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# On some potential applications of the heat equation with a repulsive point interaction to derivative pricing

*Dedichiamo questo articolo al Prof. Sergio Albeverio (Università di Bonn, USI, CERFIM, Università di Trento), senza dubbio uno dei "maestri" della Fisica Matematica e della Matematica Applicata contemporanee.*

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Abstract: *In this note we first investigate in detail the "heat equation" with the free Laplacian replaced by the one with a repulsive point interaction centred at the origin in the case where the initial condition is given by any function proportional to* $e^{x/2}\chi_{(-\infty,0]}(x)$*. The solution is expressed in terms of the cumulative function of the normal distribution in view of its direct application to derivative pricing. In the second part of the paper, with reference to the quantum mechanical approach to option pricing proposed in the last decade, we use the results in order to solve explicitly the Black-Scholes equation with a perturbing term given by a point interaction of the type* $\lambda \cdot \delta(\ln(\frac{s}{E}))$*, s being the price of the underlying asset and E the exercise price of the option.*

## 1 – Introduction

In this brief note we first investigate the following "heat equation" related to the free Hamiltonian perturbed by a repulsive point interaction centred at the

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origin:

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}\psi = -\dfrac{\sigma^2}{2}\left[-\dfrac{\partial^2}{\partial x^2} + \lambda \cdot \delta(x)\right]\psi \\ \psi(x,0) = Be^{x/2}\chi_{(-\infty,0]}(x) \end{cases}$$

with $B, \lambda, \sigma > 0, x \in \mathbf{R}$.

By citing Wilmott and collaborators (see [1]), we briefly wish to recall that *the heat equation models the diffusion of heat in one space dimension where* $u(x,\tau)$ *represents the temperature in a long, thin, uniform bar of material whose sides are perfectly insulated so that its temperature varies only with distance* $x$ *along the bar and, of course, with time* $\tau$.

In our case the initial distribution is exponential on the left half of the bar and null on the right half with a peak at the origin where a "permeable" barrier of the $\delta$-type is situated. The solution of (1.1) shows how heat flows over time penetrating the barrier represented by the point interaction. We will also consider the limiting case where the soft barrier becomes impermeable and no heat flows through it.

Motivated both by our previous work ([2], [3]) and by the relatively recent results published in [4] regarding the use of point interactions in derivative pricing as well as by those in [5] (in which a"quantum mechanical" approach to derivative pricing has been proposed), we have then decided to investigate the renowned Black-Scholes equation, found in any introductory textbook in Financial Mathematics, perturbed by a point interaction of the type $\lambda \cdot \delta(\ln(\frac{s}{E}))$ ($s$ being the price of the underlying asset and $E$ the exercise price of the option), taking advantage of the results of the first part of this paper. As is shown in detail in the second part of the paper, such a partial differential equation does lead to an *exactly solvable model* in Quantum Finance, that is to say the price of various types of options based on this perturbed PDE can be written explicitly as a function of the price of the underlying asset. In fact, it will be shown that for $\lambda = 0$ the functions reduce to the classical B-S ones, whereas for $\lambda = +\infty$ one gets those of knockout options.

It might be worth stressing here that the connection between our approach, mainly based on semigroup theory, and the probabilistic one more widely used in the theory of stochastic processes and option pricing is given by the well-known Feynman-Kac formula (see [5], [6], [7]).

Let us now briefly describe the main points of this note.

By using the semigroup (known also as propagator in the physics literature) generated by the Hamiltonian in (1.1), defined in a rigorous way from a functionally analytic point of view, it will be shown that the solution can be explicitly calculated in terms of rather simple functions, the cumulative function of the normal distribution being the least trivial among them. This is obviously related to the anticipated applications to derivative pricing models. Furthermore, the special cases in which the strength of the point interaction is equal to one and to infinity will be singled out, as for those values the general formula for the solution exhibits an indetermination.

In the second part the following perturbed B-S equation will be considered:

$$(1.2) \qquad \begin{cases} \dfrac{\partial}{\partial t}P = -\dfrac{\sigma^2}{2}\left[-s^2\dfrac{\partial^2}{\partial s^2} - 2s\dfrac{\partial}{\partial s} + \alpha + \lambda \cdot \delta\left(\ln\left(\dfrac{s}{E}\right)\right)\right] P \\ \\ P(s,0) = B\chi_{[0,E]}(s) \end{cases}$$

where the time variable represents the so-called "time to expiry" and $\alpha = \frac{2r}{\sigma^2}$, $r$ being the interest rate. As is well known to option pricing specialists, the initial datum in (1.2) is the payoff of a digital put option (also called cash-or-nothing put) paying $B$ if $s_T < E$, 0 otherwise. This choice is less restrictive than it may seem since:

$$[E-s]^+ = E\chi_{[0,E]}(s) - s\chi_{[0,E]}(s),$$

which shows that the payoff of a vanilla put can be written as the difference between a cash-or-nothing put with $B = E$ and an asset-or-nothing put. Furthermore, although the value of the coefficient of the drift term in (1.2) has been chosen in such a way to get the self-adjoint operator on the r.h.s. of (1.1), our results can be extended without any major difficulty to the case of any value of the drift coefficient.

## 2 – The heat equation in the presence of a one-dimensional repulsive point interaction with a particular type of discontinuous initial condition

The "heat equation" is certainly one of the most renowned PDE's arising in Mathematical Physics. It has also been used in Financial Mathematics, undoubtedly one of the areas of Mathematics which has exhibited a very impressive growth over the last thirty years. In fact, the most celebrated PDE of option pricing, namely the Black-Scholes equation can be shown to be "almost equivalent" to the usual heat equation (see [1], [2], [3] and [5]).

As the initial function $\psi(\cdot,0) \in L^p(\mathbf{R}), p \geq 1$ , the problem could be formulated in any function space of that type.

The cases $L^1(\mathbf{R}), L^2(\mathbf{R})$ and $L^\infty(\mathbf{R})$ are clearly the most significant ones. However, throughout the following we shall assume that our space of functions is $L^2(\mathbf{R})$, since it is precisely in such a space that a great deal of work has been carried out to show that the operator on the r.h.s of the PDE in (1.1) can be rigorously defined as an unbounded self-adjoint operator. With regard to the various ways in which such a rigorous definition can be achieved, the reader is obviously referred to the treatise by Albeverio and collaborators [8]. Here we only wish to recall that the Hamiltonian formally expressed by $\left[-\frac{d^2}{dx^2} + \lambda \cdot \delta(x)\right]$ represents the self-adjoint extension of the Laplacian operator with dense domain given by $\psi \in H^{2,1}(\mathbf{R}) \cap H^{2,2}(\mathbf{R} \setminus 0)$, (Sobolev spaces on $L^2(\mathbf{R})$ such that $\psi'(0_+) - \psi'(0_-) = \lambda\psi(0)$. As we are going to consider only $\lambda > 0$, it is also worth mentioning that the operator is positive and can thus be regarded as the infinitesimal generator of a contraction semigroup.

As a consequence, in order to solve (1.1) we can take advantage of the propagator of the operator $\frac{\sigma^2}{2}\left[-\frac{d^2}{dx^2} + \lambda \cdot \delta(x)\right]$, namely:

$$B \int_{-\infty}^{+\infty} e^{-\frac{\sigma^2}{2}[-\frac{\partial^2}{\partial x^2}+\lambda\cdot\delta(x)]t}(x,y)e^{y/2}\chi_{(-\infty,0]}(y)dy. \tag{2.1}$$

The integral kernel in (2.1) is one of the few non-trivial propagators which are explicitly known in the mathematical literature (see [8], [9], [10] and [11]). Although Gaveau and Schulman in [9] have only provided an integral formula for the interaction term of the kernel ( present only if $\lambda \neq 0$), the latter can easily be integrated to yield a simple expression in terms of the well-known cumulative function of the normal distribution $N(x) = \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}, dy$:

$$\begin{aligned} e^{-\frac{\sigma^2}{2}[-\frac{\partial^2}{\partial x^2}+\lambda\cdot\delta(x)]t}(x,y) &= \frac{e^{-\frac{(x-y)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}} - \frac{\lambda}{2}e^{\left[\frac{\lambda}{2}(|x|+|y|)+\frac{\lambda^2\sigma^2 t}{8}\right]}\cdot \\ &\cdot N\left(-\frac{(|x|+|y|)+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right). \end{aligned} \tag{2.2}$$

As will be shown later in the appendix, the integral operator whose kernel is given by the interaction term of the propagator, that is to say the second term in the above formula, is actually trace class.

Taking account of (2.2), (2.1) becomes:

$$\begin{aligned} &B\int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}} e^{\frac{y}{2}}\chi_{(-\infty,0]}(y)dy - \frac{B\lambda}{2}e^{\left[\frac{\lambda}{2}|x|+\frac{\lambda^2\sigma^2 t}{8}\right]}\int_{-\infty}^{+\infty} e^{\frac{\lambda}{2}|y|}\cdot \\ &\cdot N\left(-\frac{(|x|+|y|)+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) e^{\frac{y}{2}}\chi_{(-\infty,0]}(y)dy = \\ &= B\int_{-\infty}^{0} \frac{e^{-\frac{(x-y)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}} \exp(y/2)dy + \\ &- \frac{B\lambda}{2}e^{\left[\frac{\lambda}{2}|x|+\frac{\lambda^2\sigma^2 t}{8}\right]}\int_{-\infty}^{0} e^{\frac{(1-\lambda)}{2}y} N\left(-\frac{(|x|+|y|)+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)dy. \end{aligned} \tag{2.3}$$

Let us carry out the calculation of the two terms in (2.3) separately in order to avoid dealing with extremely lengthy expressions.

The first term is essentially the contribution resulting from the action of the "free" heat propagator on the initial function and has already been calculated in [2] and [3] under more general assumptions on the values of the parameters yielding:

$$B\frac{e^{\frac{\sigma^2 t}{8}}}{\sqrt{2\pi}}e^{\frac{x}{2}}\int_{-\infty}^{-\frac{x+\frac{\sigma^2}{2}t}{\sigma\sqrt{t}}} e^{-\frac{y^2}{2}}dy = Be^{\frac{\sigma^2 t}{8}}e^{\frac{x}{2}}N\left(-\frac{x+\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right). \tag{2.4}$$

Let us now focus on the integral inside the second term in (2.3), that is to say the interaction term. By using integration by parts we get:

$$\begin{aligned}
&\int_{-\infty}^{0} e^{\frac{(1-\lambda)}{2}y}N\left(-\frac{(|x|+|y|)+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)dy = \\
&= \frac{2}{1-\lambda}\left[e^{\frac{(1-\lambda)}{2}y}N\left(-\frac{(|x|+|y|)+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)\right]_{-\infty}^{0} + \\
&\quad -\frac{2}{1-\lambda}\int_{-\infty}^{0} e^{\frac{(1-\lambda)}{2}y}\frac{d}{dy}N\left(-\frac{(|x|+|y|)+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)dy = \\
&= \frac{2}{1-\lambda}N\left(-\frac{|x|+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) - \frac{2}{1-\lambda}\int_{-\infty}^{0} e^{\frac{(1-\lambda)}{2}y}\frac{e^{-\frac{(y-|x|-\lambda\frac{\sigma^2 t}{2})^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}}dy.
\end{aligned} \tag{2.5}$$

In order to calculate the latter integral it is convenient to set $\zeta = y - |x| - \lambda\frac{\sigma^2 t}{2}$. Then (2.5) can be rewritten as:

$$\begin{aligned}
&\frac{2}{1-\lambda}N\left(-\frac{|x|+\lambda\dfrac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) + \\
&\quad -\frac{2}{1-\lambda}e^{-\frac{\lambda(\lambda-1)\sigma^2 t}{4}}e^{\frac{(1-\lambda)}{2}|x|}\int_{-\infty}^{-\left[|x|+\lambda\frac{\sigma^2 t}{2}\right]} e^{\frac{(1-\lambda)}{2}\zeta}\frac{e^{-\frac{\zeta^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}}d\zeta.
\end{aligned} \tag{2.6}$$

By completing the square in the exponent of the exponential function inside the integral the latter becomes:

$$\begin{aligned}
&\frac{2}{1-\lambda}\left[N\left(-\frac{|x|+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) - e^{-\frac{(\lambda^2-1)\sigma^2 t}{8}}e^{\frac{(1-\lambda)}{2}|x|}\int_{-\infty}^{-\left[|x|+\frac{\sigma^2 t}{2}\right]}\frac{e^{-\frac{\zeta^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}}d\zeta\right] = \\
&= \frac{2}{1-\lambda}\left[N\left(-\frac{|x|+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) - e^{-\frac{(\lambda^2-1)\sigma^2 t}{8}}e^{\frac{(1-\lambda)}{2}|x|}N\left(-\frac{|x|+\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)\right].
\end{aligned} \tag{2.7}$$

By substituting (2.7) in place of the integral in the interaction term in (2.3) and taking account of (2.4) we obtain that, for $\lambda \neq 1$, (2.3) is equal to:

$$\begin{aligned}&Be^{\frac{\sigma^2 t}{8}}e^{\frac{x}{2}}N\left(-\frac{x+\dfrac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)+\frac{B\lambda}{\lambda-1}e^{\left[\frac{\lambda}{2}|x|+\frac{\lambda^2\sigma^2 t}{8}\right]}\times\\ &\times\left[N\left(-\frac{|x|+\lambda\dfrac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)-e^{\frac{(1-\lambda)}{2}|x|-\frac{(\lambda^2-1)\sigma^2 t}{8}}N\left(-\frac{|x|+\dfrac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)\right].\end{aligned} \tag{2.8}$$

Therefore, for $\lambda \neq 1$, the solution of (1.1) is given by:

$$\begin{aligned}\psi_\lambda(x,t)=&Be^{\frac{\sigma^2 t}{8}}e^{\frac{x}{2}}N\left(-\frac{x+\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)+\frac{B\lambda}{\lambda-1}\times\\ &\times\left[e^{\left[\frac{\lambda}{2}|x|+\frac{\lambda^2\sigma^2 t}{8}\right]}N\left(-\frac{|x|+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)-e^{\left[\frac{1}{2}|x|+\frac{\sigma^2 t}{8}\right]}N\left(-\frac{|x|+\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)\right].\end{aligned} \tag{2.9}$$

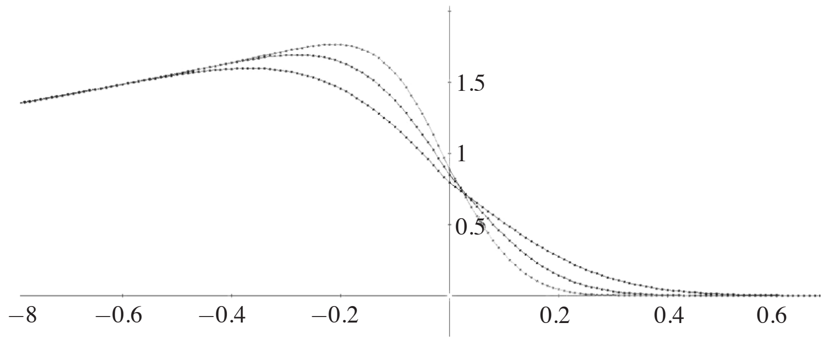


Fig. 1: The graph of the solution $\psi_\lambda(x,t)$ with $\lambda = 2, \sigma = 0.2$ at three different times $t = 1.0, 0.5, 0.25$.

The behaviour of the solution with respect to the diffusion coefficient, that is to say the volatility in option pricing, is illustrated by the next graph.

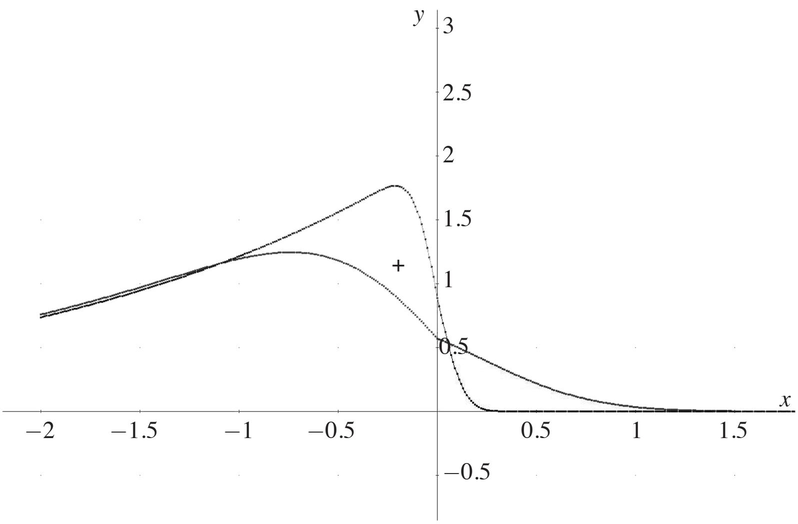


Fig. 2: The graph of the solution $\psi_\lambda(x,t)$ with $\lambda = 2$, $\sigma = 0.2$, $1.0$, $t = 0.25$.

By inspecting (2.9) it is clear that the case $\lambda = 1$ is to be handled with special care given the presence of the vanishing denominator. The analysis of that case as well as that of the limit of $\psi_\lambda(x,t)$ as $\lambda \to +\infty$ will be the main focus of the next section.

## 3 – The special cases $\lambda = 1$ and $\lambda \to +\infty$

Let us first consider the case $\lambda = 1$. It is immediate to notice that the quantity inside the square brackets in (2.9) vanishes for that particular value of the strength of the point interaction so that the interaction term exhibits an indetermination of the type $\frac{0}{0}$.

By using L'Hopital's theorem it is straightforward to show that the limit for $\lambda \to 1$ exists. Before plotting the graph of the solution for this special case, we write its final formula, namely:

$$\begin{aligned}\psi_1(x,t) = {} & Be^{\frac{\sigma^2 t}{8}} e^{\frac{x}{2}} N\left(-\frac{x+\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) + B\frac{e^{\frac{|x|}{2}+\frac{\sigma^2 t}{8}}}{2}\times \\ & \times\left[\left(|x|+\frac{\sigma^2 t}{2}\right) N\left(-\frac{|x|+\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) - \sigma\sqrt{\frac{t}{2\pi}} e^{-\frac{1}{2\sigma^2 t}\left[|x|+\frac{\sigma^2 t}{2}\right]^2}\right].\end{aligned} \tag{3.1}$$

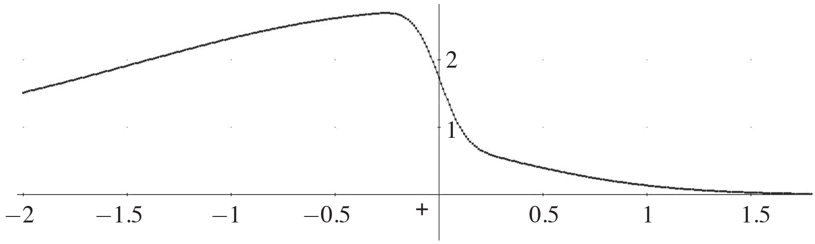


Fig. 3: The graph of the solution $\psi_1(x,t)$ with $t = 0.25$, $\sigma = 0.2$.

The behaviour of the solution with respect to the diffusion coefficient, that is to say the volatility in option pricing, is illustrated by the following graph.

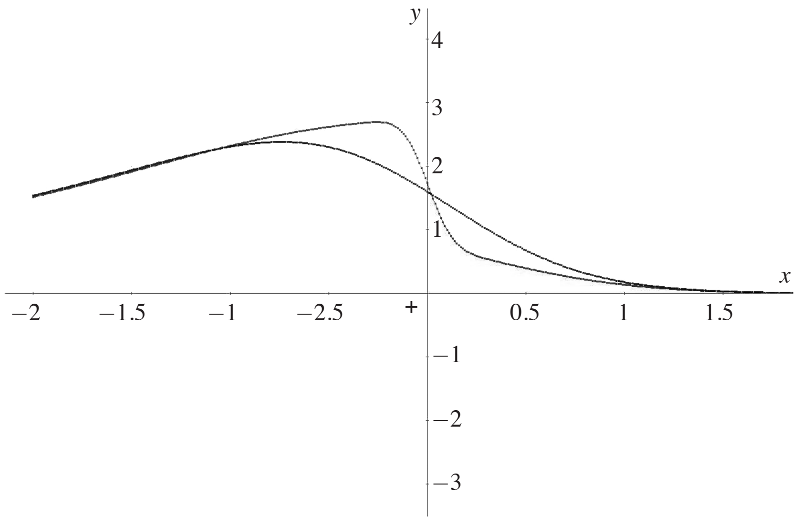


Fig. 4: The graph of the solution $\psi_1(x,t)$ with $\sigma = 0.2$, $1.0$, $t = 0.25$.

In order to deal with the limiting case $\lambda \to +\infty$ we must first recall a well-known fact regarding point interactions, that is to say

$$-\frac{d^2}{dx^2} + \lambda \cdot \delta(x) \to \left[-\frac{d^2}{dx^2}\right]_D \tag{3.2}$$

as $\lambda \to +\infty$, in the sense of resolvent convergence where the limiting operator is the Laplacian with a Dirichlet boundary condition at the origin (see [8]). By going back to the explicit formula for the propagator (2.2) it is not difficult to realise that in this case we have an indetermination of the type $\infty \cdot 0$ in the interaction term of the integral kernel. We omit the rather straightforward calculation of the limit and write only the final result:

$$e^{-\frac{\sigma^2}{2}\left[-\frac{d^2}{dx^2}\right]_D t}(x,y) = \frac{e^{-\frac{(x-y)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}} - \frac{e^{-\frac{(|x|-y)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}}. \tag{3.3}$$

Given the fact that the kernel vanishes $\forall x \geq 0$, the calculation of the integral (2.1) is quite simple in this case and the final result is:

$$\psi_\infty(x,t) = \begin{cases} Be^{\frac{\sigma^2 t}{8}}\left[e^{\frac{x}{2}} N\left(-\frac{x+\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) - e^{-\frac{x}{2}} N\left(\frac{x-\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)\right] & x < 0 \\ 0 & x \geq 0. \end{cases} \tag{3.4}$$

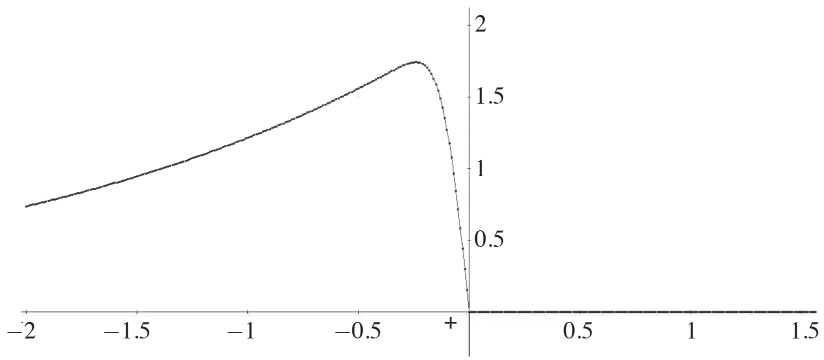


Fig. 5: The graph of the solution $\psi_\infty(x,t)$ with $t = 0.25$, $\sigma = 0.2$.

## 4 – Possible applications to Quantum Finance

In the aforementioned paper [5], what could be defined as a quantum mechanical approach to option pricing was introduced (see the less technical [13] as well as the rather recent [14]). The main idea therein is to perturb the "free" Hamiltonian, that is to say the Black-Scholes operator, by means of a "potential" function related to the payoff of a so-called barrier option, in a way that will be specified a bit later, in order to fully describe the dynamics of such a contract before expiry.

As a consequence, the entire quantum mechanical apparatus has been borrowed to build up such models. Whilst such an approach looks certainly appealing, it may be a bit risky if some functional analytic subtleties implied by the use of unbounded operators, in particular differential ones, are not handled carefully. As is well known to mathematical physicists, the use of the term Hermitian (or self-adjoint) referred to differential operators goes well beyond the merely algebraic Hermiticity since it involves rather delicate domain issues. Furthermore, it seems to us that little attention in a large portion of the existing literature on the B-S equation has been paid to the choice of the various function (or distribution) spaces in which the equation is to be solved.

Before showing how the results of the previous sections can be used to solve explicitly a particular perturbed B-S equation, some remarks are to be made with regard to the way the operator $s\frac{d}{ds}$ (respectively $s^2\frac{d^2}{ds^2}$ ) appearing in the "free" Hamiltonian is transformed into $\frac{d}{dx}$ (respectively $\frac{d^2}{dx^2}$ ) when the variable representing the underlying asset price is replaced by $x = \ln(s)$. Following [15], the transformation depends crucially on the space of functions being considered. Although it is seldom explicitly stated, the conventional choice in the case of digital or vanilla put options is $L^\infty[0, \infty)$, which leads to $L^\infty(\mathbf{R})$. According to this choice, $s\frac{d}{ds}$ trivially becomes $\frac{d}{dx}$, which then leads to the usual B-S operator:

$$H^{L^\infty}_{B-S} = \frac{\sigma^2}{2}\left[-\frac{d^2}{dx^2} + (1-\beta)\frac{d}{dx} + \alpha\right] \tag{4.1}$$

with $\alpha = \frac{2r}{\sigma^2}$ and $\beta = \alpha - \frac{2D}{\sigma^2}$, in the case of an asset paying the continuous dividend D. In accordance with both [15] and our previous work on the B-S operator [2], [3], by adopting the more "quantum mechanical" choice of the space of square summable functions (the only $L^p$ space which can be identified with its dual), the free B-S Hamiltonian gets transformed instead into:

$$H^{L^2}_{B-S} = \frac{\sigma^2}{2}\left[-\frac{d^2}{dx^2} + (2-\beta)\frac{d}{dx} + \frac{3}{2}\left(\alpha - \frac{1}{2}\right)\right] \tag{4.2}$$

as a consequence of the unitary transformation from $L^2[0, \infty)$ to $L^2(\mathbf{R})$ defined by $(Uf)(x) = e^{\frac{x}{2}} f(e^x)$ for any $f \in L^2[0, \infty)$.

Therefore, let us consider the PDE (1.2), in which the so-called "time to expiry" $t = T - \tau$ is used:

$$\begin{cases} \dfrac{\partial}{\partial t}P = -\dfrac{\sigma^2}{2}\left[-s^2\dfrac{\partial^2}{\partial s^2} - 2s\dfrac{\partial}{\partial s} + \alpha + \lambda \cdot \delta\left(\ln\left(\dfrac{s}{E}\right)\right)\right] P \\ P(s, 0) = B\chi_{[0,E]}(s). \end{cases}$$

According to [5], (1.2) is equivalent to pricing a European digital put option with $\beta = 2(r = D + \sigma^2)$ whose final payoff is given by:

$$Be^{-\frac{\sigma^2}{2}\lambda \int \delta(\ln(s(t)/E))dt}\chi_{[0,E]}(s_T) \tag{4.3}$$

where $s_T$ denotes the value of the underlying asset at expiry and the exponent obviously involves a path integral. As a consequence of the homogeneity of (1.2), by setting $\xi = \frac{s}{E}$ and $p(\xi, t) = P\left(\frac{s}{E}, t\right)$, the PDE can be rewritten as:

$$\begin{cases} \dfrac{\partial}{\partial t} p = -\dfrac{\sigma^2}{2}\left[-\xi^2 \dfrac{\partial^2}{\partial \xi^2} - 2\xi \dfrac{\partial}{\partial \xi} + \alpha + \lambda \cdot \delta(\ln(\xi))\right] p \\ p(\xi, 0) = B\chi_{[0,1]}(\xi). \end{cases} \tag{4.4}$$

Due to (4.2), for this particular value of $\beta$ the drift term vanishes and the PDE becomes:

$$\begin{cases} \dfrac{\partial}{\partial t} \psi = -\dfrac{\sigma^2}{2}\left[-\dfrac{\partial^2}{\partial x^2} + \lambda \cdot \delta(x) + \dfrac{3}{2}\left(\alpha - \dfrac{1}{2}\right)\right] \psi \\ \psi(x, 0) = Be^{x/2}\chi_{(-\infty,0]}(x) \end{cases} \tag{4.5}$$

having set $\psi(x) = (Up)(x)$ and exploited the fact that the unitary equivalent of any multiplication operator of the type $V(\ln(\cdot))$ on $L^2[0, \infty)$ is given by $V(\cdot)$ on $L^2(\mathbf{R})$. Apart from the harmless constant term inside the square brackets, (4.5) coincides with the equation (1.1) previously investigated. Hence, it is quite straightforward to write its solution taking advantage of (2.9) for any positive $\lambda \neq 1$, (3.1) for $\lambda = 1$ and (3.4) for $\lambda \to +\infty$. For sake of simplicity, let us consider the three cases only for an asset paying no dividend ( $r = \sigma^2$ ):

$$\begin{aligned} \psi_\lambda(x,t) = {} & Be^{-rt} e^{\frac{x}{2}} N\left(-\frac{x + \frac{rt}{2}}{\sqrt{rt}}\right) + \frac{B\lambda}{\lambda - 1} \times \\ & \times \left[ e^{\left[\frac{\lambda}{2}|x| - \frac{(9-\lambda^2)rt}{8}\right]} N\left(-\frac{|x| + \lambda\frac{rt}{2}}{\sqrt{rt}}\right) - e^{\frac{1}{2}|x| - rt} N\left(-\frac{|x| + \frac{rt}{2}}{\sqrt{rt}}\right)\right] \end{aligned} \tag{4.6a}$$

for any positive $\lambda \neq 1$,

$$\begin{aligned} & \psi_1(x,t) = Be^{-rt} \times \\ & \times \left[ e^{\frac{x}{2}} N\left(-\frac{x + \frac{rt}{2}}{\sqrt{rt}}\right) + \frac{e^{\frac{|x|}{2}}}{2}\left(\left(|x| + \frac{rt}{2}\right) N\left(-\frac{|x| + \frac{rt}{2}}{\sqrt{rt}}\right) - \sqrt{\frac{rt}{2\pi}} e^{-\frac{1}{2rt}\left[|x| + \frac{rt}{2}\right]^2}\right)\right] \end{aligned} \tag{4.6b}$$

and

$$\text{(4.6c)}\quad \psi_\infty(x,t)=\begin{cases} Be^{-rt}\left[e^{\frac{x}{2}}N\left(-\dfrac{x+\dfrac{rt}{2}}{\sqrt{rt}}\right)-e^{-\frac{x}{2}}N\left(\dfrac{x-\dfrac{rt}{2}}{\sqrt{rt}}\right)\right] & x<0\\ 0 & x\geq 0.\end{cases}$$

By using the inverse unitary transformation from $L^2(\mathbf{R})$ to $L^2[0,\infty)$, defined by $(U^{-1}f)(\xi)=\frac{1}{\sqrt{\xi}}f(\ln(\xi))$ the three expressions can be easily rewritten in terms of the variable $\xi=\frac{s}{E}$:

(4.7a)

$$p_\lambda(\xi,t)=Be^{-rt}N\left(-\frac{\ln(\xi)+\dfrac{rt}{2}}{\sqrt{rt}}\right)+\frac{B\lambda}{\lambda-1}\times$$
$$\times\left[\frac{e^{\left[\frac{\lambda}{2}|\ln(\xi)|-\frac{(9-\lambda^2)rt}{8}\right]}}{\sqrt{\xi}}N\left(-\frac{|\ln(\xi)|+\lambda\dfrac{rt}{2}}{\sqrt{rt}}\right)-\frac{e^{\frac{1}{2}|\ln(\xi)|-rt}}{\sqrt{\xi}}N\left(-\frac{|\ln(\xi)|+\dfrac{rt}{2}}{\sqrt{rt}}\right)\right]$$

for any positive $\lambda\neq 1$,

$$\text{(4.7b)}\quad \begin{aligned} p_1(\xi,t)&=Be^{-rt}\left[N\left(-\frac{\ln(\xi)+\dfrac{rt}{2}}{\sqrt{rt}}\right)\right]+Be^{-rt}\times\\ &\times\left[\frac{e^{\frac{|\ln(\xi)|}{2}}}{2\sqrt{\xi}}\left(\left(|\ln(\xi)|+\frac{rt}{2}\right)N\left(-\frac{|\ln(\xi)|+\dfrac{rt}{2}}{\sqrt{rt}}\right)-\sqrt{\frac{rt}{2\pi}}e^{-\frac{1}{2rt}\left[|\ln(\xi)|+\frac{rt}{2}\right]^2}\right)\right]\end{aligned}$$

and

$$\text{(4.7c)}\quad p_\infty(\xi,t)=\begin{cases} Be^{-rt}\left[N\left(-\dfrac{\ln(\xi)+\dfrac{rt}{2}}{\sqrt{rt}}\right)-\dfrac{1}{\xi}N\left(\dfrac{\ln(\xi)-\dfrac{rt}{2}}{\sqrt{rt}}\right)\right] & \xi<1\\ 0 & \xi\geq 1.\end{cases}$$

As anticipated in the introduction, it is clear from (4.7a) that for $\lambda=0$ the solution reduces to the usual B-S formula for a digital put.

As can be clearly seen from the plot shown below, the effect of the presence of the repulsive point interaction situated at $\xi=1(s=E)$ , responsible for the cusp at that point, becomes progressively less strong over time as the option approaches its expiry date.

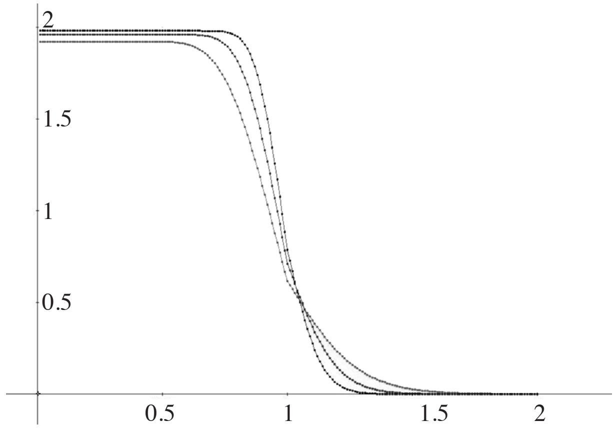


Fig. 6: The solution $p_\lambda(\xi, t)$ for $\lambda = 5$, $B = 2$ and $\sigma^2 = r = 0.04$ at three different times to expiry: $t = 1$, $t = 0.5$ and $t = 0.25$.

In the next two graphs we wish to exhibit graphically the difference between $p_\lambda(\xi, t)$, the solution of the perturbed B-S equation, and $p_0(\xi, t)$, the solution of the unperturbed one, that is to say the classical European digital put, for two different values of the strength of the point interaction with time to expiry equal to 90 days in both cases.

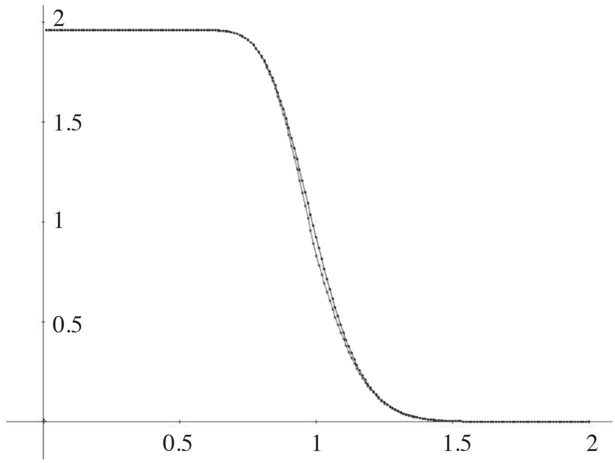


Fig. 7: The solution $p_\lambda(\xi, t)$ for $\lambda = 2$, $B = 2$ and $\sigma^2 = r = 0.04$ at $t = 0.25$ compared with $p_0(\xi, t)$, the European digital put, with the same values of the parameters.

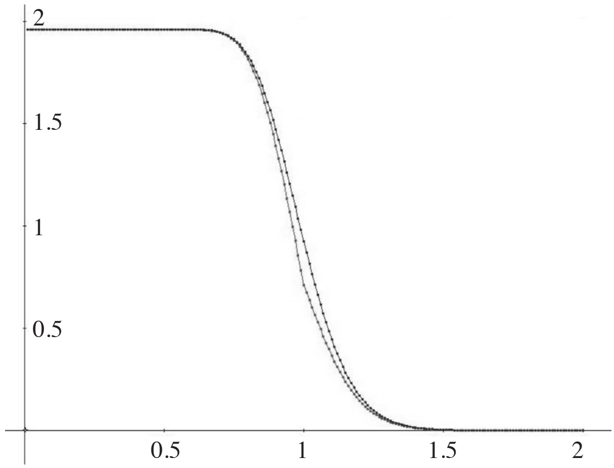


Fig. 8: The solution $p_\lambda(\xi, t)$ for $\lambda = 5$, $B = 2$ and $\sigma^2 = r = 0.04$ at $t = 0.25$ compared with $p_0(\xi, t)$, the European digital put, with the same values of the parameters.

Finally, the remaining two graphs show the behaviour of the solution for the two special cases singled out in our analysis, namely $\lambda = 1$ and the one in which the repulsive point interaction has infinite strength. As can be easily gathered from (4.7c) and Figure 10, $p_\infty(\xi, t)$ coincides with a classical digital down-and-out option vanishing at $\xi = 1 (s = E)$ (also called cash-or-nothing knockout put).

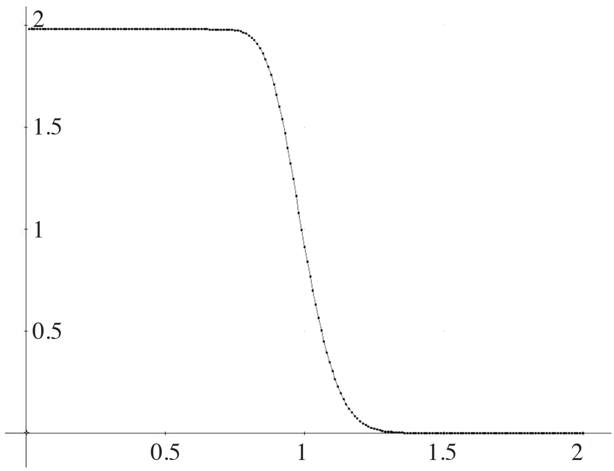


Fig. 9: The solution $p_\lambda(\xi, t)$ for $\lambda = 1$, $B = 2$ and $\sigma^2 = r = 0.04$ at $t = 0.25$.

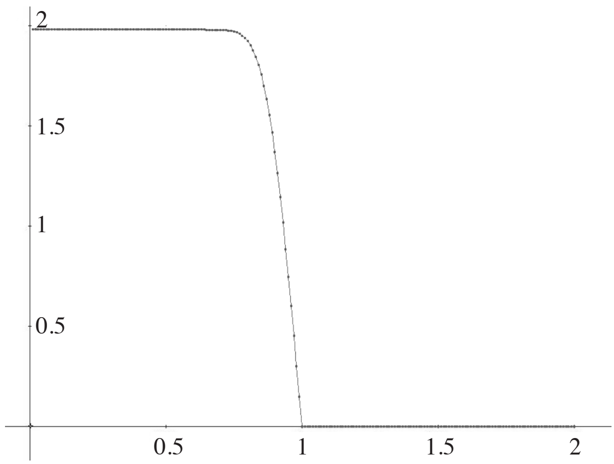


Fig. 10: The solution $p_\infty(\xi, t)$ for $B = 2$ and $\sigma^2 = r = 0.04$ at $t = 0.25$.

As stated in the introduction, we have chosen the particular value $\beta = 2$ in order to make the operator defined by (4.2) self-adjoint. Nevertheless, our results can be generalised to include a drift term given the fact that the latter can be shown to be an infinitesimally small perturbation of the operator $\left[-\frac{d^2}{dx^2} + \lambda \cdot \delta(x)\right]$, which implies that, even without self-adjointness, the contractive nature of the semigroup generated by (4.2) will be preserved (see [7]).

## – Appendix

In this appendix we wish to state and prove the following result.

Proposition A.1. *The integral operator whose kernel is given by the second term in* (2.2) *is a trace class operator on* $L^2(\mathbf{R}), \forall \lambda, \sigma, t > 0$, *and* $||K_t(\lambda, \sigma)||_{T_1} < 1/2$.

Proof. According to [12], given the fact that

$$K_t(x, y; \lambda, \sigma) = \frac{\lambda}{2} e^{\left[\frac{\lambda}{2}(|x|+|y|) + \frac{\lambda^2 \sigma^2 t}{8}\right]} \cdot N\left(-\frac{(|x| + |y|) + \lambda \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) > 0,$$

is a continuous function of the two variables $x, y$, and $K_t(x, x; \lambda, \sigma) > 0, \forall x, t > 0$ in order to prove that the associated integral operator is trace class we need only show that

$$\text{(A.1)} \quad \int_{-\infty}^{+\infty} K_t(x,x;\lambda,\sigma)dx = \frac{\lambda}{2} e^{\frac{\lambda^2\sigma^2 t}{8}} \int_{-\infty}^{+\infty} e^{\lambda|x|} \cdot N\left(-\frac{2|x|+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) dx < \infty.$$

The left hand side of the latter inequality can be rewritten as

$$\text{(A.2)} \qquad \lambda e^{\frac{\lambda^2\sigma^2 t}{8}} \int_0^{+\infty} e^{\lambda x} \cdot N\left(-\frac{2x+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) dx.$$

By using integration by parts (A.2) becomes:

$$\text{(A.3)} \qquad 2e^{\frac{\lambda^2\sigma^2 t}{8}} \left[\frac{1}{\sigma\sqrt{2\pi t}} \int_0^{+\infty} e^{\lambda x} \cdot e^{-\frac{1}{2}\left(\frac{2x+\lambda\frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right)^2} dx - \frac{1}{2} N\left(-\frac{\lambda\sigma\sqrt{t}}{2}\right)\right].$$

The latter can be further simplified by expanding the square in the exponent of the Gaussian inside the integral to get:

$$\text{(A.4)} \qquad \begin{aligned} &2e^{\frac{\lambda^2\sigma^2 t}{8}} \left[\frac{e^{-\frac{\lambda^2\sigma^2 t}{8}}}{\sigma\sqrt{2\pi t}} \int_0^{+\infty} e^{-\frac{2x^2}{\sigma^2 t}} dx - \frac{1}{2} N\left(-\frac{\lambda\sigma\sqrt{t}}{2}\right)\right] = \\ &= e^{\frac{\lambda^2\sigma^2 t}{8}} \left[\frac{1}{2} e^{-\frac{\lambda^2\sigma^2 t}{8}} - N\left(-\frac{\lambda\sigma\sqrt{t}}{2}\right)\right]. \end{aligned}$$

Therefore, we get the following bound on the trace class norm of the integral operator:

$$\text{(A.5)} \qquad \begin{aligned} \|K_t(\lambda,\sigma)\|_{T_1} &= \frac{1}{2} - e^{\frac{\lambda^2\sigma^2 t}{8}} N\left(-\frac{\lambda\sigma\sqrt{t}}{2}\right) = \\ &= \frac{1}{2} - e^{-\frac{\lambda^2\sigma^2 t}{8}} \left[\frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\frac{\lambda\sigma\sqrt{t}}{2}} e^{-\frac{y^2}{2}} dy\right]. \end{aligned}$$

As the second negative term is a slowly decreasing function for $t \geq 0$, whose range is (0, 1/2], the trace class norm is an increasing function with range [0, 1/2), which fully proves our claim. ∎

As is well known, for the first term we have the classical estimate of the convolution by means of Young's inequality

$$\text{(A.6)} \qquad \left\| \frac{e^{-\frac{(\cdot)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}} * \psi \right\|_2 \leq \left\| \frac{e^{-\frac{(\cdot)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}} \right\|_1 \|\psi\|_2 = \|\psi\|_2 .$$

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