla v \in L^2(\log L)^{2\delta-1}$ and

$$\|\nabla v\|_{L^2(\log L)^{2\delta-1}(\Omega)} \leqslant c(K)\,\|f\|_{L(\log L)^\delta(\Omega)}\,, \tag{1.10}$$

where $c(K) > 0$ depends only on K.

We prove the following

THEOREM 1.1. *Let $A = A(x, \xi)$ satisfy conditions (1.2)-(1.6) and let $f \in L(\log L)^{\frac{1}{2}}(\log\log L)^{\frac{1}{2}}(\Omega)$. Then, there exists an unique $v \in W_0^{1,2}(\Omega)$ solution to*

$$\text{(1.11)} \qquad \begin{cases} -\text{div}\, A(x, \nabla v) = f & \text{in } \Omega \\ v \in W_0^{1,2}(\Omega), \end{cases}$$

such that $\nabla v \in L^2(\log\log L)(\Omega)$ and

$$\|\nabla v\|_{L^2(\log\log L)(\Omega)} \leqslant C(K)\, \|f\|_{L(\log L)^{\frac{1}{2}}(\log\log L)^{\frac{1}{2}}(\Omega)}.$$

Note that by imbedding theorems for Orlicz-Sobolev spaces, (see [5]) we obtain in particular that the solution v in Theorem 1.1 belongs to the Orlicz space $L^\Lambda(\Omega)$ generated by the Young function $\Lambda(t) = \exp\left\{t^2 \log(e+t)\right\} - 1$.

It is worth to point out that under the assumptions of Theorem 1.1 we cannot expect the boundedness of the solution u. In fact in [2] is proved that $f \in L \log L(\Omega)$ is a sufficient condition for the boundedness (and continuity) of the solution u and in [3] there are examples where $f \in L \log^\delta L(\Omega)$, $\delta \in [\frac{1}{2}, 1[$, and the solution u is not bounded.

In Section 5 we prove that also approaching $L \log L(\Omega)$ in the scale of spaces $L \log L(\log\log L)^\alpha$, $L \log L(\log\log\ \log L)^\alpha(\Omega)$, $L \log L(\log\log\log \ldots \log L)^\alpha(\Omega)$, $\alpha < 0$, we cannot obtain the boundedness of the solution.

The case $n \geqslant 3$ is extensively treated for the n-harmonic equations in the recent papers [14] and [12].

2 – Young's functions and Orlicz spaces

Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a Young's function, i.e. a convex function of type $\Phi(t) = \int_0^t \varphi(s)ds$, $t > 0$, where $\varphi : [0, \infty[\to \mathbb{R}$ is nondecreasing, right-continuous and such that

$$\text{(2.1)} \qquad \varphi(s) > 0 \quad \forall s > 0, \qquad \varphi(0) = 0, \qquad \lim_{s\to\infty} \varphi(s) = +\infty.$$

The Young's function $\tilde{\Phi}(t)$, complementary to $\Phi(t)$, is defined by $\tilde{\Phi}(t) = \sup\{st - \Phi(s) : s > 0\}$ and it is easy to see that $\tilde{\tilde{\Phi}} = \Phi$.

In the sequel we will deal with a particular class of Young functions Φ verifying a suitable sub-homogeneity property at infinity called Δ_2-*condition*. Namely, Definition 1. A young function Φ satisfies the Δ_2-condition (we will write $\Phi \in \Delta_2$) if there exists a constant $l > 0$ such that

$$\Phi(\lambda t) \leqslant \lambda^l \Phi(t), \quad \forall \lambda \geqslant 1, \quad \forall t \geqslant t_0, \tag{2.2}$$

where $t_0 \geqslant 0$ is a suitable large constant.

Let Ω be an open and bounded set in $\mathbb{R}^n$, $n \geqslant 1$. The Orlicz class $\Lambda^\Phi(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \Phi(|u(x)|)dx < \infty$$

The Orlicz Space $L^\Phi = L^\Phi(\Omega)$ is the linear hull of $\Lambda^\Phi(\Omega)$ and the equality $L^\Phi(\Omega) \equiv \Lambda^\Phi(\Omega)$ holds if and only if $\Phi \in \Delta_2$.

Define the functional $\|u\|_{L^\Phi(\Omega)} : L^\Phi(\Omega) \to [0, +\infty[$ by

$$\|u\|_{L^\Phi(\Omega)} = \inf \left\{ K > 0 : \int_\Omega \Phi\left(\frac{|u(x)|}{K}\right) dx \leqslant 1 \right\}. \tag{2.3}$$

It is a norm, called the *Luxemburg norm*, and $L^\Phi(\Omega)$ is a Banach space when endowed with it. When no confusion arise we will simply write $\|u\|_{L^\Phi}$ or $\|u\|_\Phi$ instead of $\|u\|_{L^\Phi(\Omega)}$.

We recall that:

i) If $\Phi(t) = t^p$ and $1 \leqslant p < \infty$ then $L^\Phi(\Omega) = L^p(\Omega)$, the classical Lebesgue space and $\| \cdot \|_{L^\Phi(\Omega)} = \| \cdot \|_{L^p}$.

ii) If $\Phi(t) = t^p(\log(e+t))^q$ where either $p > 1$ and $-\infty < q < \infty$ or $p = 1$ and $q \geqslant 0$, then the Orlicz space $L^\Phi(\Omega)$ is the Zygmund space $L^p(\log L)^q(\Omega)$, and the norm (2.3) is equivalent to the quantity (see [16])

$$[v]_{L^p(\log L)^q(\Omega)} = \left[⨍_\Omega |v|^p \log^q \left(e + \frac{|v|}{\left(⨍_\Omega |v|^p dx\right)^{\frac{1}{p}}} \right) dx \right]^{\frac{1}{p}} \tag{2.4}$$

where, for all Lebesgue measurable set E with positive measure, we denote by $⨍_E f dx$ the mean value of f taken over the set E, i.e. $⨍_E f dx = f_E = \frac{1}{|E|} \int_E f dx$, where $|E|$ denotes the Lebesgue measure of E.

iii) If $\Phi(t) = e^{t^a} - 1$, $a > 0$, then the Orlicz space $L^\Phi(\Omega)$ reproduces the space of exponentially integrable functions $EXP(\Omega)$ when $a = 1$ and $EXP_a(\Omega)$ otherwise.

iv) If $\Phi(t) = t^p(\log\log(e^e + t))^q$ where either $p > 1$ and $-\infty < q < \infty$ or $p = 1$ and $q \geqslant 0$, then the Orlicz space $L^\Phi(\Omega)$ is the space $L^p(\log\log L)^q(\Omega)$.

The closure of $C_0^\infty(\Omega)$ in $L^\Phi(\Omega)$ is denoted by $E^\Phi(\Omega)$ and the inclusions

$$E^\Phi(\Omega) \subseteq \Lambda^\Phi(\Omega) \subseteq L^\Phi(\Omega) \tag{2.5}$$

are trivial with equality holding if and only if $\Phi \in \Delta_2$.

The *Orlicz-Sobolev* space $W^{1,\Phi}(\Omega)$ is defined as

$$W^{1,\Phi}(\Omega) = \left\{u \in W^{1,1}(\Omega) \cap L^\Phi(\Omega) : |Du| \in L^\Phi(\Omega)\right\},$$

and, equipped with the norm

$$\|u\|_{W^{1,\Phi}} = \|u\|_\Phi + \|Du\|_\Phi$$

it is a Banach space.

By $W_0^{1,\Phi}(\Omega)$ we denote the subspace of $W^{1,\Phi}(\Omega)$ of those functions whose continuation by 0 outside Ω belongs to $W^{1,\Phi}(\mathbb{R}^n)$. Properties of Orlicz-Sobolev spaces are presented in [7], [20].

The Orlicz space $L^\Phi(\Omega)$ is isometrically isomorphic to the dual space of $E^{\tilde\Phi}(\Omega)$ (see [17], [20]) and $[L^\Phi(\Omega)]' \simeq L^{\tilde\Phi}(\Omega)$ if and only if $\Phi \in \Delta_2$. In particular the space $L^\Phi(\Omega)$ is reflexive if and only if both Φ and $\tilde\Phi$ belong to class Δ_2.

Here below we recall the explicit expression of the dual spaces of some Orlicz space (see [4] and [8]) which will be useful in the sequel

i) for any $1 < p < \infty$ and $-\infty < q < \infty$ it is

$$(L^p(\log L)^q(\Omega))' \cong \frac{L^{p'}}{(\log L)^{\frac{q}{p-1}}}(\Omega)$$

where p' is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$

ii) for any $1 < p < \infty$ and $-\infty < q < \infty$ it is

$$(L^p(\log\log L)^q(\Omega))' \cong \frac{L^{p'}}{(\log\log L)^{\frac{q}{p-1}}}(\Omega)$$

iii) for $p = 1$ and $q > 0$ it is

$$(L(\log L)^q(\Omega))' \cong EXP_{\frac{1}{q}}(\Omega) \tag{2.6}$$

The following partial ordering relation between functions is involved in imbedding theorems between Orlicz spaces associated with different Young functions.

Definition 2. The function Ψ is said to dominate the function Φ globally (respectively near infinity) if there exists $c > 0$ such that

$$\Phi(t) \leqslant \Psi(ct) \tag{2.7}$$

for any $t \geqslant 0$ (respectively for any t greater than some positive number).

The functions Φ and Ψ are called equivalent globally (respectively near infinity) if each dominates the other globally (respectively near infinity).

Lemma 2.1. *Let* $\Theta(t) = \exp\left\{\frac{t^2}{\log(e+t)}\right\} - 1$. *Then the conjugate Young function* $\tilde{\Theta}(t)$ *of* Θ *is equivalent, near infinity, to the function*

$$\Psi(t) = t\log^{\frac{1}{2}}(e+t)(\log\log(e+t))^{\frac{1}{2}}.$$

Proof. Let us start the proof by observing that the derivative function of Θ

$$\theta(t) = \Theta'(t) = \exp\left\{\frac{t^2}{\log t}\right\}\frac{2t\log t - t}{\log^2 t}$$

is equivalent near infinity to Θ. In fact, for any t sufficiently large we have

$$\theta(t) \cong \exp\left\{\frac{t^2}{\log t}\right\}\frac{2t}{\log t}$$

and

$$\exp\left\{\frac{t^2}{\log t}\right\} \leqslant \exp\left\{\frac{t^2}{\log t}\right\}\frac{2t}{\log t} \leqslant \exp\left\{\frac{(ct)^2}{\log ct}\right\},$$

for some constant $c > 1$. On the other hand it is not hard to see that the inverse function θ^{-1} of θ is equivalent near infinity to the function

$$\psi(s) = \frac{1}{\sqrt{2}}\log^{\frac{1}{2}} s(\log\log s)^{\frac{1}{2}}.$$

Hence, near infinity we have

$$\tilde{\Theta}(y) = \int_0^y \theta^{-1}(s)ds \cong y\log^{\frac{1}{2}} y(\log\log y)^{\frac{1}{2}}$$

as we claimed. □

Theorem 2.1. *The continuous imbedding* $L^\Psi(\Omega) \to L^\Phi(\Omega)$ *holds if and only if either* Ψ *dominates* Φ *globally or* $|\Omega| < \infty$ *and* Ψ *dominates* Φ *near infinity.*

In particular, for any Young function $\Psi = \Psi(t)$ which is dominated (near infinity) by the Young function

$$\Theta(t) = \exp\left\{\frac{t^2}{\log(e+t)}\right\} - 1,$$

by Theorem 2.1 we have

$$EXP_2(\Omega) \to L^{\Theta}(\Omega) \to L^{\Psi}(\Omega). \tag{2.8}$$

Moreover, for any $0 < \varepsilon < p < \infty$ and $-\infty < a < b < \infty$ the following imbedding are obvious

$$L^{p+\varepsilon}(\Omega) \to L^p(\log L)^b(\Omega) \to L^p(\log L)^a(\Omega) \to L^{p-\varepsilon}(\Omega)$$

$$L^p(\log L)^{\varepsilon}(\Omega) \to L^p(\Omega) \to L^p(\log L)^{-\varepsilon}(\Omega).$$

The following Sobolev-Trudinger type embedding holds

$$W_0\frac{L^2}{(\log L)^a}(\Omega) \hookrightarrow EXP_{\frac{2}{1+a}}(\Omega) \qquad \text{for } a < 1, \tag{2.9}$$

(see [22], [10], [5]), where we denote by $W_0\frac{L^2}{(\log L)^a}(\Omega)$ the space $W_0^{1,\Phi}(\Omega)$ where $\Phi(t) = t^2 \log^{-a}(e+t)$. It is worth to point out that in case $a = 0$ imbedding (1.8) follows.

We will finish this section by recalling the following result (see [5], Example 2 pag. 43)

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with C^1-boundary. If we consider Young functions $\Phi(t)$ which are equivalent to $t^p(\log\log(e+t))^q$ near infinity, where either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geqslant 0$, then*

$$W^{1,\Phi}(\Omega) \to C_b(\Omega)$$

if $p > 2$ and

$$W^{1,\Phi}(\Omega) \to L^{\Phi_2}(\Omega) \tag{2.10}$$

otherwise, where Φ_2 is equivalent near infinity to

$$\begin{cases} t^{\frac{2p}{2-p}}(\log\log(t))^{\frac{2q}{2-q}} & \text{if } 1 \leqslant p < 2 \\ e^{t^2(\log(t))^q} & \text{if } p = 2 \end{cases}$$

(Here $C_b(\Omega)$ denotes the space of continuous bounded functions on Ω).

For more details and proofs of results about Young function and Orlicz spaces we refer the reader to [1], [5], [6], [17], [20], [23].

3 – Preliminaries

The results we are going to obtain in this section are true in all dimensions. Hence, here we assume $A = A(x, \xi)$ to be defined on $\Omega \times \mathbb{R}^n$, where conditions (1.2)–(1.6) hold for $x \in \Omega \subset \mathbb{R}^n$ and $\xi, \eta \in \mathbb{R}^n$. Let us recall the following regularity result for the solution to quasilinear elliptic problem with the right-hand side in divergence form (see Theorem 3.2 of [3]).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1-boundary and let $A = A(x, \xi)$ be as before. Then for $\psi_1, \psi_2 \in \frac{L^2}{(\log L)^a}(\Omega; \mathbb{R}^n)$ with $0 \leqslant a \leqslant 1$, each of the two problems*

$$\begin{cases} \operatorname{div} A(x, \nabla\varphi_1) = \operatorname{div} \psi_1 & \text{in } \Omega \\ \varphi_1 \in W_0^{1,1}(\Omega) \end{cases}$$

$$\begin{cases} \operatorname{div} A(x, \nabla\varphi_2) = \operatorname{div} \psi_2 & \text{in } \Omega \\ \varphi_2 \in W_0^{1,1}(\Omega) \end{cases}$$

has a unique solution and

$$\|\nabla\varphi_1 - \nabla\varphi_2\|_{\frac{L^2}{(\log L)^a}(\Omega)} \leqslant c(K) \|\psi_1 - \psi_2\|_{\frac{L^2}{(\log L)^a}(\Omega)} \tag{3.1}$$

where $c(K) > 0$ depends only on K.

We prove the following

3.2. *Let $A = A(x, \xi)$ satisfy hypotheses (1.2)–(1.6). Then for $\psi_1, \psi_2 \in \frac{L^2}{\log\log L}(\Omega)$ each of the two problems*

$$\begin{cases} \operatorname{div} A(x, \nabla\varphi_1) = \operatorname{div} \psi_1 & \text{in } \Omega \\ \varphi_1 \in W_0^{1,1}(\Omega) \end{cases}$$

(3.2)

$$\begin{cases} \operatorname{div} A(x, \nabla\varphi_2) = \operatorname{div} \psi_2 & \text{in } \Omega \\ \varphi_2 \in W_0^{1,1}(\Omega) \end{cases}$$

has a unique solution and

$$\|\nabla\varphi_1 - \nabla\varphi_2\|_{\frac{L^2}{\log\log L}(\Omega)} \leqslant c(K) \|\psi_1 - \psi_2\|_{\frac{L^2}{\log\log L}(\Omega)} \tag{3.3}$$

Proof. For $i = 1, 2$ let $\psi_i \in \frac{L^2}{\log\log L}(\Omega)$. Then obviously ψ_i belong to $\frac{L^2}{(\log L)^a}(\Omega)$, $0 < a \leqslant 1$. Hence, by Theorem 3.1, there exists a unique solution φ_i to the Dirichlet Problem (3.2) and the estimate

$$\|\nabla\varphi_1 - \nabla\varphi_2\|_{\frac{L^2}{(\log L)^a}(\Omega)} \leq c(K)\|\psi_1 - \psi_2\|_{\frac{L^2}{(\log L)^a}(\Omega)} \tag{3.4}$$

holds uniformly with respect to $a \in \,]0, 1]$.

Now we claim that the following inequality holds true:

$$\begin{aligned}\left(1 - \frac{1}{e}\right)\int_\Omega \frac{k(x)^2}{\log\log(k(x) + e^e)}dx &\leqslant \int_0^1 da \int_\Omega \frac{k(x)^2}{\log^a(k(x) + e^e)}dx \leqslant \\ &\leqslant \int_\Omega \frac{k(x)^2}{\log\log(k(x) + e^e)}dx.\end{aligned} \tag{3.5}$$

Indeed by

$$\int_0^1 \frac{1}{\log^a(e^e + k(x))}da = \left[1 - \frac{1}{\log(k(x) + e^e)}\right]\frac{1}{\log\log(k(x) + e^e)},$$

we have

$$\left(1 - \frac{1}{e}\right)\frac{1}{\log\log(k(x) + e^e)} \leqslant \int_0^1 \frac{1}{\log^a(e^e + k(x))}da \leqslant \frac{1}{\log\log(k(x) + e^e)}$$

so that Inequality (3.5) follows.

Integrating both sides of (3.4) with respect to $0 \leqslant a \leqslant 1$ and using suitably (2.4) and (3.5) with $k(x) = |\nabla\varphi_1 - \nabla\varphi_2|$ and $k(x) = |\psi_1 - \psi_2|$ the thesis follows. $\square$

4 – The main result

In this Section we will give the proof of Theorem 1.1. Here and below we assume

$$\Phi(t) = t\log^{\frac{1}{2}}(e + t)(\log\log(e + t))^{\frac{1}{2}}.$$

Proof of Theorem 1.1. We start the proof by using the linearization procedure contained in [15] (see also [3]) which we report for the convenience of the reader. So, let $v \in W_0^{1,2)}(\Omega)$ be the solution to quasilinear problem

$$\begin{cases} -\text{div}\, A(x, \nabla v) = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

which exists and is unique because $f \in L^{\Phi}(\Omega) \subset L^1(\Omega)$ (see [9], [15]). We will determine a symmetric measurable matrix valued function $\mathcal{A} = \mathcal{A}(x)$ such that v satisfies the linear problem

$$\begin{cases} -\text{div } \mathcal{A}(x)\nabla v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

and $\mathcal{A}$ verifying

$$\frac{|\xi|^2}{C(K)} \leqslant \langle \mathcal{A}(x)\xi, \xi\rangle \leqslant C(K)|\xi|^2, \tag{4.3}$$

for any $\xi \in \mathbb{R}^2$, a.e.$x \in \Omega$, and where $C(K)$ is a constant depending only upon K.

Setting

$$B = A(x, \nabla v(x)), \qquad E = \nabla v(x). \tag{4.4}$$

one obtain, by assumptions (1.4)-(1.6)

$$|B| \leqslant K|E|, \qquad |E|^2 \leqslant K|\langle B, E\rangle|. \tag{4.5}$$

Moreover if we set,

$$\lambda = \frac{\langle B, E\rangle}{|E|^2}, \qquad \Lambda = \frac{|B|}{|E|} \qquad (|E| > 0)$$

by (4.5) we have

$$\frac{1}{K} \leqslant \lambda \leqslant \Lambda \leqslant K \qquad \text{and} \qquad \frac{|B|^2 + |E|^2}{\langle B, E\rangle} = \frac{1+\Lambda^2}{\lambda}. \tag{4.6}$$

Define $H \geqslant 1$ by solving the equation

$$H + \frac{1}{H} = \frac{1+\Lambda^2}{\lambda}$$

that is,

$$H = \frac{1}{2}\left[\frac{1+\Lambda^2}{\lambda} + \sqrt{\left(\frac{1+\Lambda^2}{\lambda} - 4\right)^2}\,\right].$$

Then, consider the 2×2 matrix defined by

$$\mathcal{A} = HI_d + \left(\frac{1}{H} - H\right)\frac{B - HE}{|B - HE|} \otimes \frac{B - HE}{|B - HE|},$$

where for $z = (x, t)$, we have used the shorthand notation

$$z \otimes z = \begin{pmatrix} x^2 & xt \\ xt & t^2 \end{pmatrix}$$

and $I_d = (\delta_{ij})$ is identical matrix. It holds (see [15])

$$\mathcal{A}E = B \tag{4.7}$$

and

$$\frac{|\xi|^2}{H} \leqslant \langle \mathcal{A}(x)\xi, \xi\rangle \leqslant H|\xi|^2, \qquad \forall \xi \in \mathbb{R}^2. \tag{4.8}$$

By (4.4) and (4.8), we have

$$\mathcal{A}(x)\nabla v(x) = B$$

which implies (4.2). Finally, by (4.8) and observing that it holds

$$H(x) \leqslant C(K),$$

(4.3) follows, with

$$C(K) = \frac{1}{2}\left[(K + K^3) + \sqrt{(K + K^3)^2 - 4}\right].$$

Now, let

$$L\cdot = -\mathrm{div}\, \mathcal{A}(x)\nabla \cdot .$$

Since $f \in L^\Phi(\Omega)$ then v is the variational solution in $W_0^{1,2}(\Omega)$ to the equation $Lv = f$. Hence we have

$$\int_\Omega \langle \mathcal{A}(x)\nabla v, \nabla\varphi\rangle dx = \int_\Omega \varphi f dx$$

for any $\varphi \in W_0^{1,2}(\Omega)$.

Now, let us fix $\psi \in \mathcal{C}^1(\bar{\Omega}; \mathbb{R}^2)$ with

$$\|\psi\|_{\frac{L^2}{(\log\log L)}(\Omega)} \leqslant 1 \tag{4.9}$$

and let φ be the (unique) solution to the Dirichlet problem

$$\begin{cases} L\varphi = \mathrm{div}\, \psi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

given by Theorem 3.2. Note that φ verifies

$$\|\nabla\varphi\|_{\frac{L^2}{(\log\log L)}(\Omega;\mathbb{R}^2)} \leqslant c(K)\|\psi\|_{\frac{L^2}{(\log\log L)}(\Omega;\mathbb{R}^2)} \leqslant c(K). \tag{4.10}$$

We have

$$|\langle\nabla v,\psi\rangle| = \left|\int_\Omega \langle\mathcal{A}(x)\nabla v,\nabla\varphi\rangle\,dx\right| = \left|\int_\Omega \varphi f\,dx\right|. \tag{4.11}$$

On the other hand, using Lemma 2.2 with $p=2$ and $q=-1$, the Orlicz-Sobolev imbedding

$$W_0^{1,}\frac{L^2}{\log\log L}(\Omega) \to L^\Theta(\Omega) \qquad \text{where} \qquad \Theta(t) = \exp\frac{t^2}{\log(e+t)} - 1$$

holds. Moreover, by Lemma 2.1 the conjugate Young function $\tilde\Theta$ of Θ is equivalent (near infinity) to the Young function Φ and then

$$L^{\tilde\Theta}(\Omega) = L^\Phi(\Omega). \tag{4.12}$$

Thus, for any $\psi \in \mathcal{C}^1(\bar\Omega,\mathbb{R}^2)$ verifying (4.9), by (4.11) and using Hölder inequality between associated Orlicz spaces (see for example [1]), we obtain

$$|\langle\nabla v,\psi\rangle| \leqslant c\|\varphi\|_{L^\Theta(\Omega)}\|f\|_{L^\Phi(\Omega)} \tag{4.13}$$

Taking the supremum under conditions $\psi \in \mathcal{C}^1(\bar\Omega;\mathbb{R}^2)$ and $\|\psi\|_{\frac{L^2}{(\log\log L)}(\Omega)} \leqslant 1$, the estimates (4.10) and (4.13) give

$$\sup\left\{|\langle\nabla v,\psi\rangle| : \psi \in \mathcal{C}^1(\bar\Omega;\mathbb{R}^2) \text{ and } \|\psi\|_{\frac{L^2}{(\log\log L)}(\Omega)} \leqslant 1\right\} \leqslant c(K,|\Omega|)\|f\|_{L^\Phi(\Omega)}$$

and the thesis follows. In fact it is now sufficient to observe that

$$\|\nabla v\|_{L^2(\log\log L)(\Omega)} = \sup_{\|\psi\|_{\frac{L^2}{(\log\log L)}(\Omega)}\leqslant 1} |\langle\nabla v,\psi\rangle|.$$

and that by (2.5) the space $C^1(\bar\Omega)$ is dense in $\frac{L^2}{\log\log L}(\Omega)$. ∎

Remark 4.1 It is evident that the thesis of Theorem 1.1 remains invaried whenever $f \in L^\Psi(\Omega)$, Ψ any Young function verifying

$$\Psi(t) \geqslant t\log^{\frac12}(e+t)(\log\log(e+t))^{\frac12}$$

for any $t > 0$ sufficiently large.

5 – On the boundedness of the solution

In this section we show with an example that we cannot expect the boundedness of the solution under the assumptions of Theorem 1.1 (see also [3], [14]).

Example 1. Let

$$u(x) = \log\log\log\frac{1}{|x|}$$

and let $\Omega = \left\{x \in \mathbb{R}^2 : |x| < e^{-e}\right\}$. Then, the unbounded function u verifies $|\nabla u| \in L^2 \log\log L(\Omega)$ and solves the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{5.1}$$

where

$$f := \frac{1}{|x|^2 \log^2 \frac{1}{|x|} \log\log \frac{1}{|x|}} \left(1 + \frac{1}{\log\log\frac{1}{|x|}}\right) \in L(\log L)(\log\log L)^\alpha(\Omega), \quad \forall \alpha < 0.$$

Proof. We have

$$\nabla u(x) = \frac{-x}{|x|^2 \log\frac{1}{|x|} \log\log\frac{1}{|x|}}, \qquad \forall\, x \neq 0,$$

so that

$$|\Delta u(x)| = |\mathrm{div}\ \nabla u(x)| = \frac{1}{|x|^2 \left(\log\frac{1}{|x|}\right)^2 \log\log\frac{1}{|x|}} \left(1 + \frac{1}{\log\log\frac{1}{|x|}}\right).$$

Hence, by $|f| = |\Delta u|$ we have, for any $\alpha < 0$,

$$\begin{aligned} \int_\Omega |f| \log(|f|)\,(\log\log|f|)^\alpha\, dx \leqslant \\ &\leqslant c\int_\Omega \frac{1}{|x|^2\log\frac{1}{|x|}\left(\log\log\frac{1}{|x|}\right)^{1-\alpha}} dx = \\ &= c\int_0^{e^{-e}} \frac{1}{\rho\log\frac{1}{\rho}(\log\log\frac{1}{\rho})^{1-\alpha}} d\rho = \\ &= \frac{c}{-\alpha}\left[\left(\log\log\frac{1}{\rho}\right)^\alpha\right]_0^{e^{-e}} < \infty, \end{aligned}$$

so that f belongs to $L\log L(\log\log L)^\alpha(\Omega)$ for any $\alpha < 0$. Note that for $\alpha = 0$ first integral in last inequality is infinite. ∎

In a similar way we have the following

Example 2. Let

$$u(x) = \log\log\log\log \frac{1}{|x|}$$

and let $\Omega = \left\{x \in \mathbb{R}^2 : |x| < e^{-e^e}\right\}$. Then, the unbounded function u verifies $\nabla u \in L^2 \log\log L(\Omega)$ and solves the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ v \in W_0^{1,2}(\Omega), \end{cases} \tag{5.2}$$

where

$$f := \frac{1}{|x|^2 \log^2 \frac{1}{|x|} \log\log \frac{1}{|x|} \log\log\log \frac{1}{|x|}} \left(1 + \frac{1}{\log\log \frac{1}{|x|}} + \frac{1}{\log\log \frac{1}{|x|} \log\log\log \frac{1}{|x|}}\right)$$

and holds

$$f \in L(\log L)(\log\log\log L)^{\alpha}(\Omega), \quad \forall \alpha < 0.$$

By continuing in the same way, we can conclude that if by one hand $f \in L\log L$ is a sufficient condition to obtain the boundedness of the solution u (see [2]) by the other hand slightly weaker condition $f \in L\log L(\log\log\log \ldots \log L)^{\alpha}(\Omega)$, $\alpha < 0$, is insufficient.

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