The Chow ring of double EPW sextics

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This article has been selected by the Editors for the prize of the Rendiconti di Matematica e delle sue Applicazioni for the best PhD Thesis of the Academic Year 2009/2011.

– **Introduction**

Irreducible symplectic varieties are a class of Kähler manifolds introduced by A. Beauville in [Bea83b]. They arise naturally, together with Calabi-Yau and abelian varieties, in the classification of Kähler manifolds with vanishing first Chern class. Their study has been carried on by many authors, notably Beauville himself, C. Voisin, F. Bogomolov, D. Huybrechts, A. Fujiki, K. O'Grady, S. Mukai, Y. Namikawa, E. Markman and many others. In many respects (period map, structure of the ample cone, behaviour of linear systems, Chow groups...) the theory mimics that of the $K3$ surfaces, which are exactly the irreducible symplectic varieties of dimension 2.

Often this parallel is only conjectural. One outstanding difficulty in testing these conjectures is that few examples of irreducible symplectic varieties are currently known. This true in the sense of deformation theory: except in dimension 6 and 10 there are only two distinct known examples, up to deformation. Of course deformation equivalent varieties can have quite different geometric properties, so this does not render the theory poor. But it is also true in the sense of projective'families.

In the surface case, we have a complete description of the families of polarized $K3$ when the polarization has low degree. Namely (in the generic case

- a degree 2 $K3$ is a double covering of $\mathbb{P}^2$ ramified over a smooth sextic curve;
- a degree 4 $K3$ is just a smooth quartic in $\mathbb{P}^3$;
- a degree 6 $K3$ is the intersection of a quadric and a cubic in $\mathbb{P}^4$;
- a degree 8 $K3$ is the intersection of three quadrics in $\mathbb{P}^5$;

and the list goes on (see [Muk88]). This is partly due to the good understanding that we have of linear systems on $K3$ surfaces ([SD74]).

In contrast the known families of polarized irreducible symplectic varieties of higher dimension are rarely complete, even locally. Actually only few examples are known, all in dimension 4. The first are the Fano varieties of smooth cubic fourfolds, introduced by Beauville and Donagi in [BD85]. A closely related example has been given by Iliev and Ranestad in [IR01]; the remark that their construction yields a family actually distinct from the previous one is due to Markman. The third is the example studied in the present thesis, namely the double coverings of EPW sextics. Finally a new construction by Debarre and Voisin has appeared in [DV09].

EPW sextics were introduced by Eisenbud, Popescu and Walter, as a side example, in [EPW00]. O'Grady has shown in [O'G08a] that, in the generic case, they admit a smooth double covering which is an irreducible symplectic variety, and that the family thus obtained is complete.

The aim of this thesis has been to test a conjecture of Beauville and Voisin about the Chow groups of irreducible symplectic varieties, on the case of the double EPW family. The conjecture is the following:

CONJECTURE. Let X be an irreducible symplectic variety. Any polynomial relation

$$P(D_1, \ldots, D_k, c_i(X)) = 0$$

in the fundamental classes of divisors and in the Chern classes of X which holds in $H^*(X)$ already holds in $CH^*(X)$.

For a discussion of the meaning of this, and its motivation, we refer to Section 1.5.

We were able to test the truth of the conjecture for a very general double EPW sextic; this is the content of Theorem 4.9. In doing so we have proved a number of auxiliary results concerning the local structure of EPW sextics and the existence of special subvarieties. In particular it was crucial for the proof of the theorem the construction of an Enriques surface inside the generic EPW sextic.

We explain this in more detail. Claire Voisin was able to prove her conjecture in the case where X is the Fano variety of a cubic fourfold in [Voi08]. We partly follow her proof; the difficulty is that the geometry of cubics in $\mathbb{P}^5$ is quite concrete and thoroughly studied, while that of EPW sextics is still rather mysterious. In particular the point of departure for her analysis is a standard construction of a Lagrangian surface in X which is singular enough to be rational. This is part of a family of Lagrangian surfaces on X, which are simply the Fano varieties of hyperplane sections of the cubic.

In our case this construction is the most delicate point: it turns out that the right analog for this surface is Enriques, but exhibiting such a surface is complicated. This is mostly because the Lagrangian surfaces which are analog of those on the Fano variety are not sections of a global Lagrangian vector bundle. We have to turn to a degeneration argument instead.

We now introduce the content of the various chapters in more detail.

Chapter 1 introduces irreducible symplectic varieties. We start from the Iitaka fibration as a motivation to study varieties of Kodaira dimension 0, and then we specialize to the case where the canonical class is trivial, giving the classification of Beauville in Section 1.2.

Next we move to the real content of the Chapter, which is the study of irreducible symplectic manifolds: we introduce their main properties and the fundamental examples, and we end with a discussion of the conjectures of Beauville and Voisin and some other related conjectures. This chapter is introductory and nothing here is new.

In Chapter 2 we introduce EPW sextics and their double coverings, following O'Grady. Section 2.6 is rather long: we analyze the local structure of EPW sextics, especially in degenerate cases, and of some related subvarieties. Most of the content of this chapter is due to O'Grady; some local descriptions are new, following ideas in [O'G]. The local description we give allows us to explicitly desingularize a surface

which sits inside these sextics; this should allow us to prove that it is birational to an Enriques surface in the case we need, although we don't follow this path.

Chapter 3 is about the geometry of quartic surfaces $S \subset \mathbb{P}^3$ and their surface of bitangents, with a special regard to the case where S acquires finitely many nodes. The motivation is the following. One can exhibit a degeneration of double EPW sextic to the Hilbert scheme $S^{[2]}$ of 2 points on a quartic. Under this degeneration, the surface we are interested in is sent to the surface $\mathrm{Bit}(S)$ of bitangent lines to S. These matters are explained in Sections 3.1 and 3.2.

The rest of the chapter studies the geometry of these objects, which are quite more concrete than EPW sextics. First we recall some general results due to Welters ([Wel81]), mostly in the smooth case; these should be the results needed to prove that the surface we are interested in is Enriques (in particular the computations of the canonical class and the irregularity).

Next we study the singular case: in Section 3.6 we give examples of a quartic S with 11 nodes such that $\mathrm{Bit}(S)$ is rational, while in Section 3.7 we give an example of a quartic S with 10 nodes such that $\mathrm{Bit}(S)$ is birational to an Enriques surface. The rational case would be even better for our aims, but unfortunately it holds on a too small locus, so we use the Enriques case instead.

These sections are preceded by a discussion of the relationship between quartic surfaces and cubic threefold, using the classical case of quartic curves and cubic surfaces as a motivation. I believe that the construction in Section 3.5 is new, at least this explicit; although it is sometimes hinted, I could not find any reference in the literature.

Finally Chapter 4 puts it all together to give the proof of the main theorem. We begin with some cohomological computations on a double EPW sextics, mostly following [O'G08b]. The geometric argument is given in Sections 4.2 and 4.3.

I have tried to give proper credit for the existing results and conjectures which appear in this thesis; any error or omission in this respect should be attributed to my ignorance.

– **Notation**

Here we briefly recall the notation we use. *Everything we do is over the field* $\mathbb{C}$ *of complex numbers.*

– Linear algebra

Given vectors $v_1, \ldots, v_n$ in a vector space V,

$$\langle v_1, \ldots, v_n \rangle$$

denotes their span. The class of a vector v in the associated projective space $\mathbb{P}(V)$ is instead $[v]$. For us the projective space is the Grassmannian of *lines* in V, so we don't follow Grothendieck's convention.

The Grassmannian $\mathrm{Gr}(k, V)$ denotes the set of k-dimensional vector subspaces of V; when we write $\mathrm{Gr}(k, n)$ we mean $\mathrm{Gr}(k, \mathbb{C}^n)$. When we want to see this as a set of *projective* subspaces of $\mathbb{P}(V)$ we write $\mathrm{Gr}(k - 1, \mathbb{P}(V))$ (notice the shift of index).

– Quadratic forms

If q is a quadratic form on V, the symmetric bilinear form from which it is obtained is written as $\widetilde{q}$. Usually the class of q up to multiples is identified with the quadric defined in $\mathbb{P}(V)$, and denoted by Q. When we could not avoid it we denote by $\widetilde{q}$ the *linear* map

$$\widetilde{q} \colon V \to V^{\vee}$$

which corresponds to the bilinear form. The *kernel* of q is by definition $V^{\perp}$, which is just $\ker \widetilde{q}$. We have tried to avoid to write $\ker q$. When more than one bilinear form is involved, the orthogonal with respect to q is denoted $V^{\perp q}$.

– Cohomology and Chow rings

Given a subvariety $Y \subset X$, the fundamental class of Y in cohomology is denoted $[Y]$ too; this should not cause any trouble. When we write $H^n(X)$ without specifying any coefficient ring, it means that the ring is irrelevant in that context.

The Chow groups of X are

$$CH^n(X) = \{\text{cycles on } X \text{ of codimension } k\}/\text{rational equivalence,}$$

in particular they are comparable with integral cohomology. The Chow groups with coefficients in $\mathbb{Q}$ are denoted $CH^n(X)_{\mathbb{Q}} = CH^n(X) \otimes \mathbb{Q}$. When we perform an intersection product between classes of subvarieties Y_1 and Y_2 of X and we think that the ambient space is not clear from the context we write

$$(Y_1 \cdot Y_2)_X$$

– EPW sextics

Next we come to notation specific to EPW sextics. Given a symplectic vector space U of dimension $2n$, the subvariety of the Grassmannian $\mathrm{Gr}(n, U)$ consisting of Lagrangian subspaces is denoted by $\mathbb{LG}(U)$. Usually V refers to a fixed vector space of dimension 6. Inside $\mathbb{LG}(\bigwedge^3 V)$ we have the open set $\mathbb{LG}(\bigwedge^3 V)^0$ consisting of those Lagrangian subspaces A such that

- A does not contain any decomposable form and
- the locus $Y_A[3]$ is empty.

This is the locus where X_A is smooth; it is defined in Section 2.3. At the other end we will need to consider the closure of the set of Lagrangian subspaces containing 10 independent decomposable forms; this is denoted Σ_{10}.

The subbundle F of the constant vector bundle $\bigwedge^3 V$ over $\mathbb{P}(V)$ is defined in Section 2.1. It has fiber

$$F_{[v]} = v \wedge \bigwedge{}^2 V.$$

Given a $A \in \mathbb{LG}(\bigwedge^3 V)$ we let

$$\lambda_A \colon F \to \mathcal{O}_{\mathbb{P}(V)} \otimes (\bigwedge{}^3 V/A)$$

be the inclusion of F in the trivial bundle $\bigwedge^3 V$ followed by projection. The k-th degeneracy locus of λ_A is

$$Y_A[k] = \left\{ [v] \in \mathbb{P}(V) \mid \dim\left(A \cap F_{[v]} \right) \geq k \right\}$$

and $Y_A = Y_A[1]$ is called an *EPW sextic*, see Section 2.1.

Under mild hypothesis it admits a double covering, denoted X_A; this is constructed in Section 2.2. This double covering is ramified over $Y_A[2]$, hence it contains an isomorphic copy of $Y_A[2]$; to avoid confusion this is usually denoted Z_A.

When A contains a decomposable form, say $A \supset \bigwedge^3 W$ for some $W \subset V$ of dimension 3, the plane $\mathbb{P}(W)$ is contained in Y_A. The intersection

$$\mathbb{P}(W) \cap Y_A[2]$$

is a sextic curve in $\mathbb{P}(W)$ and we denote it by $C_{A,W}$, see Section 2.5.

Finally let $U \subset A$ be a subspace of dimension 9. Given any other Lagrangian subspace B with

$$A \cap B = U,$$

the sextics Y_A and Y_B are tangent along a divisor $D = D_U$ of Y_A, which only depends on U. We define it in Subsection 4.4.2. It comes equipped with a rational map

$$r_D \colon D \dashrightarrow \ell_D,$$

where ℓ_D is the pencil of Lagrangian subspaces containing U.

1 – Holomorphic symplectic manifolds

1.1 – A glimpse of the classification of algebraic varieties

The most ambitious task in algebraic geometry would be to classify all algebraic varieties up to isomorphism. Experience with successful cases of such classifications shows that this breaks up in several steps. If one wants to classify all varieties of dimension n, the first task is to identify a set of discrete invariants, the main example being the genus of a smooth curve. In higher dimension we have more invariants, like the plurigenera, and a first rough invariant which actually gives an idea of the geometric properties of a variety is its Kodaira dimension.

Having fixed a set of such discrete invariants, one would want to create a *reasonable* bijective correspondence between the set of isomorphism classes of varieties with given invariants and some object with more structure, usually another algebraic variety. The meaning of *reasonable* can be adequately specified, giving rise to the concept of coarse and fine moduli spaces.

Usually before being able to create a moduli space one has to relax the condition under which two varieties should be identified, and work with birational isomorphism instead. This difference cannot be seen at the level of curves, where birational maps are actually regular, but is a typical feature already of surface classification, where the concept of minimal surface (a preferred representative of its birational class, at least when the Kodaira dimension is non-negative) arises. So a further step would be to describe what varieties do appear inside a given birational class. This is in a sense the object of minimal model theory.

A different approach to the problem of classification, due to Iitaka, is the following. Given a variety X we try to fit it into a fibration

$$\pi \colon X \to B$$

where the base B and the typical fiber F have a particularly simple structure. Of course then we can expect to recover information on X only if we have a good understanding of the properties of fibrations, given the base and the general fiber; a first instance of this question is the classification of elliptic surfaces by Kodaira in [Kod63].

The construction that we have in mind in this case is the Kodaira-Iitaka fibration. To give a precise statement let us first recall the definition of Kodaira dimension, which we have already mentioned.

DEFINITION 1.1. Let X be a smooth algebraic variety and denote by K_X or simply K its canonical divisor. Assume that for some $n > 0$ the line bundle $\mathcal{O}_X(nK)$ has some nonzero section; then we have a rational map

$$\varphi_n \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}(nK))^\vee).$$

The Kodaira dimension of X, denoted by $\kappa(X)$, is the maximum of the dimensions of the images $\iota_n(X)$ for $n \in \mathbb{N}_{>0}$. If $H^0(X, \mathcal{O}(nK)) = 0$ for all $n > 0$, we agree to put $\kappa(X) = -\infty$.

It is easily seen that the Kodaira dimension is a birational invariant for smooth varieties, so we can extend the notion to the singular case as follows. Given a singular variety X we choose some smooth desingularization Y and define the Kodaira dimension of X by $\kappa(X) = \kappa(Y)$. Our previous remark implies that this does not depend on the choice of the desingularization Y.

By definition the Kodaira dimension is $-\infty$ or an integer between 0 and $\dim(X)$. A priori we should study all this cases separately. The Iitaka fibration allows us, for some purposes, to restrict to the extreme cases. The next theorem allows us to put together all the maps φ_n and build some kind of *limit* map. The precise result is the following (see [Laz04, sec. 2.1.C]):

THEOREM 1.1 (Iitaka). *Let X be a smooth projective variety of* positive *Kodaira dimension. Then there exists a fibration*

$$\varphi \colon X' \to B$$

such that X and X' are birational and for every sufficiently big n such that

$$h^0(X, nK_X) \neq 0$$

the pluricanonical map φ_n is identified with φ up to birational equivalence. This means that, if Y_n is the image of φ_n, we have a commutative diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi_n} \\
B & \dashrightarrow & Y_n,
\end{array}
$$

where the horizontal maps are birational.

Moreover one has

$$\kappa(X) = \kappa(B) = \dim(B)$$

and the very general fiber F of φ has $\kappa(F) = 0$.

The theorem tells us that, in a sense, the fundamental cases to study are those of varieties with $\kappa(X) = -\infty, 0$ or $\dim(X)$.

The case where $\kappa(X) = -\infty$ is studied in Mori theory: there varieties are conjectured to be Mori spaces (see [KM98] for this topic). Of particular interest inside this class are the varieties X such that $-K_X$ is ample, the so-called *Fano* varieties; a huge literature is devote to their study.

At the other extreme there is the case of varieties of *general type*: those having $\kappa(X) = \dim(X)$. Although they have received much attention, this class has proved to be too wide to be studied systematically, even in the case of surfaces, where only partial results on the classification are available.

So we shall concentrate on the case where $\kappa(X) = 0$, and actually a sub-case of it. A typical situation when one finds $\kappa(X) = 0$ is when the canonical K_X is trivial, or more generally it is a torsion class. If K_X is torsion, there exists a unramified finite covering $\pi \colon Y \to X$ such that K_Y is trivial, so, up to finite coverings, we can deal with this simpler case.

In the next section we will give a description, due to Beauville, of these varieties.

1.2 – Varieties with trivial first Chern class

So far we should have given some motivation to study varieties with trivial canonical class, and why they occupy a relevant place between all varieties. But where do we go from here? The fundamental result here is a classification, due to Beauville ([Bea83b]), of the *bricks* that are used to build all such varieties. The technique here is highly non-algebraic, so we'd better remark again that we're working over $\mathbb{C}$. Actually the classification is achieved through the study of holonomy, and the main ingredient is the famous solution due to Yau of the Calabi conjecture in [Yau78]. I could not improve upon the presentation in [Bea83b], hence the following is nothing more than a short digest of it.

The idea is roughly to translate a classification of holonomy groups due to Berger in algebro-geometric terms. Recall that given a Riemannian manifold M and a point $p \in M$, one has the holonomy group

$$H_p < O(T_p X, g);$$

if M is connected of (real) dimension n, the conjugacy class of H_p in $O(n)$ is independent of p. One can view H_p as an abstract Lie group with a given n-dimensional representation.

One can assume that this is representation is irreducible, thanks to the

THEOREM (De Rham). *Let M be a complete, simply connected Riemannian manifold. Then M splits uniquely as a product of manifolds*

$$M = M_0 \times M_1 \times \cdots \times M_k,$$

where M_0 is the Euclidean space and each factor M_i for $i \geq 1$ has irreducible holonomy representation.

REMARK. It follows immediately that M has the product representation of the M_i. So, the reducibility of the holonomy representation is reflected in the reducibility of M itself.

The classification of holonomy subgroups with irreducible representation was achieved by Berger:

THEOREM (Berger). *Let M be Riemannian manifold, and assume that M is not locally a symmetric space. Then the identity component of the holonomy subgroup H is one of the following list:*

$$SO(n) \qquad\qquad U(m),\ n = 2m$$
$$SU(m),\ n = 2m \qquad\qquad SP(r),\ n = 4r$$
$$Sp(1) \times SP(r),\ n = 4r \qquad\qquad Spin(9),\ n = 16$$
$$Spin(7),\ n = 8 \qquad\qquad G_2,\ n = 7.$$

The fundamental remark now is that if H leaves some tensor invariant, one can obtain a tensor field on M by parallel transport, and this will not depend on the choice of a path. For instance if $H = U(m)$, H leaves invariant a complex structure J on T_pM. By extending this to M one obtains a quasi-complex structure which is invariant for parallel transport. It is well known that the structure is then integrable, so M becomes a complex manifold. Moreover the compatibility condition between the complex structure and the metric says that M has a Kähler metric.

Since we work in the realm of algebraic geometry we are interested in the cases where this happens, so that we obtain a Kähler manifold. The only groups in the Berger list which are subgroups of $U(m)$ are $SU(m)$ and $Sp(r)$.

In the first case, the elements of $SU(m)$ also preserve a complex volume form, hence we obtain on M a parallel volume form. The compatibility condition between the metric and the complex structure then implies that this form is actually holomorphic.

In the second case the elements of $Sp(r)$ also preserve a complex 2-form in p. Again, by extending it we obtain a parallel 2-form, which is then holomorphic.

The last ingredient in the story is the following result of Bochner, together with Yau's theorem.

THEOREM (Bochner). *Let M be a compact Kähler manifold, and assume that M has zero Ricci curvature. Then every tensor field on M which is holomorphic is parallel.*

Now if M is Kähler and has $c_1(M) = 0$, by Yau's theorem there exists on M a Kähler metric with zero Ricci curvature. Hence one can apply the above result of Bochner and classify the holomorphic section of Ω_X^p for every p exclusively in terms of the holonomy group.

With some work one then translates the Berger list in a classification of the possible Kähler manifold with $c_1(M) = 0$. Before stating the result, we give a couple of definitions.

DEFINITION 1.2. Let X be a compact Kähler variety of complex dimension n. We say that X is a *Calabi-Yau* variety if $K_X = \Omega_X^n$ is trivial and

$$H^0(X, \Omega_p) = 0 \text{ for } p < n.$$

DEFINITION 1.3. Let X be a compact Kähler variety of complex dimension $2n$. We say that X is an *irreducible symplectic* variety if $\pi_1(X) = 0$ and

$$H^0(X, \Omega_X^2) = \mathbb{C}\sigma,$$

where σ is a holomorphic symplectic 2-form.

REMARK. According to the above list, the Calabi-Yau case corresponds to the holonomy group $SU(m)$, while the irreducible symplectic case corresponds to the holonomy groups $Sp(r)$.

REMARK. It is not difficult to see that an irreducible symplectic variety has

$$H^0(X, \Omega_X^p) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \mathbb{C}\sigma^q & \text{if } p = 2q \text{ is even.} \end{cases}$$

REMARK. When the dimension of X is 2, the definitions of Calabi-Yau variety and irreducible symplectic variety agree, and characterize he so-called $K3$ surfaces.

We can finally enounce the main result of this section.

THEOREM (Beauville). *Let X be a compact Kähler variety such that $c_1(X) = 0$. Then X admits a finite unramified covering $X' \to X$ such that X' splits (uniquely) as a product*

$$X' = X_1 \times \cdots \times X_n,$$

where each factor X_i is either

  i) *a complex torus, or*
 ii) *a Calabi-Yau variety, or*
iii) *an irreducible symplectic variety.*

Of course if one wants uniqueness, all the factor which are complex tori have to be merged into a single factor, since a product of complex tori is again a torus.

The story now splits into three directions with rather different flavours. While the study of complex tori and Abelian varieties is quite old, the subject of Calabi-Yau varieties is younger, and the third type of varieties, the irreducible symplectic, only began to be studied thoroughly after the appearance of this classification theorem. In the rest of this chapter we shall be concerned with irreducible symplectic varieties, and in later chapters we will specialize to a rather peculiar family of examples.

1.3 – Irreducible symplectic manifolds

Recall that we have defined in the previous section an irreducible symplectic variety to be a compact complex, simply connected, manifold X endowed with a symplectic holomorphic form σ which spans $H^0(\Omega_X^2)$. Here we want to recall some general facts about this class of varieties, and underline their similitude with the $K3$ surfaces.

The main tool we will use in studying irreducible symplectic manifolds is the local *period map*. Let X be any irreducible symplectic manifolds, and let

$$\pi \colon \mathcal{X} \to U$$

be a semiuniversal deformation of X; this always exist by a general result of Kuranishi, but the base U of the deformation is a priori not smooth. It is a result of Bogomolov ([Bog79]) that for an irreducible symplectic manifold we can always take U to be smooth.

REMARK. It is not difficult to prove ([Bea83b, Proposition 9]) that the fibers X_t of π are still irreducible symplectic varieties, in particular on each X_t we have a holomorphic 2-form σ_t, unique up to multiples.

By general deformation theory U has then dimension

$$\dim U = h^1(X, T_X) = h^1(X, \Omega_X^1),$$

since the symplectic form induces an isomorphism $T_X \cong \Omega_X^1$. Since by hypothesis

$$h^{2,0}(X) = h^0(X, \Omega_X^2) = 1,$$

we find $\dim U = b_2(X) - 2$.

The local period map is then constructed as follows. First, we can shrink U and assume that it is contractible; it follows that the local coefficient system

$$R^2\pi_*(\mathbb{Z})$$

is trivial. This allows us to identify the cohomology of X with that of a nearby fiber X_t. To be precise about notation, it gives isomorphisms

$$\varphi_t \colon H^2(X, \mathbb{Z}) \to H^2(X_t, \mathbb{Z}).$$

For each $t \in U$ we have the distinguished line $H^{2,0}(X_t) \subset H^2(X_t, \mathbb{C})$, so we define the period map

$$P_X \colon U \longrightarrow \mathbb{P}(H^2(X, \mathbb{C})).$$

$$t \longrightarrow \varphi_t^{-1}(H^{2,0}(X_t))$$

General results of Griffiths ([Voi02, Sec. 10.2.3]) allow us to compute the differential of P_X at 0, in particular we have the following result (for more details we refer to [O'G05a]).

THEOREM. *The differential $d(P_X)_0$ is injective. In particular the period map is a local embedding.*

We now want to identify the image of the local period map. Since U has dimension $b_2(X) - 2$ and $\mathbb{P}(H^2(X, \mathbb{C}))$ has dimension $b_2(X) - 1$, this image will locally be a hypersurface. We want to identify what this is; in doing so we will introduce the fundamental *Beauville-Bogomolov form*.

We recall that the Theorem above is well-known in the case of $K3$ surfaces, and the image of P_X in that case is the quadric defined by the equation

$$(\omega, \omega) = 0,$$

where $(\cdot, \cdot)$ is the intersection form on the surface X.

THEOREM (Beauville-Fujiki). *Let X be an irreducible symplectic variety of dimension $2n$. There exist unique:*

- *an integral indivisible quadratic form $(\cdot, \cdot)$ on $H^2(x, \mathbb{Z})$ and*
- *a positive number $c_X \in \mathbb{Q}$*

such that for every class $\alpha \in H^2(X, \mathbb{Z})$ we have

$$\alpha^{2n} = c_X (\alpha, \alpha)^n.$$

In other words the intersection form on $H^2(X)$ is, up to a constant, the n-th power of a quadratic form.

DEFINITION 1.4. The form $(\cdot, \cdot)$ is called the *Beauville-Bogomolov form* of X. When we want to specify X from the context we will write $(\cdot, \cdot)_X$. The number c_X is called the *Fujiki constant* of X.

The Beauville-Bogomolov form of X allows us to identify the image of the period map. Indeed we have

THEOREM (Local Torelli). *Let X be as above. The period map P_X is locally an isomorphism of U on the quadric*

$$Q = \{\omega \in H^2(X, \mathbb{C}) \mid (\omega, \omega)_X = 0\}.$$

More precisely the image of P_X is contained in the analytic open set of Q defined by $(\omega, \overline{\omega})_X > 0$.

The characterization of the image of P_X won't be of much concern to us, but the Beauville-Bogomolov form is a fundamental tool. Still the two theorems are easier

to prove together, as in [O'G05a, Sec. 3.2]. The idea is that the quadratic form is just a local equation of the image of P_X, which we already know is a hypersurface.

REMARK. Since the work of Kodaira it is known that all $K3$ surfaces are deformation of each other. In higher dimensions we have various families. Moreover there exist examples of birational irreducible symplectic varieties which are not isomorphic, as the next example shows.

We describe a so-called *Mukai flop*. This is the first instance of a birational map between irreducible symplectic varieties which is not an isomorphism.

Let X be an irreducible symplectic variety of dimension $2n$ ($n \geq 2$) and assume that we have a subvariety $P \subset X$ isomorphic to $\mathbb{P}^n$. Then we claim that

$$\mathcal{N}_{P/X} \cong \Omega^1_P. \tag{1.1}$$

To see this, start from the exact sequence

$$0 \longrightarrow TP \longrightarrow TX\big|_P \longrightarrow \mathcal{N}_{P/X} \longrightarrow 0$$

and dualize it to obtain

$$0 \longrightarrow \mathcal{N}^\vee_{P/X} \longrightarrow \Omega^1_X\big|_P \longrightarrow \Omega^1_P \longrightarrow 0 \ .$$

The symplectic form defines an isomorphism

$$TX \xrightarrow{\cong} \Omega^1_X;$$

by composition we obtain a morphism

$$TP \longrightarrow TX\big|_P$$

$$\longrightarrow \;\tilde{}\; \longrightarrow$$

$$\Omega^1_X\big|_P \longrightarrow \Omega^1_P.$$

This has to be identically zero, because $\mathbb{P}^n$ does not have any nontrivial 1-form. So the morphism

$$TP \to \Omega^1_X\big|_P$$

lifts to $\mathcal{N}^\vee_{P/X}$; since it is injective and the two vector bundles have the same rank n, it is an isomorphism. Finally its dual gives rise to the desired isomorphism (1.1).

More precisely it fits into an isomorphism of short exact sequences

$$0 \longrightarrow TP \longrightarrow TX\big|_P \longrightarrow \mathcal{N}_{P/X} \longrightarrow 0$$
$$\xrightarrow{\cong} \qquad \xrightarrow{\cong} \qquad \xrightarrow{\cong}$$
$$0 \longrightarrow \mathcal{N}_{P/X}^\vee \longrightarrow \Omega_X^1\big|_P \longrightarrow \Omega_P^1 \longrightarrow 0.$$

Now let $\widetilde{X}$ be the blowup of X along P, with exceptional divisor E. The projection $E \to P$ is the projective fiber bundle $\mathbb{P}\mathcal{N}_{P/X}$, but thanks to (1.1) this is identified with the projective fiber bundle

$$\mathbb{P}\Omega_P^1 \to P.$$

In particular the fiber bundle structure is independent of X and the embedding $P \hookrightarrow X$. We can recognize it as the fiber bundle structure of the incidence divisor

$$H = \big\{ (x, \ell) \mid x \in \ell \big\} \subset P \times P^\vee.$$

$$\longrightarrow$$

$$P$$

In particular $E \cong H$ has an obvious fiber bundle structure over $P^\vee$ too.

Thanks to a criterion of Nakano (see [Nak70]) we can contract E the other way round to obtain a smooth complex variety $\widehat{X}$. By construction X and $\widehat{X}$ are isomorphic outside the image of E, in particular they are birational. The subvariety $P \subset X$ has been replaced by $P^\vee$.

If $\widehat{X}$ is again Kähler, it is itself an irreducible symplectic variety. Indeed the birational isomorphism with X shows that $\widehat{X}$ is simply connected, and one obtains an isomorphism

$$H^0(X, \Omega_X^2) \cong H^0(\widehat{X}, \Omega_{\widehat{X}}^2).$$

Indeed the two varieties are isomorphic outside a set of codimension $n \geq 2$, and a holomorphic 2-form on the complementary extends over this set by Hartog's theorem.

There exist many cases where such a projective space can be found inside an irreducible symplectic variety, and one can also exhibit examples where both X and $\widehat{X}$ are algebraic.

In view of the preceding construction, the following result from Huybrechts ([Huy03]) comes unexpected. For instance for Calabi-Yau varieties the analog statement is false.

THEOREM (Huybrechts). *Let X and Y be irreducible symplectic manifold, and assume that they are birational. Then they are deformation equivalent, that is, there exists a family over a smooth connected base such that X and Y appear as fibers of the family. In particular X and Y are diffeomorphic.*

1.4 – Hilbert schemes of points

In this section we introduce the main example of higher-dimensional holomorphic symplectic varieties. The idea is simple: we just take a symmetric product of a given $K3$ surface S. This turns out to be the wrong candidate, since it is singular, but a suitable desingularization will do the trick. Indeed a simple computation proves the following

THEOREM. *Let X be a smooth variety and let $Y = X^{(n)}$ be the n-th symmetric product of X, for some $n \geq 2$. Then Y is smooth if and only if $\dim X = 1$.*

In the case where X is a surface, we are lucky enough to find a standard resolution of the singularities of $X^{(n)}$. Indeed we can consider the *Hilbert scheme* of n points of X. This is built as follows: let $\mathcal{I}_Z \subset \mathcal{O}_X$ be the ideal sheaf of a finite subscheme $Z \subset X$, and let n be its length. If the support of Z is

$$\operatorname{Supp} Z = \{x_1, \ldots, x_k\}$$

we just define the length of Z to be

$$l(Z) = \sum_{i=1}^{k} l\left(\mathcal{O}_{X,x_i}/\mathcal{I}_{Z,x_i}\right),$$

where on the right hand side l denotes the length as a $\mathcal{O}_{X,x_i}$-module.

Then the Hilbert polynomial of the quotient $\mathcal{O}_Z$ of $\mathcal{O}_X$ is just n, so there is a Quot scheme which parametrizes such quotients (for the construction of Hilbert and Quot schemes see [Ser06]).

DEFINITION 1.5. The Quot schemes of such quotients is denoted by $X^{[n]}$; we will refer to it as the Hilbert schemes of n points on X

In case X is not algebraic we can replace the above construction with the so-called *Douady space* of n points on X (see [CP94]).

We have the following result (see [Fog73]):

THEOREM (Fogarty). *If X is a smooth surface, then $X^{[n]}$ is smooth for all n.*

REMARK. Of course $X^{[n]}$ is always birational to $X^{(n)}$: they are isomorphic on the open set which parametrizes distinct points. They are different in the way they treat non-reduced schemes. On $X^{[n]}$ we have a point for each scheme Z, while on $X^{(n)}$ we are only able to parametrize the support of Z, together with the multiplicity at each point. In particular we have a natural forgetful map (the *cycle map*)

$$c\colon X^{[n]} \longrightarrow X^{(n)}.$$
$$Z \longrightarrow \sum_{i=1}^{k} l(\mathcal{O}_{Z,x_i}) x_i$$

The simplest example where they differ is when Z is not reduced of length 2. Then Z is equivalent to the datum of a point and a tangent direction at that point, while $c(Z)$ is just a point counted with multiplicity 2.

We do not prove Fogarty's result here. We just remark that smoothness of $S^{[n]}$ is clear away from the diagonal, so we only need to check smoothness locally around a point in the diagonal. So it is not restrictive to assume that $S = \mathbb{C}^2$, and then one can rely on an explicit description of $(\mathbb{C}^2)^{[n]}$, as in [Nak99, Sec. 1.3].

After this preliminaries we can introduce Beauville's examples. Let S be a $K3$ surface, and let ω be its symplectic form. On the product S^n we consider the 2-form

$$\overline{\sigma} = \sum_{i=1}^{n} \pi_i^*(\omega),$$

where

$$\pi_i\colon S^n \to S$$

is the projection on the i-th factor. Then σ is invariant under the action of the symmetric group $\mathfrak{S}_n$, hence it descends to a 2-form on $S^{(n)}$, at least on the open set $S^{(n)} \setminus \Delta$ where $S^{(n)}$ is smooth. Here we have denoted

$$\Delta = \{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for at least one choice of } i \neq j\}$$

the big diagonal. We can pullback this form via the cycle map c to obtain a holomorphic 2-form σ on the open set of reduced schemes on $S^{[n]}$.

We remark that the set of non-reduced subschemes is a divisor on $S^{[n]}$, hence we cannot just appeal to Hartog's theorem to extend σ on the whole Hilbert scheme $S^{[n]}$. Nevertheless we have the following

THEOREM (Beauville, [Bea83b]). *The form σ extends to a holomorphic 2-form on the whole $S^{[n]}$, and the form thus obtained is symplectic. Moreover this gives $S^{[n]}$ the structure of an* irreducible *symplectic variety.*

REMARK. The construction of Beauville works fine even if one starts with an abelian surface T. The only difference is that $T^{[n]}$, endowed with the symplectic form defined above, is not irreducible.

In order to obtain an irreducible symplectic variety, Beauville considers the kernel $K_n(T)$ of the summation map

$$T^{[n]} \to T^{(n)} \xrightarrow{\Sigma} T.$$

In the same paper [Bea83b] Beauville shows that $K_n(T)$ is indeed an irreducible symplectic variety, called a *generalized Kummer variety*.

We will not be concerned with generalized Kummer varieties in this work.

To get some feeling about this varieties, we can start by proving the following very simple proposition, which gives an alternative picture of $S^{[2]}$.

PROPOSITION 1.2. *Let S be a smooth surface (or more generally a smooth variety) and let $\Delta \subset S^{(2)}$ be the diagonal. Then $S^{[2]}$ is the blowup of $S^{(2)}$ along Δ.*

PROOF. We just have to identify the map

$$c \colon S^{[2]} \to S^{(2)}$$

with the blowup map. Since $S^{(2)}$ is singular, it will be easier to do things the other way round: first we blow up the diagonal of S^2, then we quotient by the action of $\mathfrak{S}_2$.

Let

$$\epsilon \colon X \to S^2$$

be the blowup of the diagonal $\Delta \subset S^2$. Over a point $(x,x) \in \Delta$ the fiber of ϵ is

$$\epsilon^{-1}(x,x) = \mathbb{P}(\mathcal{N}_{\Delta/S^{[2]}}).$$

Moreover over $\Delta \cong S$ we have the isomorphism of vector bundles

$$TS \cong \mathcal{N}_{\Delta/S^2},$$

hence we can identify the fiber

$$\epsilon^{-1}(x,x) \cong \mathbb{P}(T_x S) = \{(x,\ell) \mid \ell \text{ is a tangent direction at } x\}.$$

The involution of S^2 gives rise to an involution of X which fixes pointwise the exceptional divisor. It is easy to identify the quotient of X by this involution with $S^{[2]}$. Hence we obtain a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & S^{[2]} \\
{\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle c} \\
S^2 & \longrightarrow & S^{(2)}.
\end{array}
$$

It follows easily that c is the blowup map too.

We end this section with a study of the cohomology of the symplectic variety $S^{[n]}$. We begin by building some natural classes in $H^2(S^{[n]}, \mathbb{Z})$. The first one is the fundamental class of the diagonal.

Let $\Delta \subset S^{[n]}$ be the big diagonal. We have already remarked that this is a divisor on $S^{[n]}$, hence we have a class

$$[\Delta] \in H^2(S^{[n]}, \mathbb{Z})$$

Using the action of the alternating group on S^n, one can exhibit a double cover of $S^{[n]}$ ramified exactly over Δ. A little bit of care is needed to resolve the singularities of the obvious quotient.

The existence of such a double cover implies that $[\Delta]$ is divisible by 2 inside $H^2(S^{[n]}, \mathbb{Z})$. So we let

$$\delta \in H^2(S^{[n]}, \mathbb{Z})$$

be a class such that $2\delta = [\Delta]$.

Other cohomology classes can be obtained from those of S. Indeed let $\alpha \in H^2(S)$, then the class

$$\mu(\alpha) = \sum_{i=1}^{n} \pi^*(\alpha)$$

is a class on S^n invariant under the action of $\mathfrak{S}_n$. This descends to a cohomology class on $S^{(n)}$ (this requires a little care), and we let

$$\widetilde{\mu}(\alpha) \in H^2(S^{[n]})$$

be its pullback. This construction yields a homomorphism

$$\widetilde{\mu} \colon H^2(S, \mathbb{Z}) \to H^2(S^{[n]}, \mathbb{Z}).$$

This two examples are enough, as is shown by the following

THEOREM.

i) *The homomorphism $\widetilde{\mu}$ is injective, and we have the isomorphism*

$$H^2(S^{[n]}, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta, \tag{1.2}$$

   where the first factor is embedded in $H^2(S^{[n]}, \mathbb{Z})$ via $\widetilde{\mu}$.
ii) *If δ is given type $(1,1)$, the decomposition (1.2) is actually an isomorphism of Hodge structures.*

iii) Moreover if $(\cdot, \cdot)$ denotes the Beauville-Bogomolov form, we have

$$(\delta, \delta) = -2(n-1),$$

and the decomposition (1.2) is an isomorphism of lattices, where the two factors on the right are considered orthogonal.

The above theorem gives a complete description of $H^2(S^{[n]}, \mathbb{Z})$ as a *polarized Hodge structure*, where the lattice structure is given by the Beauville-Bogomolov form. In many occasions it is useful to know the Fujiki constant of $S^{[n]}$, and this is given by the following:

THEOREM. *The Fujiki constant of $S^{[n]}$ is $\frac{(2n)!}{n!2^n}$.*

The above results are probably due to Beauville, anyway they are now common folklore.

We can use the above description to compute the Euler characteristic of $S^{[2]}$. There are more general results by Göttsche describing the Betti numbers of $S^{[n]}$ for all n (see [Göt90]) but the following will be enough for our needs.

PROPOSITION 1.3. *The Euler characteristic of $S^{[2]}$ is 324.*

PROOF.[Sketch of proof] Let b_i denote the i-th Betti number of $S^{[2]}$. Of course $b_0 = 1$, and since $S^{[2]}$ is simply connected we have $b_1 = 0$. We have a complete description of $H^2(S^{[2]}, \mathbb{Z})$, in particular we find

$$b_2 = 1 + b_2(S) = 23.$$

Moreover it is known that $b_3 = 0$ and that the intersection product

$$\mathrm{Sym}^2 H^2(S^{[2]}, \mathbb{Z}) \to H^4(S^{[2]}, \mathbb{Z})$$

is actually an isomorphism. This implies that

$$b_4 = \frac{b_2 \cdot (b_2 + 1)}{2} = 276.$$

Finally Poincaré duality implies that for the other Betti numbers we have

$$b_5 = b_7 = 0, \qquad b_6 = 23, \qquad b_8 = 1,$$

so the thesis follows.

1.5 – Conjectures on symplectic varieties

In this section we briefly recall some conjectures about holomorphic symplectic varieties which were the motivation for this work. By no means this is a complete survey about open problems on irreducible symplectic varieties; indeed we just mention the conjectures related to the present thesis.

Actually we start one step before, from the cycle map

$$c\colon CH^*(X) \to H^*(X,\mathbb{Z})$$

for a smooth variety X over $\mathbb{C}$. Although Chow groups were introduces to mimic the topological construction of cohomology, very little is known about this map in general. Of course it is a homomorphism of rings, since cup products and intersection products agree.

The problem is to characterize the image and the kernel of c. We can extend c to a homomorphism

$$c_{\mathbb{Q}}\colon CH^*(X)_{\mathbb{Q}} \to H^*(X,\mathbb{Q});$$

then we have the well-known

CONJECTURE 1 (Hodge). *Assume that X is projective. The image of $c_{\mathbb{Q}}$ is generated by the classes in $H^*(X,\mathbb{Q})$ of type (p,p).*

We won't enter in detail about this, for a survey see [Bea08].

The problem of characterizing the kernel, if less rewarded, is no less mysterious. Rational equivalence is indeed very fine, as shown for example by the following result (see [Voi03, Thm. 10.15]).

DEFINITION 1.6. We let

$$CH^*_{hom}(X) = \ker c$$

be the ideal of $CH^*(X)$ consisting of classes of cycles homologically equivalent to 0.

THEOREM (Mumford). *Let S be a surface with*

$$H^0(S,K_S) \neq 0.$$

Then the map

$$S^m \times S^m \longrightarrow CH^2_{hom}(S)$$

$$(Z_1, Z_2) \longmapsto Z_1 - Z_2$$

has general countable fiber, and is never surjective.

The meaning of the above theorem is that there is that every sensible definition of dimension for Chow groups would lead us to define $CH^2_{hom}(S)$ infinite-dimensional.

REMARK. On any variety X of we have by definition

$$CH^1_{hom}(X) = \text{Pic}^0(X),$$

which is an abelian variety. So the pathological behaviour above can only happen in codimension at least 2. Indeed Mumford's theorem shows that the simplest nontrivial case, that of point on surfaces, can already be quite subtle.

By the way, also Hodge conjecture is known to hold in codimension 1, and is just Lefschetz theorem on $(1,1)$ classes.

So we see that CH^*_{hom} can be rather large: only a small part of the information is retained by homological equivalence. The conjectural characterization of the kernel of c is much less precise given by the following

CONJECTURE 2 (Bloch-Beilinson). *For every variety X there exists a (decreasing) functorial filtration* of rings

$$CH^p(X) = F^0CH^p(X) \supset F^1CH^p(X) \supset \cdots \supset F^{p+1}CH^p(X) = 0$$

such that $F^1CH^p(X)$ is exactly the kernel of the cycle map c.

Here functorial *means that it is compatible both with pull-backs f^* and push-forwards f_*, when they are defined.* Of rings *means that it respects the intersection product on the Chow ring.*

REMARK. We should emphasize that the requirement

$$F^{p+1}CH^p(X) = 0$$

is far from trivial; indeed this is the delicate point of many proposed constructions. Moreover the conjecture is somewhat more precise. For a discussion of the conjecture and some of its consequences see [GP03].

Beauville and Voisin have proved in [BV04] the following simple but rather remarkable result:

THEOREM (Beauville-Voisin). *Let S be a K3 surface, C, D two divisors on it. Then there exists a 0-cycle θ on S such that*

$$C \cdot D = \deg(C \cdot D)\theta$$

in $CH^2(S)$. Moreover $c_2(S) = 24\theta$.

This is in sharp contrast with Mumford's theorem. Although the group $CH^2(S)$ is infinite-dimensional, the image of the intersection product

$$\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to CH^2(S)$$

is just a line inside it. This has lead Beauville to formulate in [Bea07] the following addendum to the Bloch-Beilinson conjecture:

CONJECTURE 3 (Beauville). *If X is an irreducible symplectic variety, the Bloch-Beilinson filtration for X splits, that is, it is the filtration associated to a graduation.*

An easy, and more testable, consequence of this (second order) conjecture is then

CONJECTURE 4 (Beauville). *Let X be an irreducible symplectic variety, and let $DCH(X) \subset CH^*(X)$ be the* subring *generated by the divisors. Then the cycle map*

$$c\colon CH^*(X) \to H^*(X, \mathbb{Z})$$

is injective when restricted to $DCH(X)$.

That conjecture 3 implies Conjecture 4 is easy, and is shown in [Bea07]. Indeed if X is symplectic we have

$$F^1 CH^1(X) = \mathrm{Pic}^0(X) = 0$$

since X is simply connected, in particular $q(X) = 0$. But then Conjecture 3 implies that no product of divisors can lie in

$$F^1 CH^*(X) = \ker c.$$

REMARK. We should mention that Huybrechts in [Huy09] has given an alternative interpretation of the result of Beauville and Voisin in the framework of derived categories. Namely let S be a $K3$ and let

$$R(S) = CH^0(S) \oplus CH^1(S) \oplus \mathbb{Z}p,$$

where p is a point on any rational curve on S. Then the first claim of the theorem of Beauville and Voisin can be restated by saying that $R(S)$ is a subring of the Chow ring. Huybrechts gives the following equivalent formulation: for any line bundle L on S one has

$$v(L) \in R(S),$$

where v is the Mukai vector; this includes the second half of the statement. In this form, assuming $\mathrm{rk}\,\mathrm{Pic}(S) \geq 2$, the result is generalized from line bundles on S to

spherical objects in the derived category of S (these are objects E in the derived the derived category which satisfy

$$\mathrm{Ext}^*(E, E) \cong H^*(S^2, \mathbb{C}),$$

see [Huy09] for details).

We come back to the more elementary formulation.

REMARK. Conjecture 4 explicitly means the following: any polynomial relation

$$P(D_1, \ldots, D_k) = 0$$

in the fundamental classes of divisors which holds in $H^*(X)$ already holds inside $CH^*(X)$.

The state of the conjecture is as follows. It has been proved by Beauville himself in the first cases. Then it has been extended by Voisin in [Voi08] in the following form:

CONJECTURE 5 (Voisin). *Let X be an irreducible symplectic variety. Any polynomial relation*

$$P(D_1, \ldots, D_k, c_i(X)) = 0$$

in the fundamental classes of divisors and in the Chern classes of X which holds in $H^*(X)$ already holds in $CH^*(X)$.

In the same paper Voisin proves

THEOREM (Voisin). *Conjecture 5 holds true when*

- *$X = S^{[n]}$, for some K3 S, and $n \leq 2b_2(S)_{tr} + 4$, where $b_2(S)_{tr}$ is the rank of the transcendental part of $H^2(S)$, that is, the orthogonal of the Néron-Severi lattice, or*
- *X is the Fano variety of lines on a cubic fourfold $Y \subset \mathbb{P}^5$.*

The proof is quite complicated in both cases: one has to produce a number of relations in $CH^*(X)$. In this thesis we propose to study Conjecture 5 in the case where X is a double EPW sextic, to be defined in next chapter. Roughly we follow the ideas of the proof of Voisin for the case of the Fano variety.

The problem is that the definition of EPW sextic is quite involved, hence many geometric constructions which are easy to produce from the projective geometry of cubic fourfolds become far less trivial to replicate on EPW sextics. The geometry of EPW sextics is indeed rather subtle; in particular inside the Fano variety one can produce a lot of rational surfaces which are sections of a Lagrangian rank 2 vector bundle, while on EPW sextics we had to replace these with Enriques surfaces, which do not come as sections of a vector bundle.

In particular we are aiming to the following result

THEOREM. *Let X be a double EPW sextic, $f: X \to Y$ its associated double covering. Let*

$$h = f^* \mathcal{O}_Y(1)$$

be the natural polarization. Then every polynomial relation between h and the Chern classes of X which holds in $H^*(X, \mathbb{Q})$ already holds in $CH^*(X)_{\mathbb{Q}}$.

In particular if X is very general, Conjecture 5 holds for X.

The second claim follows form the fact that for the very general double EPW sextic the Picard group is generated by h.

In proving the theorem we are lead to produce a number of results about the geometry of EPW sextics. In particular in Chapter 2 we define EPW sextics and make a detailed study of the singularities that can appear. This is needed to study the degenerate cases of EPW sextics. In Chapter 3 we show that a suitable degeneration allows us to transfer problems about surfaces inside EPW sextics to the study of the surface of bitangents to a quartic surface $S \subset \mathbb{P}^3$. In particular we want to study the behaviour of such a surface when S acquires many nodes. Finally Chapter 4 puts everything together to give a proof of the main theorem.

2 – Double EPW sextics

2.1 – The construction of EPW sextics

2.1.1 – Some linear algebra constructions

EPW sextics were introduce by Eisenbud, Popescu and Walter in [EPW00]; their double covering was constructed by O'Grady, who also showed that it gives an example of an irreducible symplectic variety. Most of the results of this chapter are due to O'Grady, we have in particular relied on the work in preparation [O'G].

In this section we do some preliminary linear algebra, in order to discuss EPW sextics and their double coverings. We begin with a 6-dimensional vector space V over the field $\mathbb{C}$. The space $\bigwedge^6 V$ is 1-dimensional, so we choose once and for all an isomorphism

$$\text{vol} \colon \bigwedge\nolimits^6 V \to \mathbb{C}.$$

This endows $\bigwedge^3 V$ with a symplectic form, given by

$$(\alpha, \beta) = \text{vol}(\alpha \wedge \beta),$$

for $\alpha, \beta \in \bigwedge^3 V$.

We remark that $\dim \bigwedge^3 V = \binom{6}{3} = 20$, and as usual we say that a subspace $A \subset \bigwedge^3 V$ is *Lagrangian* if

$$\dim A = \frac{1}{2} \dim \bigwedge\nolimits^3 V = 10$$

and the form $(\cdot, \cdot)$ restricts to 0 on A. In particular for each non-zero $v \in V$ we can consider the Lagrangian subspace

$$F_v = \{v \wedge \alpha \mid \alpha \in \textstyle\bigwedge^2 V\}.$$

Of course

$$(v \wedge \alpha) \wedge (v \wedge \beta) = 0$$

for every $\alpha, \beta \in \bigwedge^2 V$. So to prove that F_v is Lagrangian it is enough to show that $\dim F_v = 10$. But it is clear that

$$F_v \cong \textstyle\bigwedge^2(V/\langle v \rangle), \tag{2.1}$$

so $\dim F_v = \binom{5}{2} = 10$.

Since the subspace F_v only depends on the class $[v] \in \mathbb{P}(V)$, the subspaces F_v fit together, giving rise to a *Lagrangian subbundle* F of the trivial symplectic bundle $\mathbb{P}(V) \times V$.

We can compute the Chern classes of F as follows. First we globalize the isomorphism in (2.1). It is immediate to see that the isomorphism

$$\varphi_v \colon \textstyle\bigwedge^2(V/\langle v \rangle) \longrightarrow F_v$$

$$[\alpha] \longrightarrow v \wedge \alpha$$

depends on the choice of v, and more precisely $\varphi_{kv} = k\varphi_v$. From this it is immediate to realize that the maps φ_v fit together and yield an isomorphism

$$F \cong \mathcal{S} \otimes \textstyle\bigwedge^2 \mathcal{Q},$$

where $\mathcal{Q}$ is the tautological quotient bundle on $\mathbb{P}(V)$ and $\mathcal{S}$ the tautological subbundle. The exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \otimes V \longrightarrow \mathcal{Q} \longrightarrow 0$$

implies that

$$c_1(\mathcal{Q}) = -c_1(\mathcal{S}) = h,$$

the hyperplane class on $\mathbb{P}(V)$. A standard computation using the splitting principle gives that

$$c_1(\mathrm{Sym}^2 \mathcal{Q}) = 6c_1(\mathcal{Q}) = 6h.$$

In a similar way we get

$$c_1(\mathcal{Q} \otimes \mathcal{Q}) = 10c_1(\mathcal{Q}) = 10h.$$

Since

$$\bigwedge^2 \mathcal{Q} \cong (\mathcal{Q} \otimes \mathcal{Q})/\operatorname{Sym}^2 \mathcal{Q}$$

we find

$$c_1(\bigwedge^2 \mathcal{Q}) = 4h,$$

and a last application of the splitting principle yields

$$c_1(F) = c_1(\bigwedge^2 \mathcal{Q}) + \operatorname{rk}(F)c_1(\mathcal{S}) = -6h. \tag{2.2}$$

2.1.2 – The definition of EPW sextics

We are now ready to define the EPW sextics, as follows. Fix a Lagrangian subspace $A \subset \bigwedge^3 V$. Note that the symplectic form gives a canonical identification

$$\bigwedge^3 V/A \cong A^\vee.$$

Let

$$\lambda_A \colon F \to \mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee \tag{2.3}$$

be the inclusion $F \hookrightarrow \mathcal{O}_{\mathbb{P}(V)} \otimes \bigwedge^3 V$ followed by the projection modulo A. The map λ_A is a map of vector bundles of equal rank 10.

DEFINITION 2.1. We set
$$Y_A = Z(\det \lambda_A),$$

the zero locus of the determinant of λ_A. This is a subscheme of $\mathbb{P}(V)$; when it is not the whole $\mathbb{P}(V)$, Y_A is called a *EPW* sextic.

REMARK. The locus Y_A is indeed a sextic. To see this, note that λ_A is a section of

$$\mathcal{H}om(F, \mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee) \cong F^\vee \otimes (\mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee) \cong F^\vee \otimes A^\vee$$

so $\det \lambda_A$ is a section of

$$\det(F^\vee) \cong \mathcal{O}_{\mathbb{P}(V)}(6)$$

thanks to Equation (2.2).

REMARK. The support of the scheme Y_A is by definition the locus

$$\big\{ [v] \in \mathbb{P}(V) \mid \dim(F_v \cap A) \geq 1 \big\}.$$

We then set

$$Y_A[k] = \big\{ [v] \in \mathbb{P}(V) \mid \dim(F_v \cap A) \geq k \big\},$$

so that $Y_A = Y_A[1]$, at least set-theoretically.

The loci $Y_A[k]$ also have a natural scheme structure, given by the vanishing of the determinants of the $(k+1) \times (k+1)$ minors of λ_A.

We can also describe Y_A locally as follows. Choose a Lagrangian subspace $B \subset \bigwedge^3 V$ complementary to A, so that

$$\bigwedge^3 V = A \oplus B. \tag{2.4}$$

Since both A and B are Lagrangian, the pairing

$$(\cdot, \cdot) \colon A \times B \to \mathbb{C}$$

is non-degenerate, allowing us to identify B with $A^\vee$. Assume that B is transversal to some F_v; then F_v is the graph of a linear map

$$\varphi_v \colon A \to B \cong A^\vee.$$

Indeed the projection $F_v \to A$ is an isomorphism, since its kernel is

$$F_v \cap B = \underline{0}.$$

Then φ_v is just the composition

$$A \to F_v \to B.$$

REMARK. The hypothesis that A, B and F_v are Lagrangian implies that φ_v is actually a *symmetric* map, meaning that for each $\alpha, \beta \in A$ we have

$$\langle \varphi_v(\alpha), \beta \rangle = \langle \varphi_v(\beta), \alpha \rangle, \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between $A^\vee$ and A.

Indeed (2.5) is equivalent, by the definition of the isomorphism between B and $A^\vee$ to

$$(\varphi_v(\alpha), \beta) = (\varphi_v(\beta), \alpha), \tag{2.6}$$

where now φ_v is seen as a map from A to B. To verify that (2.6) holds let us take $\gamma, \delta \in B$ such that

$$\alpha + \gamma \in F_v, \qquad \beta + \delta \in F_v;$$

such vectors are unique since the projection of F_v to A is an isomorphism (recall that B and F_v are transverse). By construction we have $\varphi_v(\alpha) = \gamma$ and $\varphi_v(\beta) = \delta$. So (2.6) becomes

$$(\alpha, \delta) = (\beta, \gamma),$$

and the latter is a consequence of

$$(\alpha, \beta) = (\gamma, \delta) = (\alpha + \gamma, \beta + \delta) = 0.$$

By definition $\ker \varphi_v = A \cap F_v$, hence we have the following local description of Y_A. Let

$$U_B = \big\{ [v] \mid F_v \text{ is transverse to } B \big\} \subset \mathbb{P}(V).$$

Then we have constructed a map

$$U_B \longrightarrow \mathrm{Sym}^2(A^\vee).$$

$$[v] \longmapsto \varphi_v$$

We now see that

$$Y_A \cap U_B = Z(\det \varphi).$$

We will see later a local description similar to the above, but better suited for computations.

2.2 – The double covering

In this section we propose to show that a generic EPW sextics admits a 2 : 1 covering, ramified exactly on the singular locus, which is smooth, and is indeed an irreducible symplectic variety.

We start with an exact sequence. Assume that Y_A is not the whole $\mathbb{P}(V)$. The map of vector bundles λ_A in (2.3) is an *injective homomorphism of sheaves*: indeed it is an isomorphism on $[v]$ for $[v] \in \mathbb{P}(V)$ generic (precisely when $[v] \notin Y_A$). Of course it is not an isomorphism, so it is not surjective; still it is clear that the cokernel is supported on Y_A. If we denote

$$i_A \colon Y_A \to \mathbb{P}(V)$$

the inclusion, then we have an exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee \longrightarrow i_{A*}(\xi_A) \longrightarrow 0 \qquad (2.7)$$

for some sheaf ξ_A on Y_A. We will see in a moment that for a generic Lagrangian subspace A the locus

$$Y_A[2] = \{ [v] \in \mathbb{P}(V) \mid \dim(F_v \cap A) \geq 2 \}$$

is properly contained in Y_A; it follows that ξ_A is generically free of rank 1.

We now dualize the exact sequence in (2.7). Note that

$$\mathcal{H}om(i_{A*}(\xi_A), \mathcal{O}_{\mathbb{P}(V)}) = 0 \quad \text{and} \quad \mathcal{E}xt^1(\mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee, \mathcal{O}_{\mathbb{P}(V)}) = 0 :$$

the first because $i_{A*}(\xi_A)$ is zero outside Y_A and the second because the two sheaves are free. So we the dual sequence is just

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \otimes A \longrightarrow F^\vee \longrightarrow \mathcal{E}xt^1(i_{A*}(\xi_A), \mathcal{O}_{\mathbb{P}(V)}) \longrightarrow 0 \ . \qquad (2.8)$$

The exact sequences in (2.7) and (2.8) fit together to form a commutative diagram. To see this we introduce the morphism of vector bundles

$$\mu_A \colon F \to \mathcal{O}_{\mathbb{P}(V)} \otimes A \qquad\qquad (2.9)$$

which is analogous to λ_A, the only difference being that we project on the other factor in decomposition (2.4). Taking into account the transpose of both λ_A and μ_A we obtain a diagram

$$0 \longrightarrow F \xrightarrow{\lambda_A} \mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee \longrightarrow i_{A*}(\xi_A) \longrightarrow 0$$

$$\xrightarrow{\mu_A} \qquad \xrightarrow{\mu_A^t}$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \otimes A \xrightarrow{\lambda_A^t} F^\vee \longrightarrow \mathcal{E}xt^1(i_{A*}(\xi_A), \mathcal{O}_{\mathbb{P}(V)}) \longrightarrow 0$$

The left square is commutative: indeed this means that

$$\mu_A^t \circ \lambda_A = \lambda_A^t \circ \mu_A;$$

in other words that the map

$$\mu_A^t \circ \lambda_A \colon F \to F^\vee$$

is symmetric. This follows from the fact that A, B and F_v are all Lagrangian subspaces, as in the remark in the previous section.

Standard diagram chasing shows that we can complete the diagram with a map

$$i_{A*}(\xi_A) \to \mathcal{E}xt^1(\mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee, \mathcal{O}_{\mathbb{P}(V)})$$

which gives rise to a morphism of short exact sequences.

The next lemma identifies the term $\mathcal{E}xt^1(i_{A*}(\xi_A), \mathcal{O}_{\mathbb{P}(V)})$, which is the most obscure one in the diagram.

LEMMA 2.1. *We have an isomorphism*

$$\mathcal{E}xt^1(i_{A*}(\xi_A), \mathcal{O}_{\mathbb{P}(V)}) \cong i_{A*}(\xi_A^\vee(6)),$$

where we have called

$$\xi_A^\vee = \mathcal{H}om(\xi_A, \mathcal{O}_{Y_A})$$

(pay attention that ξ_A is not locally free).

PROOF. Start from the exact sequence of Y_A: since it is a sextic we have, after a twist by 6,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(6) \longrightarrow \mathcal{O}_{Y_A}(6) \longrightarrow 0 \ .$$

Apply $\mathcal{H}om(\xi_A, \cdot)$ and take into account that

$$\mathcal{H}om(i_{A*}(\xi_A), \mathcal{O}_{\mathbb{P}(V)}(6)) = 0,$$

to obtain

$$0 \longrightarrow \mathcal{H}om(i_{A*}(\xi_A), \mathcal{O}_{Y_A}(6)) \overset{\alpha}{\longrightarrow} \mathcal{E}xt^1(i_{A*}(\xi_A), \mathcal{O}_{\mathbb{P}(V)}) \overset{\beta}{\longrightarrow}$$

$$\overset{\beta}{\longrightarrow} \mathcal{E}xt^1(i_{A*}(\xi_A), \mathcal{O}_{\mathbb{P}(V)}(6)) \longrightarrow \cdots$$

Since

$$\mathcal{H}om(i_{A*}(\xi_A), \mathcal{O}_{Y_A}(6)) \cong \mathcal{H}om(i_{A*}(\xi_A), \mathcal{O}_{Y_A})(6) = i_{A*}(\xi_A^\vee)(6),$$

it is enough to prove that α is an isomorphism, or equivalently that β vanishes.

But the map β is induced by the map

$$\mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(6),$$

which is multiplication by the section of $\mathcal{O}_{\mathbb{P}(V)}(6)$ whose zero locus is Y_A. Since $i_{A*}(\xi_A)$ is supported on Y_A, β is then zero, so the conclusion follows.

We have a look at the diagram we have obtained so far:

$$0 \longrightarrow F \overset{\lambda_A}{\longrightarrow} \mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee \longrightarrow i_{A*}(\xi_A) \longrightarrow 0$$

$$\overset{\mu_A}{\longrightarrow} \qquad \overset{\mu_A^t}{\longrightarrow} \qquad \longrightarrow$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \otimes A \overset{\lambda_A^t}{\longrightarrow} F^\vee \longrightarrow i_{A*}(\xi_A^\vee)(6) \longrightarrow 0;$$

in particular we are interested in the map

$$i_{A*}(\xi_A) \to i_{A*}(\xi_A^\vee)(6).$$

Denote

$$\zeta_A = \xi^\vee(3).$$

Then the above becomes a map

$$\zeta_A \to \zeta_A^\vee;$$

composing with the natural pairing between ζ_A and $\zeta_A^\vee$ we obtain a *multiplication map*

$$m_A \colon \zeta_A \otimes \zeta_A \to \mathcal{O}_{Y_A}.$$

One can prove (see [O'G08a])the following

LEMMA 2.2 (O'Grady). *The map m_A is symmetric and associative, and gives an isomorphism between $\zeta_A \otimes \zeta_A$ and $\mathcal{O}_{Y_A}$.*

Thanks to the lemma we see that the sheaf

$$\mathcal{O}_{Y_A} \oplus \zeta_A$$

has the structure of $\mathcal{O}_{Y_A}$-algebra, so we have an associated double covering.

DEFINITION 2.2. We denote by

$$X_A = \mathrm{Proj}_{Y_A}(\mathcal{O}_{Y_A} \oplus \zeta_A)$$

this double covering. The scheme X_A is called a *double EPW sextic*. We denote by

$$f_A \colon X_A \to Y_A$$

the covering map.

REMARK. We claim that the ramification locus of the map f_A is $Y_A[2]$. To see this we just need to observe that by construction the ramification locus is the locus where the sheaf ζ_A, or equivalently the sheaf ξ_A, is not locally free. Since $i_{A*}(\xi_A)$ is the cokernel of the map

$$\lambda_A \colon F \to \mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee,$$

we see that the rank of ξ_A jumps exactly along $Y_A[2]$, hence our claim.

We will verify in Section 3.1 that every double EPW sextic X_A is a deformation of $S^{[2]}$, where S is a quartic $K3$ surface. Since the latter is an irreducible symplectic variety, we deduce

PROPOSITION 2.3 (O'Grady). *Let $A \in \mathbb{LG}(\bigwedge^3 V)^0$. Then X_A is an irreducible symplectic variety. The polarized Hodge structure on $H^2(X, \mathbb{Z})$ is the same as that of $S^{[2]}$, where S is a $K3$ surface, and its Fujiki constant is 3.*

2.3 – The parameter space for double EPW sextics

In this section we wish to describe the geometry of the Lagrangian Grassmannian $\mathbb{LG}(\bigwedge^3 V)$, which is the natural parameter space for EPW sextic. Inside it we will be able to find some subvarieties corresponding to degenerate cases of EPW sextics. We anticipate that the generic behaviour is the following.

If $A \in \mathbb{LG}(\bigwedge^3 V)$ is generic, A does not contain any decomposable form, and the locus $Y_A[3]$ is empty. In this case we will see that Y_A is singular exactly along $Y_A[2]$, and the double covering X_A is smooth, actually an irreducible symplectic variety.

We start with the following

DEFINITION 2.3. If E is a symplectic vector space of dimension $2n$ we let

$$\mathbb{LG}(E) \subset \mathrm{Gr}(n, E)$$

be the subset of the Grassmannian parametrizing Lagrangian subspaces of E.

It is immediate to check that $\mathbb{LG}(E)$ is a closed subset of $\mathrm{Gr}(n, E)$. Since the symplectic group acts transitively on $\mathbb{LG}(E)$, we see that it is smooth. We describe its tangent. Recall that for a subspace $A \in \mathrm{Gr}(n, E)$ we have a canonical identification

$$T_A \mathrm{Gr}(n, E) \cong \mathrm{Hom}(A, E/A).$$

When A is Lagrangian we can use the symplectic form to identify

$$E/A = E/A^\perp \cong A^\vee.$$

It is easy to verify that under this identification we have

$$T_A \mathbb{LG}(E) = \mathrm{Sym}^2(A^\vee) \subset \mathrm{Hom}(A, A^\vee) = T_A \mathrm{Gr}(n, E).$$

In other words the tangent to $\mathbb{LG}(E)$ only consists of those maps $A \to A^\vee$ which are symmetric.

From this it is immediate to compute that

$$\dim \mathbb{LG}(E) = \dim \mathrm{Sym}^2(A^\vee) = \binom{n+1}{2}.$$

Now we come back to the case where $E = \bigwedge^3 V$. Following [O'G] we give the following definitions.

DEFINITION 2.4. We let

$$\Sigma = \left\{ A \in \mathbb{LG}(\textstyle\bigwedge^3 V) \mid \textstyle\bigwedge^3(W) \subset A \text{ for some } W \subset V, \dim W = 3 \right\}.$$

In other words Σ is the set of Lagrangian subspaces of $\bigwedge^3 V$ containing a decomposable form.

More generally for each $k \in \mathbb{N}$ we define Σ_k as the Zariski closure of the locus of Lagrangian subspaces $A \in \mathbb{LG}(\bigwedge^3 V)$ that contain exactly k *linearly independent* decomposable forms. In this way we have $\Sigma = \Sigma_1$ (see Proposition 2.4), and of course $\Sigma_k = \emptyset$ when $k > 10$.

We let
$$\Delta = \left\{ A \in \mathbb{LG}(\bigwedge^3 V) \mid Y_A[3] \neq \emptyset \right\} \subset \mathbb{LG}(\bigwedge^3 V).$$

Finally we define
$$\mathbb{LG}(\bigwedge^3 V)^0 = \mathbb{LG}(\bigwedge^3 V) \setminus (\Sigma \cup \Delta).$$

We shall now see that both Σ and Δ are divisors on $\mathbb{LG}(\bigwedge^3 V)$, so $\mathbb{LG}(\bigwedge^3 V)^0$ is open, and that $\mathbb{LG}(\bigwedge^3 V)^0$ is exactly the locus where the double covering X_A is smooth. More generally we study the dimension of Σ_k, since in this work we are particularly interested to the locus Σ_{10}.

PROPOSITION 2.4 (O'Grady).
i) *The set Σ is closed in $\mathbb{LG}(\bigwedge^3 V)$.*
ii) *The set Σ_k has codimension k in $\mathbb{LG}(\bigwedge^3 V)$ (when it is not empty).*
iii) *Σ_k is smooth away from Σ_{k+1}.*
iv) *Let*
$$A \in \Sigma_k \setminus \Sigma_{k+1},$$

so that A contains exactly k decomposable forms $\alpha_1, \ldots, \alpha_k$, up to multiples. Then the tangent space to Σ_k in A is

$$T_A \Sigma_k = \{ q \in \mathrm{Sym}^2(A^\vee) \mid q(\alpha_1) = \cdots = q(\alpha_k) = 0 \}.$$

PROOF. It is enough to prove that Σ is closed and that the description of the tangent space holds. Indeed once we know this, we see that the dimension of the tangent space is constant on $\Sigma_k \setminus \Sigma_{k+1}$, since linearly independent points induce independent conditions on quadrics. Hence, since this is an open subset of Σ_k, the common dimension is the dimension of Σ_k, so these are smooth points. Moreover the same description of the tangent yields that the codimension of Σ_k is exactly k.

So we turn to proving that Σ is closed. We consider the incidence variety

$$\widetilde{\Sigma} = \left\{ (W, A) \mid \bigwedge^3 W \subset A \right\} \subset \mathrm{Gr}(3, V) \times \mathbb{LG}(\bigwedge^3 V).$$

Since $\widetilde{\Sigma}$ is closed and Σ is its projection, it follows that Σ is closed too.

Finally we look at the description of the tangent space. To simplify notation, we deal with the case $k = 1$. Let $A(t)$ be a curve inside Σ with $A(0) = A$.

We choose a basis
$$\left\{ \beta_1(t), \ldots, \beta_9(t), \alpha(t) \right\}$$

of $A(t)$, such that $\alpha(t)$ is decomposable. Choose a subspace $C \subset \bigwedge^3 V$ complementary to A. Then the element of $\mathrm{Sym}^2(A)^\vee$ associated to the tangent vector $\dot{A}(0)$ is constructed as follows.

Since
$$\bigwedge^3 V = A \oplus C,$$
the subspace $A(t)$, for t small, is the graph of a map $f(t)\colon A \to C$. The vector
$$\dot{A}(0) \in T_A \mathbb{L}\mathbb{G}(\bigwedge^3 V)$$
corresponds to the symmetric homomorphism $f'(0)\colon A \to C$. Here the subspace C is identified, by the symplectic form, with $A^\vee$.

Let now $\alpha = \alpha(0)$ be the decomposable form in A. By definition of graph, $\alpha + f(t)\alpha \in A(t)$ for all t. So we can choose functions
$$\lambda_1(t), \ldots, \lambda_{10}(t)$$
such that
$$f(t)\alpha + \alpha = \lambda_1(t)\beta_1(t) + \cdots + \lambda_9(t)\beta_9(t) + \lambda_{10}(t)\alpha(t).$$
We observe that $\alpha = \alpha(0)$ implies that
$$\lambda_1(0) = \cdots = \lambda_9(0) = 0.$$
Moreover since A is Lagrangian
$$(\alpha, \beta_1(0)) = \cdots = (\alpha, \beta_9(0)) = (\alpha, \alpha(0)) = 0.$$
So, when we take derivatives, we are left with
$$(\alpha, f'(0)\alpha) = (\alpha, \alpha'(0)).$$
Now we use the fact that $\alpha(t)$ is decomposable, say
$$\alpha(t) = v_1(t) \wedge v_2(t) \wedge v_3(t),$$
and let $v_i = v_i(0)$, so that $\alpha = v_1 \wedge v_2 \wedge v_3$. Then we have
$$\alpha \wedge \alpha(t) = v_1 \wedge v_2 \wedge v_3 \wedge v_1(t) \wedge v_2(t) \wedge v_3(t),$$
and taking derivatives yields
$$(\alpha, f'(0)\alpha) = (\alpha, \alpha'(0)) = \mathrm{vol}(\alpha \wedge \alpha'(0)) = 0.$$
So we see that α lies on the quadric defined by $\dot{A}(0)$, that is
$$T_A \Sigma \subset \{q \in \mathrm{Sym}^2(A^\vee) \mid q(\alpha) = 0\}.$$
The other inclusion is proved in a similar way.

Next we turn the description of Δ. In this work it does not really come into play, so we limit ourselves to the following

Proposition 2.5 (O'Grady). Δ *is an irreducible divisor of* $\mathbb{LG}(\bigwedge^3 V)$.

Proof. As before we consider the incidence variety

$$\widetilde{\Delta} = \{([v], A) \mid \dim A \cap F_v \geq 3\} \subset \mathbb{P}(V) \times \mathbb{LG}(\textstyle\bigwedge^3 V).$$

We have the first projection

$$\pi \colon \widetilde{\Delta} \to \mathbb{P}(V),$$

and we claim that the fiber $\pi^{-1}([v])$ is irreducible of dimension 49.

Indeed to find a Lagrangian subspace A such that

$$\dim F_v \cap A \geq 3$$

we have to choose first a subspace $U \subset F_v$ of dimension 3, and then a Lagrangian subspace A with

$$U \subset A \subset U^{\perp}.$$

We remark that the symplectic form on $\bigwedge^3 V$ induces a symplectic form on $U^{\perp}/U$.

So we see that $\pi^{-1}([v])$ is itself fibered over $\mathrm{Gr}(3, F_v)$, whit fiber a Lagrangian Grassmannian $\mathbb{LG}(U^{\perp}/U)$. It follows that $\pi^{-1}([v])$ is irreducible and

$$\dim \pi^{-1}([v]) = \dim \mathrm{Gr}(3, F_v) + \dim \mathbb{LG}(U^{\perp}/U) = 21 + 28 = 49.$$

We conclude that $\widetilde{\Delta}$ is irreducible, with

$$\dim \widetilde{\Delta} = \dim \mathbb{P}(V) + \dim \pi^{-1}([v]) = 5 + 49 = 54.$$

Since Δ is the projection of $\widetilde{\Delta}$ it follows that Δ is irreducible, of dimension at most 54. The reverse inequality

$$\dim \Delta \geq 54$$

is a bit trickier to obtain, and we will not prove it here, since we don't need it in the sequel.

Next we see why these loci are relevant.

Proposition 2.6 (O'Grady). *Let* $A \in \mathbb{LG}(\bigwedge^3 V)$, *and assume that* Y_A *is not the whole* $\mathbb{P}(V)$. *Let* $[v] \in Y_A$. *Then* Y_A *is smooth at* $[v]$ *if and only if* $[v] \notin Y_A[2]$ *and* A *does not contain any decomposable form multiple of* $[v]$.

In other words the singular locus of Y_A *is the union of* $Y_A[2]$ *and the planes* $\mathbb{P}(W)$, *where* W *varies through all 3-planes of* V *such that* $\bigwedge^3 W \subset A$.

Corollary 2.7. *The double covering* X_A *is smooth if and only if*

$$A \in \mathbb{LG}(\textstyle\bigwedge^3 V)^0.$$

We postpone the proof of the proposition up to when we have a better grasp of the local structure of Y_A; it will appear again as Proposition 2.23. We sketch here the proof of the corollary. PROOF.[Proof of Corollary 2.7] In one direction assume that

$$A \in \mathbb{LG}(\textstyle\bigwedge^3 V)^0.$$

Since Y_A is smooth outside $Y_A[2]$ and the covering

$$f_A \colon X_A \to Y_A$$

is unramified outside $Y_A[2]$ we just have to check that X_A is smooth in the points of $f_A^{-1}(Y_A[2])$. This follows from the local description of $Y_A[2]$ in Proposition 2.27.

Viceversa assume that X_A is smooth. Since f_A is unramified outside $Y_A[2]$ it follows that Y_A is smooth outside $Y_A[2]$, hence by the above proposition we have

$$A \notin \Sigma.$$

A local study then shows that if $[v] \in Y_A[3]$, the point $f_A^{-1}([v])$ is not a smooth point of X_A, so we must have

$$A \notin \Delta.$$

Finally we mention the natural duality for double EPW sextics, introduced in [O'G08a]. The volume form vol on V induces one, which we shall call $\mathrm{vol}^\vee$, on $V^\vee$. Hence we have a symplectic form on $\bigwedge^3 V^\vee$, given by

$$\textstyle\bigwedge^3 V^\vee \times \bigwedge^3 V^\vee \longrightarrow \mathbb{C}$$

$$(\alpha, \beta) \longmapsto \mathrm{vol}^\vee(\alpha \wedge \beta)$$

As in the non Lagrangian case, we have a duality isomorphism

$$\delta \colon \mathbb{LG}(\textstyle\bigwedge^3 V) \longrightarrow \mathbb{LG}(\textstyle\bigwedge^3 V^\vee).$$

$$A \longrightarrow \mathrm{Ann}(A)$$

Hence, given $A \in \mathbb{LG}(\bigwedge^3 V)$ we have also a dual EPW sextic

$$Y_{\delta(A)} \subset \mathbb{P}(V^\vee).$$

The connection with Y_A is given by the following result.

THEOREM (O'Grady). *If $A \in \mathbb{LG}(\bigwedge^3 V)$ is generic, $Y_{\delta(A)}$ is the dual variety (in the sense of projective geometry) of Y_A.*

2.4 – The surface Z_A

Let $A \in \mathbb{LG}(\bigwedge^3 V)^0$, and let $f_A \colon X_A \to Y_A$ be the double covering of the corresponding EPW sextic. Let $Z_A \subset X_A$ be the fixed locus of the covering involution φ_A. The projection

$$f_A \colon Z_A \to Y_A[2]$$

is an isomorphism. In this section we want to recall some properties of the surface Z_A. We omit A from the notation, since it will be kept fixed.

REMARK. Z is a Lagrangian surface inside X. Indeed let σ be the symplectic form of X. The covering involution satisfies

$$\varphi^*(\sigma) = -\sigma. \tag{2.10}$$

This is not completely obvious, since the proof that X is symplectic relies on the fact that it can be deformed to $S^{[2]}$, where S is a $K3$; we will show this in Section 3.1. Granting this for a moment, one gets

$$\varphi\big|_Z = 0,$$

because Z is fixed pointwise by φ.

Equation 2.10 is a consequence of a more general fact. The polarized variety $(X, \mathcal{O}_X(1))$ is a polarized deformation of $(S^{[2]}, h)$, where

$$h \in H^2(S^{[2]}, \mathbb{Z})$$

is a certain ample class. Moreover the involution φ deforms to an involution ψ on $S^{[2]}$.

Recall that $H^2(X, \mathbb{Z})$ and $H^2(S^{[2]}, \mathbb{Z})$ are endowed with the Beauville-Bogomolov form. O'Grady shows in [O'G05b, Prop. 4.1] that

$$\psi^* \colon H^2(S^{[2]}, \mathbb{Z}) \to H^2(S^{[2]}, \mathbb{Z})$$

equals the reflection in the span of h. It follows that

$$\varphi^* \colon H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

is the reflection in the span of $c_1(\mathcal{O}_X(1))$. In particular since σ and $c_1(\mathcal{O}_X(1))$ are orthogonal with respect to the Beauville-Bogomolov form, Equation 2.10 follows.

We have the following

PROPOSITION 2.8 (O'Grady). *The canonical class of Z satisfies*

$$2K_Z = \mathcal{O}_Z(6).$$

REMARK. The above proposition determines K_Z only up to 2-torsion. Namely we can rewrite it as

$$K_Z = \mathcal{O}_Z(3) + \kappa,$$

where κ is a 2-torsion class. We will see in Section 3.3 that the class κ is really non-zero.

PROOF. For simplicity let us denote $W = f(Z)$ the singular set of Y. We know that on W the map λ has constant rank 8, so we get the following exact sequence of *vector bundles* on W:

$$0 \longrightarrow \mathcal{K} \longrightarrow F \xrightarrow{\ \lambda|_W\ } \mathcal{O}_W \otimes (\textstyle\bigwedge^3 V/A) \longrightarrow \zeta|_W \longrightarrow 0. \tag{2.11}$$

Here $\mathcal{K}$ is defined to be the kernel of $\lambda|_W$; it has rank 2. Identifying W with its preimage $Z \subset X$, we claim that the following isomorphisms hold:

$$\zeta|_W \cong \mathcal{N}_{Z/X}. \tag{2.12}$$
$$\mathcal{K} \cong \mathcal{N}_{Z/X}^{\vee} \tag{2.13}$$

Assuming Equations (2.12) and (2.13) for a moment, the exact sequence in (2.11) gives

$$c_1(\mathcal{N}_{Z/X}^{\vee}) - c_1(F) - c_1(\mathcal{N}_{Z/X}) = 0,$$

hence

$$2c_1(\mathcal{N}_{Z/X}) = -c_1(F) = \mathcal{O}_Z(6).$$

Since X has trivial canonical class, it follows that

$$2K_Z = 2c_1(\mathcal{N}_{Z/X}) = \mathcal{O}_Z(6),$$

as desired.

So we now turn to the proof of (2.12) and (2.13). Let $p \in Z$; then the covering involution φ fixes p, so φ^* acts on $T_p X$. This gives a decomposition

$$T_p X = (T_p X)_+ \oplus (T_p X)_-$$

in eigenspaces for φ^*, with eigenvalues ± 1. Since Z is the fixed locus of φ,

$$(T_p X)_+ = T_p Z.$$

On the other hand, since

$$X = \mathrm{Proj}(\mathcal{O}_Y \oplus \zeta),$$

we can identify

$$(T_p X)_- \cong \zeta_{f(p)}.$$

It follows that

$$(\mathcal{N}_{Z/X})_p \cong \zeta_{f(p)}$$

and the isomorphism in (2.12) holds.

For the other, we show that $\mathcal{K} \cong \zeta^{\vee}|_W$. Indeed observe that over W we have

$$\mathcal{K}_v = F_v \cap A \text{ and}$$
$$\zeta_v = \bigwedge\nolimits^3 V/(F_v + A).$$

The symplectic form identifies $\mathcal{K}_v^{\vee}$ with the quotient $\bigwedge^3 V/(F_v \cap A)^{\perp}$, and since both A and F_v are Lagrangian we have

$$(F_v \cap A)^{\perp} = F_v^{\perp} + A^{\perp} = F_v + A,$$

thereby proving isomorphism (2.13).

COROLLARY 2.9. *For $A \in \mathbb{LG}(\bigwedge^3 V)^0$ the surface $Z_A \cong Y_A[2]$ is of general type.*

We end this brief section with the computation of some invariants of Z. We recall without proof from [O'G08b] that the topological Euler characteristic of Z is $\chi(Z) = 192$, and that $Y[2]$ has degree 40 in $\mathbb{P}^5$. It follows from Proposition 2.8 that

$$K_Z^2 = 9 \deg Y[2] = 360,$$

so by Noether formula

$$\chi(Z, \mathcal{O}_Z) = \frac{1}{12}(\chi(Z) + K_Z^2) = \frac{1}{12}(192 + 360) = 46.$$

By definition this is $1 - q(Z) + p_g(Z)$, hence $p_g(Z) = 45 + q(Z)$. By Corollary 3.19 we find

$$q(Z) = 0, \qquad p_g(Z) = 45.$$

2.5 – *EPW* sextics containing a plane

We consider in this section the case where the Lagrangian subspace A contains some nonzero completely decomposable tensor $w_1 \wedge w_2 \wedge w_3$. Let

$$W = \langle w_1, w_2, w_3 \rangle \, ;$$

then $\dim W = 3$ and $\bigwedge^3 W \subset A$.

It is clear by definition that $\mathbb{P}(W) \subset Y_A$, and as we have remarked Y_A is singular along $\mathbb{P}(W)$, as well as on $Y_A[2]$. In this section we want to show that $\mathbb{P}(W)$ and $Y_A[2]$ meet along a sextic curve on $\mathbb{P}(W)$, and that this curve comes endowed with an unramified double covering.

PROPOSITION 2.10 (O'Grady, [O'G]). *Let $W \subset V$ be a subspace with $\dim W = 3$, $A \subset \bigwedge^3 V$ a generic Lagrangian subspace containing $\bigwedge^3 W$. Then*

$$C_{A,W} = \mathbb{P}(W) \cap Y_A[2]$$

is a curve of degree 6 inside $\mathbb{P}(W)$.

PROOF. The proof is almost a repetition of the fact that Y_A is a sextic. Take any $[w] \in \mathbb{P}(W)$; then $[w] \in Y_A[2]$ if and only if

$$\dim(A \cap F_w) \geq 2.$$

By hypothesis $\bigwedge^3 W \subset A$, and it is clear that $\bigwedge^3 W \subset F_w$, so in any case

$$\dim(A \cap F_w) \geq 1.$$

To get rid of this, we quotient out the factor $\bigwedge^3 W$. So let

$$E_W = (\bigwedge^3 W)^\perp / \bigwedge^3 W,$$

where orthogonality is with respect to the symplectic form on $\bigwedge^3 V$; this is well defined because $\bigwedge^3 W$ is of course isotropic. The symplectic form on $\bigwedge^3 V$ induces an antisymmetric form on E_W, which is again non-degenerate, so E_W is a symplectic space itself. Inside E_W we find

$$B = A / \bigwedge^3 W$$

(recall that $A \subset (\bigwedge^3 W)^\perp$, since it is isotropic and contains $\bigwedge^3 W$).

The trivial symplectic vector bundle $E_W \otimes \mathcal{O}_{\mathbb{P}(W)}$ has a Lagrangian vector sub-bundle G defined by

$$G_w = F_w / \bigwedge^3 W.$$

By the remark at the beginning of the proof,

$$C_{A,W} = \{[w] \in \mathbb{P}(W) \mid \dim(B \cap G_w) > 0\}.$$

The inclusion of G inside $E_W \otimes \mathcal{O}_{\mathbb{P}(W)}$, followed by projection, gives a map of vector bundles

$$\nu_{A,W} \colon G \to E_W / B,$$

and $C_{A,W}$ is the locus where $\nu_{A,W}$ has not maximal rank. Since

$$c_1(G) = c_1(F) = \mathcal{O}_{\mathbb{P}(W)}(-6),$$

and E_W/B is trivial, $\det \nu_{A,W}$ is a section of

$$\det(\mathcal{H}om(G, E_W/B)) = \mathcal{O}_{\mathbb{P}(W)}(6),$$

and $C_{A,W}$, being the zero locus of $\det \nu_{A,W}$, is a sextic curve, unless $\det \nu_{A,W}$ vanishes identically.

In Section 2.6 we will compute local equations for $C_{A,W}$. In particular we will see that for A generic, it is smooth, so certainly it is not the whole $\mathbb{P}(W)$.

We can also mimic the construction of the double covering X_A of Y_A, to obtain a double covering $\widetilde{C}_{A,W}$ of $C_{A,W}$. First we define

$$C_{A,W}[k] = \{[w] \in \mathbb{P}(W) \mid \dim(B \cap G_w) \geq k\},$$

so that $C_{A,W} = C_{A,W}[1]$. Different from the case of Y_A, we expect for dimensional reasons that $C_{A,W}[k]$ is empty for $k \geq 2$, and this is actually the case for A generic in Σ. Indeed, being a degeneracy locus, $C_{A,W}$ is singular along $C_{A,W}[2]$. But it follows from the results of Section 2.6 that $C_{A,W}$ is smooth for A generic in Σ.

Let us choose a complementary Lagrangian subspace in E_W to B; as in Section 2.1 the symplectic form allows one to identify this subspace to $B^\vee$, so one obtains the decomposition

$$E_W = B \oplus B^\vee.$$

Let

$$\lambda \colon G \to B^\vee \otimes \mathcal{O}_{P(W)}$$

denote the inclusion of G inside E_W followed by the projection on B, and similarly for

$$\mu \colon G \to B \otimes \mathcal{O}_{P(W)}.$$

These maps and their adjoints fit into a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G & \xrightarrow{\lambda} & \mathcal{O}_{\mathbb{P}(W)} \otimes B^\vee & \longrightarrow & j_*(\xi) & \longrightarrow & 0 \\
& & \xrightarrow{\mu} & & \xrightarrow{\mu^t} & & \longrightarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}(W)} \otimes B & \xrightarrow{\lambda^t} & G^\vee & \longrightarrow & \mathcal{E}xt^1(j_*(\xi), \mathcal{O}_{\mathbb{P}(W)}) & \longrightarrow & 0.
\end{array}
\tag{2.14}
$$

Here $j \colon C_{A,W} \to \mathbb{P}(W)$ denotes the inclusion, and the lower line is obtained by dualizing the upper one. Details are completely analogous to Section 2.2.

LEMMA 2.11. *We have an isomorphism*

$$\mathcal{E}xt^1(j_*(\xi), \mathcal{O}_{\mathbb{P}(W)}) \cong j_*(\xi^\vee(6))$$

PROOF. Start from the exact sequence of $C_{A,W}$: since it is a sextic we have, after a twist by 6,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(W)} \longrightarrow \mathcal{O}_{\mathbb{P}(W)}(6) \longrightarrow \mathcal{O}_{C_{A,W}}(6) \longrightarrow 0 \ .$$

Apply $\mathcal{H}om(j_*(\xi), \cdot)$ and take into account that

$$\mathcal{H}om(j_*(\xi), \mathcal{O}_{\mathbb{P}(V)}(6)) = 0,$$

to obtain

$$0 \longrightarrow \mathcal{H}om(j_*(\xi), \mathcal{O}_{C_{A,W}}(6)) \xrightarrow{\ \alpha\ } \mathcal{E}xt^1(j_*(\xi), \mathcal{O}_{\mathbb{P}(W)}) \xrightarrow{\ \beta\ }$$

$$\xrightarrow{\ \beta\ } \mathcal{E}xt^1(j_*(\xi), \mathcal{O}_{\mathbb{P}(W)}(6)) \longrightarrow \cdots \ .$$

Since

$$\mathcal{H}om(j_*(\xi), \mathcal{O}_{C_{A,W}}(6)) \cong \mathcal{H}om(j_*(\xi), \mathcal{O}_{C_{A,W}})(6) = j_*(\xi^\vee)(6)$$

it enough to prove that α is an isomorphism, or equivalently that β vanishes.

But the map β is induced by the map

$$\mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(6),$$

which is multiplication by a section of $\mathcal{O}_{\mathbb{P}(W)}(6)$ which vanishes on $C_{A,W}$. Since $j_*(\xi)$ is supported on $C_{A,W}$, β is then zero, so the conclusion follows.

The preceding lemma, together with the commutative diagram 2.14 gives us a map

$$j_*(\xi) \to j_*(\xi^\vee(6))$$

or equivalently

$$\xi \to \xi^\vee(6).$$

Let $\zeta = \xi(3)$; then this map can be viewed as a map $\zeta \to \zeta^\vee$. Composing it with the natural pairing between ζ and its dual we get a map

$$m \colon \zeta \otimes \zeta^\vee \to \mathcal{O}_{C_{A,W}}.$$

We have the analog

LEMMA 2.12 (O'Grady). *The map m is symmetric and associative, so it gives the sheaf*

$$\mathcal{O}_{C_{A,W}} \oplus \zeta$$

the structure of a $\mathcal{O}_{C_{A,W}}$-algebra.

After this lemma we can define $\widetilde{C}_{A,W}$ as the double covering of $C_{A,W}$ associated to the algebra $\mathcal{O}_{C_{A,W}} \oplus \zeta$. If $C_{A,W}[2]$ is empty, ζ is locally free, hence the covering is unramified.

We compute the genera. For $A \in \Sigma$ generic the curve $C_{A,W}$ is a smooth sextic, so it has genus

$$g(C_{A,W}) = \binom{5}{2} = 10.$$

The genus of $\widetilde{C}_{A,W}$ can be computed by Riemann-Hurwitz and it turns out that $g(\widetilde{C}_{A,W}) = 19$.

REMARK. Let $A \in \Sigma$ be a Lagrangian subspace, such that $A \supset \bigwedge^3 W$ for exactly one subspace $W \subset V$ of dimension 3. Then we can construct an irreducible symplectic variety in the following way.

Let X_A be the double covering of Y_A ramified over $Y_A[2]$; then X_A is singular along the double covering S of $\mathbb{P}(W)$. The surface S is a double covering of $\mathbb{P}(W)$ ramified along the smooth sextic $C_{A,W}$, hence it is a $K3$. Let $\widetilde{X}_A$ be the blowup of X_A along S. Then it is not difficult to see that $\widetilde{X}_A$ is an irreducible symplectic variety, deformation equivalent to a smooth double EPW sextic.

2.6 – Local study

We wish to understand the local description of the surface $Y_A[2]$, in particular when A contains $\bigwedge^3 W$ for some 3-plane $W \subset V$. In particular we will see that $Y_A[2]$ is smooth away from $\mathbb{P}(W)$ and we will describe its singularities along the curve $C_{A,W} = Y_A[2] \cap \mathbb{P}(W)$. The most delicate analysis will be needed near point where two planes $\mathbb{P}(W)$ and $\mathbb{P}(W')$ meet.

In doing so we will also describe the singularities of the curve $C_{A,W}$ itself; actually we shall tackle this problem first, since it will be needed to study Y_A. Our final goal is to be able to produce an explicit sequence of blow-ups that resolves the singularities of $Y_A[2]$. This will be needed in the sequel to show that when Y_A contains 10 planes and is generic with respect to this condition, the surface $Y_A[2]$ is Enriques: we will be able to show this by studying sections of the canonical bundle of our explicit desingularization.

To start our program let us fix some point $[v_0] \in Y = Y_A$ and choose a complement V_0 in V, so

$$V = \langle v_0 \rangle \oplus V_0. \tag{2.15}$$

LEMMA 2.13. *Assume $Y_{A^\perp}$ is not the whole $\mathbb{P}(V^\vee)$. Then we can choose the complement V_0 in such a way that*

$$\bigwedge^3 V = A \oplus \bigwedge^3 V_0. \tag{2.16}$$

By construction we have $\bigwedge^3 V_0 \cap F_{v_0} = \underline{0}$; since both subspaces have dimension 10 we then have

$$\bigwedge^3 V = F_{v_0} \oplus \bigwedge^3 V_0. \tag{2.17}$$

Let us call

$$\pi \colon \bigwedge^3 V \to F_{v_0}$$

the projection relative to this decomposition. Since $A \cap \bigwedge^3 V_0 = \underline{0}$, the restriction

$$\pi\big|_A \colon A \to F_{v_0}$$

is an isomorphism; its inverse followed by projection on $\bigwedge^3 V_0$ gives a well-defined map

$$\tau_A \colon F_{v_0} \to \bigwedge^3 V_0$$

whose graph is A. Moreover, since F_{v_0} and $\bigwedge^3 V_0$ are complementary Lagrangian subspaces, the symplectic form identifies $\bigwedge^3 V_0$ with $F_{v_0}^\vee$: namely $\beta \in \bigwedge^3 V_0$ corresponds to the functional

$$F_{v_0} \longrightarrow \mathbb{C}.$$

$$\alpha \longmapsto \mathrm{vol}(\beta \wedge \alpha)$$

The fact that A is Lagrangian too easily implies that τ_A is symmetric.

To put it another way, we also have an isomorphism

$$\bigwedge^2 V_0 \longrightarrow F_{v_0}.$$

$$\alpha \longmapsto v_0 \wedge \alpha$$

So we can consider τ_A as a map

$$\tau_A \colon \bigwedge^2 V_0 \to \bigwedge^3 V_0,$$

and again we see that τ_A is symmetric, with respect to the obvious duality between $\bigwedge^2 V_0$ and $\bigwedge^3 V_0$ given by

$$\bigwedge^2 V_0 \times \bigwedge^3 V_0 \longrightarrow \mathbb{C}.$$

$$(\alpha, \beta) \longmapsto \mathrm{vol}(v_0 \wedge \alpha \wedge \beta)$$

The quadratic form on $\bigwedge^2 V_0$ associated to the symmetric map τ_A will be denoted by q_A. Sometimes we will abuse notation and use q_A to denote a quadratic form on F_{v_0} as well. In particular we see that the kernel of q_A (as a form on F_{v_0}) is just

$$\ker \tau_A = A \cap F_{v_0}.$$

This makes the construction above look promising to study the loci $Y_A[k]$ where the dimensions of $A \cap F_v$ jump.

In particular we will describe these loci in a neighbourhood of $[v_0]$. We can use V_0 as an affine chart of $\mathbb{P}(V)$ near $[v_0]$, identifying $v \in V_0$ with the point $[v+v_0] \in \mathbb{P}(V)$. If $v \in V_0$ the subspace F_{v+v_0} remains transverse to $\bigwedge^3 V_0$. Indeed $v + v_0 \notin V_0$, so we have a decomposition

$$V = \langle v + v_0 \rangle \oplus V_0$$

analogous to 2.15, and as a consequence

$$\bigwedge{}^3 V = F_{v+v_0} \oplus \bigwedge{}^3 V_0. \tag{2.18}$$

Since the subspace F_{v+v_0} is again Lagrangian we can perform the same construction we did for A to obtain a symmetric map

$$\tau_v \colon \bigwedge{}^2 V_0 \to \bigwedge{}^3 V_0$$

and an associated quadratic form q_v on $\bigwedge^2 V_0$, or on F_{v_0}. The following lemma extends our previous remark, allowing us to get a description of $Y_A[k]$ on the affine chart V_0 of $\mathbb{P}(V)$.

LEMMA 2.14 (O'Grady). *The kernel of $q_A - q_v$ on F_{v_0} is the projection of $A \cap F_{v+v_0}$ on F_{v_0}, relative to the decomposition* (2.17).

PROOF. It is immediate by the definitions that this kernel is just $\ker \tau_A - \tau_v$, so $\alpha \in F_{v_0}$ will lie in the kernel if and only if $\tau_A \alpha = \tau_v \alpha$. In this case let us call β this element. Then by construction

$$\beta \in A \cap F_{v+v_0}$$

and

$$\pi(\beta) = \alpha.$$

Viceversa if $\beta \in A \cap F_{v+v_0}$ then $\alpha = \pi(\beta)$ satisfies $\tau_A \alpha = \tau_v \alpha$.

This result suggests to consider the map that associates to every $v \in V_0$ the quadratic form $q_A - q_v$. Actually this extends to the whole projective space, so we define

$$\psi \colon \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^2(\bigwedge{}^2 V_0)^\vee) \tag{2.19}$$

by the requirements that

$$\psi(v + v_0) = q_A - q_v \text{ for all } v \in V_0,$$
$$\text{in particular } \psi(v_0) = q_A;$$
$$\psi(v) = q_v \text{ for all } v \in V_0.$$

Note that ψ is a well defined (linear) projective map. Once we require that $\psi(v) = q_v$ for all $v \in V_0$ and $\psi(v_0) = q_A$, ψ is determined up to normalization. Explicitly we have to choose a vector $v \in V_0$ and some $\lambda \in \mathbb{C}$ such that $\psi(v + v_0) = q_A + \lambda q_v$, our choice corresponding to $\lambda = -1$; once you choose λ for one such v it is the same for every element of V_0.

Inside $\mathbb{P}(\mathrm{Sym}^2(\bigwedge^2 V_0)^\vee)$ we consider the loci U_j of quadratic form of rank at most j. Then we can restate Lemma 2.14 as follows:

COROLLARY 2.15 (of Lemma 2.14). *For each $k \geq 1$ we have*

$$Y[k] \cap V_0 = \psi^{-1}(U_{11-k}),$$

where of course we regard V_0 as an affine chart of $\mathbb{P}(V)$ identifying $v \in V_0$ with the point $[v + v_0] \in \mathbb{P}(V)$ as we did before.

REMARK. Since $Y[k]$ is locally defined by the vanishing of some minors, one can check that the equality above is actually an equality of *schemes*.

It is useful to describe explicitly the quadratic forms q_v: we shall now see that these are all in Plücker form. Take $\alpha \in \bigwedge^2 V_0$; then this is identified to $v_0 \wedge \alpha \in F_{v_0}$. Now, for any $v \in V_0$ the form $v \wedge \alpha \in \bigwedge^3 V_0$, and

$$(v + v_0) \wedge \alpha = v \wedge \alpha + v_0 \wedge \alpha$$

is written as a sum of an element in F_{v_0} and one in $\bigwedge^3 V_0$. Since τ_v is defined as the map having F_{v+v_0} as a graph, we see that

$$\tau_v(v_0 \wedge \alpha) = v \wedge \alpha.$$

It follows that

$$q_v(\alpha) = \mathrm{vol}(v_0 \wedge \alpha \wedge v \wedge \alpha) = \mathrm{vol}(v_0 \wedge v \wedge \alpha \wedge \alpha). \tag{2.20}$$

If we denote by $\widetilde{q_v}$ the bilinear symmetric form associated to q_v, then the polarized form of (2.20) is

$$\widetilde{q_v}(\alpha, \beta) = \mathrm{vol}(v_0 \wedge v \wedge \alpha \wedge \beta). \tag{2.21}$$

2.6.1 – The curve $C_{A,W}$

Having set out these preliminaries we assume that A contains $\bigwedge^3 W$ for some $W \subset V$ of dimension 3, and we pass to the local study of the curve $C_{A,W}$. So we consider the construction above starting from a point $[v_0] \in C_{A,W}$. Let

$$W_0 = V_0 \cap W$$

and choose a base v_1, v_2 of W_0, so that A contains the decomposable form $v_0 \wedge v_1 \wedge v_2$.

We consider the quadrics parametrized (via the map ψ in Equation (2.19)) by $\mathbb{P}(W) \subset \mathbb{P}(V)$. The first remark is that the quadrics in $\psi(\mathbb{P}(W))$ are all singular in the point

$$p = [v_1 \wedge v_2] \in \mathbb{P}(\textstyle\bigwedge^2 V_0).$$

This amounts to proving that $v_1 \wedge v_2$ lies in the kernel of q_A and in that of q_v for every $v \in W_0$. By Equation (2.21) is clear that

$$\widetilde{q_v}(v_1 \wedge v_2, \cdot) \equiv 0,$$

so $v_1 \wedge v_2$ is in the kernel of q_v for every $v \in W_0$. Then the fact that $v_0 \wedge v_1 \wedge v_2 \in A$ shows that

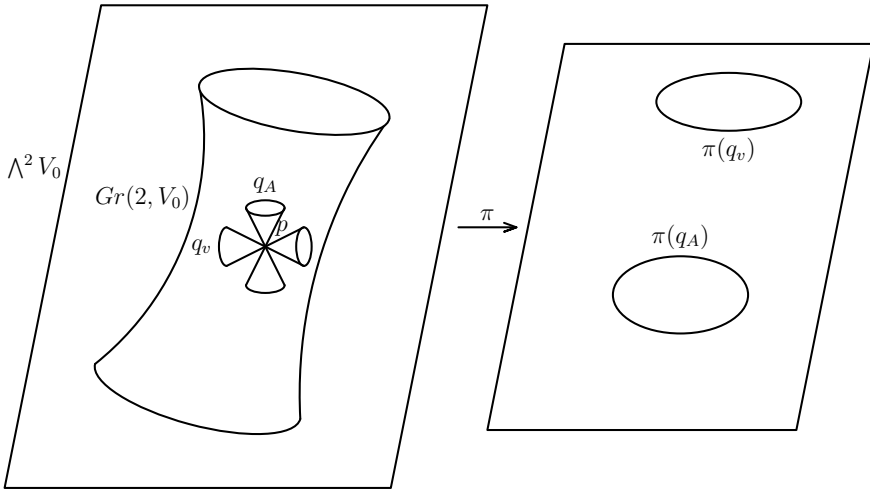$$\tau_A(v_1 \wedge v_2) = 0,$$

so the remark is proved.

Figure 2.1 The quadrics in $\psi(\mathbb{P}(W))$

Now a singular quadric is just a cone over a quadric of smaller rank, so it is natural to consider the projection π from p to some $\mathbb{P}^8 \subset \mathbb{P}(\bigwedge^2 V_0)$. In coordinate-free terms the projection will be on

$$\mathbb{P}(\bigwedge^2 V_0 / \langle v_1 \wedge v_2 \rangle).$$

Let us call

$$q_0 = \pi(q_A)$$
$$q_1 = \pi(q_{v_1})$$
$$q_2 = \pi(q_{v_2}).$$

Since $[v_0] \in C_{A,W} \subset Y_A[2]$ we have

$$\dim(A \cap F_{v_0}) \geq 2.$$

Let $K \subset \bigwedge^2 V_0$ be the subspace corresponding to $A \cap F_{v_0}$ under the identification $\bigwedge^2 V_0 \cong F_{v_0}$, which is just the kernel of the quadratic form q_A. Our first hypothesis is that $\dim K = 2$, or in other words that

$$[v_0] \in Y_A[2] \setminus Y_A[3].$$

Then it follows that q_0 has rank exactly 8.

Instead the quadrics in the pencil generated by q_1 and q_2 have all rank 6. Each such quadric has the form $\pi(q_v)$ for some $v \in W_0$. From Equation (2.21) it is immediate that the kernel of q_v is generated by the multiples of v inside $\bigwedge^2 V_0$, so it has dimension 4. The quadric $\pi(q_v)$ has the same rank of q_v, so the claim follows.

LEMMA 2.16. *Let $\mathcal{N}$ denote the net of quadrics generated by q_0, q_1 and q_2, and let*

$$\widetilde{\psi} \colon \mathbb{P}(W) \to \mathcal{N}$$

be the obvious isomorphism (in other words $\widetilde{\psi}(v)$ is the projection of $\psi(v)$). Let $\Delta \subset \mathcal{N}$ be the set of singular quadrics. Then Δ is a curve of degree 9, and we have the decomposition

$$\Delta = \widetilde{\psi}(C_{A,W}) + 3\ell,$$

where $\ell = \widetilde{\psi}(\mathbb{P}(W_0))$ is the pencil generated by q_1 and q_2.

It is useful to recall the following formula for the derivative of a determinant: if A is a $n \times n$ matrix whose entries are functions of t, then

$$\frac{d}{dt} \det A(t) = \sum_{i=1}^{n} \det \big(A_1(t) \mid \cdots \mid A_i'(t) \mid \cdots \mid A_n(t) \big), \qquad (2.22)$$

where the prime means d/dt and A_j is the j-th column of A. This follows formally from Leibniz formula, so it is valid both in the analytic and algebraic context, if the entries of $A(t)$ are polynomials.

PROOF. In coordinates Δ is given by the vanishing of the determinant of a 9×9 matrix, so it is a curve of degree 9.

Since $C_{A,W} = Y_A[2] \cap \mathbb{P}(W)$ we see by Corollary 2.15 that $\widetilde{\psi}(C_{A,W})$ is contained in Δ; more precisely Δ consists of this curve and at most some components which don't meet the affine chart

$$\mathcal{N} \setminus \widetilde{\psi}(\mathbb{P}(W_0)).$$

But then the only other component that can appear is ℓ; since $\deg(C_{A,W}) = 6$ and $\deg(\Delta) = 9$, it must appear with multiplicity 3.

We can also check this directly. Choose coordinates on $\mathbb{P}^8$ so that q_0, q_1 and q_2 are represented by matrices R, S and T respectively. We develop the equation of Δ near the point $q_1 \in \ell$. Consider the function

$$f(r,t) = \det(rR + S + tT).$$

To prove that ℓ appears with multiplicity 3 in Δ we must show that

$$\frac{\partial^i}{\partial t^i} f \big|_{r=t=0} = 0$$

for $i = 0, 1, 2$. The case $i = 0$ amounts to saying that q_1 is singular. Using (2.22) we can compute

$$\frac{\partial}{\partial t} \det(rR + S + tT)\big|_{r=t=0} = \sum_{i=1}^{9} \det\left(S_1 \mid \cdots \mid T_i \mid \cdots \mid S_9\right) = 0,$$

since all 8×8 minors of S vanish. Similarly

$$\frac{\partial^2}{\partial t^2} \det(rR + S + tT)\big|_{r=t=0} = \sum_{i,j=1}^{9} \det\left(S_1 \mid \cdots \mid T_i \mid \cdots \mid T_j \mid \cdots \mid S_9\right) = 0,$$

since all 7×7 minors of S vanish.

We can now prove that $C_{A,W}$ is smooth at $[v_0]$ provided $[v_0]$ doesn't lie on another plane $\mathbb{P}(W')$ for some $W' \subset V$ with $\dim W' = 3$ and $\bigwedge^3 W' \subset A$. But before doing this it will be useful to do a remark.

Inside $\mathbb{P}(\bigwedge^2 V_0)$ there is the Grassmannian $\mathrm{Gr}(2, V_0)$, composed of decomposable forms. It has dimension 6 and it is easy to see, by Schubert calculus, that

$$\deg \mathrm{Gr}(2, V_0) = \sigma_1^6 = (\sigma_1 \cdot (\sigma_2 + \sigma_{1,1}))^2 = (\sigma_3 + 2\sigma_{2,1})^2 = 5$$

(the notation is standard, see for example [GH78, sec. 1.5]). It is well-known that $\mathrm{Gr}(2, V_0)$ is cut out by the Plücker quadrics q_v given by (2.20). It follows that its projection $\pi(\mathrm{Gr}(2, V_0))$ has dimension 6 and degree 4; since it must be contained in both q_1 and q_2, we see that

$$q_1 \cap q_2 = \pi(\mathrm{Gr}(2, V_0)).$$

PROPOSITION 2.17 (O'Grady). *Assume that* $\dim(A \cap F_{v_0}) = 2$ *and* $[v_0] \in \mathbb{P}(W)$, *where* W *is as above. Then the following conditions are equivalent:*

i) *The curve* $C_{A,W}$ *is smooth at* $[v_0]$.
ii) $A \cap F_{v_0}$ *contains only one decomposable form* $v_0 \wedge v_1 \wedge v_2$ *up to multiples.*

PROOF. We use the notation above; in particular we use R, S, T to denote matrices for q_0, q_1 and q_2 respectively. Under our hypothesis the kernel of q_0 is one dimensional, and we choose a base $\{\alpha_1, \dots, \alpha_9\}$ of $\mathbb{P}(\bigwedge^2 V_0 / \langle v_1 \wedge v_2 \rangle)$ such that this kernel is spanned by α_9.

Using Lemma 2.16 we see that it is enough to prove that Δ is smooth at q_0. So we consider the local equation of Δ around q_0, given by

$$f(s, t) = \det(R + sS + tT) = 0.$$

With the aid of (2.22) we can compute

$$\frac{\partial}{\partial s} f \big|_{s=t=0} = \sum_{i=1}^{9} \det \left(R_1 \mid \cdots \mid S_i \mid \cdots \mid R_9 \right).$$

By construction the column R_9 is identically zero, so

$$\frac{\partial}{\partial s} f \big|_{s=t=0} = \det \left(R_1 \mid \cdots \mid R_8 \mid S_9 \right).$$

In this matrix the last row is 0 except for the last entry; hence the derivative is just $S_{9,9}$ times the determinant of the 8×8 top-left minor of R. Since the latter does not vanish we find that

$$\frac{\partial}{\partial s} f \big|_{s=t=0} = 0 \text{ if and only if } S_{9,9} = 0.$$

Similarly

$$\frac{\partial}{\partial t} f \big|_{s=t=0} = 0 \text{ if and only if } T_{9,9} = 0,$$

so $[v_0]$ is a singular point of $C_{A,W}$ if and only if both $S_{9,9}$ and $T_{9,9}$ vanish.

By the remark before the proof this means that

$$\alpha_9 \in q_1 \cap q_2 = \pi(\mathrm{Gr}(2, V_0)).$$

So $[v_0]$ is a singular point of $C_{A,W}$ if and only if K contains a decomposable form other than $v_1 \wedge v_2$.

In view of this result our next concern is to study what kind of singularity we find near a point of $C_{A,W}$ which happens to lie on another plane $\mathbb{P}(W')$. We should first remark that any two such planes intersect, and any point in the intersection $\mathbb{P}(W) \cap \mathbb{P}(W')$ actually lies on the curve.

Indeed assume we have two subspaces $W, W' \subset V$, with

$$\dim(W) = \dim(W') = 3$$

and such that

$$\bigwedge\nolimits^3 W, \bigwedge\nolimits^3 W' \subset A.$$

Choose bases $\{a_1, a_2, a_3\}$ for W and $\{a_4, a_5, a_6\}$ for W'. Since A is Lagrangian we must have

$$a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_6 = 0,$$

hence

$$\dim(W + W') \leq 5.$$

It follows that $\dim(W \cap W') \geq 1$, so $\mathbb{P}(W) \cap \mathbb{P}(W') \neq \emptyset$.

Next let $v_0 \in W \cap W'$; then $[v_0] \in Y_A[2]$. Indeed

$$F_{v_0} \supset \bigwedge\nolimits^3 W \text{ and } F_{v_0} \supset \bigwedge\nolimits^3 W',$$

so $\dim(F_{v_0} \cap A) \geq 2$. Since

$$C_{A,W} = \mathbb{P}(W) \cap Y_A[2]$$

it follows that $[v_0] \in C_{A,W}$, and by Lemma 2.17 it is a singular point.

In order to study the singularities of $C_{A,W}$ at $[v_0]$ we need an elementary lemma.

LEMMA 2.18. *Let V be a complex vector space of finite dimension and let q, q' be two non-degenerate quadratic forms on V. Assume that the quadrics*

$$Q = V(q) \text{ and } Q' = V(q')$$

in $\mathbb{P}(V)$ are transverse. Then q and q' can be diagonalized simultaneously.

PROOF. Choose a diagonal basis for q and represent q' with respect to this basis by the symmetric matrix M. Let v be any eigenvector for M, say $Mv = \lambda v$. Then the q-orthogonal of v is

$$v^{\perp q} = \{w \in V \mid (v, w) = 0\}$$

and similarly

$$v^{\perp q'} = \{w \in V \mid (Mv, w) = 0\},$$

where $(\cdot, \cdot)$ denotes standard scalar product in the chosen basis. Hence we have $v^{\perp q} = v^{\perp q'}$, and we call it simply $v^{\perp}$.

It follows that $v \notin v^{\perp}$. Indeed if $v \in v^{\perp}$ we find

$$q(v) = q'(v) = 0,$$

so $[v] \in Q \cap Q'$. Since the projective tangent space $T_v Q$ is just $\mathbb{P}v^{\perp}$ and similarly for Q', we see that $[v]$ is a tangency point between Q and Q', which is excluded by hypothesis.

Thus we have the direct sum decomposition

$$V = \langle v \rangle \oplus v^{\perp}$$

and we can conclude by induction on the dimension of V.

Let now W, W' be as above and assume that

$$\dim(W \cap W') = 1.$$

Let $\langle v_0 \rangle = W \cap W'$ and choose bases $\{v_0, v_1, v_2\}$ of W and $\{v_0, v_3, v_4\}$ of W'. Finally we complete this to a basis $\{v_0, v_1, v_2, v_3, v_4, v_5\}$ of V, where v_5 is chosen such that

$$\mathrm{vol}(v_0 \wedge v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge v_5) = 1.$$

We choose

$$V_0 = \langle v_1, v_2, v_3, v_4, v_5 \rangle.$$

Finally we make the following assumption:

ASSUMPTION 2.19. Let q be the quadratic form on $\bigwedge^2 V_0$ given by standard scalar product with respect to the monomial basis

$$v_1 \wedge v_2, v_1 \wedge v_3, \ldots, v_4 \wedge v_5.$$

Then the quadrics on $\mathbb{P}(\bigwedge^2 V_0)$ defined by q and by q_A are transverse.

We are now ready to state the

PROPOSITION 2.20. *Let A, W, W' and v_0 be as above, and suppose that*

$$\dim(A \cap F_{v_0}) = 2$$

and Assumption 2.19 is satisfied. Then $C_{A,W}$ has a node at $[v_0]$.

PROOF. We write the local equation of $C_{A,W}$, or, using the isomorphism $\widetilde{\psi}$, the local equation of Δ near q_0. Under our assumptions the quadratic form q_0 has a kernel of dimension 1, generated by (the class modulo $v_1 \wedge v_2$ of) $v_3 \wedge v_4$.

Let J be the subspace of $\bigwedge^2 V_0 / \langle v_1 \wedge v_2 \rangle$ generated by all monomials $v_i \wedge v_j$ except for $v_3 \wedge v_4$. Then the restriction of q_0 to J is non-degenerate.

By Lemma 2.18 we can choose a basis for J under which q_0 is the identity matrix, and we can do so using an orthogonal basis for for the standard scalar product (with respect to the monomial basis). In other words we can find a basis of $\bigwedge^2 V_0 / \langle v_1 \wedge v_2 \rangle$ such that

  i) The matrix for q_0 is just $R = \mathrm{diag}(1, \ldots, 1, 0)$;
  ii) the matrix M of change of basis has the block form

$$M = \begin{pmatrix} A & 0 \\ \underline{0} & 1 \end{pmatrix} \tag{2.23}$$

and satisfies $M^T \cdot M = \mathrm{I}$.

We denote by S and T respectively the matrices for q_1 and q_2 in the chosen basis. We remark that

$$q_1(v_3 \wedge v_4) = q_2(v_3 \wedge v_4) = 0,$$

so $S_{9,9} = T_{9,9} = 0$.

Let

$$f(s,t) = \det(R + sS + tT).$$

We have already seen that f vanishes of order at least 2 at $s = t = 0$.

Using Equation (2.22) we compute

$$\frac{\partial^2}{\partial s^2} f \Big|_{s=t=0} = \sum_{i,j=1}^{9} \det\big(R_1 \mid \cdots \mid S_i \cdots \mid S_j \mid \cdots \mid R_9 \big).$$

Since $R_9 = 0$ the only non-zero addends are those with $i = 9$ or $j = 9$, so

$$\frac{\partial^2}{\partial s^2} f \Big|_{s=t=0} = 2 \sum_{i=1}^{8} \det\big(R_1 \mid \cdots \mid S_i \mid \cdots \mid R_8 \mid S_9 \big) = -2 \sum_{i=1}^{8} (S_{i,9})^2,$$

where we have used the symmetry of S and the fact the determinants above can be expanded (using the fact that the top-left 8×8 minor of R is the identity) to obtain

$$\det \left(R_1 \mid \cdots \mid S_i \mid \cdots \mid R_8 \mid S_9 \right) = \det \begin{pmatrix} S_{i,i} & S_{i,9} \\ S_{9,i} & 0 \end{pmatrix}.$$

We can perform the same computation for the other second derivatives, so finally the Hessian of f at $(0,0)$ is

$$H = - \begin{pmatrix} 2 \sum_{i=1}^{8} (S_{i,9})^2 & \sum_{i=1}^{8} S_{i,9} T_{i,9} \\ \sum_{i=1}^{8} S_{i,9} T_{i,9} & 2 \sum_{i=1}^{8} (T_{i,9})^2 \end{pmatrix}.$$

Our task is thus to show that $\det H \neq 0$.

To compute $\det H$ we note that the entries of H are just the $9, 9$ coefficients in the matrix product S^2, $S \cdot T$ and T^2. Let S', T' be the matrices of q_1 and q_2 in the monomial basis; S' and T' are readily computed using (2.21). Then

$$S = M S' M^T$$
$$T = M T' M^T,$$

so we have

$$(S^2)_{9,9} = (M S' M^T M S' M^T)_{9,9} = (S' M^T M S')_{9,9}$$

where the last equality used the fact that M has the special form (2.23).

By our choice of basis $M^T M$ is diagonal, say

$$M^T M = \mathrm{diag}(\lambda_1, \ldots, \lambda_9),$$

where the $\lambda_i \neq 0$ since M is a matrix of change of basis, and $\lambda_9 = 1$. So finally we get

$$(S^2)_{9,9} = \sum_{i=1}^{9} \lambda_i (S')_{9,9}^2,$$

and similarly for T.

Using the monomial basis

$$\{v_1 \wedge v_3, v_1 \wedge v_4, \ldots, v_4 \wedge v_5, v_3 \wedge v_4\}$$

(of course any other ordering keeping $v_3 \wedge v_4$ at the end will do) we find

$$S'_{i,9} = \begin{cases} 0 & i \neq 5 \\ 1 & i = 5 \end{cases}$$

$$T'_{i,9} = \begin{cases} 0 & i \neq 3 \\ 1 & i = 3. \end{cases}$$

Finally this gives

$$H = \begin{pmatrix} -2\lambda_5 & 0 \\ 0 & -2\lambda_3 \end{pmatrix},$$

hence $\det H = 4\lambda_3\lambda_5 \neq 0$.

It will be useful to restate another way the nonvanishing of the Hessian determinant. We make a parenthesis to describe in general the first terms of the Taylor expansion of the determinant of a quadratic form. Let U be a vector space of dimension n and fix a quadratic form $q_0 \in \operatorname{Sym}^2 U^\vee$ of rank $n-1$. We let

$$f(q) = \det(q_0 + q) = f_0(q) + f_1(q) + \cdots + f_n(q),$$

where f_i is the homogeneous component of degree i of f.

Under our hypothesis we have $f_0 = \det(q_0) = 0$. Let $e \in U$ be a generator of the kernel of q_0. The term f_1 is a linear functional on $\operatorname{Sym}^2 U^\vee$, and it is easy to prove (see below) that

$$\ker f_1 = \{q \in \operatorname{Sym}^2 U^\vee \mid q(e) = 0\}. \tag{2.24}$$

We wish to describe explicitly $f_2\big|_{\ker f_1}$. Let $\bar{q}_0$ be the induced quadratic form on $U/\langle e \rangle$. This is non-degenerate by our hypothesis, so it induces a quadratic form $\bar{q}_0^\vee$ on $(U/\langle e \rangle)^\vee$. Let

$$\widetilde{q} \colon U \to U^\vee$$

be the symmetric map which induces q. If $q \in \ker f_1$ then $q(e) = 0$, hence

$$\widetilde{q}(e) \in \operatorname{Ann}(e) \cong (U/\langle e \rangle)^\vee.$$

So it makes sense to compute $\bar{q}_0^\vee(\widetilde{q}(e))$.

LEMMA 2.21 (O'Grady). *With the above notation we have*

$$f_2(q) = c\bar{q}_0^\vee(\widetilde{q}(e)) \tag{2.25}$$

for some nonzero constant c.

PROOF. Choose a basis $\mathcal{B}$ of U such that the matrix representing q_0 is just $M_0 = \operatorname{diag}(1, \ldots, 1, 0)$. Then for any symmetric matrix M we can compute

$$f(m_{i,j}) = \det(M_0 + M) =$$
$$= \det \begin{pmatrix} 1 + m_{1,1} & m_{1,2} & \cdots & m_{1,n-1} & m_{1,n} \\ m_{2,1} & 1 + m_{2,2} & \cdots & & \\ \vdots & \vdots & \ddots & & \vdots \\ m_{n-1,1} & & & 1 + m_{n-1,n-1} & \\ m_{n,1} & & \cdots & & m_{n,n} \end{pmatrix}.$$

The linear part of this expression is just $m_{n,n}$, which proves (2.24).

Assuming $m_{n,n} = 0$, the quadratic part of f is obtained as follows. Expand the determinant along the lowest row to write $f(m_{i,j})$ as a linear combination of the $m_{n,i}$, each with coefficient the corresponding cofactor. The linear part of the cofactor is just $m_{i,n} = m_{n,i}$, so at the end

$$f_2(m_{i,j}) = \sum_{i=1}^{n-1} m_{n,i}^2.$$

On the other hand the basis $\mathcal{B}$ has some multiple of e as the last element. Hence, in coordinates,

$$\widetilde{q}(c \cdot e) = (m_{n,1}, \ldots, m_{n,n-1}, 0)$$

for some nonzero constant c. This is identified with the element

$$(m_{n,1}, \ldots, m_{n,n-1}) \in (U/\langle e \rangle)^\vee,$$

where the coordinates are with respect to the $\bar{q}_0$-dual basis of the chosen basis.

Since $\mathcal{B} \setminus \{ce\}$ diagonalizes $\bar{q}_0$, its dual diagonalizes the dual form, hence

$$\bar{q}_0^\vee(\widetilde{q}(c \cdot e)) = \sum_{i=1}^{n-1} m_{n,i}^2.$$

By comparison we get the thesis.

We can apply this in our situation with $U = \mathrm{Sym}^2(V_0)/\langle v_1 \wedge v_2 \rangle$, taking $e = v_3 \wedge v_4$. In order to apply (2.25) we notice that

$$\widetilde{q_1}(v_3 \wedge v_4) = v_1 \wedge v_3 \wedge v_4,$$

hence

$$f_2(q_1) = c\bar{q}_0^\vee(v_1 \wedge v_3 \wedge v_4)$$

and similarly for the other terms. It follows that the Hessian of f can be written

$$H = c \begin{pmatrix} \bar{q}_0^\vee(v_1 \wedge v_3 \wedge v_4) & \widetilde{\bar{q}_0^\vee}(v_1 \wedge v_3 \wedge v_4, v_2 \wedge v_3 \wedge v_4) \\ \widetilde{\bar{q}_0^\vee}(v_1 \wedge v_3 \wedge v_4, v_2 \wedge v_3 \wedge v_4) & \bar{q}_0^\vee(v_2 \wedge v_3 \wedge v_4) \end{pmatrix}.$$

Let $\mathcal{B}$ be the monomial basis for

$$U/\langle e \rangle = \mathrm{Sym}^2(V_0)/\langle v_1 \wedge v_2, v_3 \wedge v_4 \rangle;$$

then the dual basis of $\mathcal{B}$ is also monomial, where of course we are using the natural identification

$$(U/\langle e \rangle)^\vee = \mathrm{Sym}^3(V_0)/\langle v_3 \wedge v_4 \wedge v_5, v_1 \wedge v_2 \wedge v_5 \rangle.$$

In particular the dual elements of $v_1 \wedge v_5$ and $v_2 \wedge v_5$ are $-v_2 \wedge v_3 \wedge v_4$ and $v_1 \wedge v_3 \wedge v_4$ respectively.

Since we have proved in Proposition 2.20 that H is nonsingular we find

COROLLARY 2.22. *Under the same hypothesis of Proposition 2.20 the form $q_A^\vee$ is non-degenerate on the subspace generated by the duals of $v_1 \wedge v_5$ and $v_2 \wedge v_5$.*

REMARK. By symmetry the form $q_A^\vee$ is non-degenerate on the subspace generated by the duals of $v_3 \wedge v_5$ and $v_4 \wedge v_5$ too.

2.6.2 – The sextic Y_A

We don't give a very detailed description of the local structure of Y_A around special points, since we won't need it. We limit ourselves to following very basic proposition, as an application of (2.24).

PROPOSITION 2.23 (O'Grady).

i) *Let $[v_0] \in Y_A$. Then Y_A is smooth at $[v_0]$ if and only if*

$$\dim F_{v_0} \cap A = 1$$

and A does not contain any decomposable form multiple of v_0.
ii) *In this case let*
$$F_{v_0} \cap A = \langle v_0 \wedge \alpha \rangle,$$

with α indecomposable, and let

$$H_{v_0} = \{v \in V \mid \mathrm{vol}(v_0 \wedge v \wedge \alpha \wedge \alpha) = 0\}.$$

Then the projective tangent space of Y_A at $[v_0]$ is

$$T_{[v_0]} Y_A = \mathbb{P}(H_{v_0}).$$

COROLLARY 2.24. *The singular locus of Y_A is the union of $Y_A[2]$ and the planes $\mathbb{P}(W)$, where W runs through all 3-planes of V such that $\bigwedge^3 W \subset A$.*

PROOF. Consider the map ψ from (2.19). By Corollary 2.15 Y_A is locally the preimage of the locus U_{10} of singular quadrics, hence $[v_0]$ will be a smooth point of Y_A if and only if U_{10} is smooth at q_A and the map ψ is a submersion at $[v_0]$.

The first condition means that q_A has rank exactly 10; since the kernel of q_A is identified with $F_{v_0} \cap A$, this means that

$$\dim F_{v_0} \cap A = 1.$$

So assume this and let

$$F_{v_0} \cap A = \langle v_0 \wedge \alpha \rangle.$$

By (2.24) we see that the tangent space $T_{q_A} U_{10}$ consists of the quadratic forms q such that $q(\alpha) = 0$.

So $[v_0]$ is a singular point of Y_A if and only if

$$q_v(\alpha) = 0 \text{ for every } v \in V_0.$$

By the explicit description of q_v in (2.20) this translates to

$$\text{vol}(v_0 \wedge v \wedge \alpha \wedge \alpha) = 0. \tag{2.26}$$

It follows that in any case (α decomposable or not) we have

$$T_{[v_0]} Y_A = \mathbb{P}(H_{v_0}).$$

Clearly this is either a hyperplane or the whole $\mathbb{P}(V)$, so we have to check that (2.26) holds for every $v \in V_0$ if and only if α is decomposable. In other words we are saying that a form satisfies all Plücker quadratic relations if and only if it lies in the Grassmannian, which is a well-known fact (see for instance [GH78, sec. 1.5]).

2.6.3 – The surface $Y_A[2]$

Now we turn to the local description of $Y_A[2]$. To see why this is more delicate, recall that we are using Corollary 2.15, so we see that locally $Y_A[2]$ is the preimage of the degeneracy locus U_8. The problem is that while U_9 is defined by the single equation $\det(q) = 0$, the variety U_8 is not a complete intersection.

For the curve $C_{A,W}$ we could overcome this difficulty by the remark that each quadric on $\bigwedge^2 V_0$ parametrized by a point of $\mathbb{P}(W)$ had to be singular in a fixed point. Hence the trick was to project from that point and consider the equation $\det(q) = 0$ for quadrics in a space of smaller dimension.

For $Y_A[2]$ we are not so lucky and we actually have to work in codimension more than 1. The first step is to perform a reduction of the dimension based on the following remark.

REMARK. Let

$$[v_0] \in Y_A[k] \setminus Y_A[k+1],$$

so that $\dim A \cap F_{v_0} = k$, and let $K \subset \bigwedge^2 V_0$ be the subspace corresponding to K under the usual identification $F_{v_0} \cong \bigwedge^2 V_0$. Let J be complementary to K, so

$$\bigwedge^2 V_0 = J \oplus K \tag{2.27}$$

and q_A is non-degenerate on J. Here the direct sum is q_A-orthogonal, just because K is the kernel of q_A.

If $v \in V_0$ is sufficiently close to 0, the form $q_A - q_v$ is still non-degenerate on J. Let

$$K(v) = J^{\perp(q_A - q_v)}$$

be the orthogonal to J with respect to the form $q_A - q_v$. Then $K(v)$ contains the kernel of $q_A - q_v$, so it is easy to see that $\dim K(v) = k$ anyway. Moreover $K(v)$ meets J only in 0 (because $q_A - q_v$ is non-degenerate on J), so we have the direct sum decomposition

$$\bigwedge^2 V_0 = J \oplus K(v),$$

and this decomposition is orthogonal with respect to $q_A - q_v$.

The remark is that in studying $Y_A[k]$ one can work inside $K(v)$, in the following sense. By Corollary 2.15

$$Y_A[k] \underset{loc}{=} \{[v_0 + v] \mid \operatorname{cork}(q_A - q_v) \geq k\},$$

where cork denotes the corank of a quadratic form. By construction the corank of $q_A - q_v$ is the same of its restriction to $K(v)$. So we get that

$$Y_A[k] \underset{loc}{=} \left\{ [v_0 + v] \mid \operatorname{cork}(q_A - q_v)\big|_{K(v)} \geq k \right\}.$$

The simplification is apparent if one considers the case of $Y_A[2]$, where we are led to study quadratic forms on a space of dimension 2 instead of 10.

Now we translate the above in matrix terms. According to decomposition (2.27) we find matrices both for q_A:

$$\begin{pmatrix} N & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix}$$

and for q_v:

$$\begin{pmatrix} Q(v) & R(v)^T \\ R(v) & P(v) \end{pmatrix}.$$

Of course here we have chosen arbitrary bases both for J and for K. We perform the change of basis in two steps. First choose any matrix S such that $S \cdot S^T = (N - Q)^{-1}$ (for readability sake we temporarily omit the dependence on v in the computations). This can be done, for instance, with an analog of the Choleski algorithm, which is valid for real symmetric positive definite matrices (the algorithm will fail on a Zariski closed subset of all symmetric complex matrices, but the formula of Equation (2.28) will still be valid by a limit argument).

Consider the block matrix

$$S' = \begin{pmatrix} S & \underline{0} \\ \underline{0} & \mathrm{I} \end{pmatrix}.$$

S' sends each of the factors J, K to itself, but in the new basis the form $q_A - q_v$ is just the standard scalar product. More precisely the matrix for $q_A - q_v$ will be

$$\begin{pmatrix} S & \underline{0} \\ \underline{0} & I \end{pmatrix} \cdot \begin{pmatrix} N - Q & -R^T \\ -R & -P \end{pmatrix} \cdot \begin{pmatrix} S^T & \underline{0} \\ \underline{0} & I \end{pmatrix} = \begin{pmatrix} I & -R^T S^T \\ -SR & -P \end{pmatrix}.$$

In this form it is easy to find the change of basis which will put the matrix in block form. Indeed

$$\begin{pmatrix} I & \underline{0} \\ SR & I \end{pmatrix} \cdot \begin{pmatrix} I & -R^T S^T \\ -SR & -P \end{pmatrix} \cdot \begin{pmatrix} I & R^T S^T \\ \underline{0} & I \end{pmatrix} = \begin{pmatrix} I & \underline{0} \\ \underline{0} & -P + R(N-Q)^{-1} R^T \end{pmatrix}.$$

We conclude that the restriction of $q_A - q_v$ to $K(v)$ is given, in a suitable basis, by the matrix

$$M(v) = -P(v) + R(v) \cdot (N - Q(v))^{-1} \cdot R(v)^T. \tag{2.28}$$

The following elementary remark will be useful:

LEMMA 2.25. *Let V be a finite-dimensional vector space endowed with a non-degenerate quadratic form q and let $q^\vee$ be the form induced on $V^\vee$.*

If N is the matrix which represents q with respect to a basis $\mathcal{B}$ of V, then the matrix of $q^\vee$ with respect to the dual basis $\mathcal{B}^\vee$ is N^{-1}.

PROOF. A non-degenerate quadratic form q induces an isomorphism

$$\tau \colon V \to V^\vee$$

defined by the condition

$$q(v, w) = \tau(v)w.$$

By construction the matrix associated to τ, with respect to the bases $\mathcal{B}$ and $\mathcal{B}^\vee$, is just N.

The dual form $q^\vee$ is defined by

$$q^\vee(\varphi, \psi) = q(\tau^{-1}(\varphi), \tau^{-1}(\psi)),$$

hence it is represented by the matrix

$$(N^{-1})^T \cdot N \cdot N^{-1} = N^{-1},$$

using the fact that N is symmetric.

We can now prove the

PROPOSITION 2.26. *Let* $W, W' \subset V$ *be two subspaces of dimension* 3 *such that* $\bigwedge^2 W, \bigwedge^2 W' \subset A$. *Let*

$$[v_0] = W \cap W'$$

(in particular we are assuming that $\dim W \cap W' = 1$*) and complete it to bases* $\{v_0, v_1, v_2\}$ *and* $\{v_0, v_3, v_4\}$ *of* W, W' *respectively. Assume that*

i) $v_0 \wedge v_1 \wedge v_2$ *and* $v_0 \wedge v_3 \wedge v_4$ *are, up to multiples, the only decomposable forms in* A *(this translates to the fact that* $[v_0]$ *lies on* $\mathbb{P}(W)$ *and* $\mathbb{P}(W')$, *but on no other planes inside* Y_A);
ii) $\dim(A \cap F_{v_0}) = 2$;
iii) *assumption 2.19 is satisfied.*

Then there exist local analytic coordinates $y_1, \ldots, y_5$ *near* $[v_0]$ *such that* $Y_A[2]$ *locally is given by*

$$Y_A[2] \underset{loc}{=} V(y_1 y_2, y_3 y_4, y_5).$$

Moreover the coordinates can be taken so that

$$C_{A,W} \underset{loc}{=} V(y_1 y_2, y_3, y_4, y_5)$$

$$C_{A,W'} \underset{loc}{=} V(y_1, y_2, y_3 y_4, y_5).$$

PROOF. We follow the notation of the Remark above. We choose

$$\{v_1 \wedge v_2, v_3 \wedge v_4\}$$

as a basis for K and

$$\{v_1 \wedge v_3, v_1 \wedge v_4, v_1 \wedge v_5, v_2 \wedge v_3, v_2 \wedge v_4, v_2 \wedge v_5, v_3 \wedge v_5, v_4 \wedge v_5\}$$

for J.

Write a vector $v \in V_0$ as

$$v = s_1 v_1 + \cdots + s_5 v_5,$$

so $s_1, \ldots, s_5$ are affine coordinates around $[v_0]$. With respect to these coordinates we can easily compute the matrices

$$P(v) = \begin{pmatrix} 0 & s_5 \\ s_5 & 0 \end{pmatrix}$$

and

$$R(v) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -s_4 & s_5 \\ 0 & 0 & -s_2 & 0 & 0 & s_1 & 0 & 0 \end{pmatrix}.$$

Let us write
$$(N - Q(v))^{-1} = (c_{i,j}(s)),$$
so that the matrix $M(v)$ of (2.28) is
$$- P(v) + R(v) \cdot (N - Q(v))^{-1} \cdot R(v)^T =$$
$$= \begin{pmatrix} s_4^2 c_{7,7} - 2 s_4 s_5 c_{7,8} + s_5^2 c_{8,8} & -s_5 + \delta(s) \\ -s_5 + \delta(s) & s_2^2 c_{3,3} - 2 s_2 s_1 c_{3,6} + s_1^2 c_{6,6} \end{pmatrix},$$
where
$$\delta(s) = s_2 s_4 c_{7,3} - s_2 s_5 c_{8,3} - s_1 s_4 c_{7,6} + s_1 s_5 c_{8,6}$$
is not really relevant since it is of higher order than s_5.

Now we claim that the matrices
$$\begin{pmatrix} c_{7,7}(0) & c_{7,8}(0) \\ c_{7,8}(0) & c_{8,8}(0) \end{pmatrix} \text{ and } \begin{pmatrix} c_{3,3}(0) & c_{3,6}(0) \\ c_{3,6}(0) & c_{6,6}(0) \end{pmatrix}$$
are both nonsingular.

Granting this, by the local inversion theorem we can find analytic coordinates $x_1, \ldots, x_5$ near $[v_0]$ such that
$$-P(v) + R(v) \cdot (N - Q(v))^{-1} \cdot R(v)^T = \begin{pmatrix} x_3^2 + x_4^2 & x_5 \\ x_5 & x_1^2 + x_2^2 \end{pmatrix}.$$

Finally, taking $y_1 = x_1 + i x_2$, $y_2 = x_1 - i x_2$ and similarly for y_3 and y_4, and $y_5 = x_5$ gives
$$-P(v) + R(v) \cdot (N - Q(v))^{-1} \cdot R(v)^T = \begin{pmatrix} y_3 y_4 & y_5 \\ y_5 & y_1 y_2 \end{pmatrix},$$
hence the thesis.

So we turn to the proof of the claim. The form $q = q_A\big|_J$ is non-degenerate; we use Lemma 2.25 to identify N^{-1} with the matrix of $q^\vee$ on $J^\vee$.

The nonsingularity of
$$\begin{pmatrix} c_{3,3}(0) & c_{3,6}(0) \\ c_{3,6}(0) & c_{6,6}(0) \end{pmatrix}$$
means that $q^\vee$ is non-degenerate when restricted to the subspace $J' \subset J$ generated by the dual elements of $v_1 \wedge v_5$ and $v_2 \wedge v_5$. This is exactly the content of Corollary 2.22.

Similarly the nonsingularity of
$$\begin{pmatrix} c_{7,7}(0) & c_{7,8}(0) \\ c_{7,8}(0) & c_{8,8}(0) \end{pmatrix}$$
is equivalent to the remark thereafter.

A similar approach, even easier, will then prove the

PROPOSITION 2.27 (O'Grady). *Let $W \subset V$ be a subspace of dimension 3 such that $\bigwedge^3 W \subset A$. Let $v_0 \in W$ and assume that*

i) *$v_0 \wedge v_1 \wedge v_2$ is, up to multiples, the only decomposable form in $A \cap F_{v_0}$ (this translates to the fact that $[v_0]$ lies on $\mathbb{P}(W)$ but on no other planes inside Y_A);*
ii) *$\dim(A \cap F_{v_0}) = 2$;*
iii) *$v_0 \wedge v_1 \wedge v_2$ is, up to multiples, the only form in $A \cap F_{v_0} \cap S_W$, where*

$$S_W = \bigwedge^2 W \wedge V \subset \bigwedge^3 V.$$

Then there exist local analytic coordinates $y_1, \ldots, y_5$ near $[v_0]$ such that $Y_A[2]$ locally is given by

$$Y_A[2] \underset{loc}{=} V(y_1 y_2, y_4, y_5).$$

Moreover the coordinates can be taken so that

$$C_{A,W} \underset{loc}{=} V(y_1, y_2, y_4, y_5).$$

This is enough to prove the result we are looking for about the desingularization of $Y_A[2]$.

COROLLARY 2.28.

i) *Let $A \subset \bigwedge^3 V$ be a Lagrangian subspace, and suppose that there exist a finite number $W_1, \ldots, W_n$ of 3-planes inside V such that $\bigwedge^3 W_i \subset A$. Then the intersection $\mathbb{P}(W_i) \cap \mathbb{P}(W_j)$ consists of a single point p_{ij}.*
ii) *Corresponding to W_i we have the curve*

$$C_i = C_{A,W_i} \subset Y_A[2].$$

Suppose that at each point p_{ij} Assumption 2.19 is satisfied. Assume moreover that

$$\dim A \cap S_{W_i} = 1 \text{ for each } i = 1, \ldots, n,$$

where

$$S_W = \bigwedge^2 W \wedge V \subset \bigwedge^3 V.$$

Finally suppose that

$$Y_A[3] = \emptyset.$$

Let $\widetilde{Y_A[2]}$ be obtained by blowing up all the points p_{ij}. Then the strict transform $\widetilde{C_i}$ of C_i is smooth for every i. Moreover the blowup of $\widetilde{Y_A[2]}$ at each of the (disjoint) curves $\widetilde{C_i}$ is smooth.

PROOF. Under the hypothesis A contains a finite number of decomposable forms, up to multiples. We already know that

$$\dim(W_i \cap W_j) \geq 1.$$

Assume that $\dim(W_i \cap W_j) = 2$ and let

$$W_i \cap W_j = \langle v_0, v_1 \rangle.$$

Complete this to bases $\{v_0, v_1, v_2\}$ and $\{v_0, v_1, v_3\}$ of W_i, W_j respectively. Then the form

$$v_0 \wedge v_1 \wedge (\lambda v_2 + \mu v_3) \in A$$

for each choice of λ, μ, contradiction. So i) is proved.

To prove ii) we just remark that under our hypothesis $Y_A[2]$ is smooth far from the curves $\{C_i\}$ and singular along them. The curve C_i is smooth outside the p_{ij} by Proposition 2.17 and has a node at each point p_{ij} by Proposition 2.20. It follows that it can be desingularized by a single blowup at each p_{ij}.

The singularities of $Y_A[2]$ are described by Propositions 2.26 and 2.27, in the points p_{ij} and in the other points respectively. A local computation shows that the process above will desingularize $Y_A[2]$.

To be sure, we show the computation around some point p_{ij}, the other case being easier. By Proposition 2.26 we can assume that $Y_A[2]$ is locally given in $\mathbb{C}^4$ by

$$y_1 y_2 = y_3 y_4 = 0,$$

while the two curves C_i and C_j meeting at p_{ij} are given by

$$y_1 = y_2 = y_3 y_4 = 0 \text{ and}$$
$$y_1 y_2 = y_3 = y_4 = 0$$

respectively. Let $\widetilde{\mathbb{C}^4}$ denote the blowup of $\mathbb{C}^4$ at the origin; then explicitly

$$\widetilde{\mathbb{C}^4} = \left\{ \big((y_1, \ldots, y_4), [Y_1, \ldots, Y_4]\big) \mid y_i Y_j = y_j Y_i \text{ for each } i, j \right\} \subset \mathbb{C}^4 \times \mathbb{P}^3.$$

This is covered by the four charts

$$U_j = \{Y_j \neq 0\}, \text{ for } j = 1, \ldots, 4.$$

We perform the verification on U_1. There we can define

$$z_j = \frac{Y_j}{Y_1}, \text{ for } j = 2, 3, 4,$$

and use (y_1, z_2, z_3, z_4) as local coordinates.

We note that C_i doesn't meet the chart U_1. As for the strict transforms of $Y_A[2]$ and C_j, they are as follows. The inverse image of $Y_A[2]$ is cut out by the equations

$$y_1^2 z_2 = y_1^2 z_3 z_4 = 0,$$

so after factoring out the exceptional divisor E given by $y_1 = 0$ we see that $\widetilde{Y_A[2]}$ is defined by

$$z_2 = z_3 z_4 = 0.$$

In a similar fashion the inverse image of C_j is defined by

$$y_1^2 z_2 = y_1 z_3 = y_1 z_4 = 0,$$

so $\widetilde{C_j}$ has equations

$$z_2 = z_3 = z_4 = 0.$$

It follows that $\widetilde{Y_A[2]}$ is just the product of $\widetilde{C_j}$ for a node, so blowing up $\widetilde{C_j}$ will desingularize it.

3 – Quartic surfaces

3.1 – An interesting degeneration of double EPW sextics

In this section we present some degenerate examples of double EPW sextics. We shall in next section that double EPW sextics can degenerate to a Hilbert scheme $S^{[2]}$, where S is a quartic surface in $\mathbb{P}^3$. In particular, since S is a $K3$, this will imply that double EPW sextics are irreducible symplectic varieties, and will allow us to compute their topological invariants.

Inside a double EPW sextic X_A we can find the surface

$$Z_A = f_A^{-1}(Y_A[2]),$$

which can also be regarded as the fixed locus of the covering involution. Under this process, Z_A degenerates to the surface $\mathrm{Bit}(S)$ of bitangents to S. This allows us to translate some questions about the geometry of X_A and Z_A, which are invariant under deformation, to questions about quartic surfaces and their bitangents, which are somewhat more concrete. This section and the following are therefore a motivation for the rest of the present chapter, where we study quartic surfaces in some more detail.

3.1.1 – An involution over $S^{[2]}$

We begin with a classical example of Beauville, from [Bea83a, sec. 6]. Let U be a vector space of dimension 4 and let

$$G = \mathrm{Gr}(2, U)$$

be the Grassmannian of lines in $\mathbb{P}(U) = \mathbb{P}^3$. Under the Plücker embedding

$$G \hookrightarrow \mathbb{P}(\textstyle\bigwedge^2 U) = \mathbb{P}^5$$

G is defined by the single equation

$$g(\alpha) = \alpha \wedge \alpha = 0,$$

hence it is a quadric.

Now let $S \subset \mathbb{P}^3 = \mathbb{P}(U)$ be a quartic. Each cycle $Z \in S^{[2]}$ determines a line $\ell_Z \subset \mathbb{P}^3$: either the line joining the two points in Z, if it is reduced, or the line passing through the unique point in Z with the given tangent direction. This yields a morphism

$$S^{[2]} \longrightarrow G,$$

$$Z \longmapsto \ell_Z$$

which is $6 : 1$; indeed the generic line ℓ cuts S in 4 distinct points, so it has $\binom{4}{2}$ preimages.

Assume that S does not contain any line. There is an involution

$$i \colon S^{[2]} \to S^{[2]}$$

which makes the following diagram commute:

$$S^{[2]} \xrightarrow{\ i\ } S^{[2]}.$$
$$\underset{\varphi}{\longrightarrow} \overset{\varphi}{\longrightarrow}$$
$$G$$

The involution i is defined as follows. Let subscheme Z determine the line ℓ_Z; then

$$\ell_Z \cdot S = Z + Z'$$

for some subscheme $Z' \subset S$ of length 2. We define $i(Z) = Z'$.

Now assume that S contains a line ℓ; then $S^{[2]}$ contains $P = \ell^{(2)}$, which is isomorphic to $\mathbb{P}^2$. In this case one can define the involution i as above, but it becomes only birational, since it is not defined along P. One can easily check that in this case i is in fact a *biregular* involution, followed by the Mukai flop along P. The construction generalizes to the case where S contains a finite number of lines; for details we refer to [Bea83a].

The case we are more interested in is when S does not contain lines, but assumes some singularity. First suppose that S_0 is a quartic with an ordinary double point p, and let S be the blowup of S_0 at p, so that S is smooth.

LEMMA 3.1. *S is a K3 surface.*

PROOF. First we check that S has trivial canonical. For this let

$$\epsilon \colon \widetilde{\mathbb{P}^3} \to \mathbb{P}^3$$

be the blowup at p, so that $S \subset \widetilde{\mathbb{P}^3}$. Let E be the exceptional divisor of ϵ, and denote

$$\mathcal{O}_{\widetilde{\mathbb{P}^3}}(1) = \epsilon^* \mathcal{O}_{\mathbb{P}^3}(1).$$

The relative canonical divisor $K_{\widetilde{\mathbb{P}^3}/\mathbb{P}^3}$ is given by

$$K_{\widetilde{\mathbb{P}^3}/\mathbb{P}^3} = V(\det \epsilon).$$

A local computation shows that $\det \epsilon$ vanishes of order 2 on E, hence

$$K_{\widetilde{\mathbb{P}^3}} = \epsilon^*(K_{\mathbb{P}^3}) + 2E = \mathcal{O}_{\widetilde{\mathbb{P}^3}}(-4) + 2E.$$

Moreover S_0 is a section of $\mathcal{O}_{\mathbb{P}^3}(4)$, and since an equation for S_0 vanishes of order 2 at p we obtain

$$S + 2E = \epsilon^*(S_0) \in |\mathcal{O}_{\widetilde{\mathbb{P}^3}}(4)|.$$

By adjunction we can compute

$$K_S = \left(K_{\widetilde{\mathbb{P}^3}} + S\right)\big|_S = \left(\mathcal{O}_{\widetilde{\mathbb{P}^3}}(-4) + S + 2E\right)\big|_S = 0.$$

We can apply Lefschetz theorem on hyperplane section to see that S_0 is simply connected. The theorem applies even if S_0 is singular, for instance in the form of [Mil69, Cor. 7.3]. Using, for instance, Mayer-Vietoris relative to a decomposition

$$S_0 = (S_0 \setminus \{p\}) \cup U,$$

where U is a small neighbourhood of p, one easily sees that

$$\epsilon_* \colon H_1(S) \to H_1(S_0)$$

is an isomorphism, hence $b_1(S) = 0$ and S is a $K3$.

We now want to exhibit a map analogous to φ, with $S^{[2]}$ in place of $S_0^{[2]}$. Let $\ell \subset S$ be the exceptional divisor; then ℓ is a conic, hence a smooth rational curve. We let

$$P = \ell^{(2)} \subset S^{[2]},$$

then P is isomorphic to $\mathbb{P}^2$. We have a rational map

$$\varphi' \colon S^{[2]} \dashrightarrow G$$

defined as above; since all points of ℓ are mapped to p, φ' is undefined exactly on P. Let X be the Mukai flop of $S^{[2]}$ along P; we claim that we have a *regular* map

$$\varphi \colon X \to G,$$

such that

$$
\begin{array}{ccc}
S^{[2]} & \dashrightarrow & X \\
 & \searrow{\scriptstyle \varphi'} \quad \swarrow{\scriptstyle \varphi} & \\
 & G &
\end{array}
$$

commutes.

We only have to define φ at points of $P^\vee$. By definition of the Mukai flop, X is obtained by $S^{[2]}$ by first blowing up along P and then contracting the exceptional divisor E along the other fibration. Let us call $\widetilde{X}$ the blowup of $S^{[2]}$ along P. Then we have

$$E = \mathbb{P}\mathcal{N}_{P/S^{[2]}},$$

so a point of E is a couple $(Z, [v])$, where

$$Z \in \ell^{(2)} \text{ and } v \in T_Z S^{[2]}/T_Z P.$$

Assume for simplicity that $Z = q_1 + q_2$ is reduced; then

$$T_Z S^{[2]}/T_Z P \cong (T_{q_1} S/T_{q_1}\ell) \oplus (T_{q_2} S/T_{q_2}\ell).$$

The kernel of the differential

$$d\epsilon_{q_i} \colon T_{q_i} S \to T_p \mathbb{P}^3$$

is exactly $T_{q_i}\ell$, so the differential identifies each factor $T_{q_1} S/T_{q_1}\ell$ with its image, which is exactly the (tangent of the) line trough p corresponding to the direction $q_i \in \ell$.

The lines corresponding to q_1 and q_2 span a plane $\Pi \subset \mathbb{P}^3$ through p, and the direction $[v]$ identifies a line $\ell_Z \subset \Pi$. The construction carries over to the case where Z is not reduced, so finally we get a regular map

$$\widetilde{\varphi} \colon \widetilde{X} \to G$$

sending Z to ℓ_Z.

Following the definitions, one can see that $\widetilde{\varphi}$ is constant along the fibers of the other blowup $\widetilde{X} \to X$, so it descends to the desired regular morphism

$$\varphi \colon X \to G.$$

One can finally extend this construction to cover the case where S_0 has finitely many ordinary double points; in this case one has to introduce a Mukai flop for each singular point. We do not describe the details, as they are only notationally heavier than in the case of one point.

3.1.2 – Triple quadrics as EPW sextics

We now discuss in which way the above examples may be seen as degenerations of double EPW sextics; this construction is present in [O'G]. Recall that our quartic surface S lives inside $\mathbb{P}(U)$, where U is a vector space of dimension 4. We take the vector space $V = \bigwedge^2 U$. Then inside $\mathbb{P}(\bigwedge^3 V)$ we have the Grassmannian $\mathrm{Gr}(3, V)$, by the Plücker embedding.

To each $[u] \in \mathbb{P}(U)$ we can associate the subspace

$$u \wedge U \in \mathrm{Gr}(3, \textstyle\bigwedge^2 U);$$

this gives an embedding

$$\iota_+ \colon \mathbb{P}(U) \hookrightarrow \mathrm{Gr}(3, V) \subset \mathbb{P}(\textstyle\bigwedge^3 V).$$

REMARK. Any two subspaces in the image of ι_+ intersect along a line; more precisely $\iota_+([u_0])$ and $\iota_+([u_1])$ intersect along the line generated by $u_0 \wedge u_1$. If we see the Grassmannian $\mathrm{Gr}(3, V)$ as a parameter space for planes in $\mathbb{P}(V)$, this means that we have a 3-dimensional family of planes, parametrized by $\mathbb{P}(U)$, such that any two planes in the family have non-empty intersection.

LEMMA 3.2 (O'Grady). *$\iota_+(\mathbb{P}(U))$ spans a subspace of $\mathbb{P}(\bigwedge^3 V)$ which corresponds to a* isotropic *subspace*

$$A_+(U) \subset \textstyle\bigwedge^3 V.$$

PROOF. Let
$$\alpha, \beta \in A_+(U) \subset \bigwedge\nolimits^3 V;$$
we have to check that $\alpha \wedge \beta = 0$. Of course it is enough to verify this on a set of generators; hence we can assume that
$$[\alpha] = \iota_+(u_0), \qquad [\beta] = \iota_+(u_1)$$
for some $[u_0], [u_1] \in \mathbb{P}(U)$. By the remark above
$$V' = \iota_+(u_0) + \iota_+(u_1) \subsetneq V,$$
so $\alpha \wedge \beta \in \bigwedge^6 V' = \underline{0}$.

We'd like to verify that $A_+(U)$ is actually Lagrangian. In order to do this we need to introduce the symmetric construction. This is easy: since $\dim V = 6$, we have a canonical isomorphism
$$\mathrm{Gr}(3, V) \cong \mathrm{Gr}(3, V^\vee).$$

Now we can repeat the construction using $U^\vee$ in place of U, and then use the identification above. In the end we find an embedding
$$\iota_- : \mathbb{P}(U^\vee) \hookrightarrow \mathrm{Gr}(3, V).$$

By the same argument, any two planes in the image of ι_- are concurrent, and so we get another isotropic subspace $A_-(U) \subset \bigwedge^3 V$. We wish to prove that
$$\bigwedge\nolimits^3 V = A_+(U) \oplus A_-(U); \tag{3.1}$$
in particular this says that both $A_+(U)$ and $A_-(U)$ have dimension 10, hence they are Lagrangian.

The above decomposition will be more apparent if one regards all involved vector space as $SL(U)$-modules. Let $\mathcal{L}$ be the line bundle on $\mathrm{Gr}(3, V)$ which induces the Plücker embedding. One checks directly that
$$\iota_+^*(\mathcal{L}) \cong \mathcal{O}_{\mathbb{P}(U)}(2).$$
By duality it follows that
$$\iota_-^*(\mathcal{L}) \cong \mathcal{O}_{\mathbb{P}(U^\vee)}(2).$$
Now the global sections of the involved line bundles are
$$H^0(\mathbb{P}(U), \mathcal{O}_{\mathbb{P}(U)}(2)) = \mathrm{Sym}^2 U^\vee,$$
$$H^0(\mathbb{P}(U^\vee), \mathcal{O}_{\mathbb{P}(U^\vee)}(2)) = \mathrm{Sym}^2 U,$$
$$H^0(\mathrm{Gr}(3, V), \mathcal{L}) = H^0(\mathbb{P}(\bigwedge\nolimits^3 V), \mathcal{O}_{\mathbb{P}(\wedge^3 V)}(1)) = \bigwedge\nolimits^3 V^\vee = \bigwedge\nolimits^3 (\bigwedge\nolimits^2 U^\vee),$$

and these are all $SL(U)$-modules. Moreover the embeddings ι_+ and ι_- are equivariant under the action of $SL(U)$, hence the induced maps on sections

$$\iota_+^* \colon H^0(\mathrm{Gr}(3,V),\mathcal{L}) \to H^0(\mathbb{P}(U),\mathcal{O}_{\mathbb{P}(U)}(2)) \text{ and}$$
$$\iota_-^* \colon H^0(\mathrm{Gr}(3,V),\mathcal{L}) \to H^0(\mathbb{P}(U^\vee),\mathcal{O}_{\mathbb{P}(U^\vee)}(2))$$

are morphisms of $SL(U)$-modules. Since both $\mathrm{Sym}^2 U^\vee$ and $\mathrm{Sym}^2 U$ are irreducible, these maps must be surjective.

Comparing the dimensions, we obtain an isomorphism of $SL(U)$-modules

$$\bigwedge\nolimits^3(\bigwedge\nolimits^2 U^\vee) \cong \mathrm{Sym}^2 U^\vee \oplus \mathrm{Sym}^2 U, \tag{3.2}$$

which must then be the decomposition into irreducible factors of $\bigwedge^3(\bigwedge^2 U^\vee)$.

It follows that any section of $\mathcal{L}$ on $\mathrm{Gr}(3,V)$ which restricts to 0 both on the image of ι_+ and on the image of ι_- is itself 0. In other words the image of ι_+ and the image of ι_- span the whole $\mathbb{P}(\bigwedge^3 V)$. We deduce that the decomposition given by (3.1) holds, and in particular $A_+(U)$ and $A_-(U)$ are both Lagrangian.

Associated to a Lagrangian subspace we have an EPW sextic. This is given by the following

PROPOSITION 3.3 (O'Grady). *Let the notation be as above. Then*

$$Y_{A_+(U)} = Y_{A_-(U)} = 3G.$$

PROOF. We omit the check that $Y = Y_{A_+(U)}$ is not the whole $\mathbb{P}(V)$. By construction Y is invariant under the action of $SL(U)$ on $V = \bigwedge^2 U$. This group acts transitively on the Grassmannian G; since Y meets G, it contains the whole G. Actually, since $SL(U)$ is connected, this holds true for every irreducible component of Y.

It follows that $Y = kG$ for some k, and comparing the degrees we find $k = 3$.

Since any two smooth quadric in $\mathbb{P}(V)$ are projectively equivalent, we see that for every smooth quadric $Q \subset \mathbb{P}(V)$ the non-reduced sextic $3Q$ is EPW.

3.2 – The deformation argument

Now we want to connect the preceding examples. Namely, with the notation of the preceding section, we want to prove that as the generic Lagrangian subspace A degenerates to $A_+(U)$, the corresponding double EPW sextic X_A deforms to $S^{[2]}$, and the fixed locus of the involution Z_A deforms to $\mathrm{Bit}(S)$.

3.2.1 – The smooth case

The result is the following.

PROPOSITION 3.4. *Let $S \subset \mathbb{P}^3$ be a smooth quartic. Then there exists a smooth complex variety U of dimension 20 with a marked point 0 and a family*

$$\pi_X \colon \mathcal{X} \to U$$

such that

  i) *$\mathcal{X}_0 \cong S^{[2]}$ and*
 ii) *there exists a divisor $D \subset U$ such that $\mathcal{X}_t = X_{A(t)}$ is a smooth double EPW sextic for each $t \in U \setminus D$.*

COROLLARY 3.5. *There exists over U a family*

$$\pi_Z \colon \mathcal{Z} \to U$$

such that $\mathcal{Z}_0 \cong \mathrm{Bit}(S)$ and for $t \in U \setminus D$

$$\mathcal{Z}_t \cong Z_{A(t)} \cong Y_{A(t)}[2].$$

COROLLARY 3.6. *Every smooth double EPW sextic is an irreducible symplectic variety.*

PROOF.[Proof of Corollary 3.5] By the proof of Proposition 3.4, the total space $\mathcal{X}$ admits an involution. It is enough to take as $\mathcal{Z}$ the fixed locus of the involution of $\mathcal{X}$.

PROOF.[Proof of Corollary 3.6] Proposition 3.4 tells us that there exist *some* smooth double EPW sextic which is deformation equivalent to $S^{[2]}$. Since $\mathbb{LG}(\bigwedge^3 V)$ is irreducible, its Zariski open set $\mathbb{LG}(\bigwedge^3 V)^0$ is connected, hence each double EPW sextic is deformation equivalent to each other. Actually a global family of double EPW sextics over $\mathbb{LG}(\bigwedge^3 V)^0$ does not exist, but such families exist locally.

It follows that *every* smooth double EPW sextic is a deformation of $S^{[2]}$, in particular it is an irreducible symplectic variety.

PROOF.[Proof of Proposition 3.4] Let V be a local semiuniversal deformation space of $S^{[2]}$; by the results of Section 1.3 it is smooth of dimension 21. Let $h \in \mathrm{Pic}(S^{[2]})$ be the divisor class associated to the map f. By the local Torelli theorem the locus $U \subset V$ parametrizing those deformations such that h remains of type $(1,1)$ (and so remains the class of a divisor) is a smooth hypersurface. After restricting U we can assume that we have a family

$$\pi_X \colon \mathcal{X} \to U$$

of polarized irreducible symplectic varieties $(\mathcal{X}_t, h_t)$ such that $(h_t, h_t) = 2$ for the Beauville-Bogomolov form.

Let φ be the Beauville involution on $S^{[2]}$. By the remark in section 4.1.3 of [O'G05b] this extends to an involution φ_t of $\mathcal{X}_t$. Consider for every $t \in U$ the quotient

$$Y_t = \mathcal{X}_t / \varphi_t.$$

There is a divisor H'_t on Y_t such that

$$\varphi_t^*(c_1(H'_t)) = h_t.$$

This is because the involution fixes h_t; more precisely

$$\varphi_t^* \colon H^2(\mathcal{X}_{\bar{t}}, \mathbb{Z}) \to H^2(\mathcal{X}_{\bar{t}}, \mathbb{Z})$$

is the reflection in the span of h_t, see [O'G05b, Sec. 4.1.3].

Since Y' has terminal singularities and $K_{Y'} = 0$, we can apply a variant of the Kodaira vanishing theorem for singular varieties, for instance Theorem $1 - 2 - 5$ in [KMM87], to conclude that

$$h^i(Y_t, H'_t) = 0$$

for all $i > 0$. It follows that $h^0(Y_t, H'_t) = \chi(Y_t, H'_t)$. Let H_t be the pullback of H'_t; we claim that

$$\chi(\mathcal{X}_t, H_t) = \chi(Y_t, H'_t)$$

for all $t \in U$. Indeed by flatness we can prove it just when $t = 0$, and in this case it is clear. Applying Kodaira vanishing on $\mathcal{X}_t$ we conclude that

$$h^0(\mathcal{X}_t, H_t) = h^0(Y_t, H'_t). \tag{3.3}$$

We claim that there is some $\bar{t} \in U$ such that $(\mathcal{X}_{\bar{t}}, h_{\bar{t}})$ satisfy the conclusions of Proposition 3.2 of [O'G08b]. Indeed we have (1) by definition, and (5) holds for every t by Proposition 3.6 of the same paper.

Moreover (2) and (4) are satisfied outside a countable union of proper subvarieties of U by the local Torelli theorem. Finally (3) and (6) follow formally from the other points, as in the proof of Proposition 3.2 of the same paper.

O'Grady then classifies polarized irreducible symplectic varieties numerically equivalent to $S^{[2]}$ (this means that their H^2, endowed with the Beauville-Bogomolov form, are isomorphic lattices, and that the Fujiki constants are the same) which satify the conclusion of Proposition 3.2. Namely let (X, H) be such a polarized variety, and consider the map

$$f \colon X \dashrightarrow |H|^\vee.$$

Then $|H|^\vee \cong \mathbb{P}^5$, and there are two cases for f. Either it is birational on the image Y, or it is everywhere defined and the map

$$f\colon X \to Y \subset |H|^\vee$$

is the quotient by an anti-symplectic involution on X, and $Y \subset |H|^\vee$ is a sextic.

Now apply all this with $X = \mathcal{X}_{\bar{t}}$. We want to exclude the first case, and we proceed as follows. Let

$$\pi\colon \mathcal{X}_{\bar{t}} \to Y_{\bar{t}} = \mathcal{X}_{\bar{t}}/\varphi_{\bar{t}}$$

be the projection. We have an injective pull-back map

$$\pi^*\colon H^0(Y_{\bar{t}}, H'_{\bar{t}}) \to H^0(\mathcal{X}_{\bar{t}}, H_{\bar{t}}).$$

By (3.3) the dimensions on the two sides are the same, so π^* is an isomorphism.

But then the map f factors through the projection

$$X \to Y',$$

so it cannot be birational.

The condition of having a $2:1$ map on a sextic of $\mathbb{P}^5$ is open by [O'G05b, Prop 3.3], so it follows that for t outside a divisor D the same conclusion holds. Finally O'Grady shows in [O'G06] that the sextics thus obtained are all EPW sextics, so we are done.

3.2.2 – The singular case

We now want to extend the result to the case where S has finitely many singular points.

PROPOSITION 3.7. *Let $S_0 \subset \mathbb{P}^3$ be a quartic with k nodes and no other singularities. Then there exists a smooth complex variety U with a marked point 0 and a family*

$$\pi_X\colon \mathcal{X} \to U$$

such that

 i) *$\mathcal{X}_0$ is birational to $S_0^{[2]}$ and*
 ii) *for $t \in U$ generic, $\mathcal{X}_t = X_{A(t)}$ is a singular double EPW sextic; more precisely $A(t)$ contains $\bigwedge^3 W_i$ for k distinct choices of $W_i \subset V$ of dimension 3.*

Moreover one has a family

$$\pi_Z\colon \mathcal{Z} \to U$$

such that

i) $\mathcal{Z}_0$ is birational to $\mathrm{Bit}(S_0)$ and

ii) $\mathcal{Z}_t$ is isomorphic to $Z_{A(t)} \cong Y_{A(t)}[2]$ for $t \in U$ generic.

Before turning to the proof, we give some reference diagrams, which summarize the diverse varieties and maps introduced in this section and in Subsection 3.1.1. To minimize the clutter, there are three different diagrams.

$$
\begin{array}{c}
\widetilde{X} \\[4pt]
\ell^{(2)} = P \xrightarrow{\ \ p_1\ \ } \xrightarrow{\ \ p_2\ \ } P^\vee \\
\downarrow \qquad\qquad\qquad\qquad \downarrow \\
S^{[2]} \dashrightarrow X \xrightarrow{\ g_0\ } \widehat{X}_0 \\
\cup \qquad\quad \cup \qquad\ f\quad \cup \\
D' \dashrightarrow D \xrightarrow{\ \ f\ \ } S
\end{array}
\tag{3.4}
$$

$$
\begin{array}{c}
X \xrightarrow{\ \varphi_H\ } \mathrm{Gr}(1,\mathbb{P}^3) \subset |H| \\
\xrightarrow{g_0}\ \xrightarrow[6:1]{\ \widehat{\varphi}_0\ } \xrightarrow{\psi_0} \\
S \subset \widehat{X}_0 \xrightarrow{\qquad\qquad} \widehat{Y}_0 = \widehat{X}_0 / \widehat{i}_0
\end{array}
\tag{3.5}
$$

$$
\begin{array}{c}
X_t \xrightarrow{\ \varphi_{H_t}\ } Y_t \subset |H_t| \\
\xrightarrow{g_0}\ \xrightarrow[2:1]{\ \widehat{\varphi}_t\ } \xrightarrow{\psi_t} \\
S_t \subset \widehat{X}_t \xrightarrow{\qquad\qquad} \widehat{Y}_t = \widehat{X}_t / \widehat{i}_t
\end{array}
\tag{3.6}
$$

We also advise the reader to have a look at Remark 2.5, since the aim of the present subsection is precisely to invert the construction described there.

LEMMA 3.8. *Let S be a smooth connected surface, $f \colon D \to S$ a fibration with fiber $\mathbb{P}^1$, and assume we have a local deformation $\pi_D \colon \mathcal{D} \to U$ of D over the base U. Then, up to restricting U, each fiber D_t has the structure of a fibration $f_t \colon D_t \to S_t$, where S_t is a deformation of S.*

PROOF. Let P be the Hilbert polynomial of a fiber of f and consider the relative Hilbert scheme

$$
\pi_H \colon \mathcal{H} = \mathrm{Hilb}^P(\mathcal{D}/U) \to U
$$

parametrizing subvarieties of the fibers of π_D with Hilbert polynomial P. It is known that $\mathcal{H}$ is proper over U.

Let ℓ be a fiber of f, and regard ℓ as a point of $\pi_H^{-1}(0)$. The fibrations f and π_D respectively show that

$$
\mathcal{N}_{\ell/D} \cong \mathcal{O}_\ell^2 \qquad \text{and} \qquad \mathcal{N}_{D/\mathcal{D}}\big|_\ell \cong \mathcal{O}_\ell.
$$

Since
$$\mathrm{Ext}^1(\mathcal{O}_\ell, \mathcal{O}_\ell^2) = 0,$$

the exact sequence

$$0 \longrightarrow \mathcal{N}_{\ell/D} \longrightarrow \mathcal{N}_{\ell/\mathcal{D}} \longrightarrow \mathcal{N}_{D/\mathcal{D}}\big|_\ell \longrightarrow 0$$

shows that $\mathcal{N}_{\ell/\mathcal{D}} \cong \mathcal{O}_\ell^3$, in particular

$$h^0(\ell, \mathcal{N}_{\ell/\mathcal{D}}) = 3 \qquad \text{and} \qquad h^1(\ell, \mathcal{N}_{\ell/\mathcal{D}}) = 0.$$

From deformation theory it follows that $\mathcal{H}$ is smooth of dimension 3 at ℓ.

Since this holds for all ℓ in the central fiber we see that $\mathcal{H}$ is smooth along the central fiber. By properness of $\mathcal{H}$, the singular locus of $\mathcal{H}$ projects to a closed subset of U not containing 0, so up to restricting U we can assume that $\mathcal{H}$ is smooth.

The Hilbert scheme $\mathcal{H}$ is endowed with a universal family $\mathcal{C}$ with maps

$$
\begin{array}{ccc}
 & \mathcal{C} & \\
 {}_\alpha\swarrow & & {}^\beta\searrow \\
\mathcal{H} & & \mathcal{D}.
\end{array}
$$

Here $\mathcal{C}$ comes with a proper map $\pi_C \colon \mathcal{C} \to U$, and the maps α and β commute with the projections to U.

By hypothesis S is isomorphic to a component of $\pi_H^{-1}(0)$; up to replacing $\mathcal{H}$ with one of its connected (hence irreducible) components we can assume that $\pi_C^{-1}(0) \cong D = \pi_D^{-1}(0)$. In other words β is an isomorphism over 0. As above we can use properness of $\mathcal{C}$ and $\mathcal{D}$ over U to assume that β is an isomorphism everywhere.

Then the map $\alpha \circ \beta^{-1} \colon \mathcal{D} \to \mathcal{H}$ is the required fibration; more precisely letting $S_t = \pi_H^{-1}(t)$ this restricts to a map $f_t \colon D_t \to S_t$ for every $t \in U$.

Assume that S_0 has only one node p, and let S be the blowup of S_0 at p, so that S is a $K3$ surface. We let ℓ be the exceptional divisor of the blowup; since p is a node it is a smooth conic, in particular isomorphic to $\mathbb{P}^1$.

The symplectic variety $S^{[2]}$ contains $P = \ell^{(2)} \cong \mathbb{P}^2$; let X be the Mukai flop of P. We want to show that X contains a divisor D with a fibration $f \colon D \to S$ with fiber $\mathbb{P}^1$.

Let $D' \subset S^{[2]}$ be the divisor given by

$$D' = \{Z \in S^{[2]} \mid \mathrm{Supp}(Z) \cap \ell \neq \emptyset\}.$$

There is a rational fibration

$$\psi \colon D' \dashrightarrow S$$

which can be described as follows. The generic point $q + r \in D'$ has $q \in \ell$ and $r \notin \ell$; we set $\psi(q + r) = r$. The generic fiber of ψ is $\ell \cong \mathbb{P}^1$.

In the notation of Section 1.4 we have

$$[D'] = \widetilde{\mu}(\ell) \in H^2(S^{[2]}).$$

We also let

$$H' = \widetilde{\mu}(\mathcal{O}_S(1)) \in H^2(S^{[2]}).$$

Then, since $\widetilde{\mu}$ is an isometry, we have

$$q(D', D') = -2, \qquad q(D', H') = 0.$$

We let D, H be the divisors on X corresponding to D', H' respectively.

LEMMA 3.9. *The rational fibration ψ induces a regular fibration*

$$f \colon D \to S.$$

PROOF. Let $\widetilde{X}$ be the blowup of $S^{[2]}$ along P, so we have a diagram

$$\widetilde{X} = \mathrm{Bl}_P \, S^{[2]}$$

$$\xrightarrow{\quad p_1 \quad} \xrightarrow{\quad p_2 \quad}$$

$$S^{[2]} \qquad\qquad X.$$

Let $\widetilde{D} \subset \widetilde{X}$ be the strict transform of D.

Let $q + q' \in P = \ell^{(2)}$ with $q \neq q'$. Then we have the identification

$$p_1^{-1}(q + q') = \mathbb{P}(\mathcal{N}_{P/S^{[2]}})_{q+q'} \cong \mathbb{P}\big((\mathcal{N}_{\ell/S})_q \oplus (\mathcal{N}_{\ell/S})_{q'}\big).$$

We have already remarked that, via the differential, $(\mathcal{N}_{\ell/S})_q$ is identified with the line r_q through p corresponding to q itself, and the same remark applies to q'. So a point $x \in p_1^{-1}(q + q')$ defines a line $l(x)$ in the plane spanned by r_q and r'_q.

When $x \in \widetilde{D}$ the line $l(x)$ is in the tangent cone to S_0 in p, hence a point of ℓ. We let $\widetilde{\psi}(x)$ be this points. If we let

$$\widetilde{\psi}(x) = \psi(p_1(x))$$

when $p_1(x) \notin P$, we obtain a map

$$\widetilde{\psi} \colon \widetilde{D} \to S$$

which resolves the indeterminacy of ψ. Actually we did not cover the case of a point $2q \in P$, but that is easy: we can just let $\widetilde{\psi}(x) = q$ for any $x \in \pi_1^{-1}(q)$; this fits well with our definition when $q \neq q'$.

It remains to check that $\widetilde{\psi}$ descends to a map from D, and in order to do this we have to identify the fibres of p_2. The dual plane $P^\vee$ can be identified with the $\mathbb{P}^2$ parametrizing lines through p; in this way the fibration over $P^\vee$ is just the map described above, sending $x \in p_1^{-1}(P)$ to the line $l(x)$.

Indeed let $E \subset \widetilde{X}$ be the exceptional divisor, so that E can be identified with the incidence variety inside $P \times P^\vee$. The map

$$l \colon E \to \{\text{lines through } p\}$$

is a $\mathbb{P}^1$ fibration over $\mathbb{P}^2$, and the only such fibrations are the projections on P and $P^\vee$.

So we see that by construction $\widetilde{\psi}$ descends to D.

Thanks to the two lemmas we conclude the following. Consider the locus U inside the local semiuniversal deformation space of X parametrizing deformations which keep D and H of type $(1,1)$. By the local Torelli theorem U is smooth of dimension 18. For $t \in U$ denote X_t the corresponding deformation of X; we have deformations D_t of D and H_t of H inside X_t.

More precisely we have a family $\pi_X \colon \mathcal{X} \to U$ with two divisors $\mathcal{D}$ and $\mathcal{H}$ which restrict to D_t and H_t respectively on each fiber. Moreover we have a fibration $f \colon \mathcal{D} \to \mathcal{S}$ with fiber $\mathbb{P}^1$, which restricts to fibrations $f_t \colon D_t \to S_t$ on each fiber; for $t = 0$ this gives the fibration $D \to S$ of Lemma 3.9.

We now analyze in more detail the family $\mathcal{X}$. We will allow ourselves to restrict U when necessary.

LEMMA 3.10. *The divisor H is big and nef. In particular*

$$H^i(X, H) = 0$$

for $i > 0$.

PROOF. We have shown in Section 3.1 that sections of H define a regular map

$$\varphi_H \colon X \to \mathbb{P}^5;$$

in particular H is base-point-free, and so it is nef. Since $q(H, H) > 0$ it is also big.

The last claim follows from Kawamata-Viehweg vanishing and the fact that K_X is trivial.

COROLLARY 3.11. *For every $t \in U$ we have*

$$h^i(X_t, H_t) = 0 \text{ for } i > 0 \qquad and \qquad h^0(X_t, H_t) = 6.$$

PROOF. We know that this holds for $t = 0$. By semicontinuity we have $h^i(X_t, H_t) = 0$ for all small t. Moreover by flatness we see that $\chi(X_t, H_t)$ is constant, and so $h^0(X_t, H_t)$ is constant too.

Now we consider the (relative) linear system defined by $\mathcal{H}$. We have just shown that the sheaf $(\pi_X)_*(\mathcal{H})$ has constant rank 6; hence it is a vector bundle on U. We have a map

$$\varphi_{\mathcal{H}} \colon \mathcal{X} \dashrightarrow \mathbb{P}((\pi_X)_*(\mathcal{H})^\vee),$$

which restricts to evaluation of sections on each fiber. We know that on the central fiber

$$\varphi_H \colon X \to \mathbb{P}(H^\vee)$$

does not have base points; since the base locus of $\varphi_{\mathcal{H}}$ is closed and the projection π_X is proper we see that

$$\varphi_{H_t} \colon X_t \to \mathbb{P}(H_t^\vee)$$

does not have base points for all small t; we restrict U accordingly, so that this holds for all $t \in U$.

Consider now the Stein factorization of φ_{H_t}, given by

$$X_t \xrightarrow{\varphi_{H_t}} |H_t| \cong \mathbb{P}^5.$$
$$\xrightarrow[g_t]{} \xrightarrow[\widehat{\varphi}_t]{}$$
$$\widehat{X}_t$$

LEMMA 3.12. *The variety $\widehat{X}_t$ is obtained from X_t by contraction of D_t along the fibers of $f_t \colon D_t \to S_t$.*

PROOF. By definition of the Stein factorization, g_t has connected fibers and $\widehat{\varphi}_t$ has finite fibers. So we just need to prove the fibers of f_t are the only curves contracted by φ_{H_t}. A curve $C \subset X_t$ is contracted by φ_{H_t} if and only if $H_t \cdot C = 0$, and this happens exactly for the fibers of f_t.

REMARK. There is another way to obtain this diagram, using the Cone theorem ([KM98, Theorem 3.7]). Since K_{X_t} is trivial we work with the pair $(X_t, \frac{1}{2}D_t)$; this is Kawamata-log-terminal since X_t and D_t are smooth. By the theorem, the D_t-negative part of the Mori cone is generated by the classes of rational curves on X_t. Any such curve C is contained in D_t, so it is either a fiber of f_t or it projects to a rational curve on S. However in the second case the intersection $H_t \cdot C > 0$.

This shows that the hyperplane $H_t = 0$ cuts the Mori cone precisely on the ray containing the class of the fibers of f_t. We can then perform the corresponding extremal contraction to obtain a variety $\widehat{X}_t$. Since $H_t = 0$ on the contracted ray, the associated line bundle $\mathcal{O}_{X_t}(H_t)$ descends to $\widehat{X}_t$. Moreover every section of $\mathcal{O}_{X_t}(H_t)$ is constant along the fibers, since these are rational curves and $\mathcal{O}_{X_t}(H_t)$ has degree

0 on them. We deduce that every section in $H^0(X_t, \mathcal{O}_{X_t}(H_t))$ comes from $\widehat{X}_t$, so φ_{H_t} factorizes through $\widehat{X}_t$.

LEMMA 3.13. *For generic* $t \in U$ *the map*

$$\widehat{\varphi}_t \colon \widehat{X}_t \to \mathbb{P}^5$$

is $2 : 1$ *on a sextic* Y_t *of* $\mathbb{P}^5$.

PROOF. We have verified that for $t = 0$ the map is $6 : 1$ on a quadric, namely the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$ embedded by the Plücker map. In particular

$$H_0^4 = 12,$$

and since this is constant with t we get $H_t^4 = 12$ for all t. So it is enough to show that $\widehat{\varphi}_t$ is $2 : 1$ for generic t.

Consider the rational involution $S^{[2]} \dashrightarrow S^{[2]}$ defined in Section 3.1. This induces a *regular* involution

$$i \colon X \to X.$$

By the remark in section 4.1.3 of [O'G05b] this extends to an involution i_t of X_t. One verifies that i_t sends each fiber of f_t to itself, thereby defining a regular involution

$$\widehat{i}_t \colon \widehat{X}_t \to \widehat{X}_t.$$

We let $\widehat{Y}_t$ be the quotient of $\widehat{X}_t$ by this involution. The same argument as in the proof of Proposition 3.4 shows that we have a factorization

$$\widehat{X}_t \xrightarrow{\ \widehat{\varphi}_t\ } |H_t| \cong \mathbb{P}^5.$$
$$\xrightarrow{\quad\quad} \xrightarrow[\psi_t]{\quad}$$
$$\widehat{Y}_t$$

Now the map

$$\psi_0 \colon \widehat{Y}_0 \to \mathbb{P}^5$$

is $3 : 1$ on a quadric, so for every t the map ψ_t can either be $3 : 1$ or birational. We only need to show that the former only happens for t in a Zariski closed subset of U.

If ψ_t is $3 : 1$ there is a ramification divisor $E_t \subset \widehat{Y}_t$; indeed $\widehat{Y}_t$ is a normal variety with $K_{\widehat{Y}_t} = 0$. Let $E'_t \subset X_t$ be the preimage of E_t. This is a divisor which is a deformation of E'_0. But by the local Torelli theorem the subset of U for which E'_0 remains of type $(1, 1)$ is a divisor in U.

COROLLARY 3.14. *For generic $t \in U$ the variety $\widehat{X}_t$ is a double covering of an EPW sextic Y_t.*

PROOF. We have constructed a map

$$\varphi_t \colon \widehat{X}_t \to \mathbb{P}^5$$

which is $2 : 1$ on a sextic. We need only to show that the sextic thus obtained is an EPW sextic.

For this we can adapt the arguments of [O'G06, Theorem 5.2].

So we see that from the smooth irreducible symplectic variety X_t one obtains a singular EPW sextic by first contracting the divisor D_t along the fibers of the fibration f_t and then taking the quotient by the involution.

We need one more

LEMMA 3.15. *Assume that the sextic Y_A contains a plane $\Pi = \mathbb{P}(W)$. If*

  i) Y_A is singular along Π;
 ii) $\Pi \not\subset Y_A[2]$;
iii) the singular locus of Y_A has dimension at most 2

then

$$A \supset \bigwedge\nolimits^3 W.$$

PROOF. Let $[w] \in \mathbb{P}(W) \setminus Y_A[2]$. By the description of the singularities of Y_A we know that there exists some $W' \subset V$ of dimension 3 such that $\bigwedge^3 W' \subset A$ and $[w] \in \mathbb{P}(W')$.

Assuming W' never equals W we find a 1-dimensional locus of subspaces $W' \subset V$ such that $\bigwedge^3 W' \subset A$; but then the singular locus of Y_t has dimension at least 3.

Now we can finish the proof of Proposition 3.7, showing that the EPW sextics Y_t obtained above are actually in Σ. First we remark that S is a degree 2 $K3$, with natural $2 : 1$ map to $\mathbb{P}^2$, namely projection from p. This map is induced by the divisor $h - \ell$, where $h \in \mathcal{O}_S(1)$. By construction both h and ℓ remain of type $(1, 1)$ in S_t, so each S_t is a degree 2 $K3$ surface.

More precisely we can observe that S_t, being the contraction of D_t, has a natural embedding in $\widehat{X}_t$.

LEMMA 3.16. *If one considers $S_t \subset \widehat{X}_t$, then the degree 2 map above is just the restriction $\varphi_t\big|_{S_t}$.*

PROOF. It is enough to check this for $t = 0$, so we only need to show that the divisor $\widehat{H}_0$ on $\widehat{X}_0$ which induces φ_0 restricts to $h - \ell$ on S. Recall that $\widehat{H}_0$ is induced by the divisor H on X. The map

$$\varphi_0\big|_D \colon D \to \mathbb{P}^2$$

is just the map l appearing in the proof of Lemma 3.9, so it contracts the fibers of the fibration

$$f \colon D \to S;$$

this gives the desired map $S \to \mathbb{P}^2$. Keeping track of the various constructions one realizes that this is just projection from p.

COROLLARY 3.17. *Let Y_t be one of the EPW sextics described above, say $Y_t = Y_A$. Consider the plane $\Pi = \varphi_t(S_t)$, say $\Pi = \mathbb{P}(W)$ for some $W \subset V$. Then*

$$\textstyle\bigwedge^3 W \subset A.$$

PROOF. We want to apply Lemma 3.15. First, we have to check that $\Pi \not\subset Y_A[2]$; this amounts to saying that S_t is not contained in the ramification locus of the projection $\widehat{X}_t \to Y_t$. This holds because the map φ_t has degree 2 both on $\widehat{X}_t$ and on S_t.

Second, we need to show that Y_t is singular along P. Indeed $\widehat{X}_t$ is singular along S_t; this can be checked locally using the fact that $\widehat{X}_t$ is the contraction of the fibers of f_t. On the generic point $x \in S_t$ the covering φ_t is not ramified, hence the germ of $\widehat{X}_t$ along x is the same as the germ of Y_t along $\varphi_t(x)$, showing that Y_t is singular in $\varphi_t(x)$. Since the singular locus is closed we deduce that Y_t is singular along Π.

Finally, the same argument shows that Y_t is singular along the branch locus of φ_t and the image of S_t. Since S_t is a surface, if we show that the branch locus of φ_t has dimension at most 2, we deduce that the singular locus of Y_t has dimension 2.

Consider the involution i_t of X_t; this is an antisymplectic involution, hence the fixed locus Z_t of i_t is an isotropic subvariety of X_t. In particular Z_t has dimension at most 2, and the branch locus of φ_t is just the image $\varphi_{H_t}(Z_t)$, so we are done.

We have now shown that Y_t is a member of Σ, thereby proving Proposition 3.7. $\square$

We want to be more precise in the case $k > 1$. Given a point $p \in \mathbb{P}^3$ consider the set H_p of lines through p. This can be regarded as a plane inside the Grassmannian

$$\mathrm{Gr}(1, \mathbb{P}^3) \subset \mathbb{P}^6,$$

so it yields a point $H_p \in \mathrm{Gr}(2, \mathbb{P}^6)$. This gives a map

$$\rho \colon \mathbb{P}^3 \longrightarrow \mathrm{Gr}(2, \mathbb{P}^6).$$

$$p \longmapsto H_p$$

By direct computation one sees that ρ is just the composition of the second Veronese map

$$v \colon \mathbb{P}^3 \to \mathbb{P}^{10}$$

and a linear embedding $\mathbb{P}^{10} \hookrightarrow \mathbb{P}^{19}$.

One can see this without computations in the following way. Let for a moment $\mathbb{P}^3 = \mathbb{P}(U)$. Then by the results of the previous Section, the map ρ is just the composite of the second Veronese map with the inclusion

$$\mathbb{P}(\mathrm{Sym}^2(U)) \hookrightarrow \mathbb{P}\big(\textstyle\bigwedge^2(\textstyle\bigwedge^3(U))\big)$$

induced by the decomposition (3.2).

COROLLARY 3.18. *Assume $S_0 \subset \mathbb{P}^3$ is a quartic with k nodes $p_1, \ldots, p_k$. Assume that the images of $p_1, \ldots, p_k$ under the second Veronese map v are projectively independent.*

Let $\widehat{X}_t$ be one of the singular double EPW sextics constructed above, say $\widehat{X}_t \cong X_{A(t)}$, and let $W_1, \ldots, W_k \subset V$ be the subspaces of dimension 3 such that

$$\textstyle\bigwedge^3 W_i \subset A(t).$$

Then $W_1, \ldots, W_k$, regarded as points on

$$\mathrm{Gr}(3, V) \subset \mathbb{P}(\textstyle\bigwedge^3 V),$$

are projectively independent.

PROOF. Let S be the blowup of S_0 at $p_1, \ldots, p_k$ and let $H_i \cong \mathbb{P}^2$ be the set of lines through p_i. Consider the projection from p_k

$$\pi_k \colon S \to H_i;$$

this is $2 : 1$ map, and we have shown that it deforms to a $2 : 1$ map $S_t \to \mathbb{P}(W_i)$; hence it is enough to verify that $H_1, \ldots, H_k$ are projectively independent. But this is exactly our hypothesis.

3.3 – The surface of bitangents to a quartic

In the present section we recall some known results about quartic surfaces in $\mathbb{P}^3$, with a special regard towards the surface of bitangents. The main source here is [Wel81]. We do not include proofs of the results, and refer to the original thesis instead.

Let $S \subset \mathbb{P}^3$ be a quartic surface, having at most ordinary double points as singularities, and let

$$\mathrm{Bit}(S) \subset \mathrm{Gr}(2, 4)$$

be the locus of lines bitangent to S. It is easy to check that $\mathrm{Bit}(S)$ is a surface.

$\mathrm{Bit}(S)$ is endowed with a natural double covering, ramified over the set of lines contained in S. To see this consider the double covering

$$\pi\colon X \to \mathbb{P}^3$$

ramified along S. This is a threefold whose singularities are in bijective correspondence with those of S. We also let

$$\mathcal{O}_X(1) = \pi^*\mathcal{O}_{\mathbb{P}^3}(1).$$

DEFINITION 3.1. A curve $\ell \subset X$ is called a *line* if it is rational and

$$\ell \cdot \mathcal{O}_X(1) = 1.$$

It easy not difficult to check that if $\ell \subset X$ is a line, then $\pi(\ell)$ is a bitangent to S. Viceversa if $r \in \mathrm{Bit}(S)$ and r is not contained in S, the inverse image $\pi^{-1}(r)$ splits as $\ell_1 \cup \ell_2$, where ℓ_1, ℓ_2 are lines on X. We will check something similar in a more geometric context in later sections; instead of X we will have a cubic threefold birational to it. For many purposes the double covering X is a good substitute of this cubic threefold.

Let F be the Hilbert scheme parametrizing lines on X. Thanks to the above facts, one has a double covering

$$\rho\colon F \to \mathrm{Bit}(S),$$

ramified over the set of lines in S. In particular if S does not contain any line, the covering ρ is unramified.

Now assume that S is *smooth*. Then Welters computes the Hodge numbers of $\mathrm{Bit}(S)$. The result is as follows:

THEOREM (Welters). *Let S be a smooth quartic surface. The Hodge numbers of* $\mathrm{Bit}(S)$ *are:*

$$h^{0,0}(\mathrm{Bit}(S)) = 1, \qquad h^{1,0}(\mathrm{Bit}(S)) = 0, \qquad h^{2,0}(\mathrm{Bit}(S)) = 45.$$

In particular one sees that $\mathrm{Bit}(S)$ is connected and *regular*. Applying Corollary 3.5 we find:

COROLLARY 3.19. *If $A \in \mathbb{LG}(\bigwedge^3 V)^0$, the surface $Y_A[2]$ is regular.*

Welters goes on and computes the Hodge numbers of F too, these are as follows.

THEOREM (Welters). *Let S be a smooth quartic surface not containing a line, F the double covering of* $\mathrm{Bit}(S)$ *described above. The Hodge numbers of F are:*

$$h^{0,0}(F) = 1, \qquad h^{1,0}(F) = 10, \qquad h^{2,0}(F) = 101.$$

Again we see that F is connected. In particular the double covering ρ is not trivial; hence it is defined by *nontrivial* 2-torsion class

$$\kappa \in H^2(\mathrm{Bit}(S), \mathbb{Z}).$$

The other information that we need is the canonical class of $\mathrm{Bit}(S)$. These is again computed by Welters:

THEOREM (Welters). *Let* $h \in H^2(\mathrm{Bit}(S), \mathbb{Z})$ *be the polarization given by the Plücker embedding. Then the canonical class of* $\mathrm{Bit}(S)$ *satisfies*

$$c_1(K_{\mathrm{Bit}(S)}) = 3h + \kappa.$$

REMARK. The above result is accordance with the result we found about the canonical class of $Y_A[2]$ in Proposition 2.8. But our result only determined the canonical class up to a 2-torsion element, and now we see (again, thanks to Corollary 3.5) that this 2-torsion element does not vanish, as we had anticipated.

There are many other results about the geometry of quartic surfaces and of the associated double covering X, but for our needs we can stop here, referring the interested reader to the original thesis of Welters, and to the papers [Cle83] and [Deb90].

3.4 − 28 bitangents on a quartic plane curve

In this section we recall the classical computation that a generic plane quartic C admits exactly 28 bitangent lines, and the correspondence with the 27 lines on a cubic surface. We do this in order to generalize the construction one dimension up and study the birational geometry of the surface of bitangents to a quartic surface.

We start with a smooth cubic surface $S \subset \mathbb{P}^3$; we recall the well-known fact that S contains exactly 27 lines. Once we choose a generic point $p \in S$ not lying on any of these lines, we can associate to (S, p) a quartic curve as follows. Projection from p to a plane $A \cong \mathbb{P}^2$ gives a rational map $\pi_p \colon S \dashrightarrow \mathbb{P}^2$ which is regular outside p. Let $\widetilde{S}$ be the blowup of S in p, with exceptional divisor E. The map π_p extends to a morphism $\pi_p \colon \widetilde{S} \to A$: to any point $x \in E$ corresponding to a line $\ell_x \subset T_pS$ through p we associate the intersection $\pi_p(x) = \ell_x \cap A$.

It is easy to check that π_p is finite (here we use the fact that p does not lie on any line in S) and that the generic point of $\mathbb{P}^2$ has 2 preimages, so it is a $2 : 1$ ramified covering. We claim that the ramification locus $C \subset \mathbb{P}^2$ is a quartic. Indeed any plane section of S through p is a a cubic curve with a $2 : 1$ morphism over a line in $\mathbb{P}^2$. The generic such curve is smooth, hence it has genus 1; by the Riemann-Hurwitz formula this is ramified over 4 points. This shows that the generic line of $\mathbb{P}^2$ cuts C in 4 points, hence C is a quartic. We will show later on how to invert

this construction to produce from a smooth quartic C a cubic surface S with a distinguished point on it.

For the moment let us restrict our attention to quartics obtained in this fashion. We wish to prove that C has exactly 28 bitangents; more precisely that these are the projections of the 27 lines in S and of the exceptional divisor E. We use the following notation.

- Given a line $r \subset A$ we denote by P_r the plane spanned by r and the point p.
- The intersection $P_r \cap S$ is a cubic curve in P_r, which we denote by D_r; it has a $2:1$ morphism over r.
- In particular we have a singular cubic $D = T_p S \cap S$.

As r varies in $A^\vee$ the curve D_r can acquire various singularities, but since p does not lie on a line in S it is never a union of lines. The key of the argument is the following proposition, relating the position of r relative to C to the singularities of D_r.

PROPOSITION 3.20. *Let $r \subset A$ be a line and let D_r be the corresponding cubic curve. Then*

i) *If D_r is smooth, r cuts C transversely in 4 distinct points.*
ii) *If D_r is irreducible with one node distinct from p, r is tangent to C and cuts C transversely in 2 other points.*
iii) *If D_r is irreducible with one cusp distinct from p, r is tangent to C in a flex and cuts C transversely in 1 other point.*
iv) *Assume $D_r = Q \cup \ell$, with Q a conic and ℓ a line. If Q and ℓ are transverse, r is bitangent to C.*
v) *Assume $D_r = Q \cup \ell$, with Q a conic and ℓ a line. If Q and ℓ are tangent, r is quadritangent to C.*
vi) *If $D_r = D = T_p S \cap S$ and p is a node of D, r is a bitangent to C.*
vii) *If $D_r = D = T_p S \cap S$ and p is a cusp of D, r is a quadritangent to C.*

Moreover for $p \in S$ generic, cases v) and vii) do not happen.

The main point is to count how many elements there are in the set-theoretic intersection $r \cap C$. To be get the precise conclusion, though, we use the following

LEMMA 3.21. *Let q be a point of D_r such that $D_r \cdot \overline{pq} = p + 2q$. Then $\pi(q) \in C$; moreover q is singular on D_r if and only if $\pi(q)$ is a point of tangency between r and C.*

PROOF.[Proof of Proposition 3.20] Assume first that $D_r \neq D$. Choose $q \in \pi(\ell)$; the intersection $\overline{pq} \cap S$ consists of 3 points, counted with multiplicity; to obtain $\pi_p^{-1}(q)$ we have to remove p from it.
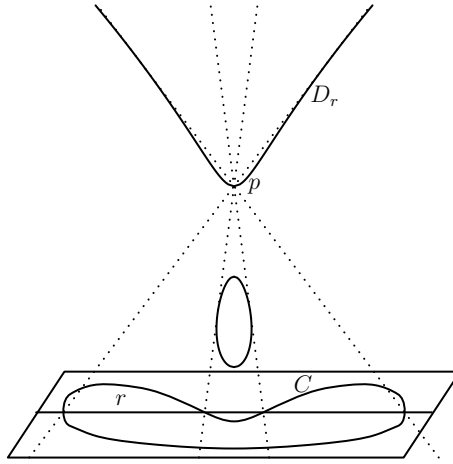
Figure 3.1 The case where D_r is a smooth cubic

i) If D_r is smooth, it is has genus 1 and by Riemann-Hurwitz the map $\pi_p\big|_{D_r}$ is ramified exactly over 4 points, so r meets C in 4 distinct points. But then the multiplicity at each of these points must be one, so r and C are not tangent.

ii) Let $\widetilde{D}$ be the normalization of D_r; then $\widetilde{D}$ is rational. The projection $\widetilde{D} \to r$ is then ramified on 2 points by the Riemann-Hurwitz formula. These two point are not above the node. So we have 3 points of intersection between C and r: two of them come from the ramification and one by the line through the node. By Lemma 3.21 the line through the node yields in a tangency point, while the other two give simple intersections.

iii) This case is like the preceding one. The only difference is that this time the point of $\widetilde{D}$ over the cusp *is* a point of ramification. So we get only 2 points of intersection between C and r, and only one is a tangent; so it must be a flex.

iv) In this case $\pi_p^{-1}(q)$ is one point exactly when the line $\overline{pq}$ passes through the singular points of $D_r = Q \cup \ell$. There are two singular points, given by $Q \cap \ell$ so r meets C twice. By Lemma 3.21 both points are of tangency.

v) This case is the same as the preceding one, but since Q and ℓ meet in just one point, the same happen for r and C, so r is a quadritangent.

vi) Now consider $\pi(E)$. First we remark that it is actually a line, since it is given by $T_pS \cap A$. As before we count, given a point $q \in r = T_pS \cap A$, how many points there are in $\pi_p^{-1}(q)$. First there is a point for the direction $\overline{pq} \in E$. Then there is a preimage for each intersection $\overline{pq} \cap D$ (apart from p itself). So $q \in C$ if and only if $\overline{pq}$ meets D only in p. If D has a node in p there are two directions which satisfy this condition. As above, these are projected to tangency points between $\pi_p(E)$ and C.

vii) This case is the same as the preceding one, but since p is a cusp, there is just
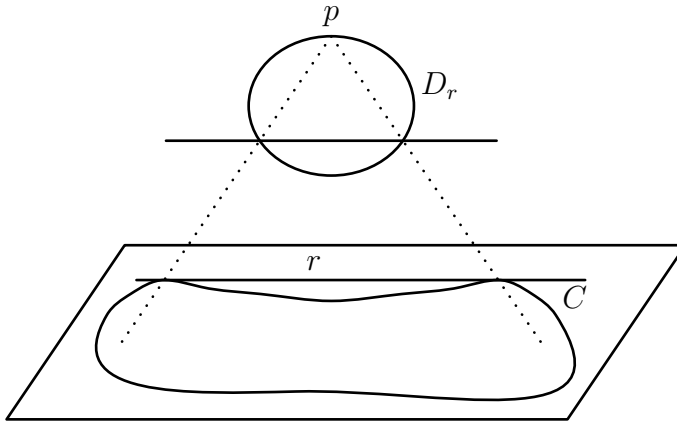
Figure 3.2 The case where D_r is reducible

one line through p which meets D thrice. So r and C meet in just one point, that is, r is a quadritangent.

COROLLARY 3.22. *If $p \in S$ is generic, the lines bitangent to C are exactly the projections of either a line in S or the exceptional divisor E. In particular there are 28 bitangents.*

So far we have counted the bitangents on a curve C obtained as the ramification locus for the projection $\pi_p \colon S \to \mathbb{P}^2$. Next we study the problem of inverting this construction, that is starting from a generic quartic $C \subset \mathbb{P}^2$, producing a cubic surface S with a distinguished point p on it.

We define the surface $\widetilde{S}$ as the double covering of $\mathbb{P}^2$ branched along C. This is uniquely defined by C and is smooth if C is ([BHPvdV04, sec. I.17]). We wish to produce S as the blow-down of a (-1)-curve in $\widetilde{S}$. Consider the case where C actually comes from the preceding construction. Then for each line r bitangent to C, $\pi_p^{-1}(r)$ is the union of two rational curves meeting in two points. Indeed it is the union of a line and quadric when $P_r \neq T_p S$; otherwise $D_r = D$ is a cubic curve with one node in p, so in the blowup its strict transform is rational and meets in two points the exceptional divisor.

This always happens. Indeed let r be a bitangent to C (these always exist by a dimension count), let $r \cap C = \{a, b\}$ and consider the ramified covering over r. By construction this is the inverse image in the total space of $\mathcal{O}_r(2)$ of the section $s \in H^0(r, \mathcal{O}_r(4))$ whose divisor is $2a + 2b$ (under the squaring map). Since s admits a global square root t, this inverse image is just $t(r) \cup -t(r)$, so we have two copies of $\mathbb{P}^1$ meeting in two points, which map to r.

In order to produce S we have to choose one such bitangent r and one of the rational curves $\ell_1, \ell_2 \subset \widetilde{S}$ mapping to it. This will be the curve contracted to a

point in S; the other one will accordingly acquire a node. For each other bitangent r', with preimages ℓ_1', ℓ_2', this choice will determine which one among ℓ_1' and ℓ_2' will become a line and which one a conic.

Let us choose a bitangent r and a rational curve ℓ_1 mapping to it. The map $f \colon \widetilde{S} \to S \subset \mathbb{P}^3$ will be given by a suitable line bundle L on $\widetilde{S}$, so we start looking for a line bundle $L \in \operatorname{Pic} \widetilde{S}$ such that

$$L^2 = 3; \qquad L \cdot \ell_1 = 0; \qquad L \cdot \ell_2 = 3. \tag{3.7}$$

The first thing to do is calculate some intersection numbers on $\widetilde{S}$. We use the following notation.

- The projection to $\mathbb{P}^2$ is denoted by $\pi \colon \widetilde{S} \to \mathbb{P}^2$.
- On $\widetilde{S}$ we have the line bundle $\mathcal{O}_{\widetilde{S}}(1) = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$.
- The branch locus of π is denoted $\widetilde{C}$; it is isomorphic to C.

In particular we can express the canonical of $\widetilde{S}$ as

$$K_{\widetilde{S}} = \pi^*(K_S) + \widetilde{C} = \mathcal{O}_{\widetilde{S}}(-3) + \widetilde{C}.$$

LEMMA 3.23. *The intersection numbers are:*

$$\mathcal{O}_{\widetilde{S}}(1)^2 = 2; \qquad \mathcal{O}_{\widetilde{S}}(1) \cdot \widetilde{C} = 4; \qquad \widetilde{C}^2 = 8.$$

PROOF. The first relation comes from the fact that π is a $2:1$ covering. For the second just use push-pull formula:

$$\mathcal{O}_{\widetilde{S}}(1) \cdot \widetilde{C} = \mathcal{O}_{\mathbb{P}^2}(1) \cdot \pi_*(\widetilde{C}) = \mathcal{O}_{\mathbb{P}^2}(1) \cdot C = 4.$$

For the last one we remark that $\widetilde{C}$ is isomorphic to C, so its genus can be computed by the genus formula in $\mathbb{P}^2$:

$$g(\widetilde{C}) = g(C) = \frac{(4-1) \cdot (4-2)}{2} = 3.$$

On the other hand, the genus formula on $\widetilde{S}$ gives

$$g(\widetilde{C}) = 1 + \frac{\widetilde{C}^2 + K_{\widetilde{S}} \cdot \widetilde{C}}{2} = 1 + \widetilde{C}^2 - \frac{3\mathcal{O}_{\widetilde{S}}(1) \cdot \widetilde{C}}{2} = \widetilde{C}^2 - 5,$$

so $\widetilde{C}^2 = 8$.

We remark that we can compute, for every $h, k \in Z$

$$(\mathcal{O}_{\widetilde{S}}(k) + h\widetilde{C})^2 = 2k^2 + 8kh + 8h^2.$$

In particular $K_{\widetilde{S}}^2 = 2$ as expected. Indeed a smooth cubic surface S is the blowup of $\mathbb{P}^2$ in six points, so $K_S^2 = 3$. From the preceding relations we see at once that $\pi^*(C) = 2\widetilde{C}$ (this is obvious also by a local computation). Indeed it is clear that $\pi^*(C) = k\widetilde{C}$ for some $k > 0$. But then

$$k^2 \widetilde{C}^2 = \pi^*(C)^2 = \pi^*(\mathcal{O}_{\mathbb{P}^2}(4))^2 = 32,$$

so $k = 2$. In particular we see that $\widetilde{C}$ is ample on $\widetilde{S}$.

Now consider $\pi^*(r) = \ell_1 + \ell_2$.

LEMMA 3.24. *We have*
$$\ell_1^2 = \ell_2^2 = -1.$$

In particular the rational curves ℓ_i are (-1)-curves.

PROOF. We have $\ell_1 + \ell_2 \sim \mathcal{O}_{\widetilde{S}}(1)$, so

$$(\ell_1 + \ell_2)^2 = 2.$$

Since ℓ_1 and ℓ_2 meet transversely in two points, $\ell_1 \cdot \ell_2 = 2$. Moreover $\ell_1^2 = \ell_2^2$, since the two curves are swapped by the covering involution, so the thesis follows.

We can now choose the line bundle $L = \mathcal{O}_{\widetilde{S}}(1) + \ell_1$. From the preceding computations we see that L satisfies the relations (3.7).

PROPOSITION 3.25. *The line bundle L has no base points and defines a morphism $\varphi_L \colon \widetilde{S} \to \mathbb{P}^3$ which contracts ℓ_1 and no other curves. Moreover $S = \varphi_L(\widetilde{S})$ is a smooth cubic surface.*

PROOF. We do this in several steps.

Step 1: The line bundle L has no higher cohomology. Let

$$M = L - K_{\widetilde{S}} = \mathcal{O}_{\widetilde{S}}(4) + \ell_1 - \widetilde{C} = \widetilde{C} + \ell_1. \qquad (3.8)$$

The last expression shows that M is the sum of an ample and an effective divisor class, so it is big. Moreover M is nef. To see this we have to intersect M with an irreducible curve D on $\widetilde{S}$. If $D \neq \ell_1$ the intersection is clearly positive; on the other hand

$$M \cdot \ell_1 = 1.$$

Since $L = K_{\widetilde{S}} + M$, the conclusion follows by Kawamata-Viehweg vanishing.

Step 2: $h^0(\widetilde{S}, L) = 4$. This can be computed by Riemann-Roch. Indeed by the preceding step

$$h^0(\widetilde{S}, L) = \chi(\widetilde{S}, L) = 1 + \frac{L^2 - K_{\widetilde{S}} \cdot L}{2} = 1 + 3 = 4.$$

Step 3: The line bundle L has no base points. Consider the incomplete linear series

$$\mathcal{L} = \{S + \ell_1 \mid S \in |\mathcal{O}_{\widetilde{S}}(1)|\} \subset |L|.$$

By Riemann-Roch and Kodaira vanishing we can compute as above that

$$h^0(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(1)) = 3,$$

so $\mathcal{L}$ has dimension 2. It follows that any element of $\mathcal{L}$ has the form $S + \ell_1$, where S is the pullback of a section of $\mathcal{O}_{\mathbb{P}^2}(1)$.

The base locus of $\mathcal{L}$ is by construction ℓ_1. Take some divisor $D \in |L| \setminus \mathcal{L}$. Then the base locus of $|L|$ is just $\ell_1 \cap D$. We remark that D cannot have ℓ_1 among its components, otherwise $D \in \mathcal{L}$. Moreover

$$D \cdot \ell_1 = L \cdot \ell_1 = 0,$$

so D is disjoint from ℓ_1 and the thesis follows.

Step 4: φ_L is birational and $\varphi_L(\widetilde{S})$ is a cubic. Let $S = \varphi_L(\widetilde{S}) \subset \mathbb{P}^3$. Then

$$\deg S \cdot \deg \varphi_L = L^2 = 3.$$

Since S is not degenerate we cannot have $\deg S = 1$, so $\deg S = 3$ and $\deg \varphi_L = 1$, that is, φ_L is birational onto S.

Step 5: φ_L contracts ℓ_1 and no other curves. Indeed $L \cdot \ell_1 = 0$, so the curve ℓ_1 is contracted by φ_L. Viceversa let D be any irreducible curve on $\widetilde{S}$ with $L \cdot D = 0$. Since $\mathcal{O}_{\widetilde{S}}(1)$ is ample we must have $\ell_1 \cdot D < 0$, so D is a multiple of ℓ_1.

Step 6: S is smooth. By the preceding steps $\varphi_L \colon \widetilde{S} \to S$ is just the contraction of ℓ_1. Since ℓ_1 is a (-1)-curve, the resulting surface is smooth.

We see at once that this construction is the inverse of the former. The map $\widetilde{S} \to \mathbb{P}^2$ is given by the incomplete linear series $\mathcal{L}$ from the proof, so it is not defined on ℓ_1, and it becomes defined outside p in S. But restricting to an incomplete linear series is the same as projecting, so we see that the map $S \dashrightarrow \mathbb{P}^2$ is given by projection. The center of the projection is necessarily p, since the map is defined elsewhere.

REMARK. We can explicitly produce the extra section in $H^0(\widetilde{S}, L)$ which does not come from a section of $\mathcal{O}_{\mathbb{P}^2}(1)$. Indeed $L = \mathcal{O}_{\widetilde{S}}(2) - \ell_2$, so we are looking for a section of $\mathcal{O}_{\widetilde{S}}(2)$ which vanishes over ℓ_2. $\widetilde{C}$ is a section of $\mathcal{O}_{\widetilde{S}}(2)$ which meets ℓ_2 in two points $\{a, b\}$; let Q be any conic in $\mathbb{P}^2$ meeting r in $\pi(a)$ and $\pi(b)$. Then $\widetilde{C}$ and $\pi^{-1}(Q)$ cut the same divisor on ℓ_2, so a suitable linear combination of the corresponding sections of $\mathcal{O}_{\widetilde{S}}(2)$ vanishes identically on ℓ_2.

We can check again how to get the bitangents.

PROPOSITION 3.26. *Let r' be any bitangent to C distinct from r, and let ℓ'_1 and ℓ'_2 be the rational curves over r. Then, up to swapping ℓ'_1 and ℓ'_2, we have that ℓ_i meets ℓ'_i transversely in one point for $i = 1, 2$ and ℓ_i is disjoint from ℓ'_j if $i \neq j$. Moreover $\varphi_L(\ell'_1)$ is a conic and $\varphi_L(\ell'_2)$ is a line.*

PROOF. The lines r and r' meet transversely in a single point, so $\ell_1 \cup \ell_2$ meets $\ell'_1 \cup \ell'_2$ transversely in two points. One of this is on ℓ_1, and up to swapping we can assume that it is on ℓ'_1. Then the other is on both ℓ_2 and ℓ'_2. This proves the first claim.

Since
$$L \cdot \ell'_1 = \mathcal{O}_{\widetilde{S}}(1) \cdot \ell'_1 + \ell_1 \cdot \ell'_1 = 2,$$

$\varphi_L(\ell'_1)$ is a plane conic. Similarly $L \cdot \ell'_1 = 1$, so the image of ℓ'_2 is a line.

COROLLARY 3.27. *Any bitangent line to C is the projection of either ℓ_1 or a line in S.*

REMARK. Let $\mathcal{S}$ denote the universal cubic surface, that is the subvariety of
$$\mathbb{P}^3 \times \mathbb{P}(\mathrm{Sym}^3(\mathbb{P}^3)^\vee) \cong \mathbb{P}^3 \times \mathbb{P}^{20}$$

of bidegree $(3, 1)$ defined by the equation
$$\sum_I a_I \mathbf{x}^I,$$

where $x_0, \ldots, x_3$ are coordinates on $\mathbb{P}^3$, $I = (i_0, \ldots, i_3)$ is a multiindex such that $i_0 + \cdots + i_3 = 3$ and the a_I are coordinates on $\mathbb{P}^{20}$.

The variety $\mathcal{S}$ is fibered over $\mathbb{P}^{20}$ and the fiber over the point
$$\mathbf{a} = \big(a_{(3,0,0,0)}, \ldots, a_{(0,0,0,3)}\big)$$

is the cubic $S \subset \mathbb{P}^3$ with equation
$$\sum_I a_I \mathbf{x}^I = 0.$$

Let $\mathcal{C} = \mathbb{P}(\mathrm{Sym}^4(\mathbb{P}^2)^\vee) \cong \mathbb{P}^{15}$ be the parameter space for quartics in $\mathbb{P}^2$. Then the above correspondence can be rephrased as follows.

PROPOSITION 3.28. *Let $\mathcal{S}'$ be the GIT quotient of $\mathcal{S}$ by the natural action of $\mathbb{P}\mathrm{GL}(3)$ (so $\mathcal{S}'$ is a moduli space for couples (S, p), where S is a cubic surface and $p \in S$ a point) and $\mathcal{C}'$ be the GIT quotient of $\mathcal{C}$ by the natural action of $\mathbb{P}\mathrm{GL}(2)$. Then we have constructed a rational map*
$$\mathcal{S}' \dashrightarrow \mathcal{C}'$$

of degree 56.

3.5 – The surface of bitangents and the Fano of a cubic

We now wish to generalize the construction of the preceding section one dimension up. Here the matters are more delicate, since every point on a cubic threefold lies on some lines, so the branch locus for the corresponding projection acquires some singularities.

As before we start with the easy direction; that is, given a cubic threefold $X \in \mathbb{P}^4$ and a generic point $p \in X$, we want to associate to (X, p) a quartic surface. We begin with a

LEMMA 3.29. *Let X be a generic cubic threefold and $p \in X$ a generic point on it. Then p lies on* 6 *lines contained in X. Moreover these lines, regarded as points of*

$$\{lines\ \ell \subset \mathbb{P}^4 \mid p \in \ell\} \cong \mathbb{P}^3,$$

lie on a conic.

PROOF. Let us choose affine coordinates in such a way that p is the origin. Then an equation for X in this patch is given by

$$f_1(\mathbf{x}) + f_2(\mathbf{x}) + f_3(\mathbf{x}) = 0,$$

where each polynomial f_i is homogeneous of degree i and

$$\mathbf{x} = (x_1, \ldots, x_4)$$

are the coordinates. Since $p \in X$,

$$f_1(0) = f_2(0) = f_3(0) = 0.$$

Let us choose some non zero vector $\mathbf{y} = (y_1, \ldots, y_4)$. Then the line trough p and $\mathbf{y}$ is contained in X if and only if

$$f_1(t\mathbf{y}) + f_2(t\mathbf{y}) + f_3(t\mathbf{y}) = 0$$

for each t. Since each f_i is homogeneous of degree i, this amounts to

$$f_1(\mathbf{y}) = f_2(\mathbf{y}) = f_3(\mathbf{y}) = 0.$$

In other words the locus of lines contained in X and passing through p is defined in $\mathbb{P}^3$ by the equations

$$f_1 = f_2 = f_3 = 0.$$

If X and p are generic, the zero loci of the f_i are transverse, so their intersection has $2 \times 3 = 6$ points.

Moreover this intersection in contained in the locus

$$f_1 = f_2 = 0,$$

which is a plane conic.

We now choose a hyperplane $A \subset \mathbb{P}^4$ not containing p and consider the projection from p

$$\pi_p \colon X \dashrightarrow A.$$

In this case π_p is no longer finite, and there are 6 points on A whose preimage is a line. Of course π_p has degree 2, and considering a generic hyperplane section we see that the ramification locus is a quartic surface $S \subset A$.

LEMMA 3.30. *Let $p_1, \ldots, p_6 \in S$ be the projection of the 6 lines through p. Then the p_i are nodes of S; moreover they all lie on a conic $Q \subset S$.*

PROOF. The fact that $p_1, \ldots, p_6$ all lie on a conic follows directly from the fact that the corresponding six lines lie on a conic, as proved in the previous lemma.

To prove the other assertion we follow the notation in the proof of Lemma 3.29. We choose as A the hyperplane $x_4 = 1$.

Let us find the equations of $S \subset A$. Given a point

$$q = (x_1, x_2, x_3, 1) \in A,$$

a point on $\overline{pq}$ has the form (tx_1, tx_2, tx_3, t). This lies on X if and only if

$$t f_1(x_1, x_2, x_3, 1) + t^2 f_2(x_1, x_2, x_3, 1) + t^3 f_1(x_1, x_2, x_3, 1) = 0.$$

Factoring out the t, which corresponds to the point $p \in X$, we are left with a second degree equation, which has a repeated root if and only if

$$g(x_1, x_2, x_3) = f_2^2 - 4 f_1 f_3 = 0.$$

This is the equation of S in A.

It is immediate to see that the partial derivatives of g vanish when $f_1 = f_2 = f_3 = 0$, so the six points $p_1, \ldots, p_6$ are indeed singularities of S. To see that they are actually nodes, one has just to compute the Hessian of g are see that its determinant is nonzero for X and p generic.

The map π_p becomes regular on the blowup $\widetilde{X}$ of X in p, sending the exceptional divisor E to the plane

$$P = T_p X \cap A.$$

REMARK. The plane P is in special position with respect to S: actually the quartic $S \cdot P$ is a double conic in P. Indeed a point $q \in P$ lies in S if and only if it

has only 1 preimage in $\widetilde{X}$. The line $\overline{pq} \in E$ is one preimage and the other are the points in the intersection $X \cap \overline{pq}$ distinct from p. So $q \in S$ if and only if

$$X \cdot \overline{pq} = 3p.$$

This means that $\overline{pq}$ is a line in the tangent cone of X at p, which is a conic cone if p is a node. So $S \cap P$, being the projection of this tangent cone, is a conic in P.

Actually $S \cap P = Q$, since of course it contains the 6 points $p_1, \ldots, p_6$.

So far we have attached to (X, p) a quartic with 6 nodes on a conic, which meets the plane spanned by the nodes in a double conic. We now want to go the other way round.

We start with an easy

PROPOSITION 3.31. *Let $S \subset \mathbb{P}^3$ be a quartic having 5 nodes $p_1, \ldots, p_5$ lying on a conic Q (but no 3 on a line). Let P be the plane spanned by Q. Then S has another node p_6 on Q and*

$$S \cdot P = 2Q. \tag{3.9}$$

PROOF. First we show that (3.9) holds. Indeed $\widehat{Q} = S \cdot P$ is a quartic in P singular in $p_1, \ldots, p_5$. So in P the curves Q and $\widehat{Q}$ meet in 5 points with multiplicity at least 2, so by Bezout $\widehat{Q}$ must contain Q. In other words

$$\widehat{Q} = Q + Q'$$

for some conic Q'. The singular points of $\widehat{Q}$ on Q are the intersections $Q \cap Q'$, so by Bezout again we must have $Q = Q'$, so (3.9) is proved.

Let us show the existence of one more singular point p_6. Fix coordinates $x_0, \ldots, x_3$ on $\mathbb{P}^3$; we can assume

$$P = \{x_3 = 0\}.$$

Let F be the polynomial defining S, then

$$F(x_0, \ldots, x_3) = B(x_0, x_1, x_2)^2 + x_3 C(x_0, \ldots, x_3),$$

where $\deg B = 2$ and $\deg C = 3$.

We show that S is singular in $P \cap V(B) \cap V(C)$. This is the intersection of Q and $V(C)$ on P, which gives 6 singular points on S.

To prove this we compute the partial derivatives:

$$\begin{cases} \dfrac{\partial F}{\partial x_3} & = C + x_3 \dfrac{\partial C}{\partial x_3} \\[2ex] \dfrac{\partial F}{\partial x_i} & = 2B \cdot \dfrac{\partial B}{\partial x_i} + x_3 \dfrac{\partial C}{\partial x_i} \quad \text{for } i = 0, 1, 2 \end{cases}$$

and all these expressions vanish when $x_3 = B = C = 0$.

We now let Y be the double covering of $\mathbb{P}^3$ ramified over S. This has 6 nodes $q_1, \ldots, q_6$ over the nodes $p_1, \ldots, p_6$ of S (we assume for a moment that these are the only singularities). Let

$$\pi \colon Y \to \mathbb{P}^3$$

be the covering map. Denote $\widetilde{S} \cong S$ the branch locus, and

$$\widetilde{Q} = \pi^{-1}(Q) \cong Q.$$

On Y we have the line bundle

$$\mathcal{O}_Y(1) = \pi^* \mathcal{O}_{\mathbb{P}(3)}(1)$$

and by construction $\widetilde{S} \in |H^0(Y, \mathcal{O}_Y(2))|$.

We would like to find, as in the one-dimensional case, a line bundle of self-intersection 3 on Y, giving the map to the cubic. This we cannot expect, since the cubic will be smooth in general, so we first have to blow up the 6 points $q_1, \ldots, q_6$. Let

$$\epsilon \colon \widetilde{X} \to Y$$

be the blowup of Y in $q_1, \ldots, q_6$; then $\widetilde{X}$ is smooth and contains six quadrics $Q_1, \ldots, Q_6$ given by the exceptional divisors.

Consider the covering over P: this is ramified over a double conic, so $\pi^{-1}(P)$ is the union of two copies of $\mathbb{P}^2$, P_1 and P_2, meeting along $\widetilde{Q}$. Let $\widetilde{P_i}$ be the strict transform of P_i in the blowup. A moment of thought shows that

$$\epsilon\big|_{\widetilde{P_i}} \colon \widetilde{P_i} \to P_i$$

is the blowup of P_i at $q_1, \ldots, q_6$. Let us fix $i = 1$ and denote by

$$E_1, \ldots, E_6 \subset \widetilde{P_1}$$

the exceptional divisors.

LEMMA 3.32. *On $\widetilde{X}$ we have the following intersection numbers:*

$$\widetilde{P_1}^2 \cdot Q_i = 0 \qquad\qquad i = 1, \ldots, 6 \qquad\qquad (3.10)$$
$$\widetilde{P_1} \cdot Q_i^2 = -1 \qquad\qquad i = 1, \ldots, 6 \qquad\qquad (3.11)$$
$$Q_i^3 = 2 \qquad\qquad i = 1, \ldots, 6 \qquad\qquad (3.12)$$

PROOF. First we note that $\widetilde{P_1}$ and Q_i meet transversely along E_i, so

$$\widetilde{P_1}^2 \cdot Q_i = \widetilde{P_1} \cdot E_i$$
$$\widetilde{P_1} \cdot Q_i^2 = Q_i \cdot E_i.$$

By adjunction

$$\mathcal{O}_{\widetilde{X}}(\widetilde{P_1})\big|_{\widetilde{P_1}} = \mathcal{N}_{\widetilde{P_1}/\widetilde{X}}$$
$$\mathcal{O}_{\widetilde{X}}(Q_i)\big|_{Q_i} = \mathcal{N}_{Q_i/\widetilde{X}}$$

To compute these normal bundles on E_i we note that we have the following exact sequence

$$0 \longrightarrow T_{E_i} \longrightarrow T_{\widetilde{P_1}}\big|_{E_i} \oplus T_{Q_i}\big|_{E_i} \longrightarrow T_{\widetilde{X}}\big|_{E_i} \longrightarrow 0 \ ,$$

so

$$\mathcal{N}_{\widetilde{P_1}/\widetilde{X}}\big|_{E_i} = \mathcal{N}_{E_i/Q_i}$$
$$\mathcal{N}_{Q_i/\widetilde{X}}\big|_{E_i} = \mathcal{N}_{E_i/\widetilde{P_1}}.$$

By adjunction again

$$\widetilde{P_1}^2 \cdot Q_i = \widetilde{P_1} \cdot E_i = \deg \mathcal{N}_{\widetilde{P_1}/\widetilde{X}}\big|_{E_i} = \deg \mathcal{N}_{E_i/Q_i} = (E_i)^2_{Q_i} = 0$$
$$\widetilde{P_1} \cdot Q_i^2 = Q_i \cdot E_i = \deg \mathcal{N}_{Q_i/\widetilde{X}}\big|_{E_i} = \deg \mathcal{N}_{E_i/\widetilde{P_1}} = (E_i)^2_{\widetilde{P_1}} = -1.$$

We can compute (3.12) in a similar way. To do this we have to understand the normal bundle $\mathcal{N}_{Q_i/\widetilde{X}}$. Since

$$Q_i \cong \mathbb{P}^1 \times \mathbb{P}^1$$

we must have

$$\mathcal{N}_{Q_i/\widetilde{X}} \cong \mathcal{O}_{Q_i}(a, b)$$

for some integers a and b. Moreover by symmetry we see that $a = b$. Let us explain this in more detail. The normal bundle $\mathcal{N}_{Q_i/\widetilde{X}}$ is determined by an analytic neighborhood of q_i inside Y, so we may perform this computation on any threefold having a singularity whose germ is isomorphic to the germ of singularity of Y near q_i. For instance we can take any ball of Y centered around q_i; but then in this case we have an automorphism of the total space of the blowup exchanging the rulings, so a and b must be equal.

It follows that $\mathcal{N}_{Q_i/\widetilde{X}}$ is determined by its restriction to a line. We have seen that

$$\mathcal{N}_{Q_i/\widetilde{X}}\big|_{E_i} \cong \mathcal{O}_{E_i}(-1)$$

so $a = b = -1$. This gives

$$Q_i^3 = (\mathcal{N}_{Q_i/\widetilde{X}})_{Q_i}^2 = \mathcal{O}_{Q_i}(-1,-1)^2 = 2.$$

Let us compute some more:

LEMMA 3.33. *On $\widetilde{X}$ we have the following intersection numbers:*

$$\widetilde{P_1}^2 \cdot \widetilde{P_2} = \widetilde{P_1} \cdot \widetilde{P_2}^2 = -2 \tag{3.13}$$

$$\widetilde{P_1} \cdot \widetilde{P_2} \cdot Q_i = 1 \qquad\qquad i = 1,\ldots,6 \tag{3.14}$$

$$\widetilde{P_i}^3 = 1 \qquad\qquad i = 1,2 \tag{3.15}$$

PROOF. In order to compute the first two products, we start with the observation that $\widetilde{P_1}$ and $\widetilde{P_2}$ meet transversely along

$$\widetilde{\widetilde{Q}} = \text{the strict transform of } \widetilde{Q} \text{ inside } \widetilde{X}.$$

The only thing which is not apparent in this claim is that the intersection is transverse. This is clear outside $Q_1,\ldots,Q_6$, since the local picture is the same of the intersection of P_1 and P_2 along $\widetilde{Q}$ inside Y, and the latter are transverse by construction of the double covering.

It follows that

$$\widetilde{P_1} \cdot \widetilde{P_2}^2 = (\widetilde{\widetilde{Q}} \cdot \widetilde{\widetilde{Q}})_{\widetilde{P_1}}$$
$$\widetilde{P_1} \cdot \widetilde{P_2} \cdot Q_i = (\widetilde{\widetilde{Q}} \cdot E_i)_{\widetilde{P_1}}.$$

These can be computed using the fact that

$$\epsilon^*(\widetilde{Q}) = \widetilde{\widetilde{Q}} + E_1 + \cdots + E_6,$$

where

$$\epsilon \colon \widetilde{P_1} \to P_1$$

is the restriction to $\widetilde{P_1}$ of the blowup map. So, inside $\widetilde{P_1}$,

$$\widetilde{\widetilde{Q}} \cdot E_i = \epsilon^*(\widetilde{Q}) \cdot E_i - (E_1 + \cdots + E_6) \cdot E_i = 1.$$

since the exceptional divisors are pairwise disjoint and contract to a point. Then

$$\widetilde{\widetilde{Q}}^{\,2} = (\epsilon^*(\widetilde{Q}) - E_1 - \cdots - E_6)^2 = (\widetilde{Q} \cdot \widetilde{Q})_{P_1} + E_1^2 + \cdots + E_6^2 = 4 - 6 = -2.$$

To compute the last term we use the fact that

$$\widetilde{P_1} + \widetilde{P_2} + Q_1 + \cdots + Q_6 \in |\mathcal{O}_{\widetilde{X}}(1)|$$

by construction, so

$$(\widetilde{P_1} + \widetilde{P_2} + Q_1 + \cdots + Q_6)^3 = 2.$$

We can use the fact that $Q_i \cdot Q_j = 0$ for $i \neq j$ and the preceding computations to expand the cube and find

$$(\widetilde{P_1} + \widetilde{P_2})^3 + 12 = 2,$$

that is

$$(\widetilde{P_1} + \widetilde{P_2})^3 = -10.$$

We expand the cube again and use the fact that $P_1^3 = P_2^3$ by symmetry to find

$$\widetilde{P_1}^3 = \widetilde{P_2}^3 = 1.$$

We are now ready to prove the following

PROPOSITION 3.34. *Let* $L = \mathcal{O}_{\widetilde{X}}(1) \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{P_1})$. *Then the line bundle L has no base points and defines a morphism*

$$\varphi_L \colon \widetilde{X} \to \mathbb{P}^4$$

such that:

   i) φ_L *is birational;*
  ii) $X = \varphi_L(\widetilde{X}) \subset \mathbb{P}^4$ *is a cubic;*
 iii) φ_L *contracts the whole $\widetilde{P_1}$ to a point $p \in X$;*
  iv) *each quadric Q_i is contracted along a ruling to a line ℓ_i through p;*
   v) *no other curves are contracted.*

We see that this construction is the exact inverse of the projection construction at the beginning of the section. This is in complete analogy with the 2-dimensional case in the preceding section. Actually, using that case, we will be able to show that the surface of bitangents to the quartic surface S is birational to the the Fano surface of lines in X.

Before proceeding with the proof of Proposition 3.34 it will be useful to write out some other intersection numbers, which can be easily derived from our previous lemmas:

$$\widetilde{P_i}^2 \cdot \mathcal{O}_{\widetilde{X}}(1) = -1 \qquad\qquad i = 1, 2 \qquad\qquad (3.16)$$

$$\widetilde{P_i} \cdot \mathcal{O}_{\widetilde{X}}(1)^2 = 1 \qquad\qquad i = 1, 2. \qquad\qquad (3.17)$$

PROOF. As in the lower dimensional case, we break up the proof in several steps.

Step 1: A linear series $\mathcal{L} \subset |L|$. Let

$$\rho = \pi \circ \epsilon \colon \widetilde{X} \to \mathbb{P}^3$$

and let

$$V = \left\{ D + \widetilde{P_1} \mid D \in |\mathcal{O}_{\widetilde{X}}(1)| \right\} \subset H^0(\widetilde{X}, L)$$

be the subspace of section which are the sum of $\widetilde{P_1}$ and some section pulled back via ρ from $\mathbb{P}^3$.

We can find a section s of L which does not lie in V. Indeed let $G \subset \mathbb{P}^3$ be a quadric such that

$$G \cdot P = Q.$$

Then $\rho^*(G)$ cuts the divisor

$$\widetilde{\widetilde{Q}} + F_1 + \ldots F_6$$

on $\widetilde{P_2}$, where $F_1, \ldots, F_6$ are the exceptional divisors of

$$\epsilon\big|_{\widetilde{P_2}} \colon \widetilde{P_2} \to P_2.$$

On the other hand

$$\epsilon^*(\widetilde{S}) \in |\mathcal{O}_{\widetilde{X}}(2)|$$

cuts on $\widetilde{P_2}$ the divisor $\widetilde{\widetilde{Q}}$, so a suitable linear combination of the two is a section of $s \in \mathcal{O}_{\widetilde{X}}(2)$ that cuts on $\widetilde{P_2}$ the divisor $F_1 + \ldots F_6$.
Subtracting $Q_1 + \cdots + Q_6$ yields a section of

$$\mathcal{O}_{\widetilde{X}}(2) \otimes \mathcal{O}_{\widetilde{X}}(Q_1 + \cdots + Q_6)^\vee \sim \mathcal{O}_{\widetilde{X}}(1) \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{P_1} + \widetilde{P_2})$$

which vanishes on $\widetilde{P_2}$, hence a new section of

$$\mathcal{O}_{\widetilde{X}}(1) \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{P_1}) = L.$$

We let $\mathcal{L}$ be the linear series generated by $|V|$ and this new section. By construction

$$\dim \mathcal{L} = 4.$$

We also see that $\mathcal{L}$ has no base points.

Step 2: $\varphi_{\mathcal{L}}$ is birational and $\varphi_{\mathcal{L}}(\widetilde{X})$ is a cubic. Let $X = \varphi_{\mathcal{L}}(\widetilde{X}) \subset \mathbb{P}^4$. Then

$$\deg X \cdot \deg \varphi_{\mathcal{L}} = L^3 = (\mathcal{O}_{\widetilde{X}}(1) \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{P_1}))^3 = 3.$$

So either $\deg X = 3$ and $\deg \varphi_{\mathcal{L}} = 1$, in which case we are done, or else $\deg \varphi_{\mathcal{L}} = 3$. The latter is excluded, since $|V| \subset \mathcal{L}$ and the map

$$\varphi_{|V|} = \rho \colon \widetilde{X} \to \mathbb{P}^3$$

has degree 2.

Step 3: $\mathcal{L} = |L|$. Since $\varphi_{\mathcal{L}}$ is birational, it is enough to show that the only sections of $\mathcal{O}_X(1)$ are the restrictions of section of $\mathcal{O}_{\mathbb{P}^4}(1)$. Bu this is clear from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

and the fact that

$$H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-2)) = 0$$

by Kodaira vanishing.

Step 4: φ_L contracts $\widetilde{P_1}$. It is enough to prove that if $C \subset \widetilde{P_1}$ is any curve, then C is contracted. In other words, if C is any such curve,

$$L \cdot C = 0. \tag{3.18}$$

By linearity it is enough to check equation (3.18) when C is a generator of $\mathrm{Pic}(\widetilde{P_1})$. Since $\widetilde{P_1}$ is just the blowup of $P_1 \cong \mathbb{P}^2$ in 6 points, C can be taken either a line or an exceptional divisor.

In the first case let $\ell \subset P_1$ be a line and $C = \epsilon^*(\ell)$. Then

$$L \cdot C = \mathcal{O}_{\widetilde{X}}(1) \cdot \epsilon^*(\ell) + \widetilde{P_1} \cdot \epsilon^*(\ell) = \mathcal{O}_{P_1}(1) \cdot \ell + \widetilde{P_1} \cdot \epsilon^*(\ell) = 1 + \widetilde{P_1} \cdot \epsilon^*(\ell).$$

We can compute the missing term by observing that

$$1 = \mathcal{O}_{\widetilde{X}}(1) \cdot \epsilon^*(\ell) = (\widetilde{P_1} + \widetilde{P_2} + E_1 + \cdots + E_6) \cdot \epsilon^*(\ell) = 2 + \widetilde{P_1} \cdot \epsilon^*(\ell).$$

We have used that

$$E_i \cdot \epsilon^*(\ell) = 0,$$

since we can assume that ℓ doesn't pass through $q_1, \ldots, q_6$ and that

$$\widetilde{P_2} \cdot \epsilon^*(\ell) = 2,$$

since we can assume that $\epsilon^*(\ell)$ meets $\widetilde{P_2}$ transversely in the 2 points where it meets $\widetilde{Q}$. It follows that

$$\widetilde{P_1} \cdot \epsilon^*(\ell) = -1$$

and so
$$L \cdot C = 0.$$

In the second case let $C = E_i$. Then
$$L \cdot C = \mathcal{O}_{\widetilde{X}}(1) \cdot E_i + \widetilde{P_1} \cdot E_i = 0,$$

thanks to the fact that E_i is contracted by ϵ and the proof of Lemma 3.32 respectively.

Step 5: φ_L contracts Q_i along a ruling, sending it to a line. Consider a line ℓ in the ruling of Q_i containing E_i. We want to show that ℓ is contracted, that is
$$L \cdot \ell = 0.$$

It is enough to prove this when $\ell = E_i$, and in this case it follows by the preceding step.

If ℓ' is a line in the other ruling, then
$$L \cdot \ell' = \mathcal{O}_{\widetilde{X}}(1) \cdot \ell' + \widetilde{P_1} \cdot \ell' = 1 + 0 = 1,$$

since ℓ' and $\widetilde{P_1}$ are disjoint, so the image of Q_i is a line.

Step 6: φ_L contracts no other curves. Let C be an irreducible curve contracted by φ_L. If C is contracted already by ϵ, it must lie on a quadric Q_i, and the preceding step shows that C must be a line on it.

Otherwise we must have
$$\mathcal{O}_{\widetilde{X}}(1) \cdot C = \mathcal{O}_Y(1) \cdot \epsilon_*(C) > 0,$$

because $\mathcal{O}_Y(1)$ is ample on Y. Since $L \cdot C = 0$ we see that C lies on $\widetilde{P_1}$.

Step 7: X is smooth. Let $p = \varphi_L(\widetilde{P_1})$ and $\ell_i = \varphi_L(Q_i)$. X is surely smooth outside the union of the ℓ_i.

That X is smooth on ℓ_i, away from p, follows by Nakano criterion ([Nak70]). It remains to be shown that X is smooth at p. By the above results it is easy to verify the X contains exactly six lines through p, namely $\ell_1, \ldots, \ell_6$. By the proof of Lemma 3.29 we see that this implies that X is smooth at p.

So we see that a quartic surface $S \subset \mathbb{P}^3$ having 6 nodes on a conic and no other singularities is the ramification locus for the projection of a smooth cubic from a point on it. Moreover:

PROPOSITION 3.35. *In this setting the surfaces* $\mathrm{Bit}(S)$ *and* $\mathrm{Fano}(X)$ *are birational.*

The proof is similar to the lower dimensional case of the previous section and will not be included.

3.6 – An example of rational $\mathrm{Bit}(S)$

In this section we give an example of a quartic surface $S \subset \mathbb{P}^3$ such that the surface of bitangents $\mathrm{Bit}(S)$ is rational. The reason why we are looking to such examples is the following.

We have seen that double EPW sextic can degenerate to varieties like $S^{[2]}$, where S is a quartic $K3$, and in this case the fixed locus of the involution degenerates to $\mathrm{Bit}(S)$. We want to make a similar procedure, using a singular quartic S, to produce rational surfaces inside an EPW sextic. Unfortunately the number of conditions is wrong. Namely, to get a rational surface, S must acquire 11 singular points, and this will produce a codimension 11 locus of Lagrangian subspaces $B \in \mathbb{LG}(\bigwedge^3 V)$ such that the surface $Y_B[2]$ is rational.

But given a generic $A \in \mathbb{LG}(\bigwedge^3 V)$, the locus of Lagrangian subspaces B such that

$$Y_B[2] \subset Y_A$$

has dimension only 10, hence the two loci may not meet. Thus this construction may fail to produce rational surfaces inside a generic EPW sextic.

Luckily there is another construction of quartics S such that $\mathrm{Bit}(S)$ is an Enriques surface, and this time the dimensions match. We present this construction in the next section. Nevertheless the example of rational surfaces may be interesting independently.

We recall from the previous section, Proposition 3.31, that if a quartic surface has 5 singular points on a plane, then there is a sixth one on the same plane.

REMARK. Let

$$U_k \subset \mathbb{P}H^0\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)\big)$$

be the (Zariski closure of the) locus of quartic surfaces having at least k singular points. Then the expected dimension of U_k is $19 - k$.

The locus of surfaces with 6 nodes *lying on a plane* has dimension 13, by direct computation, or by the results of the previous section. Hence the locus U_6 has at least 2 components: one where the six points are generic and one where they are on the same plane. By adding more and more singular points we see that we have the same reducibility for $k \geq 6$.

In our case we are interested to surfaces having 6 nodes on a plane and 5 more singular point in general position. We shall prove the following

PROPOSITION 3.36. *Let $S \subset \mathbb{P}^3$ be a quartic surface having* 11 *nodes,* 6 *of which lie on the same plane, and generic with respect to these assumptions. Then the surface of bitangents* $\mathrm{Bit}(S)$ *is rational.*

By the results of the previous section, if S has only 6 nodes on a plane, then there is a smooth cubic threefold $X \subset \mathbb{P}^4$ such that $\mathrm{Bit}(S)$ is birational to $F = \mathrm{Fano}(X)$.

It is easy to extend the argument to cover the case where S acquires more singular points. By the explicit construction of X we see that if S acquires k more nodes outside the plane, X has k nodes too.

More precisely let P be the plane containing 6 nodes of S, and let Y be the double covering of $\mathbb{P}^3$ ramified along S. We can view S as a subvariety of Y; then Y and X are isomorphic outside the intersection $S \cap P$, so every node of S outside P appears as a node of X.

Proposition 3.36 then follows from the following result about the Fano surface of a cubic threefold.

PROPOSITION 3.37. *Let X be a cubic threefold and $F = \mathrm{Fano}(X)$. Assume that X has k nodes and is generic with respect to this condition. Then*

  i) if $k = 3$, F is birational to an abelian surface;
 ii) if $k = 4$, F is rationally ruled with base an elliptic curve;
iii) if $k = 5$, F is rational.

The trick is to reduce to the case of curves, by the following

LEMMA 3.38. *Assume that X has a simple double point p and let $C_p \subset F$ be the set parametrizing lines on X passing through p. Then C_p is a curve and F is birational to the symmetric product $C_p^{(2)}$.*

PROOF. Choose affine coordinates $x_1, \ldots, x_4$ near the point p such that $p = (0, \ldots, 0)$. Since p is singular, an equation for X has the form

$$f(\mathbf{x}) = f_2(\mathbf{x}) + f_3(\mathbf{x}) = 0,$$

where f_i is homogeneous of degree i. Choose $v \in \mathbb{C}^4$; then the line

$$\{tv \mid t \in \mathbb{C}\}$$

is contained in X if and only if

$$f_2(v) = f_3(v) = 0,$$

hence C_p is the intersection of a quadric and a cubic in $\mathbb{P}^3$. Moreover f_2 and f_3 do not have common factors, since X is irreducible, so C_p is actually a curve.

The birational equivalence between F and C_p is realized as follows. Assume that we are given two distinct lines ℓ_1 and ℓ_2 on X through p, and let Λ be the plane spanned by ℓ_1 and ℓ_2. Then $X \cap \Lambda$ is a plane cubic in Λ, having ℓ_1 and ℓ_2 as two components, hence it must have a third component which is again a line on X, that is, a member of F. So we have obtained a rational map from $C_p^{(2)}$ to F.

Let us describe its inverse. We start with a line $\ell \subset X$. If ℓ is generic, then $p \notin \ell$, so ℓ and p span a plane Λ. We consider again the plane cubic $X \cap \Lambda$: this has a component ℓ, so it is the union of ℓ and a conic. The conic contains p, hence it is singular in p: it follows that it is the union of two lines (possibly non distinct) through p. This gives the rational inverse that we are looking for.

From the first part of the proof of the lemma it follows that C_p is the complete intersection of a cubic and a quartic in $\mathbb{P}^3$. By adjunction we compute its arithmetic genus $p_a(C_p) = 4$. Assume that X is singular in k points, call them $p_1, \ldots, p_k$, and is general enough with respect to this condition. The curve $C = C_{p_1}$ is singular exactly on the lines $\overline{p_1 p_j}$ for $j = 2, \ldots, k$. In our hypothesis the p_j are simple double points of X, and from this it is easy to show that the $k-1$ singularities of C are nodes. Hence C has geometric genus

$$p_g(C) = 4 - (k-1) = 5 - k.$$

From this we easily get what we want.

PROOF.[Proof of Proposition 3.37] We already know that F is birational to the twofold symmetric product of a curve C of geometric genus $5 - k$. By passing to the normalization we can assume that C is smooth. Hence the proposition is just a statement about curves.

If $k = 3$, then C has genus 2: in this case the Albanese map realizes a birational equivalence between $C^{(2)}$ and the Jacobian $\mathrm{Jac}(C)$. If $k = 4$, then C is an elliptic curve: in this case the Albanese map fibers $C^{(2)}$ over C, with rational fibers. Finally when $k = 5$, C is just $\mathbb{P}^1$, so $C^{(2)}$ is $\mathbb{P}^2$.

3.7 – An example of Enriques $\mathrm{Bit}(S)$

In the present section we shall prove the following result.

PROPOSITION 3.39. *There exists a* 9-*dimensional family of quartic surfaces with* 10 *nodes S such that the surface of bitangents $\mathrm{Bit}(S)$ is birational to an Enriques surface.*

The first step in the proof will be to give an explicit construction of quartic surfaces with 10 nodes. This construction is classical, and can be found for instance in [Cos83]; it was suggested to us by I. Dolgachev.

Let V be a vector space of dimension 4 and identify $\mathbb{P}(V) \cong \mathbb{P}^3$. Choose a generic 3-dimensional linear system of quadrics

$$\Lambda \subset |\mathcal{O}_{\mathbb{P}^3}(2)|, \quad \Lambda \cong \mathbb{P}^3.$$

Inside $|\mathcal{O}_{\mathbb{P}^3}(2)|$ we can consider the degeneracy loci

$$D_k = \{ \text{ quadrics of rank } \leq k \}.$$

It is well known that D_3 has codimension 1, D_2 has codimension 3 and D_3 is singular precisely along D_2.

We define

$$S = \{\text{singular quadrics of } \Lambda\} = \Lambda \cap D_3 \text{ and}$$
$$T = \{\text{quadrics of } \Lambda \text{ of rank} \leq 2\} = \Lambda \cap D_2.$$

If Λ is generic (transverse to all degeneracy loci), we see that S will be a surface singular along T, which is a a finite set of points. Moreover we can assume that S has only nodes at points of T.

Since S is cut out by the single equation $\det Q = 0$ we immediately see that S is a quartic. Moreover one can compute

$$\deg T = \deg D_2 = 10,$$

hence S is a surface with 10 nodes, as claimed. The degree of a symmetric determinantal variety can be computed using the results of Harris and Tu in [HT84]. In particular we use the following

THEOREM (Proposition 12(b) in [HT84]). *Let W be the space of all $n \times n$ symmetric matrices and S_r those of corank at least r. Then in $\mathbb{P}(W)$*

$$\deg \mathbb{P}(S_r) = \prod_{\alpha=0}^{r-1} \frac{\binom{n+\alpha}{r-\alpha}}{\binom{2\alpha+1}{\alpha}}.$$

In our application we have $n = 4$ and $r = 2$, hence the formula above simplifies to

$$\deg D_2 = 6 \cdot \frac{5}{3} = 10.$$

REMARK. The above construction yields a family of quartic surface with 10 nodes of dimension

$$\dim \mathrm{Gr}(3, |\mathcal{O}_{\mathbb{P}^3}(2)|) - \dim \mathbb{P}\mathrm{GL}(3) = 24 - 15 = 9.$$

As expected this family has codimension 10 in the moduli space of quartic surfaces.

Next we show how to associate an Enriques surface to Λ. For each quadratic form we can consider its associated symmetric bilinear form; this gives an embedding

$$\Lambda \hookrightarrow |\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1)| \cong \mathbb{P}(V^\vee) \times \mathbb{P}(V^\vee).$$

Each member of Λ here is seen as a divisor of type $(1, 1)$ on $\mathbb{P}(V) \times \mathbb{P}(V)$. We shall use the following notation: *for each quadric Q given by a quadratic form q, we consider the associated bilinear form $\widetilde{q}$, which gives a divisor $\widetilde{Q}$ on $\mathbb{P}(V) \times \mathbb{P}(V)$.*

Let $Q_1, \ldots Q_4$ be four quadrics spanning Λ. Then

$$S' = \bigcap_{Q \in \Lambda} \widetilde{Q} = \widetilde{Q_1} \cap \cdots \cap \widetilde{Q_4}$$

is a $K3$ surface. Indeed by adjunction we see that $K_{S'}$ is trivial, and by Lefschetz theorem on hyperplane sections we see that S' is simply connected.

By construction

$$S' \subset \mathbb{P}(V) \times \mathbb{P}(V),$$

hence we have an involution $\iota \colon S' \to S'$ interchanging the factors. We claim that ι has no fixed points. This is equivalent to saying that S' doesn't meet the diagonal. Each intersection between $\widetilde{Q_i}$ and the diagonal is a point of Q_i. For Λ generic we have

$$Q_1 \cap \cdots \cap Q_4 = \emptyset,$$

hence the claim follows.

We can then define

$$F = S'/\langle \iota \rangle;$$

by construction F admits an unramified double covering which is a $K3$, so F is an Enriques surface.

The last element that we need in order to prove Proposition 3.39 is the following explicit description of bitangents to S in terms of the web of quadrics Λ.

PROPOSITION 3.40. *Let ℓ be a pencil of quadric on $\mathbb{P}^3$, and let D_i be the degeneracy loci as above. Assume that ℓ doesn't meet D_2 (that is, every quadric in ℓ has rank at least 3) and that ℓ contains smooth quadrics. Let C be the base locus of ℓ. Then the singularities of C and the position of ℓ relative to U_3 are related as follows:*

  i) *If C is smooth, ℓ cuts D_3 in 4 distinct points;*
 ii) *if C is irreducible with a node, ℓ is a simple tangent to D_3;*
iii) *if C is irreducible with a cusp, ℓ meets D_3 in a flex and a simple point;*
 iv) *if C is the union of a line and a twisted cubic meeting in 2 distinct points, ℓ is a bitangent to D_3;*
  v) *if C is the union of a line and a twisted cubic tangent in 1 point, ℓ is a quadritangent to D_3.*

No other cases for C can arise.

COROLLARY 3.41. *The bitangents of S are exactly the pencils of quadrics containing a line.*

PROOF. We can choose two smooth quadrics $Q_1, Q_2 \in \ell$; then the base locus of ℓ is just

$$\bigcap_{Q \in \ell} Q = Q_1 \cap Q_2 = C.$$

i) If Q_1 and Q_2 are transverse, then C is smooth and K_C has degree 0 by adjunction, hence C is an elliptic curve of degree 4.

For any quadric $Q \in \ell$ the quadratic polynomial $q\big|_{T_pC}$ vanishes of order 2 at p; it follows that if Q contains any other point of T_pC it must contain the whole line. Since containing a point is a single linear condition, there is a unique quadric $Q_p \in \ell$ containing the tangent space T_pC. This gives a natural map

$$\varphi \colon C \longrightarrow \ell.$$
$$p \longrightarrow Q_p$$

We claim that φ is a $8 : 1$ covering ramified exactly along the set of singular quadrics. Indeed let $Q \in \ell$ be a quadric, and choose one of its rulings. Projection along the ruling yields a map

$$\pi \colon C \to \mathbb{P}^1;$$

let's compute its degree. If r is any line of the ruling, then

$$r \cap C = r \cap Q',$$

where Q' is any other quadric, hence it can be 2 points if r and C are transverse or 1 point if they are tangent. We should be more precise here: if Q is singular, then C does not pass through its vertex, otherwise it would be singular there. So the above argument applies for every quadric Q.

It follows that π is a double covering branched on the set of tangency points. By Riemann-Hurwitz we find that there are 4 tangents in the ruling. Since a smooth quadric has two rulings and a quadric of rank 3 just one, we find that

$$\sharp \varphi^{-1}(Q) = \begin{cases} 8 & \text{if } Q \text{ is smooth} \\ 4 & \text{if } Q \text{ is singular.} \end{cases}$$

We can apply Riemann-Hurwitz again, this time for φ. The total degree of ramification over a singular quadric is 4, so we find

$$0 = \chi(C) = 8\chi(\ell) + \sharp (\{\text{singular quadrics}\}) \cdot 4,$$

from which we deduce that ℓ contains 4 singular quadrics.

Since D_3 has degree 4, all points of intersection must be simple points. This can also be seen as follows. Let $Q \in \ell$ be a singular quadric of vertex p. By (2.24) we see that

$$T_Q D_3 = \{\text{quadrics through } p\}.$$

Hence Q is a tangency point between ℓ and D_3 if and only if its vertex lies on the base locus C. We have excluded this case because C is smooth, so ℓ is transverse to D_3.

Of course this argument is enough, and it is shorter than the one we gave above, but to analyze the next cases we will need both.

ii) Next, assume that C is singular but still irreducible. Then adjunction gives that C has arithmetic genus 1, hence it has exactly one singular point p_0 of multiplicity 2, and its normalization $\widetilde{C}$ is isomorphic to $\mathbb{P}^1$. In the first case we assume that C has a node.

We define

$$\varphi \colon C \dashrightarrow \ell$$

by sending $p \in C$ to the unique quadric Q_p containing $T_p C$ if p is a smooth point. This map becomes a regular map on $\widetilde{C}$, which we still denote by φ: indeed any point $p \in \widetilde{C}$ over p_0 corresponds to a well defined tangent line in p_0, and we can define Q_p by the same recipe.

We perform the same computation as above. If Q is a quadric *not containing any tangent to C through p_0*, we apply Riemann-Hurwitz to the map

$$\widetilde{C} \to \mathbb{P}^1,$$

and find that this ramifies in 2 points. Of these, corresponds to the line of the ruling through p_0 and one to a tangent. Moreover if Q is smooth, it certainly does not contain a tangent to C through p_0. Indeed let r be a line on Q; then

$$r \cap C = r \cap Q',$$

where Q' is any other quadric; if r was in the tangent cone to C at p_0, the intersection multiplicity with C would be at least 3, which is not possible.

The argument works also for Q singular and shows, more precisely, that if Q contains a tangent to C through p_0, it must be singular in p_0. The pencil contains at most one quadric singular in p_0: indeed the base locus of a pencil generated by two quadric singular in the same point is a union of lines. So there is exactly one quadric Q_0, singular in p_0, which contains both lines tangent to C in p_0.

In conclusion we find that if $Q \neq Q_0$

$$\sharp \varphi^{-1}(Q) = \begin{cases} 2 & \text{if } Q \text{ is smooth} \\ 1 & \text{if } Q \text{ is singular.} \end{cases}$$
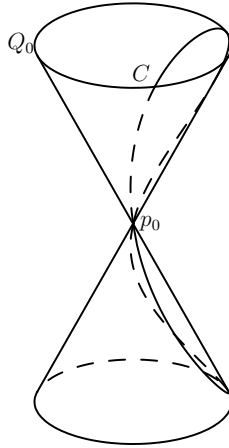
Figure 3.3 The curve C has a node

On Q_0 we each line meets C at least in p_0, so there cannot be other tangents, and

$$\sharp \varphi^{-1}(Q_0) = 2.$$

So φ has degree 2, and the total degree of ramification over a singular quadric different from Q_0 is 1. Riemann-Hurwitz yields

$$-2 = \chi(\widetilde{C}) = 2\chi(\ell) + \sharp(\{\text{singular quadrics} \neq Q_0\}),$$

from which we deduce that ℓ contains 2 singular quadrics other than Q_0.

The same argument as above shows that Q_0 is a point of tangency between ℓ and D_3, since all quadrics of ℓ pass through p_0.

iii) If C has a cusp, the argument is exactly as above. The only difference is that on Q_0 we have only 1 tangent, so φ ramifies over Q_0 too. Riemann-Hurwitz yields

$$-2 = \chi(\widetilde{C}) = 2\chi(\ell) + \sharp(\{\text{singular quadrics}\}),$$

hence ℓ contains 2 singular quadrics. Of these Q_0 is a point of tangency between ℓ and D_3 and the other is not, so Q_0 must be a flex.

iv) Now assume that the base locus C splits as

$$Q_1 \cap Q_2 = r_0 \cup C_0,$$

where r_0 is a line and $\deg C_0 = 3$. We claim that r_0 and C_0 meet in 2 points, counted with multiplicity.

Indeed, identifying Q_1 with $\mathbb{P}^1 \times \mathbb{P}^1$, we know that

$$r_0 \in |\mathcal{O}_{Q_1}(1,0)|, \qquad r_0 + C_0 \in |\mathcal{O}_{Q_1}(2,2)|,$$

so we find the intersection number

$$(r_0 \cdot C_0)_{Q_1} = 2.$$

In particular C_0 cannot be a planar cubic, so it must be a rational normal curve.

Under our hypothesis r_0 is not a tangent to C_0, so it is well defined the map

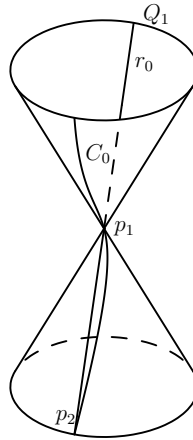$$\varphi \colon C_0 \longrightarrow \ell.$$
$$p \longrightarrow Q_p$$

Figure 3.4 The base locus splits as $C_0 \cup r_0$

Let

$$r_0 \cap C_0 = \{p_1, p_2\};$$

then the argument of the preceding step shows that if a quadric Q contains the line $T_{p_i}C_0$, it must be singular at p_i. If Q is singular with vertex in p_i, every line of Q meets C at least in p_i, hence Q contains just one tangent.

Otherwise the usual Riemann-Hurwitz argument shows that on a ruling of Q *containing* r_0 there are exactly 2 tangents to C_0. If Q is smooth, every line of the other ruling meets r_0, so it cannot be tangent to C_0. In conclusion

$$\sharp \varphi^{-1}(Q) = \begin{cases} 2 & \text{if } Q \text{ is smooth at the } p_i \\ 1 & \text{if } Q \text{ is singular in some } p_i. \end{cases}$$

This time Riemann-Hurwitz for the map φ just says that there exist two quadrics singular in some p_i. Since the two quadrics cannot have the same vertex, we have for each p_i a quadric Q_i singular at p_i.

Since all quadrics $Q \in \ell$ pass through p_1 and p_2, Q_1 and Q_2 are points of tangency between ℓ and D_3. It follows that there is no other singular quadric, and ℓ is a bitangent.

v) In this case it is enough to check that ℓ meets D_3 in just one point. Assume that C splits as in the preceding step, but now r_0 and C_0 are tangent in p. We know that there is at least one singular quadric on ℓ, and that there cannot be more than one quadric singular in p. So it is enough to show that ℓ does not contain any quadric having a singular point different from p.

Assume that Q is such a quadric, and let p_0 be its vertex. Then Q has just one ruling, which must contain r_0, in particular $p_0 \in r_0$. But C is the intersection of Q and another quadric, so it has to be singular at p_0; this is a contradiction since p_0 is a smooth point of C.

Finally we see that no other cases can arise. If C splits as

$$C = C_1 \cup C_2$$

where the C_i are irreducible conics, we can consider the map

$$\varphi \colon C_1 \longrightarrow \ell.$$
$$p \longrightarrow Q_p$$

If r is a line on $Q \in \ell$, r must meet both C_1 and C_2, so it cannot be tangent to C_1. This holds both for Q of rank 3 and 4, so

$$\varphi^{-1}(Q) = \emptyset \text{ for all } Q \in \ell$$

and we find a contradiction. This case actually arises for the pencil generated by a pair of planes and a smooth quadric.

Similarly one excludes the cases where C splits as three or more components.

After this long parenthesis, we can proceed with the proof of Proposition 3.39.

PROOF.[Proof of Proposition 3.39] We can assume that S is given by the above construction.

We explicitly show a map

$$\pi \colon S' \to \operatorname{Bit} S$$

which is generically $2 : 1$, and whose associated involution is exactly ι. This will give the birational map between F and $\operatorname{Bit}(S)$.

Let

$$(x, y) \in S' \subset \mathbb{P}(V) \times \mathbb{P}(V).$$

Then $x \neq y$, as we have remarked, so we consider the line $r = \overline{xy}$. We claim that for $(x, y) \in S'$ generic, there is a pencil ℓ of quadrics containing r. Granting this

we define $\pi(x,y) = \ell$. Indeed, by Corollary 3.41, we see that a pencil of quadrics whose base locus contains a line is in fact a bitangent to S.

To show the claim we observe that for each quadric $Q \in \Lambda$ we have $\widetilde{q}(x,y) = 0$, so if Q contains x and y it contains the whole line $\overline{xy}$. So if x and y impose independent conditions on Λ, the locus of quadrics containing r is a pencil. If this is not the case, then every quadric of Λ containing x contains y too, so there is a net $\Lambda' \subset \Lambda$ of quadrics containing r. The generic Λ does not contain such a net, by a dimension count.

Indeed the nets of quadrics containing a *fixed* line form a Grassmannian $\mathrm{Gr}(3,7)$, of dimension 12. Letting the line vary, we obtain a family of nets of quadrics containing a line of dimension 16. For every net of quadrics, the webs containing it form a $\mathbb{P}^7$, so the family

$$\{\text{webs of quadrics} \supset \mathcal{N} \mid \mathcal{N} \text{ is a net having a line in its base locus}\}$$

has dimension at most 23. Instead the family of all webs of quadrics is a Grassmannian $\mathrm{Gr}(4,10)$, so it has dimension 24.

Since by construction $\pi(x,y) = \pi(y,x)$, we obtain the desired map

$$\pi' \colon F \to \mathrm{Bit}\, S.$$

It remains to show that π' is birational.

Again, by the description of bitangents to S given above, we have to prove the following: on the generic line r contained in a pencil $\ell \subset \Lambda$ of quadrics there are exactly two points x, y with the property that

$$\widetilde{q}(x,y) = 0 \text{ for all } Q \in \Lambda. \tag{3.19}$$

Choose generators $Q_1, \ldots, Q_4$ for Λ such that Q_1 and Q_2 generate ℓ. Then it is enough to verify (3.19) for Q_3 and Q_4.

If r is generic, then Q_3 and Q_4 don't contain r, nor they are tangent, so they both cut two points on it. Moreover these points are distinct, since

$$Q_1 \cap \cdots \cap Q_4 = \emptyset,$$

for the generic choice of Λ.

Let $W \subset V$ be the subspace of dimension 2 such that $\mathbb{P}(W) = r$. Then we must prove that there is a unique (up to rescaling) basis $\{w_1, w_2\}$ of W which diagonalizes both q_3 and q_4. Indeed choosing

$$x = [w_1] \text{ and } y = [w_2]$$

yields the thesis.

The bilinear forms $\widetilde{q}_3\big|_W$ and $\widetilde{q}_4\big|_W$ are both non-degenerate, and the quadrics Q_3 and Q_4 are transverse on $\mathbb{P}(W)$ (actually they don't meet), so there exists such a basis by Lemma 2.18. Up to rescaling we can assume that in this basis $\widetilde{q}_3$ is given by the identity matrix and $\widetilde{q}_4$ by the matrix

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Then we must have $\lambda_1 \neq \lambda_2$, otherwise $Q_3 = Q_4$, so the eigenspaces have dimension 1 and the basis is unique up to multiples.

For later use we will need a more precise information. Let

$$v\colon \Lambda \to |\mathcal{O}_{\Lambda^\vee}(2)|$$

be the second Veronese map. Recall that T is the set of singular points of S. We aim to prove:

PROPOSITION 3.42. *For a generic choice of Λ, the 10 points in $v(T)$ are projectively independent.*

We can then apply Proposition 3.7 to this particular choice of S. Recalling Corollary 3.18, one obtains:

COROLLARY 3.43. *There exists a component Σ'_{10} of Σ_{10}, having codimension 10 in $\mathbb{LG}(\bigwedge^3 V)$, such that $Y_A[2]$ is birational to an Enriques surface for the generic $A \in \Sigma'_{10}$.*

In particular we note:

COROLLARY 3.44. *For the generic $A \in \Sigma'_{10}$ the 10 decomposable forms in A are linearly independent.*

So we conclude this section with a proof of Proposition 3.44. Another way to restate it is saying that T is not contained in any quadric. Recall that we have taken some 3-dimensional subspace $\Lambda \subset |\mathcal{O}_{\mathbb{P}^3(2)}|$ and defined $T = D_2 \cap \Lambda$, where

$$D_2 = \{Q \mid \mathrm{rk}\, Q \leq 2\}.$$

So our first remark is the

LEMMA 3.45. *D_2 is not contained in any quadric.*

PROOF. Indeed it is well known that the ideal of D_2 is generated by the determinants of the 3×3 minors of Q, which are cubic equations.

We now try to argue by descending induction on linear sections of D_2. We shall use the following two lemmas.

LEMMA 3.46. *Let $X \subset \mathbb{P}^n$ a variety. Assume that X is not contained in any quadric and that X is linearly normal, that is, $h^1(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$. Then for the generic hyperplane H, the linear section $H \cap X$ is not contained in any quadric of H.*

PROOF. Consider the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(2) \longrightarrow \mathcal{O}_H(2) \longrightarrow 0$$
$$\longrightarrow \qquad\qquad \longrightarrow \qquad\qquad \longrightarrow$$
$$0 \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X(2) \longrightarrow \mathcal{O}_{X \cap H}(2) \longrightarrow 0$$

obtained by twisting the defining sequences for H in $\mathbb{P}^n$ and for $X \cap H$ in X by $\mathcal{O}(2)$. These induce a commutative diagram of long exact sequences

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(2)) \longrightarrow H^0(\mathcal{O}_H(2)) \longrightarrow 0$$
$$\xrightarrow{\alpha} \qquad\qquad \xrightarrow{\beta} \qquad\qquad \xrightarrow{\gamma}$$
$$0 \longrightarrow H^0(\mathcal{O}_X(1)) \longrightarrow H^0(\mathcal{O}_X(2)) \longrightarrow H^0(\mathcal{O}_{X \cap H}(2)) \longrightarrow \cdots,$$

where we have used that $H^1(\mathcal{O}_{\mathbb{P}^n}(1)) = 0$ by Kodaira vanishing.

Our hypothesis tell that α is surjective and that β is injective, while the thesis amounts to saying that γ is injective, which is just a matter of diagram chasing.

LEMMA 3.47. *Let $X \subset \mathbb{P}^n$ a variety. Assume that X is linearly normal and regular, that is, $h^1(X, \mathcal{O}_X) = 0$. Then for the generic hyperplane H, the linear section $H \cap X$ is linearly normal.*

PROOF. We consider the same exact sequences of the previous lemma, this time twisted by $\mathcal{O}(1)$. Their associated long exact sequences yield the diagram

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \longrightarrow H^0(\mathcal{O}_H(1)) \longrightarrow 0$$
$$\longrightarrow \qquad\qquad \xrightarrow{\alpha} \qquad\qquad \xrightarrow{\beta}$$
$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_X(1)) \longrightarrow H^0(\mathcal{O}_{X \cap H}(1)) \longrightarrow 0,$$

since both $\mathbb{P}^n$ and X are regular.

This time our hypothesis is that α is surjective, and by diagram chasing we get that β is surjective too.

It is now clear how we want to use the previous lemmas to prove Proposition 3.42 by descending induction. To get from $H^1(X, \mathcal{O}_X) = 0$ to $H^1(X \cap H, \mathcal{O}_{X \cap H}) = 0$ we would like to use Lefschetz's theorem on hyperplane sections. The only obstacle is that the latter works for smooth varieties, while we are starting from the singular variety D_2.

To overcome this difficulty we pass to a smooth double cover of D_2. Namely, since every quadric of rank at most 2 is the union of two planes (maybe coincident) we can identify D_2 with the symmetric product $(\mathbb{P}^3)^{(2)}$.

In even more explicit terms consider the Segre embedding of $\mathbb{P}^3 \times \mathbb{P}^3$; this is the map

$$s\colon \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^{15} = \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))^2)$$

defined by sections of

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1).$$

If one restricts to *symmetric* sections, one obtains a map

$$t\colon \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^9 = \mathbb{P}\big(\mathrm{Sym}^2 H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))\big) = \mathbb{P}\big(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)\big),$$

which is a $2:1$ covering of D_2, ramified over D_1.

We can use this to prove the induction basis, as in the following two lemmas.

LEMMA 3.48. D_2 *is linearly normal.*

PROOF. We must show that every section $\sigma \in H^0(D_2, \mathcal{O}_{D_2}(1))$ lifts to $\mathbb{P}^9$. The section

$$t^*(\sigma) \in H^0(\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{L})$$

is of course symmetric. Since the map t is given by the linear series of *all* symmetric sections of $\mathcal{L}$ we see that σ comes from a hyperplane section of $\mathbb{P}^9$.

LEMMA 3.49. D_2 *is regular, that is,* $H^1(D_2, \mathcal{O}_{D_2}) = 0$.

PROOF. We start from the fact that $\mathbb{P}^3 \times \mathbb{P}^3$ is regular: this follows by the Hodge decomposition, since $\mathbb{P}^3 \times \mathbb{P}^3$ is simply connected. We want to apply the Leray spectral sequence to the morphism

$$t\colon \mathbb{P}^3 \times \mathbb{P}^3 \to D_2.$$

We first remark that

$$R^i t_*(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}) = 0$$

for all $i \geq 1$ by [Har77, Cor. $III.11.2$], since t is finite.

Let i be the covering involution on $\mathbb{P}^3 \times \mathbb{P}^3$. We have an action of i on $t_*(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3})$, so we can decompose

$$t_*(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}) = \mathcal{O}_{D_2} \oplus \xi,$$

where ξ is the subsheaf of eigensections with eigenvalue -1.

By what we have said the Leray spectral sequence degenerates at E_2, and we have

$$H^1(\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}) = H^1(D_2, t_* \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}) = H^1(D_2, \mathcal{O}_{D_2}) \oplus H^1(\mathcal{O}_{D_2}, \xi),$$

so we deduce that $H^1(D_2, \mathcal{O}_{D_2}) = 0$.

PROOF.[Proof of Proposition 3.42] We know that D_2 is not contained in any quadric by Lemma 3.45, and that it is linearly normal by Lemma 3.48.

Take a generic hyperplane section of D_2, call it X. By Lemma 3.46 we see that X is not contained in any quadric.

Let $Y = t^{-1}(X)$; since X is generic, Y is smooth, and we can apply Lefschetz theorem on hyperplane sections to deduce that Y is regular. We can then argue as in Lemma 3.49 to prove that X is regular too.

Finally we use Lemmas 3.49 and 3.47 to prove that X is linearly normal.

Then we pass to a hyperplane section of X and so on, as long as we are in the dimension range where we can use Lefschetz theorem. After four steps we find a surface $S \subset D_2$ which is regular, linearly normal and not contained in any quadric. In the next step we find a curve C which is only linearly normal and not contained in any quadric. Finally a last application of Lemma 3.46 yields a finite set of points T which is not contained in any quadric.

4 – The Chow ring of double EPW sextics

4.1 – Cohomology computations on X_A

Let $X = X_A$ be a smooth double EPW sextic. In this section we compute the cohomological invariants of X, partly following [O'G08b].

Let σ be the symplectic form on X. Since the canonical of X is trivial

$$H^{4,0}(X) = H^0(X, \Omega_X^4)$$

is generated by σ^2. Moreover it is known that $H^3(X) = 0$, so we can compute the Euler characteristic

$$\chi(X, \mathcal{O}_X) = h^{0,0}(X) + h^{2,0}(X) + h^{4,0}(X) = 3.$$

The symplectic form on X gives an isomorphism

$$T_X \cong \Omega_X^1,$$

hence the odd Chern classes vanish. The Hirzebruch-Riemann-Roch theorem for X simplifies to

$$3 = \chi(X, \mathcal{O}_X) = \frac{1}{240}\left(c_2(X)^2 - \frac{1}{3}c_4(X)\right). \qquad (4.1)$$

We introduce some more notation. Let us call

$$q \in \mathrm{Sym}^2(H^2(X, \mathbb{Q})^\vee)$$

the Beauville-Bogomolov form of X. Since it is non-degenerate, it allows us to give an identification

$$H^2(X, \mathbb{Q}) \cong H^2(X, \mathbb{Q})^\vee$$

hence we obtain a dual quadratic form

$$q^\vee \in \mathrm{Sym}^2(H^2(X, \mathbb{Q})).$$

Recall that the cup product yields an isomorphism between $\mathrm{Sym}^2(H^2(X, \mathbb{Q}))$ and $H^4(X, \mathbb{Q})$, so we can regard $q^\vee$ as an element of $H^4(X, \mathbb{Q})$.

O'Grady proves in [O'G08b] that we have the relation

$$q^\vee = \frac{5}{6} c_2(X), \tag{4.2}$$

and that for any $\alpha, \beta \in H^2(X, \mathbb{Q})$ we have

$$q^\vee \cdot \alpha \cdot \beta = 25 q(\alpha, \beta). \tag{4.3}$$

We now work out the relations in the cohomology of X. Let

$$h = c_1(f^* \mathcal{O}_Y(1)) \in H^2(X).$$

PROPOSITION 4.1. *In the cohomology ring $H^*(X, \mathbb{Q})$ we have*

$$h^4 = 12, \qquad h^2 \cdot c_2(X) = 60,$$
$$c_2(X)^2 = 828, \qquad c_4(X) = 324.$$

PROOF. The first and the last relations are easily handled. Indeed

$$h^4 = 2 \deg(Y) = 12.$$

As for the last one we have

$$c_4(X) = \chi(X),$$

and since X is a deformation of $S^{[2]}$, where S is a $K3$, we have

$$\chi(X) = \chi(S^{[2]}) = 324$$

by Proposition 1.3.

By O'Grady's computations (4.3) and (4.2) we also have

$$c_2(X) \cdot h^2 = \frac{6}{5} q^\vee \cdot h^2 = \frac{25 \cdot 6}{5} q(h, h) = 60.$$

Finally we can use Equation (4.1) to obtain $c_2(X)^2 = 828$.

In degree 6 the only possible relation is a linear dependency between h^3 and $c_2(X) \cdot h$, and indeed we have:

PROPOSITION 4.2. *There is a relation*

$$c_2(X) \cdot h = 5h^3$$

$H^6(X, \mathbb{Q})$.

PROOF. From O'Grady's relation (4.3) we get

$$6q^\vee \cdot h \cdot \alpha = 6 \cdot 25q(h, \alpha)$$

for all $\alpha \in H^2(X)$. On the other hand, by polarization of Fujiki's relation we obtain

$$25h^3 \cdot \alpha = 25 \cdot 3 \cdot q(h, h)q(h, \alpha) = 6 \cdot 25q(h, \alpha).$$

So Poincaré duality implies that

$$25h^3 = 6q^\vee \cdot h$$

modulo torsion, and using (4.2) we get the thesis.

We can instead exclude relations in degree 4:

LEMMA 4.3. *The classes h^2 and $c_2(X)$ are linearly independent inside $H^2(X)$.*

PROOF. We can substitute $c_2(X)$ with its multiple $q^\vee$. Assume that we have a relation

$$h^2 + \lambda q^\vee = 0$$

for some $\lambda \in \mathbb{C}$. Then we get

$$h^2\alpha^2 = -25\lambda q(\alpha, \alpha)$$

for all $\alpha \in H^2(X)$. By polarization of the Fujiki formula we also obtain

$$h^2\alpha^2 = q(\alpha, \alpha)q(h, h) + 2q(h, \alpha)^2.$$

So if $q(\alpha, \alpha) = 0$ we obtain $q(h, \alpha) = 0$. This means that q is degenerate (the quadric defined by q would be contained in a hyperplane of $\mathbb{P}H^2(X)$), contradiction.

Finally, it will be useful to write out the explicit form of Hirzebruch-Riemann-Roch, using the above computations for the characteristic classes of X. We let

$$\mathcal{O}_X(1) = f^*\mathcal{O}_Y(1).$$

Then $\mathcal{O}_X(n)$ is ample on X, and since K_X is trivial, Kodaira vanishing yields

$$\chi(X, \mathcal{O}_X(n)) = h^0(X, \mathcal{O}_X(n)).$$

The formula of Hirzebruch-Riemann-Roch then reads

$$h^0(X, \mathcal{O}_X(n)) = \frac{h^4}{24}n^4 + \frac{c_2(X) \cdot h^2}{24}n^2 + \chi(\mathcal{O}_X) = \frac{1}{2}n^4 + \frac{5}{2}n^2 + 3. \qquad (4.4)$$

We have also used a similar computation in Section 3.1:

LEMMA 4.4. *Let X be numerically equivalent to $S^{[2]}$, where S is a $K3$, and let $e \in H^2(X)$ be a class with $q(e, e) = -2$. Let L be a line bundle on X with $c_1(L) = e$. Then*

$$\chi(X, L) = 1.$$

PROOF. By Fujiki relation we obtain

$$e^4 = 3 \cdot q(e, e)^2 = 12.$$

Moreover Equations (4.2) and (4.3) yield

$$c_2(X) \cdot e^2 = \frac{6}{5} q^\vee \cdot e^2 = 30 q(e, e) = -60.$$

So Hirzebruch-Riemann-Roch reads

$$\chi(X, L) = \frac{e^4}{24} + \frac{c_2(X) \cdot e^2}{24} + \chi(\mathcal{O}_X) = \frac{1}{2} - \frac{5}{2} + 3 = 1.$$

4.2 – Everywhere tangent EPW sextics

Let $X = X_A$ be a double covering of an EPW sextic, endowed with ample line bundle

$$\mathcal{O}_X(1) = f^* \mathcal{O}_Y(1),$$

where as usual

$$f \colon X \to Y$$

is the double covering. In this section we wish to understand from a geometric point of view the odd sections of $\mathcal{O}_X(3)$.

Consider the decomposition

$$H^0(X, \mathcal{O}_X(n)) = H^0(X, \mathcal{O}_X(n))_+ \oplus H^0(X, \mathcal{O}_X(n))_-,$$

where $H^0(X, \mathcal{O}_X(n))_\pm$ are the eigenspaces relative to the eigenvalue ± 1 for the action of the covering involution φ. We call the sections in the eigenspaces even or odd respectively.

LEMMA 4.5. *The number of odd sections is given by*

$$h^0(X, \mathcal{O}_X(3))_- = 10.$$

PROOF. This is actually a simple computation using the theorem of Riemann-Roch-Hirzebruch. First we remark that even sections of $\mathcal{O}_X(3)$ descend to sections of $\mathcal{O}_Y(3)$, so

$$h^0(X, \mathcal{O}_X(3))_+ = h^0(Y, \mathcal{O}_Y(3)).$$

By Lemma 4.6 below we see that

$$h^0(Y, \mathcal{O}_Y(3)) = h^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)) = \binom{5+3}{3} = 56.$$

On the other hand we have computed in Equation (4.4) that

$$h^0(X, \mathcal{O}_X(3)) = 66,$$

hence the thesis.

LEMMA 4.6. *The restriction*

$$H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)) \to H^0(Y, \mathcal{O}_Y(3))$$

is an isomorphism.

PROOF. We just need to show that $H^0(\mathbb{P}^5, \mathcal{I}_Y(3))$ and $H^1(\mathbb{P}^5, \mathcal{I}_Y(3))$ vanish. Since Y is a sextic, $\mathcal{I}_Y \cong \mathcal{O}_{\mathbb{P}^5}(-6)$, so

$$H^0(\mathbb{P}^5, \mathcal{I}_Y(3)) = H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(-3)) = 0.$$

On the other hand $K_{\mathbb{P}^5} = \mathcal{O}_{\mathbb{P}^5}(-6)$, so

$$H^1(\mathbb{P}^5, \mathcal{I}_Y(n)) = 0$$

for every $n > 0$ by Kodaira vanishing.

Given $\eta \in H^0(X, \mathcal{O}_X(3))_-$ we obtain the even section

$$\eta \otimes \eta \in H^0(X, \mathcal{O}_X(6))_+ \cong H^0(Y, \mathcal{O}_Y(6)),$$

since even sections descend to Y. The proof of Lemma 4.6 shows that

$$H^1(\mathbb{P}^5, \mathcal{I}_Y(6)) = 0,$$

hence this section lifts to a sextic Y' of $\mathbb{P}^5$. Where Y and Y' meet the intersection is at least double: this is easily seen locally.

Indeed let $y \in Y$ be a point where $\eta \otimes \eta$ vanishes. Then for every point $x \in X$ such that $f(x) = y$ we must have

$$\eta(x) = 0,$$

so $\eta \otimes \eta$ has a double zero in x (hence in y).

This construction yields a sextic Y' everywhere tangent to Y. We now want to describe explicitly such special sextics; in particular we will show that they are again EPW sextics.

PROPOSITION 4.7. *Let* $A, A' \in \mathbb{LG}(\bigwedge^3 V)$ *be two Lagrangian subspaces such that*

$$\dim(A \cap A') = 9.$$

Then Y_A *and* $Y_{A'}$ *are everywhere tangent.*

PROOF. Let

$$[v] \in Y_A^{sm} \cap Y_{A'}^{sm}$$

be a smooth point of both Y_A and $Y_{A'}$. Then we claim that

$$F_v \cap A = F_v \cap A'. \tag{4.5}$$

Indeed both $F_v \cap A$ and $F_v \cap A'$ are 1-dimensional, because Y_A and $Y_{A'}$ are smooth in $[v]$. By symmetry it is enough to show that

$$F_v \cap A' \subset A.$$

If this does not happen, then

$$A' = (F_v \cap A') \oplus (A \cap A').$$

Let α be a generator for $F_v \cap A$. Then, since F_v and A are isotropic, α is orthogonal to both $F_v \cap A'$ and $A \cap A'$. It follows that

$$\alpha \in (A')^{\perp} = A'.$$

This is a contradiction, so (4.5) is proved.

By Proposition 2.23 this implies that

$$T_{[v]} Y_A = T_{[v]} Y_{A'}.$$

Since this is true for any smooth point of intersection, the thesis is proved.

REMARK. If A'' is any other Lagrangian subspace with

$$A \cap A' = A \cap A''$$

it is easy to see that

$$Y_A \cap Y_{A'} = Y_A \cap Y_{A''},$$

where the latter is an equality *of schemes*. So this intersection only depends on

$$U = A \cap A'.$$

In other words we can associate to every $U \in \mathbb{P}(A^{\vee})$ a section

$$\tau \in H^0(Y_A, \mathcal{O}_{Y_A}(6)) \cong H^0(X_A, \mathcal{O}_{X_A}(6)_+).$$

REMARK. In the last remark we have implicitly used the fact that every $U \in \mathbb{P}(A^{\vee})$ is contained in some other Lagrangian subspace A'. This is easy: if U is as above, then

$$U^{\perp} \supset A^{\perp} = A,$$

and every hyperplane of $U^{\perp}$ containing U is such a Lagrangian subspace. Indeed let $U \subsetneq A' \subsetneq U^{\perp}$, so that

$$A' = U \oplus \langle v \rangle$$

for some v. Then v is orthogonal both to U and to itself, so A' is isotropic.

In particular we see that there is a pencil of Lagrangian subspaces containing U.

PROPOSITION 4.8. *The above construction yields an isomorphism*

$$g \colon \mathbb{P}(A^{\vee}) \to \mathbb{P}H^0(X_A, \mathcal{O}_{X_A}(3)_-).$$

The divisors

$$D' \in |H^0(X_A, \mathcal{O}_{X_A}(3)_-)|,$$

or better their images in Y_A, are endowed with a natural rational function.

Let $U \in \mathbb{P}(A^{\vee})$ such that $g(U) = D'$, and let $D = f(D')$. We also let ℓ_D be the pencil of Lagrangian subspaces containing U. Then there is a rational function

$$r_D \colon D \dashrightarrow \ell_D$$

defined as follows.

Let A, A' be generators of ℓ_U, and x a generic point of $D \subset X_A$. Then

$$[v] = f_A(x) \in Y_A^{sm} \cap Y_{A'}^{sm},$$

and by Equation (4.5) we have

$$F_v \cap A = F_v \cap A',$$

both of dimension 1. We claim that

$$\dim(F_v \cap (A + A')) = 2. \tag{4.6}$$

Indeed we start by the simple remark that

$$(F_v + A)^{\perp} = (F_v)^{\perp} \cap A^{\perp} = F_v \cap A \subset A' = (A')^{\perp}.$$

We can dualize it to obtain

$$A' \subset F_v + A,$$

so we find that

$$\dim(F_v + A + A') = \dim F_v + A = 19$$

by Grassmann. Since

$$\dim(A + A') = 11, \qquad \dim F_v = 10,$$

Grassmann's formula applied to F_v and $A + A'$ yields Equation (4.6).

By Equation (4.6) we see that there is exactly one member $A_v \in \ell_D$ such that

$$F_v \cap (A + A') \subset A_v.$$

Indeed all members of the pencil contain $F_v \cap A$, so containing $F_v \cap (A + A')$ is just one more linear condition. We can explicitly see that

$$A_v = (A \cap A') + \big(F_v \cap (A + A')\big).$$

We then define

$$r_D \colon D \dashrightarrow \ell_D.$$
$$[v] \longrightarrow A_v$$

It is easy to describe the divisors in the linear system on D whose associated rational map is r_D. Indeed by construction we see that, given $B \in \ell_D$, we have $r_D([v]) = B$ if and only if

$$\dim(F_v \cap B) = 2,$$

hence the map r_D is defined by the pencil of divisors

$$\big\{Y_B[2] \mid B \in \ell_D\big\}.$$

REMARK. We should note that indeed if $B \in \ell_D$, then

$$\dim(B \cap A) = 9,$$

and this implies that Y_A contains $Y_B[2]$. In fact if $\dim(F_v \cap B) = 2$, then $\dim(F_v \cap A) \geq 1$.

4.3 – Proof of the main theorem

In this section we shall prove the following theorem, which is the main result of this work.

THEOREM 4.9. *Let $A \in \mathbb{LG}(\bigwedge^3 V)^0$ and let X_A be the associated double covering of the EPW sextic Y_A. Then any polynomial relation*

$$P\big(h_A, c_2(X_A), c_4(X_A)\big) = 0$$

in the classes of divisors on X_A and in the Chern classes of X_A which holds in the cohomology ring already holds at the level of the Chow ring.

REMARK. The symplectic form on $X = X_A$ gives an isomorphism

$$T_X \cong \Omega_X^1,$$

hence the odd Chern classes vanish. This is why they do not appear in the statement. Moreover if A is generic in $\mathbb{LG}(\bigwedge^3 V)$, the group $\mathrm{Pic}(X_A)$ is cyclic, generated by h_A, so the theorem above yields exactly Voisin's conjecture.

REMARK. The only relations in cohomology can be in degree 4, 6 or 8. Lemma 4.3 excludes the existence of relations of degree 4, hence we are left with relations in degree 6 or 8; these are listed in Propositions 4.1 and 4.2.

Proving the theorem will involve some steps, so we briefly outline the plan. First, we define a special 0-cycle class θ on X; we aim to prove that every 0-cycle which can be expressed as a polynomial in h, $c_2(X)$ and $c_4(X)$ is a rational multiple of θ. It turns out that it is easier to define a class $\overline{\theta}$ on Y as the class of any point on a suitable Enriques surface inside Y, and to pull back this to X

The first two subsections are devoted to the definition of $\overline{\theta}$ and the proof that it is actually well-defined. The next section carries out some geometrical construction on X which yield relations in the Chow ring. Finally the last section uses these relations to conclude the proof of Theorem 4.9.

4.3.1 – Definition of the class θ

Let $X = X_A$ as usual. Our first task is to define a class

$$\theta \in CH^4(X)$$

of degree 1. Then we will show that the relations

$$h^4 = 12\theta, \qquad h^2 c_2(X) = 60\theta, \qquad c_2(X)^2 = 828\theta, \qquad c_4(X) = 324\theta$$

hold.

It will actually be easier to work on Y, so we'd better find out the relationship between $CH(X)$ and $CH(Y)$.

REMARK. The map $f\colon X \to Y$ induces a push-forward morphism

$$f_*\colon CH(X) \to CH(Y),$$

because f is proper (for the construction of Chow rings and morphisms between them see [Ful84, Chap. 1]). On the other hand f^* is usually defined for flat maps with fibers of constant dimension, and f is not flat.

Following Example 1.7.6 on [Ful84] we can define f^* in our situation. Indeed Fulton shows that if

$$Y = X/G$$

is the quotient of X by the action of a finite group G, we have a canonical isomorphism

$$CH(Y)_{\mathbb{Q}} \cong CH(X)_{\mathbb{Q}}^G,$$

where as usual $CH(Y)_{\mathbb{Q}} = CH(Y) \otimes \mathbb{Q}$. So if f is the quotient map we can define f^* by the composition

$$CH(Y)_{\mathbb{Q}} \xrightarrow{\cong} CH(X)_{\mathbb{Q}}^G \hookrightarrow CH(X)_{\mathbb{Q}}.$$

Fulton also shows that the composition

$$CH(Y)_{\mathbb{Q}} \xrightarrow{f^*} CH(X)_{\mathbb{Q}} \xrightarrow{f_*} CH(Y)_{\mathbb{Q}}$$

is the multiplication map by $\sharp G$.

In our situation $G = \langle \varphi \rangle$, where φ is the covering involution, and the composition above is multiplication by 2.

Recall that we have defined

$$\Sigma_{10} \subset \mathbb{LG}(\textstyle\bigwedge^3 V)$$

as the (zariski closure of the) set of Lagrangian subspaces such that there exist 10 independent subspaces

$$W_1, \ldots, W_{10} \subset V$$

of dimension 3 with $\bigwedge^3 W_i \subset A$, and Σ_{10}' is a particular component defined in Section 3.7. By Corollary 3.43 we know that for $B \in \Sigma_{10}'$ generic $Y_B[2]$ is birational to an Enriques surface.

We now recall a result about Chow groups of surfaces ([Voi03, Thm. 11.10])

THEOREM (Bloch, Kas, Lieberman). *Let S be a smooth projective surface with $H^{2,0}(S) = 0$, and assume that S is not of general type. Then the Albanese map*

$$alb_S \colon CH^2_{hom}(S) \to \mathrm{Alb}(S)$$

is an isomorphism. In particular if moreover $H^{1,0}(S) = 0$, then $CH^2_{hom}(S) = 0$.

By this result we see that if S is an Enriques surface,

$$CH^2(S) \cong \mathbb{Z}.$$

In particular this conclusion is true for $Y_B[2]$, when $B \in \Sigma'_{10}$ is generic.

To handle the case where B is not generic we use the following result (the proof is the same of [Voi03, Lemma 10.7]):

THEOREM. *Consider an algebraic family of cycles $(Z_t)_{t \in U}$ on a variety X parametrized by a basis U. Then the set*

$$\{u \in U \mid Z_t \text{ is rationally equivalent to zero}\}$$

is a countable union of Zariski closed subsets of U.

By the above result, the fact that $CH^2(Y_B[2]) = \mathbb{Z}$ for B generic extends to the case where B is not generic. In conclusion we have the

PROPOSITION 4.10. *Let $B \in \Sigma'_{10}$; then*

$$CH^2(Y_B[2]) \cong \mathbb{Z}.$$

That said, we define a class

$$\overline{\theta} \in CH^4(Y_A)$$

as follows. Let $A \in \mathbb{LG}(\bigwedge^3 V)^0$. We have the

PROPOSITION 4.11. *There exists $B \in \Sigma'_{10}$ such that*

$$\dim(A \cap B) \geq 9. \tag{4.7}$$

The proof of the above Proposition is quite involved, and will be given in the following subsection.

By the remark at the end of the previous section we know that

$$Y_B[2] \subset Y_A,$$

so we define $\overline{\theta}$ as the class of a point of $Y_B[2]$. We need to do some checks in order to show that this is actually well-defined. We also define

$$\theta = \frac{1}{2} f^*(\overline{\theta}) \in CH^4(X).$$

LEMMA 4.12. *Let* $B, B' \in \mathbb{LG}(\bigwedge^3 V)$ *such that* (4.7) *holds. Then*

$$Y_B[2] \cap Y_{B'}[2] \neq \emptyset \tag{4.8}$$

PROOF. It is enough to show that

$$Y_B[2] \cdot Y_{B'}[2] \neq 0$$

in $CH^*(Y_A)$. Recall that in Section 4.2 we have exhibited a rational equivalence between $Y_B[2]$ and Y_A, given by the rational function r_U on D_U, where $U = A \cap B$. So it will be enough to prove that

$$Y_A[2]^2 \neq 0.$$

By the definition of the ring structure on $CH^*(Y_A)$ we need to prove that

$$Z_A^2 \neq 0 \text{ in } CH^*(X_A).$$

But actually $Z_A^2 \neq 0$ already in cohomology. Indeed, using the fact that Z_A is Lagrangian, we have

$$Z_A^2 = c_2(\mathcal{N}_{Z_A/X_A}) = c_2(\Omega_{Z_A}^1) = c_2(Z_A) = \chi_{top}(Z_A) = 192$$

by the results cited in Section 2.4.

By the previous Lemma we see that the class of $\overline{\theta} \in CH^4(Y)$ is actually independent of the chosen $B \in \Sigma'_{10}$ such that (4.7) holds.

4.3.2 – Proof of Proposition 4.11

We now prove that given any Lagrangian subspace $A \subset \bigwedge^3 V$ there exists $B \in \Sigma'_{10}$ such that

$$\dim A \cap B \geq 9.$$

This will be done in several steps.

We begin with the construction of a suitable incidence variety. For the present purposes it is irrelevant that the symplectic space is $\bigwedge^3 V$, so we just consider any symplectic vector space E of dimension $2n$. We define the incidence variety

$$\Omega = \big\{(A, B) \mid \dim(A \cap B) \geq n - 1\big\} \subset \mathbb{LG}(E) \times \mathbb{LG}(E).$$

This has two projections π_1 and π_2 over the Lagrangian Grassmannian $\mathbb{LG}(E)$. We can find the dimension of Ω by studying the fibers of these morphisms. Let

$$\Omega_A = \pi_1^{-1}(A)$$

be a fiber of π_1. We consider the Plücker embedding, and let $v_A \in \bigwedge^n E$ be a vector such that $[v_A] = A$.

LEMMA 4.13. *Under the Plücker embedding, Ω_A is a cone of vertex A over* $\mathbb{P}(A^\vee)$. *The latter is embedded in*

$$\mathbb{P}(\textstyle\bigwedge^n E/\langle v_A\rangle)$$

by the complete linear system $\mathcal{O}_{\mathbb{P}(A^\vee)}(2)$.

PROOF. It is easier to consider the non Lagrangian case first. So consider the bigger incidence variety

$$\widetilde{\Omega} = \big\{(A, B) \mid \dim(A \cap B) \geq n - 1\big\} \subset \mathrm{Gr}(n, E) \times \mathrm{Gr}(n, E).$$

Accordingly we have the fiber

$$\widetilde{\Omega}_A = \big\{B \in \mathrm{Gr}(n, E) \mid \dim(A \cap B) \geq n - 1\big\}.$$

We claim that this is a cone of vertex A over

$$\mathbb{P}(A^\vee) \times \mathbb{P}(E/A).$$

First, we give the embedding

$$\varphi \colon \mathbb{P}(A^\vee) \times \mathbb{P}(E/A) \hookrightarrow \mathbb{P}(\textstyle\bigwedge^n E/\langle v_A\rangle).$$

This is done as follows. Let $(U, U') \in \mathbb{P}(A^\vee) \times \mathbb{P}(E/A)$, so

$$U \subset A \subset U'$$

with

$$\dim U = n - 1, \qquad \dim U' = n + 1$$

We choose a basis $\{u_1, \ldots, u_{n+1}\}$ of U' such that $\{u_1, \ldots, u_n\}$ is a basis of A and $\{u_1, \ldots, u_{n-1}\}$ of U. We the set

$$\varphi(U, U') = [u_1 \wedge \cdots \wedge u_{n-1} \wedge u_{n+1}].$$

It is immediate to see that another choice of basis doesn't change the class of $\varphi(U, U')$ modulo

$$v_A = u_1 \wedge \cdots \wedge u_n,$$

so φ is well-defined.

Moreover, for fixed U, $\varphi(U, \cdot)$ gives a linear embedding of $\mathbb{P}(E/A)$ and viceversa. Hence we get a bilinear embedding of the product.

Now we have the projection of center A

$$\pi_A \colon \mathbb{P}(\textstyle\bigwedge^n E) \dashrightarrow \mathbb{P}(\textstyle\bigwedge^n E/\langle v_A\rangle),$$

and we can restrict this projection to $\widetilde{\Omega}_A \setminus \{A\}$. One checks easily that this is just

$$\pi_A \colon \widetilde{\Omega}_A \setminus \{A\} \longrightarrow \mathbb{P}(A^\vee) \times \mathbb{P}(E/A),$$
$$B \longrightarrow (B \cap A, B + A)$$

thereby proving the claim.

Now assume that A is Lagrangian. The symplectic form on E identifies E/A with $A^\vee$. A given subspace $B \in \widetilde{\Omega}_A$ is Lagrangian if and only if, under this identification, $B \cap A$ is identified with $B + A$. We can consider the diagonal embedding

$$\mathbb{P}(A^\vee) \to \mathbb{P}(A^\vee) \times \mathbb{P}(E/A) \to \mathbb{P}(\textstyle\bigwedge^n E/\langle v_A \rangle);$$

this is given by sections of $\mathcal{O}_{\mathbb{P}(A^\vee)}(2)$ because φ is bilinear.

Moreover Ω_A is exactly the cone above the image of this embedding, and this proves the lemma.

The above lemma allows us to compute the dimension of Ω. Indeed we see that the fibers of π_1 are irreducible of dimension n. Since

$$\dim \mathbb{LG}(E) = \binom{n+1}{2},$$

it follows that Ω is irreducible of dimension

$$\dim \Omega = n + \binom{n+1}{2}.$$

Next we study the tangent space to Ω. Recall that the tangent space $T_A \mathbb{LG}(E)$ is canonically identified with $\mathrm{Sym}^2(A^\vee)$. We describe the tangent space to Ω inside the product

$$T_A \mathbb{LG}(E) \times T_B \mathbb{LG}(E).$$

LEMMA 4.14. *Let* $(A,B) \in \Omega$ *with* $A \neq B$, *and let* $U = A \cap B$. *Then* Ω *is smooth at* (A,B), *with tangent space*

$$T_{(A,B)}\Omega = \Big\{ (q_A, q_B) \in \mathrm{Sym}^2(A^\vee) \times \mathrm{Sym}^2(B^\vee) \mid q_A\big|_U = q_B\big|_U \Big\}. \qquad (4.9)$$

PROOF. The points of Ω outside the diagonal form an orbit under the action of the symplectic group. Since this orbit is open, every point $(A,B) \in \Omega$ with $A \neq B$ has to be smooth, and this proves the first assertion.

To describe explicitly the tangent space we start by remarking that the two sides of Equation (4.9) have the same dimension $n + \binom{n+1}{2}$. We have verified that this is the dimension of Ω, hence the dimension of its tangent space at (A,B) by the

first part of the proof. That this is also the dimension of the right hand side is an immediate computation.

So we just check that we have one inclusion. Again, it is easier to work out the non Lagrangian case first. Namely consider the incidence variety

$$\widetilde{\Omega} \subset \mathrm{Gr}(n, E) \times \mathrm{Gr}(n, E).$$

The corresponding statement, that we shall now prove, is the following.

Let $(A, B) \in \widetilde{\Omega}$ with $A \neq B$, and let

$$U = A \cap B, \qquad U' = A + B,$$

so that $\dim U = n - 1$, $\dim U' = n + 1$. Given any

$$f \in T_A \mathrm{Gr}(n, E) \cong \mathrm{Hom}(A, E/A)$$

we can consider the composition $f_{A,B} \in \mathrm{Hom}(U, E/U')$ given by

$$U \hookrightarrow A \to E/A \twoheadrightarrow E/U'.$$

Similarly for B: given $g \in T_B \mathrm{Gr}(n, E)$ we consider $g_{A,B} \in \mathrm{Hom}(U, E/U')$. Then the claim is that

$$T_{A,B}\widetilde{\Omega} = \big\{(f, g) \mid f_{A,B} = g_{A,B}\big\} \subset \mathrm{Hom}(A, E/A) \times \mathrm{Hom}(B, E/B). \qquad (4.10)$$

Let us see how the lemma follows from Equation (4.10). In case E has a symplectic form and A and B are both Lagrangian, it is immediate to check that $U' = U^\perp$. In this case we can identify

$$E/U' = E/U^\perp \cong U^\vee.$$

If $f \in T_A \mathbb{LG}(E)$, the homomorphism

$$f \colon A \to E/A \cong A^\vee$$

is symmetric, so it restricts to a symmetric homomorphism $f_{A,B}$. The same remark holds for B, so Equation (4.10) implies Equation (4.9).

Let us now prove Equation (4.10). By the same dimensional count, it is enough to prove one inclusion. Now it is just a matter of unwinding the identification of $T_A \mathrm{Gr}(n, E)$ with $\mathrm{Hom}(A, E/A)$.

Let $(A(t), B(t))$ be a curve on $\widetilde{\Omega}$ with

$$A(0) = A, \qquad B(0) = B.$$

We let $U(t) = A(t) \cap B(t)$; this has dimension $n - 1$ for all t sufficiently small. So we can choose vectors

$$u_1(t), \ldots, u_{n-1}(t), a(t), b(t)$$

such that

$$U(t) = \langle u_1(t), \ldots, u_{n-1}(t) \rangle,$$
$$A(t) = \langle u_1(t), \ldots, u_{n-1}(t), a(t) \rangle,$$
$$B(t) = \langle u_1(t), \ldots, u_{n-1}(t), b(t) \rangle.$$

Choose a subspace $C \subset E$ complementary to both A and B. Then the homomorphism associated to the tangent vector $\dot{A}(0)$ is constructed as follows.

Since

$$E = A \oplus C,$$

the subspace $A(t)$, for t small, is the graph of a map $f(t) \colon A \to C$. The vector

$$\dot{A}(0) \in T_A \operatorname{Gr}(n, E)$$

corresponds to $f'(0) \colon A \to C$. Similarly $B(t)$ is seen as the graph of a map $g(t) \colon B \to C$, and we identify $\dot{B}(0)$ with $g'(0)$. The subspace C is then identified, by projection, with E/A in the first case and with E/B in the second.

Now we take a vector $v \in U$. We can choose functions

$$\lambda_1(t), \ldots, \lambda_n(t), \mu_1(t), \ldots, \mu_n(t)$$

such that

$$f(t)v + v = \lambda_1(t)u_1(t) + \cdots + \lambda_{n-1}(t)u_{n-1}(t) + \lambda_n(t)a(t)$$
$$g(t)v + v = \mu_1(t)u_1(t) + \cdots + \mu_{n-1}(t)u_{n-1}(t) + \mu_n(t)b(t),$$

so that

$$f(t)v - g(t)v = \big(\lambda_1(t) - \mu_1(t)\big)u_1(t) + \cdots +$$
$$+ \big(\lambda_{n-1}(t) - \mu_{n-1}(t)\big)u_{n-1}(t) + \lambda_n(t)a(t) - \mu_n(t)b(t).$$

Taking derivatives and using the fact that $\lambda_i(0) = \mu_i(0) = 0$ for every i, we find

$$f'(0)v - g'(0)v = \big(\lambda_1'(0) - \mu_1'(0)\big)u_1(0) + \cdots +$$
$$+ \big(\lambda_{n-1}'(0) - \mu_{n-1}'(0)\big)u_{n-1}(0) + \lambda_n'(0)a(0) - \mu_n'(0)b(0).$$

So $f'(0)v \equiv g'(0)v$ modulo U'; in other words the two homomorphisms $f_{A,B}$ and $g_{A,B}$ agree.

Now we are ready to prove the main lemma of this subsection. Of course we choose $E = \bigwedge^3 V$. We let Σ_{10}' be any irreducible component of Σ_{10} of codimension 10 in $\mathbb{LG}(\bigwedge^3 V)$. We consider the restricted incidence variety

$$\Gamma = \Omega \cap \Big(\mathbb{LG}(\textstyle\bigwedge^3 V) \times \Sigma_{10}' \Big) = \big\{ (A, B) \mid B \in \Omega_A \big\}.$$

As before we have the two projections

$$\Gamma$$

$$\xrightarrow{\ \pi\ } \xrightarrow{\ \rho\ }$$

$$\mathbb{LG}(\textstyle\bigwedge^3 V) \qquad \Sigma'_{10}.$$

Since ρ is a fibration over Σ'_{10} with fiber Ω_B, and since we have proved that

$$\dim \Omega_B = 10 = \operatorname{codim}_{\mathbb{LG}(\bigwedge^3 V)} \Sigma'_{10},$$

we deduce that

$$\dim \Gamma = \dim \mathbb{LG}(\textstyle\bigwedge^3 V).$$

Our ultimate goal is to prove that π is a generically finite map. The lemma that we shall use is the following.

LEMMA 4.15. *Let $(A, B) \in \Gamma$ and assume that*

i) *B contains exactly 10 decomposable forms $\alpha_1, \ldots, \alpha_{10}$, which are linearly independent;*
ii) *for $i = 1, \ldots 10$ the form $\alpha_i \notin A$.*

Then the differential $d\pi_{(A,B)}$ is an isomorphism.

PROOF. By our hypothesis and Proposition 2.4, we see that the tangent to Σ'_{10} at B is the subspace T of $\operatorname{Sym}^2(B^\vee)$ consisting of those quadratic forms q such that

$$q(\alpha_i) = 0 \text{ for } i = 1, \ldots, 10.$$

Let $U = A \cap B$; we claim that the composition

$$T \hookrightarrow \operatorname{Sym}^2(B^\vee) \to \operatorname{Sym}^2(U^\vee)$$

is injective. Here the second map is the restriction on quadratic forms.

Indeed assume that a quadratic form $q \in T$ vanishes identically on U; then its zero locus is the union of two hyperplanes

$$U \cup U' \subset B.$$

We have assumed that $\alpha_i \notin U$ for every i; it follows that U' has to contain all α_i. But this is impossible, since we have assumed that they are linearly independent, and the contradiction proves the claim.

We then consider the following diagram

$$T_{(A,B)}\Gamma \xrightarrow{\quad d\rho_{(A,B)} \quad} T \subset \mathrm{Sym}^2(B^\vee)$$

$$\xrightarrow{\quad d\pi_{(A,B)} \quad} \hookleftarrow \longrightarrow \longrightarrow$$

$$\mathrm{Sym}^2(A^\vee) \xrightarrow{\hspace{3cm}} \mathrm{Sym}^2(U^\vee)$$

This is commutative by Equation (4.9), since $\Gamma \subset \Omega$.

Assume that

$$d\pi_{(A,B)}v = 0$$

for some $v \in T_{(A,B)}\Gamma$. Then the diagram shows that we have also

$$d\rho_{(A,B)}v = 0.$$

Since

$$\Gamma \subset \mathbb{LG}(\textstyle\bigwedge^3 V) \times \Sigma'_{10},$$

we find that $v = 0$.

COROLLARY 4.16. Under the same hypothesis, the map π is generically finite, in particular it is surjective.

PROOF. Since we already know that Γ and $\mathbb{LG}(\bigwedge^3 V)$ have the same dimension, it is enough to show surjectivity. Assuming that π is not surjective, the image has positive codimension in $\mathbb{LG}(\bigwedge^3 V)$.

By the theorem on the dimension of the fibers it follows that every component of every fiber of π has dimension at least 1. But Lemma 4.15 implies that the fiber of π above A has an isolate point, contradiction.

Now we see that in order to prove Proposition 4.11 it is enough to show a couple of Lagrangian subspaces (A, B) which satisfy the hypothesis of Lemma 4.15. For then the assertion that the fiber of π over any A is not empty is exactly the thesis of the proposition.

By Corollary 3.44 we know that the generic $B \in \Sigma'_{10}$ contains exactly 10 independent decomposable forms, up to multiples. Let $U \subset B$ be any hyperplane which doesn't contain any of them. Then we can find a pencil of Lagrangian subspaces A such that

$$A \cap B = U;$$

then the pair (A, B) satisfies the hypothesis of Lemma 4.15, and we are done.

4.3.3 – Some geometric constructions

We now want to show that the expected relations hold in $CH(Y)_{\mathbb{Q}}$.

REMARK. In the following we need to perform intersection products on the Chow ring of Y, and this may seem not well-defined, since Y is singular. But recall that we have the isomorphism

$$CH(Y)_{\mathbb{Q}} \cong CH(X)_{\mathbb{Q}}^G,$$

and $CH(X)_{\mathbb{Q}}^G$ is a subring of $CH(X)_{\mathbb{Q}}$, so we can multiply cycle classes on Y.

Let $\overline{h} = c_1(\mathcal{O}_Y(1))$ be the hyperplane class on Y. We start to prove relations in $CH(Y)$ analogous to those found in Proposition 4.1. In order to do this, we need another geometric lemma.

LEMMA 4.17. *There exists a line $L_0 \subset Y$ which meets $Y_B[2]$.*

PROOF. Let V be the union of lines contained in Y.

Step 1: $\dim V \geq 2$. Let $R \subset \mathrm{Gr}(2, V)$ be the locus of lines $\ell \subset Y_A$. We can obtain R as follows. Let

$$Y_A = V(g),$$

where g is a degree 6 polynomial, and let $\mathcal{S}$ be the tautological subbundle on $\mathrm{Gr}(2, V)$, so that $\mathrm{Sym}^6(\mathcal{S}^\vee)$ is the fiber bundle whose fiber at ℓ is the vector space of homogeneous polynomials of degree 6 on ℓ.
Then we can define a section

$$s \in H^0\big(\mathrm{Gr}(2, V), \mathrm{Sym}^6(\mathcal{S}^\vee)\big)$$

by the condition

$$s(\ell) = g\big|_\ell.$$

By definition R is the zero locus of s. It follows that

$$\dim R \geq \dim \mathrm{Gr}(2, V) - \mathrm{rk}\,\mathrm{Sym}^6(\mathcal{S}^\vee) = 8 - 7 = 1,$$

provided R is not empty. But we can show that $R \neq \emptyset$ by computing the fundamental class

$$[R] = c_7\big(\mathrm{Sym}^6(\mathcal{S}^\vee)\big) = 432 \cdot 134\sigma_{4,3}.$$

Here the notation is that of Schubert calculus, see for instance [GH78, Sec. 1.5]. Since

$$V = \bigcup_{\ell \in R} \ell$$

is birational to a $\mathbb{P}^1$ bundle over R, it follows that $\dim V \geq 2$.

Step 2: There exists B' such that $A \cap B = A \cap B'$ and $Y_{B'}[2]$ meets V. Let

$$U = A \cap B$$

and let D_U be its associated divisor on Y_A, under the isomorphism of Proposition 4.8. Then D_U has dimension 3; since two varieties of dimension 2 and 3 in $\mathbb{P}^5$ always meet, it follows that

$$D_U \cap V \neq \emptyset.$$

So there exists a Lagrangian subspace B' such that $B' \cap A = U$ and

$$Y_{B'}[2] \cap V \neq \emptyset.$$

Step 3: B meets V. We lift everything to X, which is smooth, so intersection theory applies. Let

$$\widetilde{V_1} = f^{-1}(V) \text{ and } \widetilde{V_2} = f^{-1}(Y_{B'}[2]).$$

One easily sees that on X

$$\widetilde{V_1} \cdot \widetilde{V_2} \neq 0.$$

Since $f^{-1}(Y_B[2])$ and $\widetilde{V_2}$ have the same homology class, it follows that

$$\widetilde{V_1} \cdot f^{-1}(Y_B[2]) \neq 0,$$

in particular $\widetilde{V_1}$ must meet $f^{-1}(Y_B[2])$, and so

$$V \cap Y_B[2] \neq \emptyset.$$

The other relations come from the following

LEMMA 4.18. *Let*

$$Z = Z_A = f^{-1}(Y_A[2])$$

be the fixed locus of the involution φ on X. Then the following relation holds in $CH(X)$:

$$3Z_A = 15h^2 - c_2(X).$$

PROOF. We consider f as a map $X \to \mathbb{P}^5$, so that it induces a morphism of vector bundles over X

$$df \colon T_X \to f^*T_{\mathbb{P}^5}.$$

We notice that df in injective outside Z, so we can see Z as a degeneracy locus for this morphism. We then apply Thom-Porteous formula in the precise form stated in [Ful84, sec. 14.4]. In their notation we have $e = 4$, $f = 5$ and $k = 3$.

This yields a cycle class

$$\mathbb{D}_3(df) \in CH^2(Z)$$

whose support is Z, and such that the image of $\mathbb{D}_3(df)$ in $CH^2(X)$ is

$$\Delta_2^{(1)}(c(f^*T_{\mathbb{P}^5} - T_X)) = c_2(f^*T_{\mathbb{P}^5} - T_X).$$

Here the total Chern class

$$c(f^*T_{\mathbb{P}^5} - T_X)$$

is defined formally in such a way that Whitney's formula holds, i. e.

$$c(T_X) \cdot c(f^*T_{\mathbb{P}^5} - T_X) = c(f^*T_{\mathbb{P}^5}).$$

From the last equation and the fact that $c_1(T_X) = 0$ (since X is symplectic) we can obtain

$$c_2(f^*T_{\mathbb{P}^5} - T_X) = f^*c_2(T_{\mathbb{P}^5}) - c_2(T_X) = 15h^2 - c_2(X).$$

Here

$$c_2(T_{\mathbb{P}^5}) = 15c_1\big(\mathcal{O}_{\mathbb{P}^5}(1)\big)^2$$

can easily be derived from the usual Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5} \longrightarrow \mathbb{C}^6 \otimes \mathcal{O}_{\mathbb{P}^5}(1) \longrightarrow T_{\mathbb{P}^5} \longrightarrow 0 \ .$$

Since $\mathbb{D}_3(df)$ has support on Z, which is irreducible, we find that

$$kZ = 15h^2 - c_2(X) \tag{4.11}$$

for some $k \in \mathbb{Z}$. To find the right k, we observe that again by [Ful84, Theorem 14.4(c)] we have

$$\mathbb{D}_3(df) = [D_3(df)],$$

where $D_3(df)$ is the degeneracy locus of df. In other words $D_3(df)$ is just Z, with the scheme structure given by the vanishing of all 4×4 minors of df.

By the local study in section 2.2.6 we see that the map

$$f \colon X \to Y \subset \mathbb{P}^5$$

has, in suitable analytic coordinates around a point of Z, the local form

$$f(x, y, z, t) \underset{loc}{=} (x^2, xy, y^2, z, t).$$

The differential of f is then

$$df \underset{loc}{=} \begin{pmatrix} 2x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

equating to 0 the determinants of its 3×3 minors yields

$$D_3(df) \underset{loc}{=} V(x^2, xy, y^2).$$

So we see that $D_3(df)$ has multiplicity 3 at each point of Z, hence $k = 3$.

Alternatively we could multiply Equation (4.11) by h^2 to find

$$kZ \cdot h^2 = 15h^4 - c_2(X) \cdot h^2.$$

If we look at this relation in cohomology it becomes, thanks to Proposition 4.1,

$$40k = 15 \cdot 12 - 60,$$

so $k = 3$.

We have a closer look at the differential of

$$f \colon X \to \mathbb{P}^5.$$

As a map of vector bundles, this is not injective exactly on Z. Hence it is always injective on stalks; in other words

$$df \colon T_X \to f^*T_{\mathbb{P}^5}$$

is an injective map *of sheaves*. Let $\mathcal{R}$ denote its cokernel; this is locally free of rank 1 outside Z. So we have the exact sequence

$$0 \longrightarrow T_X \longrightarrow f^*T_{\mathbb{P}^5} \longrightarrow \mathcal{R} \longrightarrow 0 \ . \tag{4.12}$$

We now dualize it applying $\mathcal{H}om(\cdot, \mathcal{O}_X)$. We remark that

$$\mathcal{H}om(\mathcal{R}, \mathcal{O}_X)$$

is torsion-free, of rank one, and one can check in local coordinates that it is a line bundle. By (4.12) we get $c_1(\mathcal{R}) = 6h$, hence

$$\mathcal{H}om(\mathcal{R}, \mathcal{O}_X) \cong \mathcal{O}_X(-6).$$

Then we note that

$$\mathcal{E}xt^1(f^*(T_{\mathbb{P}^5}), \mathcal{O}_X) = 0,$$

because both sheaves are locally free. So if we let

$$\mathcal{Q} = \mathcal{E}xt^1(\mathcal{R}, \mathcal{O}_X),$$

the dual of (4.12) becomes

$$0 \longrightarrow \mathcal{O}_X(-6) \longrightarrow f^*(\Omega^1_{\mathbb{P}^5}) \xrightarrow{df^T} \Omega^1_X \longrightarrow \mathcal{Q} \longrightarrow 0. \qquad (4.13)$$

We remark that $\mathcal{Q}$ is set-theoretically supported on Z, because both $\mathcal{R}$ and $\mathcal{O}_X$ are locally free outside Z. Actually the schematic support of $\mathcal{Q}$ is $2Z$, that is the subscheme of X defined by the ideal $\mathcal{I}^2_Z$. This follows from the

LEMMA 4.19. *Let $\mathcal{Q}$ be as above; then* $\mathrm{Ann}(\mathcal{Q}) = \mathcal{I}^2_Z$.

PROOF. We only need to prove this locally. As in the proof of Lemma 4.18 we can choose local coordinates on X such that

$$f(x, y, z, t) \underset{loc}{=} (x^2, xy, y^2, z, t);$$

then df^T has the matrix

$$df^T \underset{loc}{=} \begin{pmatrix} 2x & y & 0 & 0 & 0 \\ 0 & x & 2y & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

hence we have the presentation

$$\mathcal{Q} \underset{loc}{=} \frac{\langle dx, dy \rangle}{\langle xdx, xdy + ydx, ydy \rangle}.$$

A given $h(x, y) \in \mathbb{C}[x, y]$ then annihilates $\mathcal{Q}$ if and only if both hdx and hdy belong to the $k[x, y]$-module generated by xdx, $xdy + ydx$ and ydy.

Let us make this more explicit. Assume that

$$h(x, y)dx = a(x, y)xdx + b(x, y) \cdot (xdy + ydx) + c(x, y)ydy.$$

This yields

$$h(x, y) = xa(x, y) + yb(x, y)$$
$$0 = xb(x, y) + yc(x, y)$$

The second equation implies $b(x, y) = yb'(x, y)$, so the first becomes

$$h(x, y) = xa(x, y) + y^2 b'(x, y).$$

If h can be written this way, then we can choose c so that the second condition is satisfied. In short

$$h(x, y)dx \in \langle xdx, xdy + ydx, ydy \rangle_{k[x,y]}$$

if and only if $h \in (x, y^2)$.

We have the symmetric condition for $h(x, y)dy$, so we conclude that $h \in \mathrm{Ann}(\mathcal{Q})$ if and only if

$$h \in (x, y^2) \cap (x^2, y) = (x^2, xy, y^2).$$

The last equality between ideals can be proved for instance by the remark that both $(x, y^2) \cap (x^2, y)$ and (x^2, xy, y^2) consist of the polynomials h such that

$$h(0,0) = \frac{dh}{dx}(0,0) = \frac{dh}{dy}(0,0) = 0.$$

Finally (x^2, xy, y^2) is exactly the square of the ideal (x, y) which locally defines Z.

We now produce another exact sequence involving $\mathcal{Q}$. Let

$$i \colon Z \hookrightarrow X$$

denote the inclusion. Recall that we have a canonical identification

$$\mathcal{I}_Z / \mathcal{I}_Z^2 \cong i_* \mathcal{N}_{Z/X}^\vee : \tag{4.14}$$

locally the function g vanishing on Z corresponds to the normal covector dg. Consider the natural projection

$$\pi \colon \Omega_X^1 \big|_Z \to \mathcal{N}_{Z/X}^\vee;$$

we see this as a map on X

$$\pi \colon \Omega_X^1 \to \mathcal{I}_Z / \mathcal{I}_Z^2.$$

LEMMA 4.20. *We have $\pi \circ df^T = 0$.*

PROOF. We keep the notation of the proof of Lemma 4.19. We need only to verify the thesis on Z. The image of df^T is generated by

$$xdx, \ xdy + ydx, \ ydy, \ dz, \ dt.$$

The first three elements vanish on Z, while the latter two are in the kernel of π.

The above lemma and the exact sequence in (4.13) provide us a surjective map

$$\alpha \colon \mathcal{Q} \to i_*(\mathcal{N}_{Z/X}^\vee).$$

LEMMA 4.21. *The kernel of α is $i_*(\det T_Z)$.*

PROOF. We can see this explicitly in local coordinates. Keeping the notation of the above proofs, $\mathcal{Q}$ is locally generated, on Z, by dx, dy and $xdy = -ydx$. The conormal bundle $\mathcal{N}_{Z/X}^{\vee}$ is generated by dx and dy, and α is the obvious projection.

The kernel of α is then generated by xdy. Under the identification in (4.14) this corresponds to the generator $dx \wedge dy$ of $\bigwedge^2 \mathcal{N}_{Z/X}^{\vee}$.

So

$$\ker \alpha = i_*(\det \mathcal{N}_{Z/X}^{\vee}) \cong i_*(\det T_Z),$$

since Z is Lagrangian.

Thanks to the lemma we get the exact sequence we are looking for:

$$0 \longrightarrow i_*(\det T_Z) \longrightarrow \mathcal{Q} \longrightarrow i_* T_Z \longrightarrow 0 \ . \tag{4.15}$$

We can now find new relations in the Chow ring of X.

PROPOSITION 4.22. In $CH(X)_{\mathbb{Q}}$ we have

$$c_2(X) \cdot h = 5h^3$$

and $c_4(X)$ is a linear combination of h^4, $c_2(X) \cdot h^2$ and $c_2(X)^2$.

PROOF. This is just a matter of putting together the relations that come from the exact sequences (4.13) and (4.15).

We start from (4.13), which yields

$$(1 - 6h) \cdot (1 + c_2(X) + c_4(X)) = (1 - h)^6 \cdot (1 + c_1(\mathcal{Q}) + c_2(\mathcal{Q}) + c_3(\mathcal{Q}) + c_4(\mathcal{Q})).$$

Comparing the terms in degree up to 2 we get:

$$\begin{aligned} c_1(\mathcal{Q}) &= 0 \\ c_2(\mathcal{Q}) &= c_2(X) - 15h^2 = -3Z, \end{aligned} \tag{4.16}$$

where the last equality is Lemma 4.18. Then in degree 3 we have

$$\begin{aligned} c_3(\mathcal{Q}) &= 6h(c_2(\mathcal{Q}) - c_2(X)) + 20h^3 = \\ &= 6h \cdot (-15h^2) + 20h^3 = -70h^3, \end{aligned} \tag{4.17}$$

using the second of (4.16). Finally in degree 4 we get, using (4.16) and (4.17),

$$\begin{aligned} c_4(X) &= 15h^4 + 15h^2 \cdot c_2(\mathcal{Q}) - 6h \cdot c_3(\mathcal{Q}) + c_4(\mathcal{Q}) = \\ &= 15h^4 - 45h^2 \cdot Z + 420h^4 + c_4(\mathcal{Q}), \end{aligned}$$

hence

$$c_4(\mathcal{Q}) = c_4(X) - 435h^4 + 45h^2 \cdot Z. \tag{4.18}$$

Next we look at the relations coming from (4.15). To do this we shall use Grothendieck-Riemann-Roch, which for the closed embedding

$$i \colon Z \hookrightarrow X$$

takes the form

$$ch(i_*\mathcal{F}) = i_*(ch(\mathcal{F}) \cdot td(\mathcal{N}_{Z/X})^{-1}),$$

for any $\mathcal{F} \in \mathrm{Coh}(Z)$. This is because in our situation we have

$$R^k i_*(\mathcal{F}) = 0$$

for all such $\mathcal{F}$, thanks to [Har77, Cor. III.11.2].

Using that Z is Lagrangian we have $\mathcal{N}_{Z/X} \cong \Omega^1_Z$, so we can compute

$$td(\mathcal{N}_{Z/X}) = 1 - \frac{1}{2}c_1(Z) + \frac{1}{12}(c_1(Z)^2 + c_2(Z));$$
$$td(\mathcal{N}_{Z/X})^{-1} = 1 + \frac{1}{2}c_1(Z) + \frac{1}{6}c_1(Z)^2 - \frac{1}{12}c_2(Z).$$

Then we have

$$ch(\det T_Z) = 1 + c_1(Z) + \frac{1}{2}c_1(Z)^2;$$
$$ch(T_Z) = 2 + c_1(Z) + \frac{1}{2}(c_1(Z)^2 - c_2(Z)).$$

So Grothendieck-Riemann-Roch for these sheaves becomes

$$ch(i_* \det T_Z) = i_*\left(1 + \frac{3}{2}c_1(Z) + \frac{7}{6}c_1(Z)^2 - \frac{1}{12}c_2(Z)\right);$$
$$ch(i_* T_Z) = i_*\left(2 + 2c_1(Z) + \frac{4}{3}c_1(Z)^2 - \frac{7}{6}c_2(Z)\right).$$

Next we use the fact that in $CH(Z)_{\mathbb{Q}}$ we have

$$c_1(Z) = -K_Z = -3i^*(h),$$

thanks to Proposition 2.8. So we obtain

$$ch(i_* \det T_Z) = Z - \frac{9}{2}h \cdot Z + \frac{21}{2}h^2 \cdot Z - \frac{1}{12}Z^2;$$
$$ch(i_* T_Z) = 2Z - 6h \cdot Z + 12h^2 \cdot Z - \frac{7}{6}Z^2.$$

We can use this to recover the Chern classes of $i_*(\det T_Z)$ and $i_*(T_Z)$. These are:

$$c_1(i_* \det T_Z) = 0$$
$$c_2(i_* \det T_Z) = -Z$$
$$c_3(i_* \det T_Z) = -9h \cdot Z$$
$$c_4(i_* \det T_Z) = Z^2 - 63h^2 \cdot Z$$

and

$$c_1(i_* T_Z) = 0$$
$$c_2(i_* T_Z) = -2Z$$
$$c_3(i_* T_Z) = -12h \cdot Z$$
$$c_4(i_* T_Z) = 9Z^2 - 72h^2 \cdot Z.$$

Finally we use the exact sequence (4.15) to get the Chern classes of $\mathcal{Q}$. The first two are

$$c_1(\mathcal{Q}) = 0$$
$$c_2(\mathcal{Q}) = -3Z,$$

in accordance with (4.16). Then we get

$$c_3(\mathcal{Q}) = -21h \cdot Z,$$

and comparing with (4.17) we obtain

$$-3h \cdot Z = -10h^3.$$

Using Lemma 4.18 this is exactly

$$c_2(X) \cdot h = 5h^3.$$

Finally we get

$$c_4(\mathcal{Q}) = 12Z^2 - 135h^2 \cdot Z;$$

comparing with (4.18) this becomes

$$12Z^2 - 135h^2 \cdot Z = c_4(X) - 435h^4 + 45h^2 \cdot Z,$$

and using again Lemma 4.18 to write Z as a rational combination of $c_2(X)$ and h^2, we get the second claim of the thesis.

4.3.4 – Conclusion of the proof

First we recall that we have defined the class

$$\theta = \frac{1}{2} f^*(\overline{\theta}).$$

Here $\overline{\theta}$ is the class of any point on $Y_B[2] \subset Y_A$.

We also recall from the previous section that on Y_A we have a divisor $D = D_U$, where $U = A \cap B$, together with a rational map

$$r_D \colon D \dashrightarrow \ell_D,$$

which exhibits a linear equivalence on D between $Y_A[2]$ and $Y_B[2]$. In other words

$$\big[Y_A[2]\big] = \big[Y_B[2]\big] \text{ in } CH^2(Y_A).$$

Using Lemma 4.17 we can start proving that

$$h^4 = 6\theta \tag{4.19}$$

in $CH(X)$.

Indeed let Λ be any plane containing L_0. Then $\overline{h}^3$ is represented by the intersection

$$\Lambda \cdot Y = L_0 + C,$$

where C is a quintic on Λ. Multiplying by $\overline{h}$ we obtain

$$\overline{h}^4 = L_0 \cdot h + C \cdot h.$$

We claim that this is represented by a 0-cycle supported on L_0. This is clear for the first addend; for the second we represent h by a hyperplane containing L_0 and transverse to Λ. It follows that $C \cdot h$ is supported on $C \cap L_0$.

Since L_0 is rational, $CH^1(L_0) \cong \mathbb{Z}$, so $\overline{h}^4$ is rationally equivalent to a multiple of a point of L_0. Finally Lemma 4.17 assures us that

$$L_0 \cap Y_B[2] \neq \emptyset,$$

so we get

$$\overline{h}^4 = k\overline{\theta} \text{ in } CH^4(Y)_{\mathbb{Q}}$$

for some $k \in \mathbb{Q}$.

Pulling back this relation to X and using $f^*(\overline{h}) = h$, $f^*(\overline{\theta}) = 2\theta$ we obtain

$$h^4 = 2k\theta \text{ in } CH^4(X)_{\mathbb{Q}}.$$

Since in cohomology we have $h^4 = 12$ we must have $k = 6$, and so (4.19) is proved.

Next we show that

$$h^2 \cdot c_2(X) = 60\overline{\theta}. \tag{4.20}$$

We start from Lemma 4.18; pushing forward that relation we get

$$3\big[Y_A[2]\big] = 15 \cdot 4\overline{h}^2 - f_* c_2(X) \text{ in } CH^2(Y). \tag{4.21}$$

Multiplying (4.21) by $\overline{h}^2$ we get

$$\overline{h}^2 \cdot f_* c_2(X) = 60\overline{h}^4 - 3\overline{h}^2 \cdot \big[Y_A[2]\big].$$

We already proved that $\overline{h}^4$ is a multiple of $\overline{\theta}$, and the cycle class

$$\overline{h}^2 \cdot \big[Y_A[2]\big] = \overline{h}^2 \cdot \big[Y_B[2]\big]$$

is supported on $Y_B[2]$, hence it is a rational multiple of $\overline{\theta}$ too.

We conclude that the relation (4.20) holds up to a multiple, that is

$$\overline{h}^2 \cdot f_* c_2(X) = k\overline{\theta}.$$

As before, we pull back this relation to X in order to make computations in cohomology. We get

$$h^2 \cdot 2c_2(X) = 2k\overline{\theta}.$$

Since in cohomology we have

$$h^2 \cdot c_2(X) = 60,$$

we must have $k = 60$, and Equation (4.20) is proved.

In a similar way, we can rewrite Equation (4.21) as

$$f_* c_2(X) = 15\overline{h}^2 - 3\big[Y_A[2]\big]$$

and take squares to write $(f_* c_2(X))^2$ as a combination of $\overline{h}^4$ and a 0-cycle supported on $Y_B[2]$. This shows that $(f_* c_2(X))^2$ is a rational multiple of $\overline{\theta}$.

As usual a cohomology computation yields the precise form

$$c_2(X)^2 = 828\theta.$$

Now we can use Proposition 4.22 to conclude that

$$c_4(X) = k\theta,$$

and finally we get $k = 324$ by comparison with the analogous computation in cohomology. This takes care of all relations in degree 8.

The only relation in degree 6 comes from Proposition 4.2, and is

$$c_2(X) \cdot h = 5h^3.$$

We already proved that the same holds in $CH^*(X)$ in Proposition 4.22. This ends the proof of the main theorem.

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