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Approximation of classes of analytic functions by de la Vallée Poussin sums in uniform metric

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Abstract: *In this paper asymptotic equalities are found for the least upper bounds of deviations in the uniform metric of de la Vallée Poussin sums on classes of 2π-periodic (ψ,β)-differentiable functions admitting an analytic continuation into the given strip of the complex plane. As a consequence, asymptotic equalities are obtained on classes of convolutions of periodic functions generated by the Neumann kernel and the polyharmonic Poisson kernel.*

Let L_s, $1 \leqslant s < \infty$, be the space of sth power summable 2π-periodic functions f with the norm $\|f\|_s := \|f\|_{L_s} = \left(\int_0^{2\pi} |f(t)|^s \, dt \right)^{1/s}$, let L_∞ be the space of measurable essentially bounded 2π-periodic functions f with the norm $\|f\|_\infty := \|f\|_{L_\infty} = \operatorname{ess\,sup}_t \, |f(t)|$ and let C be the space of continuous 2π-periodic functions f with the norm $\|f\|_C = \max_t |f(t)|$.

Suppose that $f \in L_1$ and

$$S[f] := \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx)$$

is the Fourier series of f. If a sequence of real numbers $\psi(k)$, $k \in \mathbb{N}$ and a real number β ($\beta \in \mathbb{R}$) are such that there exists a function $\varphi \in L_1$ with Fourier series

$$S[\varphi] = \sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k(f) \cos\left(kx + \frac{\beta\pi}{2}\right) + b_k(f) \sin\left(kx + \frac{\beta\pi}{2}\right)\right),$$

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then this function φ is called (see [13, p. 120]) the (ψ,β)-derivative of the function $f(\cdot)$ and is denoted by $f^{\psi}_{\beta}(\cdot)$. If $f^{\psi}_{\beta}\in\mathfrak{N}\subset L_1$, then we write $f\in L^{\psi}_{\beta}\mathfrak{N}$. Moreover, we set $C^{\psi}_{\beta}\mathfrak{N}=C\cap L^{\psi}_{\beta}\mathfrak{N}$.

By D_q we denote the set of sequences $\psi(k)>0$, $k\in\mathbb{N}$, such that

$$\lim_{k\to\infty}\frac{\psi(k+1)}{\psi(k)}=q,\quad q\in(0,1). \tag{1}$$

It is known [13, p. 130] that the class $C^{\psi}_{\beta}\mathfrak{N}$ with $\psi\in D_q$ consists of 2π-periodic functions that admit an analytic continuation into the strip $|\mathrm{Im}\, z|\leqslant \ln 1/q$ of the complex plane.

As follows from proposition 8.3 [13, p. 127], if $\psi\in D_q$, $q\in(0,1)$, $\beta\in\mathbb{R}$ and $\mathfrak{N}\subset L_s$, $1\leqslant s\leqslant\infty$, then $C^{\psi}_{\beta}\mathfrak{N}$ is the class of functions $f(x)$ representable at each point $x\in\mathbb{R}$ by the equality

$$f(x)=\frac{a_0(f)}{2}+\frac{1}{\pi}\int_{-\pi}^{\pi} f^{\psi}_{\beta}(x-t)\Psi_{\beta}(t)\,dt, \tag{2}$$

where

$$\Psi_{\beta}(t)=\sum_{k=1}^{\infty}\psi(k)\cos\Big(kt-\frac{\beta\pi}{2}\Big). \tag{3}$$

An example of the class $C^{\psi}_{\beta}\mathfrak{N}$ for which $\psi\in D_q$, $q\in(0,1)$, is the class of Poisson integrals, i.e. a class consisting of functions of the form

$$f(x)=A_0+\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x-t)P_{q,\beta}(t)\,dt,\quad A_0\in\mathbb{R},\quad \varphi\in\mathfrak{N}, \tag{4}$$

where

$$P_{q,\beta}(t)=\sum_{k=1}^{\infty}q^k\cos\Big(kt-\frac{\beta\pi}{2}\Big),\quad q\in(0,1),\quad \beta\in\mathbb{R},$$

is the Poisson kernel with parameters q and β. In this case the class $C^{\psi}_{\beta}\mathfrak{N}$ we will denote by $C^{q}_{\beta}\mathfrak{N}$.

In the current paper we take as $\mathfrak{N}$ the sets

$$U^0_s=\{\varphi\in L_s:\ \ \|\varphi\|_s\leqslant 1,\ \ \varphi\perp 1\},\quad 1\leqslant s\leqslant\infty,$$

and

$$H_{\omega}=\{\varphi\in C:\ \ \omega(\varphi;t)\leqslant\omega(t),\ t\geqslant 0\},$$

where $\omega(\varphi;t)$ is the modulus of continuity of φ and $\omega(t)$ is a fixed majorant of the modulus of continuity type. In what follows, we use the notation:

$$C^{\psi}_{\beta,s}=C^{\psi}_{\beta}U^0_s,\quad C^{q}_{\beta,s}=C^{q}_{\beta}U^0_s.$$

Denote by $V_{n,p}(f;\cdot)$ the de la Vallée Poussin sums [16] of the function $f \in L_1$:

$$V_{n,p}(f;x) = \frac{1}{p}\sum_{k=n-p}^{n-1} S_k(f;x), \qquad (5)$$

where $S_k(f;x)$ is the kth partial sum of the Fourier series of f, and $p = p(n)$ is a given natural parameter, $p \leqslant n$.

The aim of the present work is to obtain the asymptotic equalities as $n - p \to \infty$ for the quantity

$$\mathcal{E}(C^{\psi}_{\beta}\mathfrak{N}; V_{n,p}) = \sup_{f\in C^{\psi}_{\beta}\mathfrak{N}} \|f(\cdot) - V_{n,p}(f;\cdot)\|_C, \qquad (6)$$

where $\psi \in D_q$, $q \in (0,1)$, and $\mathfrak{N} = U^0_s$, $1 \leqslant s \leqslant \infty$, or $\mathfrak{N} = H_\omega$.

This paper is nearly related to works [2, 3, 5, 6, 7, 9, 10] and [14]. In [10] the asymptotic equalities were obtained for $\mathcal{E}(C^{\psi}_{\beta,s}; V_{n,p})$, $1 \leqslant s \leqslant \infty$ and $\mathcal{E}(C^{\psi}_{\beta}H_\omega; V_{n,p})$ in the case where the sequence $\psi(k)$, that defines the classes, satisfies the condition

$$\lim_{k\to\infty} \frac{\psi(k+1)}{\psi(k)} = 0 \quad (\psi \in D_0).$$

This restriction on ψ implies $C^{\psi}_{\beta,s}$ and $C^{\psi}_{\beta}H_\omega$ are the classes of entire functions. The case $\psi \in D_q$, $q \in (0,1)$ also hasn't been omitted. The solution of the problem under consideration for $\psi \in D_q$, $q \in (0,1)$, and $p = 1$ $(V_{n,1}(f;\cdot) = S_{n-1}(f;\cdot))$ was found in [6] and [14]. The main idea of paper [14] (see, also, [13, Chapt. 5, Sect. 20]) consists of reduction of the problem of obtaining asymptotic equalities for $\mathcal{E}(C^{\psi}_{\beta}\mathfrak{N}; S_{n-1})$ to solving an analogous problem for the quantity $\mathcal{E}(C^{q}_{\beta}\mathfrak{N}; S_{n-1})$ by means of the next equalities:

$$\mathcal{E}(C^{\psi}_{\beta,s}; S_{n-1}) = \psi(n)\left(q^{-n}\mathcal{E}(C^{q}_{\beta,s}; S_{n-1}) + O(1)\frac{\varepsilon_n}{(1-q)^2}\right), \quad 1 \leqslant s \leqslant \infty, \qquad (7)$$

$$\mathcal{E}(C^{\psi}_{\beta}H_\omega; S_{n-1}) = \psi(n)\left(q^{-n}\mathcal{E}(C^{q}_{\beta}H_\omega; S_{n-1}) + O(1)\frac{\varepsilon_n\omega(1/n)}{(1-q)^2}\right), \qquad (8)$$

where $\varepsilon_n := \sup_{k\geqslant n}\left|\frac{\psi(k+1)}{\psi(k)} - q\right|$, and $O(1)$ are the quantities uniformly bounded in n, s, q, $\psi(k)$ and β. Since the asymptotic equalities for $\mathcal{E}(C^{q}_{\beta,s}; S_{n-1})$ and $\mathcal{E}(C^{q}_{\beta}H_\omega; S_{n-1})$ are known (see, for example, [13, p. 295, 310], [6, p. 1278]), formulas (7) and (8) allow us to obtain the asymptotic equalities for $\mathcal{E}(C^{\psi}_{\beta,s}; S_{n-1})$ and $\mathcal{E}(C^{\psi}_{\beta}H_\omega; S_{n-1})$, respectively, with arbitrary $\beta \in \mathbb{R}$ and $\psi \in D_q$, $q \in (0,1)$.

As for the general case $p = 1, 2, \dots, n$, the analogs of (7) (with $s = \infty$) and (8) were derived in [2] and have the form

$$\mathcal{E}(C^{\psi}_{\beta,\infty}; V_{n,p}) = \psi(n-p+1)\Big(\frac{\mathcal{E}(C^{q}_{\beta,\infty}; V_{n,p})}{q^{n-p+1}} + O(1)\frac{\varepsilon_{n-p+1}}{(1-q)^4}\Big), \tag{9}$$

$$\mathcal{E}(C^{\psi}_{\beta}H_\omega; V_{n,p}) = \psi(n-p+1)\Big(\frac{\mathcal{E}(C^{q}_{\beta}H_\omega; V_{n,p})}{q^{n-p+1}} + O(1)\omega\Big(\frac{1}{n-p+1}\Big)\frac{\varepsilon_{n-p+1}}{(1-q)^4}\Big), \tag{10}$$

where $\psi \in D_q$, $q \in (0,1)$, $\beta \in \mathbb{R}$,

$$\varepsilon_{n-p+1} := \sup_{k \geqslant n-p+1}\Big|\frac{\psi(k+1)}{\psi(k)} - q\Big|, \tag{11}$$

$\omega(t)$ is an arbitrary modulus of continuity and $O(1)$ are the quantities uniformly bounded in n, p, q, ψ and β.

By using the known asymptotic equalities as $n - p \to \infty$ of the quantities $\mathcal{E}(C^{q}_{\beta,\infty}; V_{n,p})$ and $\mathcal{E}(C^{q}_{\beta}H_\omega; V_{n,p})$ (see [2] and [3]), V. I. Rukasov obtained from (9) and (10) the next formulas that make up the main result of paper [2]:

$$\begin{aligned}\mathcal{E}(C^{\psi}_{\beta,\infty}; V_{n,p}) = \frac{\psi(n-p+1)}{p}\Big(&\frac{4}{\pi(1-q^2)}\\ &+O(1)\Big(\frac{q^{p-1}}{(1-q^2)} + \frac{1}{(1-q)^3(n-p)} + \frac{p\varepsilon_{n-p}}{(1-q)^4}\Big)\Big),\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{E}(C^{\psi}_{\beta}H_\omega; V_{n,p}) = \frac{\psi(n-p+1)}{p}\Big(&\frac{2\theta_\omega}{\pi(1-q^2)}\int_0^{\pi/2}\omega\Big(\frac{2t}{n-p}\Big)\sin t\,dt\\ &+O(1)\omega\Big(\frac{1}{n-p}\Big)\Big(\frac{q^{p-1}}{(1-q^2)} + \frac{1}{(1-q)^3(n-p)} + \frac{p\varepsilon_{n-p}}{(1-q)^4}\Big)\Big),\end{aligned} \tag{13}$$

where $\psi \in D_q$, $q \in (0,1)$, $\beta \in \mathbb{R}$, $\varepsilon_{n-p} = \sup_{k\geqslant n-p}\big|\frac{\psi(k+1)}{\psi(k)} - q\big|$, $\theta_\omega \in [1/2, 1]$ ($\theta_\omega = 1$ if $\omega(t)$ is a convex (upwards) modulus of continuity) and the quantities $O(1)$ are uniformly bounded in n, p, q, ψ and β.

Formula (12), as well as formula (13) in the case of convexity of modulus of continuity $\omega(t)$, is an asymptotic equality as $n - p \to \infty$ only if the additional conditions

$$\lim_{n\to\infty} p = \infty, \tag{14}$$

$$\lim_{n\to\infty} p\varepsilon_{n-p} = 0, \tag{15}$$

are fulfilled.

In the present work we have been able to do away with restrictions (14) and (15); this means that the strong asymptotic as $n-p\to\infty$ of $\mathcal{E}(C^{\psi}_{\beta,s};V_{n,p})$ and $\mathcal{E}(C^{\psi}_{\beta}H_{\omega};V_{n,p})$ is obtained for arbitrary $\psi\in D_q$, $q\in(0,1)$, $1\leqslant s\leqslant\infty$, $\beta\in\mathbb{R}$ even in the case where at least one of (14) or (15) isn't carried out. It's essential to note that reasoning from relations (9) and (10), restrictions (14) and (15) can't be removed in principle. Thus, for the final solution of our problem, it needs to improve formulas (9) and (10), which we shall do finding more refined estimates of the remainder terms with subsequent generalization of (9) to the case of arbitrary $s\in[1,\infty]$. The sought-for relations are provided by the following assertion, which plays a key role in this paper.

Theorem 1. *Let $\psi\in D_q$, $q\in(0,1)$, $1\leqslant s\leqslant\infty$, $n,p\in\mathbb{N}$, $p\leqslant n$, $\beta\in\mathbb{R}$ and let $\omega(t)$ be an arbitrary modulus of continuity. Then, as $n-p\to\infty$,*

$$\mathcal{E}(C^{\psi}_{\beta,s};V_{n,p})=\frac{\psi(n-p+1)}{p}\left(\frac{p\mathcal{E}(C^{q}_{\beta,s};V_{n,p})}{q^{n-p+1}}+O(1)\frac{\varepsilon_{n-p+1}}{(1-q)^2}\min\left\{p,\frac{1}{1-q}\right\}\right),\quad(16)$$

$$\begin{aligned}\mathcal{E}(C^{\psi}_{\beta}H_{\omega};V_{n,p})=\frac{\psi(n-p+1)}{p}\Bigg(&\frac{p\mathcal{E}(C^{q}_{\beta}H_{\omega};V_{n,p})}{q^{n-p+1}}\\&+O(1)\omega\Big(\frac{1}{n-p+1}\Big)\frac{\varepsilon_{n-p+1}}{(1-q)^2}\min\left\{p,\frac{1}{1-q}\right\}\Bigg),\end{aligned}\quad(17)$$

where ε_{n-p+1} is defined by (11) and $O(1)$ are the quantities uniformly bounded in n, p, q, s, ψ, β and ω.

Proof. Let $f\in C^{\psi}_{\beta}\mathfrak{N}$, $\psi\in D_q$, $q\in(0,1)$ and $\mathfrak{N}=U^0_s$, $1\leqslant s\leqslant\infty$, or $\mathfrak{N}=H_{\omega}$. By (2) and (5), the deviation

$$\rho_{n,p}(f;x):=f(x)-V_{n,p}(f;x),$$

satisfies at each point $x\in\mathbb{R}$ the equality

$$\rho_{n,p}(f;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^{\psi}_{\beta}(x-t)\sum_{k=n-p+1}^{\infty}\tau_{n,p}(k)\psi(k)\cos\Big(kt-\frac{\beta\pi}{2}\Big)\,dt,\quad f^{\psi}_{\beta}\in\mathfrak{N},\quad(18)$$

where

$$\tau_{n,p}(k)=\begin{cases}1-\dfrac{n-k}{p}, & n-p+1\leqslant k\leqslant n-1,\\ 1, & k\geqslant n.\end{cases}\quad(19)$$

Setting

$$r_{n,p}(t):=\sum_{k=n-p+2}^{\infty}\tau_{n,p}(k)\left(\frac{\psi(k)}{\psi(n-p+1)}-\frac{q^k}{q^{n-p+1}}\right)\cos\Big(kt-\frac{\beta\pi}{2}\Big),\quad(20)$$

we rewrite (18) thus:

$$\rho_{n,p}(f;x)=\psi(n-p+1)\bigg(\frac{q^{p-n-1}}{\pi}\int_{-\pi}^{\pi} f_\beta^\psi(x-t)\sum_{k=n-p+1}^{\infty}\tau_{n,p}(k)q^k\cos\Big(kt-\frac{\beta\pi}{2}\Big)dt \\ +\frac{1}{\pi}\int_{-\pi}^{\pi} f_\beta^\psi(x-t)r_{n,p}(t)\,dt\bigg). \tag{21}$$

Since, by virtue of (18),

$$\mathcal{E}(C_\beta^q\mathfrak{N};V_{n,p}) := \sup_{f\in C_\beta^q\mathfrak{N}}\|\rho_{n,p}(f;\cdot)\|_C \\ = \sup_{\varphi\in\mathfrak{N}}\bigg\|\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\cdot-t)\sum_{k=n-p+1}^{\infty}\tau_{n,p}(k)q^k\cos\Big(kt-\frac{\beta\pi}{2}\Big)\,dt\bigg\|_C, \quad \mathfrak{N}\subset L_1, \tag{22}$$

it follows from (21) and (22) that

$$\mathcal{E}(C_\beta^\psi\mathfrak{N};V_{n,p})=\psi(n-p+1)\bigg(\frac{\mathcal{E}(C_\beta^q\mathfrak{N};V_{n,p})}{q^{n-p+1}}+O(1)\sup_{\varphi\in\mathfrak{N}}\bigg\|\int_{-\pi}^{\pi}\varphi(\cdot-t)r_{n,p}(t)\,dt\bigg\|_C\bigg). \tag{23}$$

If $\mathfrak{N}=U_s^0$, $1\leqslant s\leqslant\infty$, we get from the Hölder inequality (see, *e.g.*, [1, p. 410])

$$\sup_{\varphi\in U_s^0}\bigg|\int_{-\pi}^{\pi}\varphi(x-t)r_{n,p}(t)\,dt\bigg|\leqslant\|r_{n,p}(\cdot)\|_{s'}, \quad \frac{1}{s}+\frac{1}{s'}=1. \tag{24}$$

If $\mathfrak{N}=H_\omega$, then considering that the function $r_{n,p}(t)$ (see (20)) and a random trigonometric polynomial $T_{n-p}(\cdot)$ of order not more than $n-p$ are orthogonal, we can write

$$\sup_{\varphi\in H_\omega}\bigg|\int_{-\pi}^{\pi}\varphi(x-t)r_{n,p}(t)\,dt\bigg|=\sup_{\varphi\in H_\omega}\bigg|\int_{-\pi}^{\pi}(\varphi(x-t)-T_{n-p}(x-t))r_{n,p}(t)\,dt\bigg| \\ \leqslant\sup_{\varphi\in H_\omega}\|\varphi(\cdot)-T_{n-p}(\cdot)\|_C\|r_{n,p}(\cdot)\|_1. \tag{25}$$

Let $T_{n-p}^*(\cdot)$ be the polynomial of best uniform approximation of the function $\varphi\in H_\omega$ by means of trigonometric polynomials of order $\leqslant n-p$:

$$\|\varphi(\cdot)-T_{n-p}^*(\cdot)\|_C=\inf_{T_{n-p}}\|\varphi(\cdot)-T_{n-p}(\cdot)\|_C =: E_{n-p+1}(\varphi).$$

Then, by choosing $T_{n-p}^*(\cdot)$ as the polynomial $T_{n-p}(\cdot)$ in (25) and using the well-known Jackson inequality (see, for example, [1, p. 266])

$$E_{n-p+1}(\varphi)\leqslant K\omega\Big(\varphi,\frac{1}{n-p+1}\Big), \quad K=\text{const},$$

we get from (25) the estimate

$$\sup_{\varphi\in H_\omega}\left|\int_{-\pi}^{\pi}\varphi(x-t)r_{n,p}(t)\,dt\right| = O(1)\omega\Big(\frac{1}{n-p+1}\Big)\|r_{n,p}(\cdot)\|_1. \tag{26}$$

We show that

$$r_{n,p}(t) = O(1)\frac{\varepsilon_{n-p+1}}{(1-q)^2}\min\left\{1,\frac{1}{p(1-q)}\right\},\quad n-p\to\infty, \tag{27}$$

where $O(1)$ is the quantity uniformly bounded in t, n, p, q, ψ and β.

To do this we first rewrite (20) in the form

$$r_{n,p}(t) = \sum_{k=n-p+2}^{\infty}\tau_{n,p}(k)\Bigg(\prod_{l=0}^{k-n+p-2}\frac{\psi(n-p+2+l)}{\psi(n-p+1+l)} - \frac{q^k}{q^{n-p+1}}\Bigg)\cos\Big(kt-\frac{\beta\pi}{2}\Big).$$

Since $\tau_{n,p}(k) > 0$,

$$|r_{n,p}(t)| \leqslant \sum_{k=1}^{\infty}\tau_{n,p}(n-p+1+k)\left|\prod_{l=0}^{k-1}\frac{\psi(n-p+2+l)}{\psi(n-p+1+l)} - q^k\right|.$$

By the estimate

$$\left|\prod_{l=0}^{k-1}\frac{\psi(m+l+1)}{\psi(m+l)} - q^k\right| \leqslant (q+\varepsilon_m)^k - q^k,\quad m\in\mathbb{N}, \tag{28}$$

proved in [14, p. 438], this implies that

$$|r_{n,p}(t)| \leqslant \sum_{k=1}^{\infty}\tau_{n,p}(n-p+1+k)\Big((q+\varepsilon_{n-p+1})^k - q^k\Big). \tag{29}$$

The sequence ε_m tends monotonically to zero. Hence, for sufficiently large $n-p+1$,

$$\varepsilon_{n-p+1} < \frac{1-q}{2}. \tag{30}$$

Therefore, taking into account the fact that $\tau_{n,p}(k) \leqslant 1$ and using the formula

$$\sum_{k=1}^{\infty}x^k = \frac{x}{1-x},\quad 0<x<1, \tag{31}$$

from (29) we have

$$|r_{n,p}(t)| \leqslant \frac{\varepsilon_{n-p+1}}{(1-q)(1-q-\varepsilon_{n-p+1})} < 2\frac{\varepsilon_{n-p+1}}{(1-q)^2}. \tag{32}$$

On the other hand, from (29) and (19) we obtain

$$\begin{aligned} |r_{n,p}(t)| &\leqslant \sum_{k=1}^{\infty} \tau_{n,p}(n-p+1+k)\Big((q+\varepsilon_{n-p+1})^k - q^k\Big) \\ &= \sum_{k=1}^{p-2} \frac{k+1}{p}\Big((q+\varepsilon_{n-p+1})^k - q^k\Big) + \sum_{k=p-1}^{\infty}\Big((q+\varepsilon_{n-p+1})^k - q^k\Big) \\ &< \sum_{k=1}^{\infty} \frac{k+1}{p}\Big((q+\varepsilon_{n-p+1})^k - q^k\Big). \end{aligned} \tag{33}$$

Estimate (33) together with the equality

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}, \quad 0 < x < 1$$

and (30) imply that for sufficiently large $n-p+1$

$$|r_{n,p}(t)| < 2\frac{\varepsilon_{n-p+1}}{p}\frac{\Big(1-q-\dfrac{\varepsilon_{n-p+1}}{2}\Big)}{(1-q)^2(1-q-\varepsilon_{n-p+1})^2} < \frac{8}{p}\frac{\varepsilon_{n-p+1}}{(1-q)^3}.$$

In combination with (32) this yields estimate (27).

Gathering together (23), (24), (26) and (27) we obtain Theorem 1. □

The quantity $pq^{-(n-p+1)}\mathcal{E}(C^q_{\beta,s}; V_{n,p})$ is bounded above and below by some positive numbers, possibly depending only on q and s. Indeed, on the strength of (22),

$$\mathcal{E}(C^q_{\beta,s}; V_{n,p}) = \sup_{\varphi\in U^0_s}\left\|\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\cdot - t)\sum_{k=n-p+1}^{\infty}\tau_{n,p}(k)q^k\cos\Big(kt-\frac{\beta\pi}{2}\Big)\,dt\right\|_C$$

$$\leqslant C_s^{(1)}\left\|\sum_{k=n-p+1}^{\infty}\tau_{n,p}(k)q^k\cos\Big(kt-\frac{\beta\pi}{2}\Big)\right\|_{s'}.$$

Since

$$\sum_{k=n-p+1}^{\infty}\tau_{n,p}(k)q^k < \frac{1}{p}\sum_{k=1}^{\infty}kq^{k+n-p} = \frac{1}{p}\frac{q^{n-p+1}}{(1-q)^2} \tag{34}$$

by (19), we conclude that

$$pq^{-(n-p+1)}\mathcal{E}(C^q_{\beta,s};V_{n,p}) \leqslant \frac{C^{(1)}_s}{(1-q)^2}.$$

To find a lower estimate of the quantity $pq^{-(n-p+1)}\mathcal{E}(C^q_{\beta,s};V_{n,p})$, it is sufficient to consider the function

$$f_{n-p+1}(x) = q^{n-p+1}\|\sin t\|_s^{-1}\sin\Big((n-p+1)x+\frac{\beta\pi}{2}\Big).$$

The function $f_{n-p+1}(x)$ belongs to $C^q_{\beta,s}$ and so

$$pq^{-(n-p+1)}\mathcal{E}(C^q_{\beta,s};V_{n,p}) \geqslant pq^{-(n-p+1)}\|\rho_{n,p}(f_{n-p+1};\cdot)\|_C = \frac{\|\sin t\|_C}{\|\sin t\|_s} = C^{(2)}_s > 0.$$

Thus,

$$C^{(1)}_s \leqslant pq^{-(n-p+1)}\mathcal{E}(C^q_{\beta,s};V_{n,p}) \leqslant C^{(2)}_s\frac{1}{(1-q)^2},\quad C^{(i)}_s>0,\quad i=1,2. \tag{35}$$

An analogous estimate also holds for $pq^{-(n-p+1)}\mathcal{E}(C^q_\beta H_\omega;V_{n,p}):$

$$C^{(1)}_s\omega\Big(\frac{1}{n-p+1}\Big) \leqslant pq^{-(n-p+1)}\mathcal{E}(C^q_\beta H_\omega;V_{n,p}) \leqslant \frac{C^{(2)}_s}{(1-q)^2}\omega\Big(\frac{1}{n-p+1}\Big), \tag{36}$$

where $C^{(i)}_s > 0$, $i=1,2$.

Indeed, since the function $\sum_{k=n-p+1}^{\infty}\tau_{n,p}(k)q^k\cos\Big(kt-\frac{\beta\pi}{2}\Big)$ is orthogonal to every trigonometric polynomial $T_{n-p}(\cdot)$ of order $\leqslant n-p$, from (22) we have

$$\mathcal{E}(C^q_\beta H_\omega;V_{n,p}) \leqslant C^{(1)}_s\sup_{\varphi\in H_\omega}\|\varphi(\cdot)-T_{n-p}(\cdot)\|_C\Big\|\sum_{k=n-p+1}^{\infty}\tau_{n,p}(k)q^k\cos\Big(kt-\frac{\beta\pi}{2}\Big)\Big\|_1. \tag{37}$$

Choosing the polynomial of best approximation of the function $\varphi\in H_\omega$ as $T_{n-p}(\cdot)$ in (37) and applying the Jackson inequality and (34), we obtain

$$pq^{-(n-p+1)}\mathcal{E}(C^q_\beta H_\omega;V_{n,p}) \leqslant \frac{C^{(1)}_s}{(1-q)^2}\omega\Big(\frac{1}{n-p+1}\Big). \tag{38}$$

On the other hand,

$$\begin{aligned} pq^{-(n-p+1)}\mathcal{E}(C^q_\beta H_\omega;V_{n,p}) &\geqslant q^{-(n-p+1)}\mathcal{E}(C^q_\beta H_\omega;V_{n,p}) \\ &\geqslant q^{-(n-p+1)}E_{n-p+1}(C^q_\beta H_\omega), \end{aligned} \tag{39}$$

where $E_{n-p+1}(C^q_\beta H_\omega) = \sup_{f\in C^q_\beta H_\omega} \inf_{T_{n-p}} \|f(\cdot) - T_{n-p}(\cdot)\|_C$. As follows from formula (8) in [12], the next estimate holds for the quantity $E_{n-p+1}(C^q_\beta H_\omega)$:

$$E_{n-p+1}(C^q_\beta H_\omega) \geqslant C^{(2)}_s q^{n-p+1}\omega\Big(\frac{1}{n-p+1}\Big). \tag{40}$$

Comparing (38)–(40), we get (36).

Since $\varepsilon_{n-p+1} \to 0$ as $n-p\to\infty$, in view of (35) and (36) we conclude that in all cases where the asymptotic equalities for $\mathcal{E}(C^q_{\beta,s}; V_{n,p})$ and $\mathcal{E}(C^q_\beta H_\omega; V_{n,p})$ are known, relations (16) and (17) let us write the analogous equalities for the quantities $\mathcal{E}(C^\psi_{\beta,s}; V_{n,p})$ and $\mathcal{E}(C^\psi_\beta H_\omega; V_{n,p})$, respectively, for any $\psi\in D_q$, $q\in(0,1)$.

This fact enables us to give some important corollaries from Theorem 1. With this aim, we cite one of the results from [7, p. 1943], where it was shown that for $q\in(0,1)$, $\beta\in\mathbb{R}$, $1\leqslant s\leqslant\infty$ and $n,p\in\mathbb{N}$, $p\leqslant n$, the following asymptotic equality holds as $n-p\to\infty$:

$$\mathcal{E}(C^q_{\beta,s}; V_{n,p}) = \frac{q^{n-p+1}}{p}\left(\frac{\|\cos t\|_{s'}}{\pi^{1+1/s'}}K_{q,p}(s') + \frac{O(1)}{(n-p+1)(1-q)^{\sigma(s',p)}}\right), \tag{41}$$

in which

$$K_{q,p}(s') := 2^{-1/s'}\left\|\frac{\sqrt{1-2q^p\cos pt+q^{2p}}}{1-2q\cos t+q^2}\right\|_{s'}, \quad s' = \frac{s}{s-1}, \tag{42}$$

$$\sigma(s',p) = \begin{cases} 1, & s'=1, \quad p=1,\\ 2, & 1<s'\leqslant\infty, \quad p=1,\\ 3, & 1\leqslant s'\leqslant\infty, \quad p\in\mathbb{N}\setminus\{1\},\end{cases} \tag{43}$$

and $O(1)$ is the quantity uniformly bounded in n, p, q, β and s.

For $s=\infty$ asymptotic equality (41) was obtained in [5].

Combining (16) and (41), we have.

THEOREM 2. *Let $\psi\in D_q$, $q\in(0,1)$, $1\leqslant s\leqslant\infty$, $\beta\in\mathbb{R}$, $n,p\in\mathbb{N}$, $p\leqslant n$. Then the following asymptotic equality holds as $n-p\to\infty$:*

$$\begin{aligned}\mathcal{E}(C^\psi_{\beta,s}; V_{n,p}) = \frac{\psi(n-p+1)}{p}\Bigg(&\frac{\|\cos t\|_{s'}}{\pi^{1+1/s'}}K_{q,p}(s')\\ &+O(1)\Big(\frac{1}{(n-p+1)(1-q)^{\sigma(s',p)}} + \frac{\varepsilon_{n-p+1}}{(1-q)^2}\min\Big\{p, \frac{1}{1-q}\Big\}\Big)\Bigg),\end{aligned} \tag{44}$$

where $K_{q,p}(s')$ and $\sigma(s',p)$ are defined by (42) and (43), respectively, $s'=\frac{s}{s-1}$, $\varepsilon_{n-p+1} = \sup_{k\geqslant n-p+1}\left|\frac{\psi(k+1)}{\psi(k)} - q\right|$, and $O(1)$ is the quantity uniformly bounded in n, p, q, s, ψ and β.

Note that in the case where $p=1$ and $s\in[1,\infty]$, equality (44) was established in [6, p. 1289].

From the obvious relations

$$1-q^p\leqslant\sqrt{1-2q^p\cos pt+q^{2p}}\leqslant 1+q^p$$

we can write that for $s=\infty$

$$K_{q,p}(s')=K_{q,p}(1)=\int_0^\pi\frac{\sqrt{1-2q^p\cos pt+q^{2p}}}{1-2q\cos t+q^2}\,dt=\frac{1}{1-q^2}(\pi+O(1)q^p). \tag{45}$$

Thus, from (44) and (45) we obtain the next asymptotic equality as $n-p\to\infty$ and $p\to\infty$:

$$\begin{aligned}\mathcal{E}(C^\psi_{\beta,\infty};V_{n,p})=\frac{\psi(n-p+1)}{p}\Bigg(&\frac{4}{\pi(1-q^2)}\\&+O(1)\bigg(\frac{q^p}{1-q}+\frac{1}{(n-p+1)(1-q)^{\sigma(1,p)}}+\frac{\varepsilon_{n-p+1}}{(1-q)^3}\bigg)\Bigg),\end{aligned}\tag{46}$$

where $\psi\in D_q$, $q\in(0,1)$, $\beta\in\mathbb{R}$, $\sigma(1,p)$ is defined by (43) and $O(1)$ is the quantity uniformly bounded in n,p,q,ψ and β. Equality (46) improves (12) at the cost of more precise estimate of the remainder term, it still remains asymptotic even though restriction (15) doesn't hold.

In the case of arbitrary $p=1,2,\dots,n$ the behavior of the constant $K_{q,p}(1)$ could be inferred by the next identity, proved in [4, p. 215]:

$$K_{q,p}(1)=2\frac{1-q^{2p}}{1-q^2}\mathbf{K}(q^p), \tag{47}$$

where $\mathbf{K}(\rho)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-\rho^2\sin^2 t}}$ is the complete elliptic integral of the first kind. Taking (44) and (47) together, we get that for any $\psi\in D_q$, $q\in(0,1)$, $\beta\in\mathbb{R}$ and $n,p\in\mathbb{N}$ the asymptotic equality

$$\begin{aligned}\mathcal{E}(C^\psi_{\beta,\infty};V_{n,p})=\frac{\psi(n-p+1)}{p}\Bigg(&\frac{8}{\pi^2}\frac{1-q^{2p}}{1-q^2}\mathbf{K}(q^p)\\&+O(1)\bigg(\frac{1}{(n-p+1)(1-q)^{\sigma(1,p)}}+\frac{\varepsilon_{n-p+1}}{(1-q)^2}\min\Big\{p,\frac{1}{1-q}\Big\}\bigg)\Bigg)\end{aligned}\tag{48}$$

is true as $n-p\to\infty$.

In the case $p=1$ equality (48) was proved in [14, p. 443].

An analog of (48) can be obtained for the class $C^{\psi}_{\beta}H_{\omega}$ given by convex modulus of continuity $\omega(t)$. To this end, we use the following equality (see [9, p. 5])

$$\begin{aligned}\mathcal{E}(C^{q}_{\beta}H_{\omega};V_{n,p}) = \frac{q^{n-p+1}}{p}\Bigg(\frac{4}{\pi^2}\frac{1-q^{2p}}{1-q^2}\mathbf{K}(q^p)\int_0^{\pi/2}\omega\Big(\frac{2t}{n-p+1}\Big)\sin t\,dt \\ + \frac{O(1)\omega(\pi)}{(1-q)^{\gamma(p)}(n-p+1)}\Bigg), \quad n-p\to\infty,\end{aligned} \tag{49}$$

valid for every $q\in(0,1)$, $\beta\in\mathbb{R}$ and every convex modulus of continuity $\omega(t)$, in which

$$\gamma(p) = \begin{cases} 2, & p=1,\\ 3, & p=2,3,\ldots,n,\end{cases} \tag{50}$$

and the quantity $O(1)$ is uniformly bounded in n, p, q, β and ω.

Noting that (49) is an asymptotic equality if and only if $\omega(t)$ satisfies the condition

$$\lim_{t\to 0}\frac{\omega(t)}{t} = \infty, \tag{51}$$

on the basis of (17) and (49) we arrive at the following assertion.

THEOREM 3. *Let $\psi\in D_q$, $q\in(0,1)$, $n,p\in\mathbb{N}$, $p\leqslant n$ and let $\omega(t)$ be a convex modulus of continuity satisfying condition* (51). *Then, as $n-p\to\infty$,*

$$\begin{aligned}\mathcal{E}(C^{\psi}_{\beta}H_{\omega};V_{n,p}) = \frac{\psi(n-p+1)}{p}\Bigg(\frac{4}{\pi^2}\frac{1-q^{2p}}{1-q^2}\mathbf{K}(q^p)\int_0^{\pi/2}\omega\Big(\frac{2t}{n-p+1}\Big)\sin t\,dt \\ +O(1)\Bigg(\frac{\omega(\pi)}{(1-q)^{\gamma(p)}(n-p+1)} + \frac{\varepsilon_{n-p+1}}{(1-q)^2}\min\Big\{p,\frac{1}{1-q}\Big\}\omega\Big(\frac{1}{n-p+1}\Big)\Bigg)\Bigg),\end{aligned} \tag{52}$$

where $\mathbf{K}(\rho)$ is the complete elliptic integral of the first kind, $\gamma(p)$ is defined by (50), $\varepsilon_{n-p+1} = \sup_{k\geqslant n-p+1}\big|\frac{\psi(k+1)}{\psi(k)} - q\big|$, *and $O(1)$ is the quantity uniformly bounded in n, p, q, ω and β.*

Examples of convex moduli of continuity $\omega(t)$ satisfying condition (51) are the functions $\omega(t)=t^{\alpha}$, $\alpha\in(0,1)$, $\omega(t)=\ln^{\beta}(t+1)$, $\beta\in(0,1)$ and others. If $\omega(t)=t^{\alpha}$, $\alpha\in(0,1)$, the class H_{ω} turns into the well-known Hölder class H^{α}. In this case equality (52) has the form:

$$\begin{aligned}\mathcal{E}(C^{\psi}_{\beta}H^{\alpha};V_{n,p}) = \frac{\psi(n-p+1)}{p(n-p+1)^{\alpha}}\Bigg(\frac{2^{2+\alpha}}{\pi^2}\frac{1-q^{2p}}{1-q^2}\mathbf{K}(q^p)\int_0^{\pi/2}t^{\alpha}\sin t\,dt \\ +O(1)\Bigg(\frac{1}{(1-q)^{\gamma(p)}(n-p+1)^{1-\alpha}} + \frac{\varepsilon_{n-p+1}}{(1-q)^2}\min\Big\{p,\frac{1}{1-q}\Big\}\Bigg)\Bigg), \quad n-p\to\infty.\end{aligned}$$

Along with the Poisson kernel $P_{q,\beta}(t)$, the important examples of the kernels $\Psi_\beta(t)$ (see (3)) whose coefficients $\psi(k)$ belong to D_q, $q\in(0,1)$, are the Neumann kernel

$$N_{q,\beta}(t)=\sum_{k=1}^{\infty}\frac{q^k}{k}\cos\Big(kt-\frac{\beta\pi}{2}\Big),\quad q\in(0,1),\quad \beta\in\mathbb{R} \tag{53}$$

and the polyharmonic Poisson kernel [15, p. 256, 257]

$$P_{q,\beta}(m,t)=\sum_{k=1}^{\infty}\psi_m(k)\cos\Big(kt-\frac{\beta\pi}{2}\Big),\quad \beta\in\mathbb{R}, \tag{54}$$

where

$$\psi_m(k)=q^k\bigg(1+\sum_{j=1}^{m-1}\frac{(1-q^2)^j}{j!2^j}\prod_{l=0}^{j-1}(k+2l)\bigg),\quad m\in\mathbb{N},\quad q\in(0,1).$$

It's easy to verify that for the coefficients $\psi(k)=\frac{q^k}{k}$ of the Neumann kernel $N_{q,\beta}(t)$ the equality

$$\varepsilon_{n-p+1}=\sup_{k\geqslant n-p+1}\frac{q}{k+1}=\frac{q}{n-p+2}, \tag{55}$$

holds. As shown in [11, p. 180] (see, also, [8, p. 132]), in the case where $\psi(k)$ are the coefficients $\psi_m(k)$ of the polyharmonic Poisson kernel $P_{q,\beta}(m,t)$,

$$\varepsilon_{n-p+1}\leqslant\frac{(2m-3)q}{n-p+1},\quad m=2,3,\ldots \tag{56}$$

(if $m=1$, then $\psi_m(k)=\psi_1(k)=q^k$ and so $\varepsilon_{n-p+1}=0$).

Thus from Theorem 2 and Theorem 3 we obtain the next assertions.

Corollary 4. *Let $C^{\psi}_{\beta,s}$, $1\leqslant s\leqslant\infty$, and $C^{\psi}_{\beta}H_\omega$ be the classes generated by the coefficients $\psi(k)=q^k/k$ of the Neumann kernel $N_{q,\beta}(t)$, $n,p\in\mathbb{N}$, $p\leqslant n$, and a convex modulus of continuity $\omega(t)$ satisfies condition* (51). *Then the following asymptotic equalities hold as $n-p\to\infty$*

$$\mathcal{E}(C^{\psi}_{\beta,s};V_{n,p})=\frac{q^{n-p+1}}{p(n-p+1)}\Bigg(\frac{\|\cos t\|_{s'}}{\pi^{1+1/s'}}K_{q,p}(s')$$
$$+\frac{O(1)}{(n-p+1)}\bigg(\frac{1}{(1-q)^{\sigma(s',p)}}+\frac{q}{(1-q)^2}\min\Big\{p,\frac{1}{1-q}\Big\}\bigg)\Bigg),$$
$$\mathcal{E}(C^{\psi}_{\beta}H_\omega;V_{n,p})=\frac{q^{n-p+1}}{p(n-p+1)}\Bigg(\frac{4}{\pi^2}\frac{1-q^{2p}}{1-q^2}\mathbf{K}(q^p)\int_0^{\pi/2}\omega\left(\frac{2t}{n-p+1}\right)\sin t\,dt$$
$$+\frac{O(1)}{n-p+1}\bigg(\frac{\omega(\pi)}{(1-q)^{\gamma(p)}}+\frac{q}{(1-q)^2}\min\Big\{p,\frac{1}{1-q}\Big\}\omega\Big(\frac{1}{n-p+1}\Big)\bigg)\Bigg),$$

where $K_{q,p}(s')$, $\sigma(s',p)$ *and* $\gamma(p)$ *are defined by* (42), (43) *and* (50), *respectively,* $s' = \frac{s}{s-1}$, *and the quantities* $O(1)$ *are uniformly bounded in* n, p, q, s, ω *and* β.

Corollary 5. *Let* $C^{\psi}_{\beta,s}$, $1 \leqslant s \leqslant \infty$, *and* $C^{\psi}_{\beta}H_{\omega}$ *be the classes generated by the coefficients* $\psi(k) = \psi_m(k)$ *of the polyharmonic Poisson kernel* $P_{q,\beta}(m,t)$, $m \in \mathbb{N}$, $n, p \in \mathbb{N}$, $p \leqslant n$, *and a convex modulus of continuity* $\omega(t)$ *satisfies condition* (51). *Then the following asymptotic equalities hold as* $n - p \to \infty$

$$\begin{aligned}
&\mathcal{E}(C^{\psi}_{\beta,s}; V_{n,p}) \\
&= \frac{q^{n-p+1}}{p}\Bigg(1 + \sum_{j=1}^{m-1}\frac{(1-q^2)^j}{j!2^j}\prod_{l=0}^{j-1}(n-p+1+2l)\Bigg)\Bigg(\frac{\|\cos t\|_{s'}}{\pi^{1+1/s'}}K_{q,p}(s') \\
&\quad + \frac{O(1)}{(n-p+1)}\bigg(\frac{1}{(1-q)^{\sigma(s',p)}} + \frac{mq}{(1-q)^2}\min\Big\{p, \frac{1}{1-q}\Big\}\bigg)\Bigg),
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}(C^{\psi}_{\beta}H_{\omega}; V_{n,p}) &= \frac{q^{n-p+1}}{p}\Bigg(1 + \sum_{j=1}^{m-1}\frac{(1-q^2)^j}{j!2^j}\prod_{l=0}^{j-1}(n-p+1+2l)\Bigg) \\
&\times \Bigg(\frac{4}{\pi^2}\frac{1-q^{2p}}{1-q^2}\mathbf{K}(q^p)\int_0^{\pi/2}\omega\left(\frac{2t}{n-p+1}\right)\sin t\,dt \\
&+ \frac{O(1)}{n-p+1}\bigg(\frac{\omega(\pi)}{(1-q)^{\gamma(p)}} + \frac{mq}{(1-q)^2}\min\Big\{p, \frac{1}{1-q}\Big\}\omega\Big(\frac{1}{n-p+1}\Big)\bigg)\Bigg),
\end{aligned}$$

where $K_{q,p}(s')$, $\sigma(s',p)$ *and* $\gamma(p)$ *are defined by* (42), (43) *and* (50), *respectively,* $s' = \frac{s}{s-1}$, *and the quantities* $O(1)$ *are uniformly bounded in* n, p, q, m, s, ω *and* β.

REFERENCES

[1] N. P. Korneichuk: *Exact Constant in Approximation Theory,* Cambridge University Press, Cambridge, 1991.

[2] V. I. Rukasov:*Approximation of classes of analytic functions by de la Vallée Poussin sums* Ukr. Math. J., **55** (6) (2003), 974–986.

[3] V. I. Rukasov – S. O. Chaichenko:*Approximation of analytic periodic functions by de la Vallée Poussin sums,* Ukr. Math. J., **54** (12) (2002), 2006–2024.

[4] V. V. Savchuk – M. V. Savchuk – S. O. Chaichenko: *Approximation of analytic functions by de la Vallée Poussin sums,* Mat. Stud., **34** (2) (2010), 207–219 (in Ukrainian).

[5] A. S. Serdyuk: *Approximation of Poisson integrals by de la Vallée Poussin sums,* Ukr. Math. J., **56** (1) (2004), 122–134.

[6] A. S. Serdyuk: *Approximation of classes of analytic functions by Fourier sums in uniform metric,* Ukr. Math. J., **57** (8) (2005), 1275–1296.

[7] A. S. Serdyuk: *Approximation of Poisson integrals by de la Vallée-Poussin sums in uniform and integral metrics,* Ukr. Math. J., **62** (12) (2011), 1941–1957.

[8] A. S. Serdyuk – S. O. Chaichenko: *Approximation of classes of analytic functions by a linear method of special form,* Ukr. Math. J., **63** (1) (2011), 125–133.

[9] A. S. Serdyuk – Ie. Yu. Ovsii: *Uniform approximation of Poisson integrals of functions from the class H_ω by de la Vallée Poussin sums,* arXiv:1104.3060.

[10] A. S. Serdyuk – Ie. Yu. Ovsii: *Approximation on classes of entire functions by de la Vallée Poussin sums,* Zb. Pr. Inst. Mat. NAN Ukr., **5** (1) (2008), 334–351 (in Ukrainian).

[11] A. S. Serdyuk – I. V. Sokolenko: *Asymptotic behavior of best approximations of classes of periodic analytic functions defined by moduli of continuity,* In: "Mathematical analysis, differential equations and their applications", Bulgarian-Turkish-Ukrainian scientific conference, Sunny Beach, September 15–20, 2010, Academic Publishing House "Prof. Marin Drinov", Sofia, 2011, pp. 173–182.

[12] A. S. Serdyuk – I. V. Sokolenko: *Asymptotic behavior of best approximations of classes of Poisson integrals of functions from H_ω,* Journal of Approximation Theory, **163** (11) (2011), 1692–1706.

[13] A. I. Stepanets: *Methods of Approximation Theory,* VSP, Leiden, 2005.

[14] A. I. Stepanets – A. S. Serdyuk: *Approximation by fourier sums and best approximations on classes of analytic functions,* Ukr. Math. J., **52** (3) (2000), 433–456.

[15] M. F. Timan: *Approximation and properties of periodic functions,* Naukova Dumka, Kiev, 2009 (in Russian).

[16] Ch. La Vallée Poussin: *Sur la meilleure approximation des fonctions d'une variable réelle par des expressions d'ordre donné,* Comptes rendus de l'Académie des Sciences, **166** (4) (1918), 799–802.

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