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A length type functional for curves in probability spaces

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Abstract: *We propose a new length-type functional defined on (set-valued) curves in probability spaces and we give, under suitable conditions, an integral representation formula.*

1 – Introduction

The aim of this paper is to study a type of functional length for (set-valued) curves in probability spaces. Such a definition is motivated from recent results on the definition of length of curves ([2], [3], [6]) and from a well-know formula in probability spaces, the conditional probability: $p(B|A) = \frac{p(A\cap B)}{p(A)}$.

A (set-valued) curve ψ defined on the interval $[a, b]$ such that $\psi(a) = A$ and $\psi(b) = B$ can be interpreted as a particular procedure to prove that $A \Rightarrow B$, so the given definition can be considered as the cost to derive B from A with the procedure ψ.

We remark that $p(B|A) = \frac{p(A\cap B)}{p(A)}$, as function of the pair (A, B), is not symmetric, non negative, it does not satisfy, in general, a triangle inequality and $p(A|A) = \frac{p(A\cap A)}{p(A)} = 1$.

The proposed functional seem to be new and we can consider for it the usual problems of the Calculus of Variations.

The plan of the paper is the following. In Section 2, we consider, with a simplified condition on the curve, the main idea about the functional and we give the

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main property. Section 3 is devoted to studying, in a very general situation, two different functionals length's type for (set-valued) curves in probability spaces. For the functionals we establish some properties and relations. In Section 4, we give an integral representation formula for one of the functional of Section 3. To obtain this formula a suitable definition of derivative for (set-valued) curves in probability space is necessary. Finally we point out some problems which seem to be of interest, also in the spirit of the Calculus of Variation.

2 – A particular situation

In this section we consider particular curves in probability spaces.

Let (Ω, p, Θ) be a probability space; then Ω is non empty set, Θ is a $\sigma-$algebra and p a probability defined on Θ.

Now, let $\psi : [a, b] \longrightarrow \Theta$ be a set-valued curve, such that:

$$\forall t \in [a, b] : \qquad p(\psi(t)) > 0; \tag{2.1}$$

$$\exists \epsilon_0 > 0 : \quad \forall h \in [0, \epsilon_0[\Rightarrow p(\psi(t + h) \cap \psi(t)) > 0; \tag{2.2}$$

$$\exists \delta_0 > 0 : (t_1 < t_2 < t_3, \quad t_3 - t_1 < \delta_0 \Rightarrow \tag{2.3}$$

$$p(\psi(t_3)|\psi(t_1)) \geq p(\psi(t_2)|\psi(t_1))p(\psi(t_3)|\psi(t_2))).$$

Condition (2.3) can be interpreted as a "triangular inequality" here considered only as a technical tool, relaxed in the following sections.

Let $\sigma = \{a = t_0 < t_2 < \ldots < t_n = b\}$ with $t_{i+1} - t_i < \min\{\delta_0, \epsilon_0\}$ $(i = 0, 1, \ldots, n-1)$, be a partition of the interval $[a, b]$; then we can consider the following product:

$$\Lambda^+(\sigma, \psi, [a, b], p) = \Pi_0^{n-1} p(\psi(t_{i+1})|\psi(t_i)).$$

We remark that: $\Lambda^+(\sigma, \psi, [a, b], p) > 0$. Furthemore, from (2.3), we can consider, also:

$$L_0^+(\psi, [a, b], p) = \inf \left\{\Lambda^+(\sigma, \psi, [a, b], p)|\sigma\right\} (= \inf_\sigma \Lambda^+(\sigma, \psi, [a, b], p)). \tag{2.4}$$

The non-negative number $L_0^+(\psi, [a, b], p)$ exists certainly and:

$$0 \leq L_0^+(\psi, [a, b], p) \leq 1; \tag{2.5}$$

$$[c, d] \subseteq [a, b] \Rightarrow L_0^+(\psi, [a, b], p) \leq L_0^+(\psi, [c, d], p). \tag{2.6}$$

To prove (2.6), we consider $\sigma_n = \{c = t_0^n < t_1^n < \ldots < t_m^n = d\}$ such that: $\lim_n \Lambda^+(\sigma_n, [c, d], \psi, p) = L_0^+(\psi, [c, d], p)$. Then if we consider the partition of $[a, b]$ given by $\sigma_n^* = \sigma_n \cup \{a, b\}$, we have:

$$\begin{aligned} L_0^+(\psi, [a, b], p) &\leq \Lambda^+(\sigma_n^*, \psi, [a, b], p) = p(\psi(c)|\psi(a))\Lambda^+(\sigma_n, \psi, [c, d], p)p(\psi(b)|\psi(d)) \\ &\leq \Lambda^+(\sigma_n, \psi, [c, d], p); \end{aligned}$$

hence inequality (2.6) is true.

Now, we shall call $L_0^+(\psi,[a,b],p)$ the *probabilistic cost of the curve* ψ *from* a *to* b, *with respect to* p.

The cost $L_0^+(\psi,[a,b],p)$ is "productive" with the meaning given in the following theorem.

THEOREM 1. *Let* $\psi_1 : [a,b] \longrightarrow \Theta$ *and* $\psi_2 : [b,c] \longrightarrow \Theta$ *be two set-valued curves such that* $\psi_1(b) = \psi_2(b)$; *assume also that conditions* (2.1), (2.2), *and* (2.3) *hold for both* ψ_1 *and* ψ_2 *and define the set-valued curve* $\psi_3 : [a,c] \longrightarrow \Theta$ *as follows*:

$$\psi_3(t) = \left\{ \begin{array}{l} \psi_1(t), \ \textit{if } t \in [a,b] \\ \psi_2(t), \ \textit{if } t \in [b,c] \end{array} \right\} \tag{2.7}$$

assume that, also, ψ_3 *satisfies conditions* (2.1), (2.2) *and* (2.3), *then*:

$$L_0^+(\psi_3,[a,c],p) = L_0^+(\psi_1,[a,b],p)L_0^+(\psi_2,[b,c],p). \tag{2.8}$$

PROOF. First of all we point out that:

$$\begin{aligned}
&L_0^+(\psi_1,[a,b],p)L_0^+(\psi_2,[b,c],p) = \\
&= \inf\left\{\Lambda^+(\sigma,\psi_1,[a,b],p) | \sigma = \{t_0 = a < t_1 < \cdots < t_n = b\}\right\} = \\
&= \inf\left\{\Lambda^+(\sigma^*,\psi_2,[b,c],p) | \sigma^* = \{s_0 = b < s_1 < \cdots < s_m = c\}\right\} = \\
&= \inf\left\{\Lambda^+(\sigma,\psi_1,[a,b],p)\Lambda^+(\sigma^*,\psi_2,[b,c],p) | \sigma \cup \sigma^*\right\} \geq \\
&\geq \inf\left\{\Lambda^+(\sigma,\psi_3,[a,c],p) | \sigma = \{r_0 = a < r_1 < \cdots < r_q = c\}\right\} = L_0^+(\psi_3,[a,c],p).
\end{aligned}$$

Hence:

$$L_0^+(\psi_3,[a,c],p) \leq L_0^+(\psi_1,[a,b],p)L_0^+(\psi_2,[b,c],p). \tag{2.9}$$

Consider, now, a sequence (σ_n) with $\sigma_n = \{t_0^n = a < t_1^n < \cdots < t_m^n = c\}$ for wich:

$$L_0^+(\psi_3,[a,c],p) = \lim_{n\to+\infty} \Lambda^+(\sigma_n,\psi_3,[a,c],p).$$

We consider $\sigma_n^* = \sigma_n \cup \{b\}$ and j such that $t_j \leq b < t_{j+1}$; we remark that, again by (2.3):

$$\begin{aligned}
&\Lambda^+(\sigma_n,\psi_3,[a,c],p) \geq \Lambda^+(\sigma_n^*,\psi_3,[a,c],p) = \\
&= \Pi^+_{i\leq j-1} p(\psi_3(t_{i+1})|\psi_3(t_i)) p(\psi_3(b)|\psi_3(t_j)) p(\psi_3(t_{j+1})|\psi_3(b)) \cdot \\
&\quad \cdot \Pi^+_{i\geq j+1} p(\psi_3(t_{i+1})|\psi_3(t_i)) = \\
&= \Pi^+_{i\leq j-1} p(\psi_1(t_{i+1})|\psi_1(t_i)) p(\psi_1(b)|\psi_1(t_j))] [p(\psi_2(t_{j+1})|\psi_2(b)) \cdot \\
&\quad \cdot \Pi^+_{i\geq j+1} p(\psi_2(t_{i+1})|\psi_2(t_i))] \geq \\
&\geq L_0^+(\psi_1,[a,b],p)L_0^+(\psi_2,[b,c],p).
\end{aligned}$$

Then, passing to the limit $n \to \infty$, the proof is complete.

We can notice that condition (2.3) is satisfied in a very particular situation and is, in general, not fullfilled. Then we should proceed in a different way, as in the following section.

3 – A more general situation

In this section we consider a different way to introduce the cost when (2.3) is not assumed; the procedure is directly related in the spirit to [2], [3], [6].

Let (Ω, p, Θ) be a probability space; then Ω is a non empty set, Θ is a $\sigma-$algebra and p a probability defined on Θ. Now, let $\psi : [a,b] \longrightarrow \Theta$ be a set-valued curve, such that only conditions (2.1) and (2.2) hold.

If we denote a partition by $\sigma = \{t_0 = a < t_1 < \dots t_n = b\}$, with $t_{i+1} - t_i < \epsilon_0$ $(i = 0, 1, \dots, n-1)$, we can consider:

$$\Lambda_1^+(\psi, [a,b], p) = \sup\left\{\Pi_0^{n-1} p(\psi(t_{i+1})|\psi(t_i))|\sigma\right\} (= \sup_\sigma \Pi_0^{n-1} p(\psi(t_{i+1})|\psi(t_i))); \tag{3.1}$$

if we denote $\Gamma = \{T_0 = a < T_1 < \dots < T_m = b\}$, we can also consider:

$$\begin{aligned} L_1^+(\psi, [a,b], p) = \inf\left\{\Pi_0^{m-1}(\Lambda_1^+(\psi, [T_i, T_{i+1}], p))|\Gamma\right\} \cdot \\ \cdot (= \inf_\Gamma \Pi_0^{m-1}(\Lambda_1^+(\psi, [T_i, T_{i+1}], p))). \end{aligned} \tag{3.2}$$

Whenever we consider a partition σ, we can always assume that

$$|\sigma| = \max_i |t_{i+1} - t_i| < \epsilon_0.$$

For $L_1^+(\psi, [a,b], p)$ we have a "productive"' property as in the following result.

Theorem 2. *Let $\psi_1 : [a,b] \longrightarrow \Theta$ and $\psi_2 : [b,c] \longrightarrow \Theta$ be set-valued curves such that $\psi_1(b) = \psi_2(b)$; assume also conditions* (2.1), (2.2) *for both curves and define the set-valued curve $\psi_3 : [a,c] \longrightarrow \Theta$ as follows:*

$$\psi_3(t) = \begin{cases} \psi_1(t), & \text{if } t \in [a,b] \\ \psi_2(t), & \text{if } t \in [b,c] \end{cases}$$

and assume that ψ_3 satisfies conditions (2.1) *and* (2.2)*, then*:

$$L_1^+(\psi_3, [a,c], p) = L_1^+(\psi_1, [a,b], p) L_1^+(\psi_2, [b,c], p). \tag{3.4}$$

Proof. From:

$$
\begin{aligned}
&L_1^+(\psi_1,[a,b],p)L_1^+(\psi_2,[b,c],p) = \\
&= \inf\left\{\Pi_0^{m-1}(\Lambda_1^+(\psi_1,[T_i,T_{i+1}],p)) | T_0 = a < \ldots < T_m = b\right\} = \\
&= \inf\left\{\Pi_0^{k-1}(\Lambda_1^+(\psi_2,[S_j,S_{j+1}],p)) | S_0 = b < \ldots < S_k = c\right\} = \\
&= \inf\Big\{[\Pi_0^{m-1}(\Lambda_1^+(\psi,[T_i,T_{i+1}],p))][\Pi_0^{k-1}(\Lambda_1^+(\psi_2,[S_j,S_{j+1}],p))]| \\
&\quad T_0 = a < \ldots < T_m = b = S_0 < \ldots < S_k = c\Big\} \geq \\
&\geq \inf\left\{\Pi_0^{h-1}(\Lambda_1^+(\psi_3,[R_j,R_{j+1}],p)) | R_0 = a < \ldots < R_h = c\right\} = L_1^+(\psi_3,[a,c],p).
\end{aligned}
$$

Hence:

$$
L_1^+(\psi_3,[a,c],p) \leq L_1^+(\psi_1,[a,b],p)L_1^+(\psi_2,[b,c],p). \tag{3.5}
$$

Moreover, let (σ_n) such that $\sigma_n = \{T_0^n = a < \ldots T_m^n = c\}$ and:

$$
\lim_n \Pi_0^{m-1}(\Lambda_1^+(\psi_3,[T_i^n,T_{i+1}^n],p)) = L_1^+(\psi_3,[a,c],p);
$$

consider $j = j_n$ such that: $T_j^n \leq b < T_{j+1}^n$.

Then, we have the following equality:

$$
\begin{aligned}
&\Pi_0^{m-1}(\Lambda_1^+(\psi_3,[T_i^n,T_{i+1}^n],p)) = \\
&= [\Pi_{i\leq j-1}(\Lambda_1^+(\psi_2,[T_i^n,T_{i+1}^n],p))][\Pi_{i\geq j-1}(\Lambda_1^+(\psi_2,[T_i^n,T_{i+1}^n],p))]\cdot \\
&\quad \cdot \Lambda_1^+(\psi_3,[T_j^n,T_{j+1}^n],p) = \\
&= [\Pi_{i\leq j-1}(\Lambda_1^+(\psi_2,[T_i^n,T_{i+1}^n],p))\Lambda_1^+(\psi_1,[T_j^n,b],p)]\cdot \\
&\quad \cdot [\Lambda_1^+(\psi_2,[b,T_{j+1}^n],p)\Pi_{i\geq j-1}(\Lambda_1^+(\psi_2,[T_i^n,T_{i+1}^n],p))]\cdot \\
&\quad \cdot \frac{\Lambda_1^+(\psi_3,[T_j^n,T_{j+1}^n],p)}{\Lambda_1^+(\psi_1,[T_j^n,b],p)\Lambda_1^+(\psi_2,[T_i^n,T_{i+1}^n],p)}.
\end{aligned}
$$

But, by (3.1), we have

$$
\frac{\Lambda_1^+(\psi_3,[T_j^n,T_{j+1}^n],p)}{\Lambda_1^+(\psi_1,[T_j^n,b],p)\Lambda_1^+(\psi_2,[T_i^n,T_{i+1}^n],p)} \geq 1 \tag{3.6}
$$

hence:

$$
\begin{aligned}
&\Pi_0^{m-1}(\Lambda_1^+(\psi_3,[T_i^n,T_{i+1}^n],p)) \geq \\
&\geq [\Pi_{i\leq j-1}(\Lambda_1^+(\psi_2,[T_i^n,T_{i+1}^n],p))\Lambda_1^+(\psi_1,[T_j^n,b],p)]\cdot \\
&\quad \cdot [\Lambda_1^+(\psi_2,[b,T_{j+1}^n],p)\Pi_{i\geq j-1}(\Lambda_1^+(\psi_2,[T_i^n,T_{i+1}^n],p))],
\end{aligned}
$$

and the following inequality is true:

$$L_1^+(\psi_3, [a,c], p) \geq L_1^+(\psi_1, [a,b], p) L_1^+(\psi_2, [b,c], p), \tag{3.7}$$

so the thesis is achieved.

We remark that, in general, the following inequality holds:

$$L_0^+(\psi_1, [a,b], p) \leq L_1^+(\psi_1, [a,b], p).$$

Strictly related to $L_0^+(\psi_1, [a,b], p)$, as proved in the sequel, is the following more interesting definition that generated still a "productive" function of ψ.

With the same conditions (2.1), (2.2), we can consider (see also (2.4)):

$$\begin{aligned} \Lambda_2^+(\psi, [a,b], p) &= \inf\left\{\Pi_0^{n-1} p(\psi(t_{i+1}|\psi(t_i))|\sigma\right\} \cdot \\ &\cdot [= \inf\left\{\Lambda^+(\sigma, \psi, [a,b], p)|\sigma\right\} = L_0^+(\psi, [a,b], p)], \end{aligned} \tag{3.8}$$

where $\sigma = \{t_0 = a < t_1 < \ldots t_n = b\} \quad (|\sigma| < \epsilon_0)$ and

$$L_2^+(\psi, [a,b], p) = \sup\left\{\Pi_0^{m-1}(\Lambda_2^+(\psi, [T_i, T_{i+1}], p))|\Gamma\right\}, \tag{3.9}$$

where $\Gamma = \{T_0 = a < T_1 < \ldots < T_m = b\}$.

We remark that: $\Lambda_2^+(\psi, [a,b], p) \leq \Lambda_1^+(\psi, [a,b], p)$; moreover for $L_2^+(\psi, [a,b], p)$ we have also a "productive" result.

Theorem 3. *Let $\psi_1 : [a,b] \longrightarrow \Theta$ and $\psi_2 : [b,c] \longrightarrow \Theta$ be set-valued curves such that $\psi_1(b) = \psi_2(b)$; assume also conditions* (2.1), (2.2) *for both curves and define the set-valued curve $\psi_3 : [a,c] \longrightarrow \Theta$ as follows*:

$$\psi_3(t) = \begin{cases} \psi_1(t) & \text{if } t \in [a,b] \\ \psi_2(t) & \text{if } t \in [b,c] \end{cases} \tag{3.10}$$

and assume that ψ_3 satisfies conditions (2.1), (2.2), *then*:

$$L_2^+(\psi_3, [a,c], p) = L_2^+(\psi_1, [a,b], p) L_2^+(\psi_2, [b,c], p). \tag{3.11}$$

The proof of the theorem is similar to the proof of previous theorem; in particular one can prove at first step that:

$$L_2^+(\psi_3,[a,c],p) \geq L_2^+(\psi_1,[a,b],p)L_2^+(\psi_2,[b,c],p).$$

The reverse inequality is obtained arguing as in the second step of the proof of the Theorem 2.

It is easy, whenever L_0^+ is defined, to show the equality:

$$L_2^+(\psi,[a,b],p) = L_0^+(\psi,[a,b],p). \tag{3.12}$$

We remark, indeed, that for every $[c,d] \subseteq [a,b]$:

$$L_0^+(\psi,[c,d],p) = \inf\left\{\Lambda^+(\sigma,\psi,[c,d],p)|\sigma\right\} = \Lambda_2^+(\psi,[c,d],p);$$

hence for every $T = \{T_0 = a < T_1 < \ldots < T_m = b\}$ we have, by the "productivity":

$$L_0^+(\psi,[a,b],p) = \Pi_0^{m-1} L_0^+(\psi,[T_i,T_{i+1}],p) = \Pi_0^{m-1}(\Pi_2^{n-1}(\psi,[T_j,T_{j+1}],p))$$

and so equality follows.

For such equality we shall call $L_2^+(\psi,[a,b],p)$ the *probabilistic cost of the curve ψ from a to b, with respect to p.*

For the function $Ł_i^+(\psi,[c,d],p)$ we have the following general result:

$$[a,c] \subseteq [a,b] \Rightarrow L_i^+(\psi,[a,b],p) \leq L_i^+(\psi,[a,c],p) \quad \forall i = 1,2. \tag{3.13}$$

Moreover, for $i = 1,2$:

$$L_i^+(\psi,[a,b],p) \leq \inf\left\{L_i^+(\psi,[a,c],p); L_i^+(\psi,[c,b],p)|c \in [a,b]\right\} \tag{3.14}$$

and

$$\begin{aligned} [a,c] \subseteq [a,b] \Rightarrow 0 \leq L_i^+(\psi,[a,c],p) - L_i^+(\psi,[a,b],p) = \\ = L_i^+(\psi,[a,c],p)(1 - L_i^+(\psi,[c,b],p)). \end{aligned} \tag{3.15}$$

By (3.15), with $i = 2$, we deduce that:

$$\begin{aligned} 0 \leq L_2^+(\psi,[a,c],p) - L_2^+(\psi,[a,b],p) = L_2^+(\psi,[a,c],p)(1 - L_2^+(\psi,[c,b],p)) = \\ = L_2^+(\psi,[a,c],p)\left(1 - \sup_\Gamma \Pi_0^{m-1}(\Lambda_2^+(\psi,[T_i,T_{i+1}],p))\right); \end{aligned}$$

and assuming $\Gamma = \{c,b\}$ we have:

$$0 \leq L_2^+(\psi,[a,c],p) - L_2^+(\psi,[a,b],p) \leq L_2^+(\psi,[a,c],p)(1 - p(\psi(b)|\psi(c))).$$

Hence we can also prove the following inequality:

$$
\begin{aligned}
(3.16) \qquad [c,d] \subseteq [a,b] \Rightarrow 0 \leq L_2^+(\psi,[c,d],p) - L_2^+(\psi,[a,b],p) \leq \\
\leq L_2^+(\psi,[a,c],p)(1 - p(\psi(c)|\psi(a))p(\psi(b)|\psi(d))).
\end{aligned}
$$

The previous formula can be interpreted as a "Lagrange formula" for the cost.

Finally we point out that by considering conditions (2.1), (2.2) and there exists $\delta_0 > 0$ such that

$$
(3.17) \qquad t_3 - t_1 < \delta_0 \Rightarrow p(\psi(t_1)|\psi(t_3)) \geq p(\psi(t_2)|\psi(t_3))p(\psi(t_1)|\psi(t_2))),
$$

one can define:

$$
\Lambda^-(\sigma,\psi,[a,b],p) = \Pi_0^{n-1} p(\psi(t_i)|\psi(t_{i+1}))
$$

and so we can define $L_0^-(\psi,[a,b],p)$, $L_i^-(\psi,[a,b],p)$ $i = 1,2$, obtaining similar results of "productivity" for these new objects.

4 – An integral representation formula

The aim of this section is to give an integral representation formula for $L_2^+(\psi_1,[a,b],p)$; for this an appropriate definition of derivative occurs. Consider, as usual, (Ω,p,Θ) a probability space and let $\psi : [a,b] \to \Theta$ be a set valued curve for which conditions (2.1), (2.2) hold. For $t \in [a,b[$ assume that the limit:

$$
(4.1) \qquad \lim_{h\to 0^+} \frac{1 - p(\psi(t+h)|\psi(t))}{h} \equiv \dot{\psi}_p^+(t) \in R
$$

exists.

We remark that:

$$
\dot{\psi}_p^+(t) = \lim_{h\to 0^+} \frac{p(\psi(t) \setminus \psi(t+h))}{hp(\psi(t))} = \lim \frac{1 - \dfrac{p(\psi(t+h)\cap\psi(t))}{p(\psi(t))}}{h};
$$

then, when $\dot{\psi}_p^+(t)$ exists finite:

$$
\lim_{h\to 0^+} [p(\psi(t) - \psi(t+h))] = 0, \quad \lim_{h\to 0^+} [p(\psi(t+h)\cap\psi(t)] = p(\psi(t)).
$$

Moreover if ψ is non decreasing (with respect the inclusion), we have: $\dot{\psi}_p^+(t) = 0$.

Assuming some condition on $\dot{\psi}_p^+(t)$ and ψ, we can give an integral formula for $L_2^+(\psi,[a,b],p)$ as in the following theorem which can be considered the analogous of the integral representation for the length of C^1-path in R^n (see concluding remarks).

THEOREM 4. *Let $\psi : [a,b] \longrightarrow \Theta$ be a set-valued curve such that condition* (2.1) *and* (2.2) *is satisfied. Assume also that*:

$$\forall t \in [a,b[\quad \exists \dot{\psi}_p^+(t) \in R; \tag{4.2}$$

$$t \in [a,b[\to \dot{\psi}_p^+(t) \text{ is a continuous function}; \tag{4.3}$$

$$\lim_{h\to 0^+} [p(\psi(t+h)|\psi(t))] = 1, \text{ uniformly in } [a,b[. \tag{4.4}$$

Then we have:

$$\forall c \in [a,b[\quad : L_2^+(\psi,[a,c],p) = e^{-\int_a^c \dot{\psi}_p^+(t)dt}. \tag{4.5}$$

PROOF. We remark that, by definition:

$$L_2^+(\psi,[a,c],p) = \sup\left\{\Pi_0^{m-1}(\inf\left\{\Pi_0^{n-1}\psi((\eta_{j+1})|\psi(\eta_j))|\sigma\right\})|\Gamma\right\};$$

hence, due to the continuity of *log* function:

$$\begin{aligned}\log[L_2^+(\psi,[a,c],p)] &= \sup\left\{\sum_0^{m-1}\log(\inf\left\{\Pi_0^{n-1}\psi((\eta_{j+1})|\psi(\eta_j))|\sigma\right\})|\Gamma\right\} = \\ &= \sup\left\{\sum_0^{m-1}\left(\inf\left\{\sum_0^{n-1}\log(\psi((\eta_{j+1})|\psi(\eta_j)))|\sigma\right\}\right)|\Gamma\right\}.\end{aligned}$$

Now, by the "productivity" of $L_2^+(\psi,[a,c],p)$, we can assume that $T_{i+1}-T_i$ is as small as possible; in fact $L_2^+(\psi,[a,c],p) = \Pi_0^{n-1}L_2^+(\psi,[T_i,T_{i+1}],p)$ and so proceed. If we consider conditions (4.3), (4.4), we have that for every $\epsilon > 0$ there exists $\delta > 0$:

$$0 \le h < \delta \Rightarrow \forall t \in [a,c] \quad 1 - p(\psi(t+h)|\psi(t)) < \epsilon, \quad |\dot{\psi}_p^+(t+h) - \dot{\psi}_p^+(t)| < \epsilon.$$

Consider a partition of the following type:

$$\begin{aligned}T_0 = a = \eta_0 < \eta_{0,1} < \dots \eta_{0,n_0} = T_1 = \eta_{1,0} < \eta_{1,1} < \dots \eta_{1,n_1} = T_2 < \dots \\ < \dots T_{m-1} = \eta_{m-1,0} < \dots < \eta_{m-1,n_{m-1}} = T_m = c.\end{aligned}$$

and remark (in the sequel we consider simplified indices) that:

$$\begin{aligned}&\log[p(\psi(\eta_{j+1}|\psi(\eta_j))] = \\ &= \frac{\log(1-(1-p(\psi(\eta_{j+1})|\psi(\eta_j))))}{(1-p(\psi(\eta_{j+1})|\psi(\eta_j)))}\frac{(1-p(\psi(\eta_{j+1})|\psi(\eta_j)))}{\eta_{j+1}-\eta_j}[\eta_{j+1}-\eta_j].\end{aligned}$$

Then, if the partition (T_r) is well chosen, we can assume:

$$-1-\epsilon < \frac{\log(1-(1-p(\psi(\eta_{j+1})|\psi(\eta_j))))}{(1-p(\psi(\eta_{j+1})|\psi(\eta_j)))} < -1+\epsilon;$$
$$\dot{\psi}_p^+(\eta_j)-\epsilon < \frac{(1-p(\psi(\eta_{j+1})|\psi(\eta_j)))}{\eta_{j+1}-\eta_j} < \dot{\psi}_p^+(\eta_j)+\epsilon.$$

Hence:

$$(-1-\epsilon)(\dot{\psi}_p^+(\eta_j)+\epsilon)[\eta_{j+1}-\eta_j] \leq \log[p(\psi(\eta_{j+1}|\psi(\eta_j))] \leq$$
$$\leq (-1+\epsilon)(\dot{\psi}_p^+(\eta_j)-\epsilon)[\eta_{j+1}-\eta_j].$$

Moreover:

$$(-1-\epsilon)\left[\int_{\eta_j}^{\eta_{j+1}} \dot{\psi}_p^+(\eta)d\eta + 2\epsilon(\eta_{j+1}-\eta_j)\right] \leq \log[p(\psi(\eta_{j+1}|\psi(\eta_j))] \leq$$
$$\leq (-1+\epsilon)\left[\int_{\eta_j}^{\eta_{j+1}} \dot{\psi}_p^+(\eta)d\eta - 2\epsilon(\eta_{j+1}-\eta_j)\right].$$

Then we have:

$$(-1-\epsilon)\left[\int_{T_i}^{T_{i+1}} \dot{\psi}_p^+(\eta)d\eta + 2\epsilon(T_{i+1}-T_i)\right] \leq \sum \log[p(\psi(\eta_{j+1}|\psi(\eta_j))] \leq$$
$$\leq (-1+\epsilon)\left[\int_{T_i}^{T_{i+1}} \dot{\psi}_p^+(\eta)d\eta - 2\epsilon(T_{i+1}-T_i)\right].$$

Finally, by considering a sequence of partition $(T_n), (\eta_n)$ such that previous consideration hold and such that:

$$\log[L_2^+(\psi,[a,c],p)] = \lim_n \sum_{T_n}\sum_{\eta_n} \log p(\psi(\eta_{j+1})|\psi(\eta_j)),$$

we conclude that:

$$\log[L_2^+(\psi,[a,c],p)] \leq (-1+\epsilon)\left[\int_a^c \dot{\psi}_p^+(\eta)d\eta - 2\epsilon(c-a)\right].$$

We have, by the arbitrarity of ϵ, the inequality:

$$L_2^+(\psi,[a,c],p) \leq e^{-\int_a^c \dot{\psi}_p^+(\eta)d\eta}.$$

For the reverse inequality we proceed as follows. Consider a sequence of partitions T^n such that $T^n_{i+1} - T^n_i$ is as small as possible for $i = 0, 1, \dots, m_n$ and for which, with the obvious meaning of the symbols:

$$\log[L_2^+(\psi, [a,c], p)] = \lim_n \sum_0^{m_n - 1} \inf\left\{\sum \log p(\psi(\eta_{j+1})|\psi(\eta_j))|\sigma\right\}.$$

For a fixed $\epsilon > 0$ we can consider $\nu > 0$ such that, for $n > \nu$:

$$\begin{aligned}&\sum \inf\left\{\sum \log p(\psi(\eta_{j+1})|\psi(\eta_j))|\sigma\right\} - \epsilon < \\ &< \log[L_2^+(\psi, [a,c], p)] < \sum \inf\left\{\sum \log p(\psi(\eta_{j+1})|\psi(\eta_j))|\sigma\right\} + \epsilon.\end{aligned}$$

Now, we consider, for every $[T^n_i, T^n_{i+1}]$ a partition σ^n such that:

$$\begin{aligned}&\inf\left\{\sum \log p(\psi(\eta_{j+1})|\psi(\eta_j))|\sigma\right\} \leq \\ &\leq \sum \log p(\psi(\eta^n_{j+1})|\psi(\eta^n_j)) \leq \inf\left\{\sum \log p(\psi(\eta_{j+1})|\psi(\eta_j))|\sigma\right\} + \frac{\epsilon}{m_n}.\end{aligned}$$

Hence, from the previous consideration:

$$\log[p(\psi(\eta_{j+1})|\psi(\eta_j))] \geq (-1-\epsilon)(\dot{\psi}_p^+(\eta_j) + \epsilon)[\eta_{j+1} - \eta_j],$$

we have:

$$\begin{aligned}&\inf\left\{\sum \log p(\psi(\eta_{j+1})|\psi(\eta_j))|\sigma\right\} \geq \sum \log p(\psi(\eta_{j+1})|\psi(\eta_j)) - \frac{\epsilon}{m_n} \geq \\ &\geq \sum [(-1-\epsilon)(\dot{\psi}_p^+(\eta^n_j) + \epsilon)(\eta_{j+1} - \eta_j)] - \frac{\epsilon}{m_n} = \\ &= (-1-\epsilon)[\int_{T^n_i}^{T^n_{i+1}} \dot{\psi}_p^+(\eta^n_j) d\eta + \epsilon(T^n_{i+1} - T^n_i)] - \frac{\epsilon}{m_n}.\end{aligned}$$

Hence:

$$\begin{aligned}&\sum_0^{m_n - 1} \inf\left\{\sum \log p(\psi(\eta_{j+1})|\psi(\eta_j))|\sigma\right\} \geq \\ &\geq \sum_0^{m_n - 1} \left[(-1-\epsilon)\left(\int_{T^n_i}^{T^n_{i+1}} \dot{\psi}_p^+(\eta) d\eta + 2\epsilon(T^n_{i+1} - T^n_i)\right) - \frac{\epsilon}{m_n}\right] = \\ &= (-1-\epsilon)\left[\int_a^c \dot{\psi}_p^+(\eta) d\eta + 2\epsilon(c-a)\right] - \epsilon.\end{aligned}$$

Finally, passing to limit for $\epsilon \to 0$:

$$\log[L_2^+(\psi, [a,c], p)] \geq (-1-\epsilon)\left[\int_a^c \dot{\psi}_p^+(\eta)d\eta + 2\epsilon(c-a)\right] - \epsilon;$$

then we have, by the arbitrarity of ϵ:

$$\log[L_2^+(\psi, [a,c], p)] \geq e^{-\int_a^c \dot{\psi}_p^+(\eta)d\eta}.$$

Hence the thesis is achieved.

In the situation of previuous theorem, we can define in a natural way:

$$L_2^+(\psi, [a,b], p) = \lim_{c\to b} e^{-\int_a^c \dot{\psi}_p^+(\eta)d\eta}.$$

Remark. We now consider some particular situations where an explicit computation of $\dot{\psi}_p^+(.)$ is given. In the following examples, we consider subsets of R or R^2, p as the Lebesgue measure normalized with respect to a fixed measurable bounded subset Z containing the values of the considered set-valued function and Θ is the family of the Lebesgue measurable set in Z.

1. Let $\psi : [0, \frac{1}{2}] \to \Theta$ be given by: $\psi(t) = [t, 1-t]$. In such a situation we have:

   $$p(\psi(t)) = 1 - 2t; \quad \dot{\psi}_p^+(t) = \frac{2}{1-2t}.$$

   We remark that the usual derivative of $p(\psi(t))$ is -2, but:

   $$\dot{\psi}_p^+(t) = -\frac{d}{dt}\log p(\psi(t)).$$

2. Let $\psi(t) = B(0, r-t)$, $t \in [0,r]$ be the ball centered at the origin of R^2 with radius $r-t$. Hence:

   $$p(\psi(t)) = \pi(r-t)^2; \quad \dot{\psi}_p^+(t) = \frac{2}{r-t}.$$

   We have: $\quad \dot{\psi}_p^+(t) = -\frac{d}{dt}\log p(\psi(t))$.
3. Let $\psi(t) = [t, 1+t] \times [0,1]$, we have:

   $$p(\psi(t)) = 1; \quad \dot{\psi}_p^+(t) = 1.$$

   In this situation it turns out that: $1 = \dot{\psi}_p^+(t) \neq -\frac{d}{dt}\log p(\psi(t)) = 0$.
4. Let $\psi(t) = [0, 1+t] \times [0, 1-t]$, we have:

   $$p(\psi(t)) = (1-t^2); \quad \dot{\psi}_p^+(t) = \frac{1}{1-t}.$$

   In this situation we still have: $\quad \dot{\psi}_p^+(t) \neq -\frac{d}{dt}\log p(\psi(t))$.

5. Let $\psi(t) = [0, 1+t] \times [0, 1-t^2]$, then:

$$p(\psi(t)) = (1+t)(1-t^2); \quad \dot{\psi}_p^+(t) = \frac{2t}{1-t^2}.$$

Moreover: $\dot{\psi}_p^+(t) \neq -\frac{d}{dt} \log p(\psi(t)) = \frac{3t-1}{1-t^2}$.

6. Let $\psi(t) = [0, 1+t^p] \times [0, 1-t^q], \quad pq > 1$, then:

$$p(\psi(t)) = (1+t^p)(1-t^q); \quad \dot{\psi}_p^+(t) = q\frac{t^{q-1}}{1-t^q}.$$

In such situation there still holds: $\dot{\psi}_p^+(t) \neq -\frac{d}{dt} \log p(\psi(t)) = -p\frac{t^{p-1}}{1+t^p} + q\frac{t^{q-1}}{1-t^q}$.

5 – Concluding remarks

Many questions are still open; we consider some of them which seem to be interesting:

- Consider algebraic properties of $\dot{\psi}_p^+$.
- Obtain the integral representation formula given in Theorem 4 with weaker hypotheses than (4.3) and/or (4.4).
- Obtain an integral representation formula for $L_1^+(\psi, [a,b], p)$.
- Having in mind the introduction, we wish to obtain results on the upper/lower semicontinuity of $L_1^{+/-}(\psi, [a,b], p)$ and/or $L_2^{+/-}(\psi, [a,b], p)$ with respect to some appropriate convergence of the sequence $(p(\psi_n(.)))$. Then we can consider some natural questions of the calculus of the variation as the existence of a curve connecting A and B and with maximal cost (if at least a curve exists connecting A to B and verifying conditions (2.1), (2.2)).

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