Reciprocal formulae for convolutions
of Bernoulli and Euler polynomials

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ABSTRACT: *By computing the bivariate exponential generating functions for the Ω-functions, we investigate the reciprocal sums concerning Bernoulli and Euler polynomials. Their convolutions with different weight factors lead to numerous strange identities on Bernoulli and Euler polynomials, such as Miki's identity and the recent generalizations due to Pan and Sun (2006).*

There have been vast mathematical literature dedicated to Bernoulli and Euler numbers. In 1978, Miki [16] discovered through p-adic analysis the following surprising identity on binomial and ordinary convolutions of Bernoulli numbers:

$$\sum_{k=2}^{\ell-2} \frac{B_k B_{\ell-k}}{k(\ell-k)} - \sum_{k=2}^{\ell-2} \binom{\ell}{k} \frac{B_k B_{\ell-k}}{k(\ell-k)} = \frac{2H_\ell}{\ell} B_\ell.$$

With the aid of *Mathematica*, Matiyasevich [15] found two other similar formulae

$$\sum_{k=1}^{\ell-1} \frac{B_k B_{\ell-k}}{k} - \sum_{k=2}^{\ell-1} \binom{\ell}{k} \frac{B_k B_{\ell-k}}{k} = H_\ell B_\ell,$$

$$\sum_{k=0}^{\ell} B_k B_{\ell-k} - 2\sum_{k=2}^{\ell} \binom{\ell+1}{k+1} \frac{B_k B_{\ell-k}}{k+2} = (\ell+1)B_\ell;$$

where it is not hard to check that the former is equivalent to Miki's identity as shown by Pan and Sun [19].

These identities have arisen much interest and been investigated extensively.

- Dilcher [8] made a systematic research on multiple convolutions for Bernoulli and Euler polynomials as well as Riemann's zeta function through recurrence relations and generating functions. Chen [2] went even further on more general convolutions involving Dirichlet characters.
- Dunne-Schubert [9] presented an elegant approach to deal with multiple convolutions of Bernoulli numbers by means of the asymptotic generating functions from quantum field theory.
- Gessel [11] proved and extended Miki's identity through ordinary and exponential generating functions for Stirling numbers of the second kind. Further different and elementary proofs have been given by Crabb [7], Gadiyar-Padma [12] and Shirantani-Yokoyama [18].
- Pan-Sun [19] succeeded in generalizing Miki and Matiyasevich's identities into Bernoulli and Euler polynomials. However, their zig-zag approach was mysterious and hard for the reader to understand how they have discovered these extensions.

Generating functions have been considered as traditional method to treat combinatorial identities (see Wilf [23] for example). To the author's opinion, for convolutions of Bernoulli and Euler polynomials, generating function method is also a natural choice (*cf.* Chu and Wang [3, 4]). Motivated by the convolution structures of Bernoulli and Euler polynomials appeared in the recent work of Pan and Sun [19], we shall define the bivariate $\Omega_{m,n}(x,y)$-sequence and the dual $\Omega^\star_{m,n}(x,y)$-sequence. Their symmetric difference will be investigated through bivariate exponential generating functions, which will lead us to four classes of reciprocal summation formulae. Then the convolutions on these reciprocal summations will not only recover the known formulae due to Pan-Sun [19], but also give rise to several further Miki-like identities concerning both Bernoulli and Euler polynomials.

The paper will be organized as follows. In the first section, both $\Omega_{m,n}(x,y)$ and $\Omega^\star_{m,n}(x,y)$ sequences will be defined through binomial sums. Their symmetric differences will be expressed through bivariate exponential generating functions. The second section will briefly review basic properties of Bernoulli and Euler polynomials. Two binomial sums will be introduced in the third section where their reduction formulae will be derived as preparation for computing convolutions of Bernoulli and Euler polynomials. The main body of the paper will range from Section 4 to Section 7, where we shall establish several reciprocal summation formulae as well as Miki-like identities for Bernoulli and Euler polynomials. Finally, different weight factors on binomial convolutions will be considered in the last section, where further summation formulae will be derived.

1 − Bivariate exponential generating functions

For two arbitrary sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$, define the associated polynomials by

$$F_n(x) := \sum_{k=0}^{n} \binom{n}{k} f_k x^{n-k}, \qquad F_n^{\star}(x) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_k x^{n-k}; \qquad (1.1a)$$

$$G_n(x) := \sum_{k=0}^{n} \binom{n}{k} g_k x^{n-k}, \qquad G_n^{\star}(x) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} g_k x^{n-k}. \qquad (1.1b)$$

Then it is almost trivial to check the following equations:

$$F_n^{\star}(x) = (-1)^n F_n(-x) \quad \text{and} \quad G_n^{\star}(x) = (-1)^n G_n(-x).$$

For two integers $\alpha, \gamma \in \mathbb{Z}$, we introduce further the following convolutions

$$\Omega_{m,n}(x,y) := \Omega_{m,n}[\substack{\alpha,\gamma \\ F,G}](x,y) = \sum_{k=0}^{m} \binom{m}{k} \frac{F_{m-k+\alpha}(x-y)}{(m-k+1)_{\alpha}} \frac{G_{n+k+\gamma}(y)}{(n+k+1)_{\gamma}}, \quad (1.2a)$$

$$\Omega_{m,n}^{\star}(x,y) := \Omega_{m,n}^{\star}[\substack{\alpha,\gamma \\ F,G}](x,y) = \sum_{k=0}^{m} \binom{m}{k} \frac{F_{m-k+\alpha}^{\star}(x-y)}{(m-k+1)_{\alpha}} \frac{G_{n+k+\gamma}(y)}{(n+k+1)_{\gamma}}; \quad (1.2b)$$

where the shifted factorial is defined by means of the Γ-function

$$(x)_{\ell} = \frac{\Gamma(x+\ell)}{\Gamma(x)} \quad \text{where} \quad \Gamma(x) = \int_0^{+\infty} \tau^{x-1} e^{-\tau} d\tau. \qquad (1.3)$$

When ℓ is an integer, it can explicitly be displayed as follows:

$$(x)_{\ell} = \begin{cases} x(x+1)\cdots(x+\ell-1), & \ell = 1, 2, \cdots; \\ 1, & \ell = 0; \\ \dfrac{1}{(x-1)(x-2)\cdots(x+\ell)}, & \ell = -1, -2, \cdots. \end{cases}$$

According to the definition, we can verify without difficulty the following alternative expression:

$$\Omega_{m,n}^{\star}(x,y) = \sum_{k=0}^{m} (-1)^{m-k+\alpha} \binom{m}{k} \frac{F_{m-k+\alpha}(y-x)}{(m-k+1)_{\alpha}} \frac{G_{n+k+\gamma}(y)}{(n+k+1)_{\gamma}}.$$

For $\alpha = 0$, $\gamma = \ell$ and $f_k = \delta_{k,0}$ with $\delta_{i,j}$ being the Kronecker symbol, Chu and Magli [5] have recently established the following relation on symmetric difference and investigated its applications to the summation formulae on Bernoulli, Fibonacci, Lucas and Genocchi numbers.

Theorem 1 (Symmetric difference). *For two natural numbers m, $n \geq 0$ and an integer ℓ, there holds the algebraic identity:*

$$\sum_{k=0}^{m} \binom{m}{k}(x-y)^{m-k} \frac{G_{n+k+\ell}(y)}{(n+k+1)_\ell} - \sum_{k=0}^{n} \binom{n}{k}(y-x)^{n-k} \frac{G_{m+k+\ell}(x)}{(m+k+1)_\ell}$$

$$= \frac{(-1)^m m!\, n!}{(m+n+\ell)!} \sum_{k=1}^{\ell} \binom{m+n+\ell}{\ell-k} \binom{m+k-1}{m} G_{\ell-k}(x)\,(y-x)^{m+n+k}$$

$$= \frac{(-1)^{n+1} m!\, n!}{(m+n+\ell)!} \sum_{k=1}^{\ell} \binom{m+n+\ell}{\ell-k} \binom{n+k-1}{n} G_{\ell-k}(y)\,(x-y)^{m+n+k}$$

where for $\ell \leq 0$, the empty sum with respect to k is assumed to be zero by convention.

This theorem has extended and unified the related results appeared in [20, 21, 25]. Encouraged by the recent work due to Pan and Sun [19], we shall study the more general convolution functions $\Omega_{m,n}(x,y)$ and $\Omega_{m,n}^{\star}(x,y)$ and explore their applications to the Miki-like convolution identities on Bernoulli and Euler polynomials. For this purpose, consider the bivariate exponential generating function

$$\sum_{m,n\geq 0} \frac{u^m v^n}{m!n!} \Omega_{m,n}(x,y) = \sum_{m,n\geq 0} \frac{u^m v^n}{m!n!} \sum_{k=0}^{m} \binom{m}{k} \frac{F_{m-k+\alpha}(x-y)}{(m-k+1)_\alpha} \frac{G_{n+k+\gamma}(y)}{(n+k+1)_\gamma}$$

$$= \sum_{k,n\geq 0} \frac{u^k v^n}{k!n!} \frac{G_{n+k+\gamma}(y)}{(n+k+1)_\gamma} \sum_{m\geq k} \frac{u^{m-k} F_{m-k+\alpha}(x-y)}{(m-k+\alpha)!}.$$

For the internal sum with respect to m, making replacement $m \to m+k$, we get an infinite series independent of k:

$$\sum_{m\geq k} \frac{u^{m-k} F_{m-k+\alpha}(x-y)}{(m-k+\alpha)!} = \sum_{m\geq 0} \frac{u^m F_{m+\alpha}(x-y)}{(m+\alpha)!}.$$

While for the external double sum with respect to k and n, letting $n+k=\ell$ and then appealing to the binomial theorem, we can reduce it to the following series:

$$\sum_{k,n\geq 0} \frac{u^k v^n}{k!n!} \frac{G_{n+k+\gamma}(y)}{(n+k+1)_\gamma} = \sum_{\ell\geq 0} \frac{G_{\ell+\gamma}(y)}{(\ell+\gamma)!} \sum_{k=0}^{\ell} \binom{\ell}{k} u^k v^{\ell-k} = \sum_{\ell\geq 0} \frac{(u+v)^\ell}{(\ell+\gamma)!} G_{\ell+\gamma}(y).$$

Therefore we have established the following generating function:

$$\sum_{m,n\geq 0} \frac{u^m v^n}{m!n!} \Omega_{m,n}(x,y) = \sum_{m\geq 0} \frac{u^m F_{m+\alpha}(x-y)}{(m+\alpha)!} \sum_{n\geq 0} \frac{(u+v)^n}{(n+\gamma)!} G_{n+\gamma}(y) \qquad (1.4a)$$

$$= \frac{1/u^\alpha}{(u+v)^\gamma} \sum_{m\geq\alpha} \frac{u^m}{m!} F_m(x-y) \sum_{n\geq\gamma} \frac{(u+v)^n}{n!} G_n(y). \quad (1.4b)$$

Analogously, we can derive another dual equation:

$$\sum_{m,n\geq 0}\frac{u^m v^n}{m!n!}\Omega^{\star}_{m,n}(x,y) = \sum_{m\geq 0}(-1)^{m+\alpha}\frac{u^m F_{m+\alpha}(y-x)}{(m+\alpha)!}\sum_{n\geq 0}\frac{(u+v)^n}{(n+\gamma)!}G_{n+\gamma}(y).$$

Rewrite the last equation by making the exchanges $x \rightleftharpoons y$ and $u \rightleftharpoons v$:

$$\sum_{m,n\geq 0}\frac{u^m v^n}{m!n!}\Omega^{\star}_{n,m}(y,x) = \sum_{m\geq 0}(-1)^{m+\alpha}\frac{v^m F_{m+\alpha}(x-y)}{(m+\alpha)!}\sum_{n\geq 0}\frac{(u+v)^n}{(n+\gamma)!}G_{n+\gamma}(x) \quad (1.5\text{a})$$

$$= \frac{1/v^{\alpha}}{(u+v)^{\gamma}}\sum_{m\geq\alpha}\frac{(-v)^m}{m!}F_m(x-y)\sum_{n\geq\gamma}\frac{(u+v)^n}{n!}G_n(x). \quad (1.5\text{b})$$

Denote by $[\frac{u^m v^n}{m!n!}]\Phi(u,v)$ the coefficient of $\frac{u^m v^n}{m!n!}$ in the formal power series $\Phi(u,v)$. Then by combining (1.4b) with (1.5b), we have the following theorem.

THEOREM 2 (SYMMETRIC DIFFERENCE).

$$\lambda\,\Omega_{m,n}\big[{}^{\alpha,\gamma}_{F,G}\big](x,y) + \mu\,\Omega^{\star}_{n,m}\big[{}^{\beta,\rho}_{\mathcal{F},\mathcal{G}}\big](y,x)$$

$$= \Big[\frac{u^m v^n}{m!n!}\Big]\bigg\{\frac{\lambda/u^{\alpha}}{(u+v)^{\gamma}}\sum_{m\geq\alpha}\frac{u^m}{m!}F_m(x-y)\sum_{n\geq\gamma}\frac{(u+v)^n}{n!}G_n(y)$$

$$+ \frac{\mu/v^{\beta}}{(u+v)^{\rho}}\sum_{m\geq\beta}(-1)^m\frac{v^m}{m!}\mathcal{F}_m(x-y)\sum_{n\geq\rho}\frac{(u+v)^n}{n!}\mathcal{G}_n(x)\bigg\}.$$

According to this theorem, we shall choose properly parameters and polynomials so that the main mixed term of the generating functions for $\lambda\Omega_{m,n}(x,y)+\mu\Omega^{\star}_{n,m}(y,x)$ will be canceled. By manipulating further the reduced bivariate exponential generating functions, we can derive alternative expressions for the symmetric differences, which result in identities of equivalent convolutions. Computing the convolutions on the resulting identity with respect to $m+n=\ell$, we shall further establish several Miki-like identities of Bernoulli and Euler polynomials, as discovered recently by Pan and Sun [19].

2 – Bernoulli and Euler polynomials

This section reviews some basic facts about Bernoulli and Euler polynomials. More comprehensive coverage can be found in Abramowitz-Stegun [1, §23] and Comtet [6, §1.14] and Rosen [17, §3.1].

The Bernoulli and Euler numbers are defined respectively by the exponential generating functions:

$$\frac{u}{e^u - 1} = \sum_{n \geq 0} B_n \frac{u^n}{n!} \qquad \text{and} \qquad \frac{2e^u}{e^{2u} + 1} = \sum_{n \geq 0} E_n \frac{u^n}{n!}.$$

The generating functions for the corresponding polynomials read as

$$\frac{ue^{ux}}{e^u - 1} = \sum_{n \geq 0} B_n(x) \frac{u^n}{n!} \qquad \text{and} \qquad \frac{2e^{ux}}{e^u + 1} = \sum_{n \geq 0} E_n(x) \frac{u^n}{n!}.$$

Both polynomials are expressed through the respective numbers

$$B_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad \text{and} \quad E_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}.$$

They satisfy the binomial relations

$$B_n(x+y) \;=\; \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k} \quad \text{and} \quad E_n(x+y) \;=\; \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}$$

differential equations

$$B_n'(x) \;=\; nB_{n-1}(x) \quad \text{and} \quad E_n'(x) \;=\; nE_{n-1}(x)$$

reciprocal relations

$$B_n(1-x) \;=\; (-1)^n B_n(x) \quad \text{and} \quad E_n(1-x) \;=\; (-1)^n E_n(x)$$

as well as difference equations

$$B_n(x) - (-1)^n B_n(-x) = -nx^{n-1} \quad \text{and} \quad E_n(x) + (-1)^n E_n(-x) = 2x^n.$$

There is also an expression of Euler polynomials in terms of Bernoulli polynomials

$$E_n(x) = \frac{2}{n+1} \left\{ B_{n+1}(x) - 2^{n+1} B_{n+1}(x/2) \right\}.$$

Furthermore by means of the binomial relations for Euler and Bernoulli polynomials, it is not hard to derive the convolution formulae.

LEMMA 3 ($\varepsilon \geq 0$, $\gamma \geq 0$ AND $\ell \geq 0$). *Letting $\mathcal{P}_k(x)$ be $E_k(x)$ or $B_k(x)$, then the following identity holds for both Euler and Bernoulli polynomials:*

$$\frac{\mathcal{P}_{\ell+\gamma+\varepsilon}(y) - \mathcal{P}_{\ell+\gamma+\varepsilon}(x)}{(y-x)^\varepsilon} - \sum_{k=1}^{\varepsilon-1} \binom{\ell+\gamma+\varepsilon}{\varepsilon-k}(y-x)^{-k}\mathcal{P}_{\ell+\gamma+k}(x)$$

$$= \sum_{k=0}^{\ell} \binom{\ell+\gamma+\varepsilon}{k+\varepsilon}(y-x)^k \mathcal{P}_{\ell-k+\gamma}(x)$$

$$+ \sum_{k=1}^{\gamma} \binom{\ell+\gamma+\varepsilon}{\ell+k+\varepsilon}(y-x)^{k+\ell}\mathcal{P}_{\gamma-k}(x).$$

PROOF. For $\varepsilon \in \mathbb{N}_0$, consider the binomial sum

$$\mathcal{S}_\varepsilon := \sum_{k=0}^{\ell+\gamma} \binom{\ell+\gamma+\varepsilon}{k+\varepsilon}(y-x)^k \mathcal{P}_{\ell-k+\gamma}(x) = \sum_{k=0}^{\ell} \binom{\ell+\gamma+\varepsilon}{k+\varepsilon}(y-x)^k \mathcal{P}_{\ell-k+\gamma}(x)$$

$$+ \sum_{k=1}^{\gamma} \binom{\ell+\gamma+\varepsilon}{\ell+k+\varepsilon}(y-x)^{k+\ell}\mathcal{P}_{\gamma-k}(x).$$

According to the binomial relations satisfied by Euler and Bernoulli polynomials, we have also the following expression:

$$\mathcal{S}_\varepsilon = \frac{\mathcal{P}_{\ell+\gamma+\varepsilon}(y) - \mathcal{P}_{\ell+\gamma+\varepsilon}(x)}{(y-x)^\varepsilon} - \sum_{k=1}^{\varepsilon-1} \binom{\ell+\gamma+\varepsilon}{\varepsilon-k}(y-x)^{-k}\mathcal{P}_{\ell+k+\gamma}(x)$$

which leads us to the formula displayed in the lemma.   □

By means of the binomial relation

$$(k+1)\binom{\ell+\gamma+\varepsilon}{k+\varepsilon} = (\ell+\gamma+\varepsilon)\binom{\ell+\gamma+\varepsilon-1}{k+\varepsilon-1} - (\varepsilon-1)\binom{\ell+\gamma+\varepsilon}{k+\varepsilon}$$

we can also derive another identity, whose details will not be reproduced.

LEMMA 4 ($\varepsilon \geq 0$, $\gamma \geq 0$ AND $\ell \geq 0$). *Letting $\mathcal{P}_k(x)$ be $E_k(x)$ or $B_k(x)$, then the following identity holds for both Euler and Bernoulli polynomials:*

$$(1 - \varepsilon)\frac{\mathcal{P}_{\ell+\gamma+\varepsilon}(y) - \mathcal{P}_{\ell+\gamma+\varepsilon}(x)}{(y-x)^{\varepsilon}} + \sum_{k=2}^{\varepsilon-1}(k-1)\binom{\ell+\gamma+\varepsilon}{\varepsilon-k}(y-x)^{-k}\mathcal{P}_{\ell+\gamma+k}(x)$$

$$+(\ell + \gamma + \varepsilon)\frac{\mathcal{P}_{\ell+\gamma+\varepsilon-1}(y)}{(y-x)^{\varepsilon-1}} = \sum_{k=0}^{\ell}(k+1)\binom{\ell+\gamma+\varepsilon}{k+\varepsilon}(y-x)^k\mathcal{P}_{\ell-k+\gamma}(x)$$

$$+ \sum_{k=1}^{\gamma}(\ell + k + 1)\binom{\ell+\gamma+\varepsilon}{\ell+k+\varepsilon}(y-x)^{k+\ell}\mathcal{P}_{\gamma-k}(x).$$

3 – Finite sums of binomial coefficients

Throughout this section, we suppose that ε and δ are two integer parameters with $\varepsilon, \delta \in \{0, 1\}$. The reduction formulae will be established for binomial sums, which will be used subsequently for evaluating convolutions of Bernoulli and Euler polynomials.

First, there holds the following relatively easy identity on binomial coefficients.

LEMMA 5 ($0 \leq k \leq \ell$: SEE [13, §5.1] FOR EXAMPLE).

$$\sum_{n=k}^{\ell}\binom{n}{k}(1 + \varepsilon n)\{1 + (\ell - n)\delta\} = \binom{\ell + \varepsilon + \delta + 1}{k + \varepsilon + \delta + 1}(1 + \varepsilon k).$$

Then for four nonnegative integers k, ℓ, γ, σ, define two binomial sums by

$$W_{\varepsilon,\delta}(k, \ell, \gamma) := \sum_{n=k}^{\ell}\binom{-\gamma}{n-k}(\varepsilon + n)!(\delta + \ell - n)!, \qquad (3.1a)$$

$$\lambda_{\sigma}(k, \ell, \gamma) := \sum_{i=1}^{\gamma}\frac{(-1)^{k+\ell+\gamma+i}}{\ell + i + \sigma}\binom{k}{\gamma - i}\frac{(\ell + i)!}{(i - 1)!}; \qquad (3.1b)$$

where the λ-sum admits obviously the initial values:

$$\lambda_{\sigma}(k, \ell, 0) = 0 \quad \text{and} \quad \lambda_{\sigma}(k, \ell, 1) = (-1)^{k+\ell}\frac{(\ell + 1)!}{\ell + \sigma + 1}.$$

These two binomial sums are tied together by the following relation.

LEMMA 6 ($\gamma \geq 0$ AND $k \leq \ell$).

$$W_{\varepsilon,\delta}(k, \ell, \gamma) = \frac{(k + \varepsilon)!(\delta + \ell + \gamma - k)!}{(2 + \ell + \varepsilon + \delta)_{\gamma}} + \lambda_{1+\delta}(k + \varepsilon, \ell + \varepsilon, \gamma).$$

PROOF.   According to Chu-Vandermonde convolution formula

$$\binom{-\gamma}{n-k} = (-1)^{n-k}\binom{n+\gamma-k-1}{\gamma-1} = (-1)^{n-k+\gamma-1}\binom{k-n-1}{\gamma-1}$$

$$= (-1)^{n-k+\gamma-1}\sum_{i=1}^{\gamma}\binom{-n-\varepsilon-1}{i-1}\binom{k+\varepsilon}{\gamma-i}$$

we can reformulate the following binomial sum:

$$W_{\varepsilon,\delta}(k,\ell,\gamma) \;=\; \sum_{n=k}^{\ell}\binom{-\gamma}{n-k}(\varepsilon+n)!(\delta+\ell-n)!$$

$$= \sum_{i=1}^{\gamma}(-1)^{k+\gamma-i}\binom{k+\varepsilon}{\gamma-i}\frac{(\ell+\delta)!}{(i-1)!}\sum_{n=k}^{\ell}\frac{(\varepsilon+n+i-1)!}{(-\delta-\ell)_n}.$$

By means of the relation

$$\frac{(\varepsilon+n+i-1)!}{(-\delta-\ell)_n} = \frac{\dfrac{(\varepsilon+n+i)!}{(-\delta-\ell)_n} - \dfrac{(\varepsilon+n+i-1)!}{(-\delta-\ell)_{n-1}}}{\varepsilon+\delta+\ell+i+1}$$

the last sum can be telescoped as follows:

$$\sum_{n=k}^{\ell}\frac{(\varepsilon+n+i-1)!}{(-\delta-\ell)_n} = \frac{\dfrac{(\ell+\varepsilon+i)!}{(-\delta-\ell)_\ell} - \dfrac{(k+\varepsilon+i-1)!}{(-\delta-\ell)_{k-1}}}{\varepsilon+\delta+\ell+i+1}$$

$$= \frac{(-1)^\ell\dfrac{(\ell+\varepsilon+i)!}{(\delta+\ell)!} + (-1)^k\dfrac{(k+\varepsilon+i-1)!}{(2+\delta+\ell-k)_{k-1}}}{\varepsilon+\delta+\ell+i+1}.$$

This allows us to express $W_{\varepsilon,\delta}(k,\ell,\gamma)$ function as two finite sums:

$$W_{\varepsilon,\delta}(k,\ell,\gamma) = \sum_{i=1}^{\gamma}(-1)^{\gamma-i}\frac{(1+\ell+\delta-k)!}{\varepsilon+\delta+\ell+i+1}\binom{k+\varepsilon}{\gamma-i}\frac{(k+\varepsilon+i-1)!}{(i-1)!}$$

$$+ \sum_{i=1}^{\gamma}\frac{(-1)^{\ell+k+\gamma-i}}{\varepsilon+\delta+\ell+i+1}\binom{k+\varepsilon}{\gamma-i}\frac{(\ell+\varepsilon+i)!}{(i-1)!}.$$

For the last two sums, evaluating the former through the partial fraction decomposition

$$\frac{(2+\delta+\ell-k)_{\gamma-1}}{(2+\ell+\varepsilon+\delta)_\gamma} = \sum_{i=1}^{\gamma}\frac{(-1)^{\gamma-i}}{\varepsilon+\delta+\ell+i+1}\binom{k+\varepsilon}{\gamma-i}\binom{k+\varepsilon+i-1}{i-1}$$

and expressing the latter in terms of the λ-function, we have

$$W_{\varepsilon,\delta}(k,\ell,\gamma) = \frac{(k+\varepsilon)!(\delta+\ell+\gamma-k)!}{(2+\ell+\varepsilon+\delta)_\gamma} + \lambda_{1+\delta}(k+\varepsilon,\ell+\varepsilon,\gamma).$$

This is exactly the reduction formula stated in the lemma.   □

Taking into account the two special values

$$\lambda_\sigma(0,\ell,\gamma) = \frac{(-1)^\ell}{(\gamma-1)!}\frac{(\ell+\gamma)!}{\ell+\gamma+\sigma} \quad \text{and} \quad \lambda_\sigma(1,\ell,\gamma) = \frac{(-1)^{\ell-1}}{(\gamma-1)!}\frac{(\ell+\gamma-1)!(\ell^2+\ell\gamma+\ell\sigma+\sigma)}{(\ell+\gamma+\sigma)(\ell+\gamma+\sigma-1)}$$

we have no difficulty to derive the following summation formula.

LEMMA 7 ($\gamma \geq 0$ AND $\ell \geq 0$).

$$W_{\varepsilon,\delta}(0,\ell,\gamma) = \frac{(\delta+\ell+\gamma)!}{(2+\ell+\varepsilon+\delta)_\gamma} + \frac{(-1)^\ell(\ell+\gamma)!}{(\gamma-1)!(1+\delta+\ell+\gamma)}\left\{1+\varepsilon\frac{1+\delta+\ell(2+\delta+\ell+\gamma)}{2+\delta+\ell+\gamma}\right\}.$$

For $k = -1$, we use the following slightly modified notation

$$W'_{\varepsilon,\delta}(\ell,\gamma) := \sum_{n=0}^{\ell}\binom{-\gamma}{n+1}(\varepsilon+n)!(\delta+\ell-n)!. \tag{3.2}$$

Following the same proof as for Lemma 6, we can also derive the reduction formulae.

LEMMA 8 ($\gamma \geq 0$ AND $\ell \geq 0$).

$$W'_{0,\delta}(\ell,\gamma) = \lambda_{1+\delta}(-1,\ell,\gamma) - (1+\ell+\delta)!\{H_{1+\ell+\gamma+\delta} - H_{1+\ell+\delta}\}, \tag{3.3a}$$

$$W'_{1,\delta}(\ell,\gamma) = -\frac{\gamma(1+\ell+\delta)!}{2+\ell+\gamma+\delta} - \frac{(-1)^\ell}{(\gamma-1)!}\frac{(\ell+\gamma+1)!}{2+\ell+\gamma+\delta}. \tag{3.3b}$$

Now we are ready to investigate four classes of symmetric differences on convolutions of Bernoulli and Euler polynomials. According to Theorem 1, we shall label each section by symmetric combination $\lambda\,\Omega_{m,n}[\begin{smallmatrix}\alpha,\gamma\\F,G\end{smallmatrix}](x,y) + \mu\,\Omega^\star_{n,m}[\begin{smallmatrix}\beta,\rho\\\mathcal{F},\mathcal{G}\end{smallmatrix}](y,x)$ with F,G and $\mathcal{F},\mathcal{G}$ being specified by B and E (Bernoulli and Euler polynomials).

$4 - \Omega_{m,n}[\begin{smallmatrix}1,\gamma\\B,B\end{smallmatrix}](x,y) + \Omega^\star_{n,m}[\begin{smallmatrix}1,\gamma\\B,B\end{smallmatrix}](y,x)$

For both sequences $\Omega_{m,n}[\begin{smallmatrix}\alpha,\gamma\\F,G\end{smallmatrix}](x,y)$ and $\Omega^\star_{n,m}[\begin{smallmatrix}\alpha,\gamma\\F,G\end{smallmatrix}](y,x)$, we specify with $\alpha = 1$, $F_m(x) = B_m(x)$ and $G_n(y) = B_n(y)$. According to Theorem 2, the bivariate

exponential generating function for the corresponding symmetric sum $\Omega_{m,n}(x,y) + \Omega^\star_{n,m}(y,x)$ reads as follows:

$$\frac{1/u}{(u+v)^\gamma}\left\{\frac{ue^{u(x-y)}}{e^u-1}-1\right\}\times\left\{\frac{(u+v)e^{y(u+v)}}{e^{u+v}-1}-\sum_{i=0}^{\gamma-1}\frac{(u+v)^i}{i!}B_i(y)\right\}$$

$$-\frac{1/v}{(u+v)^\gamma}\left\{\frac{ve^{v(y-x)}}{e^{-v}-1}+1\right\}\times\left\{\frac{(u+v)e^{x(u+v)}}{e^{u+v}-1}-\sum_{j=0}^{\gamma-1}\frac{(u+v)^j}{j!}B_j(x)\right\}.$$

4.1 – Reducing the main term of the last expression

$$\frac{(u+v)^{1-\gamma}e^{ux+vy}}{(e^u-1)(e^{u+v}-1)}-\frac{(u+v)^{1-\gamma}e^{ux+vy}}{(e^{-v}-1)(e^{u+v}-1)}=\frac{(u+v)^{1-\gamma}e^{ux+vy}}{(e^u-1)(e^v-1)}$$

we can reformulate the generating function as

$$\frac{(u+v)^{1-\gamma}e^{ux+vy}}{(e^u-1)(e^v-1)}-\frac{1/u}{(u+v)^{\gamma-1}}\frac{e^{y(u+v)}}{e^{u+v}-1}-\frac{1/v}{(u+v)^{\gamma-1}}\frac{e^{x(u+v)}}{e^{u+v}-1} \qquad (4.1\text{a})$$

$$+\frac{1}{u}\sum_{i=1}^{\gamma}\frac{(u+v)^{-i}}{(\gamma-i)!}B_{\gamma-i}(y)-\frac{e^{u(x-y)}}{e^u-1}\sum_{i=1}^{\gamma}\frac{(u+v)^{-i}}{(\gamma-i)!}B_{\gamma-i}(y) \qquad (4.1\text{b})$$

$$+\frac{1}{v}\sum_{j=1}^{\gamma}\frac{(u+v)^{-j}}{(\gamma-j)!}B_{\gamma-j}(x)+\frac{e^{v(y-x)}}{e^{-v}-1}\sum_{j=1}^{\gamma}\frac{(u+v)^{-j}}{(\gamma-j)!}B_{\gamma-j}(x). \qquad (4.1\text{c})$$

Now that u and v are two arbitrary indeterminate, there hold the following two binomial expansions:

$$(u+v)^{-\beta}=\sum_{k\geq0}\binom{-\beta}{k}u^kv^{-k-\beta}\quad\text{if}\quad|u|<|v|; \qquad (4.2\text{a})$$

$$(u+v)^{-\beta}=\sum_{k\geq0}\binom{-\beta}{k}v^ku^{-k-\beta}\quad\text{if}\quad|u|>|v|. \qquad (4.2\text{b})$$

Recalling the generating function of Bernoulli polynomials and then extracting the coefficients of $[\frac{u^mv^n}{m!n!}]$ according to the last two expansions, we can respectively establish the two corresponding expressions for the symmetric sum $\Omega_{m,n}(x,y) + \Omega^\star_{n,m}(y,x)$, which are explicitly displayed as the following theorem.

Theorem 9 (Symmetric sum).

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)B_{n+k+\gamma}(y)}{(m-k+1)(n+k+1)_\gamma} \tag{4.3a}$$

$$-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)B_{m+k+\gamma}(x)}{(n-k+1)(m+k+1)_\gamma} \tag{4.3b}$$

$$=m!n!\sum_{k=0}^{m+1}\binom{1-\gamma}{k}\frac{B_{m-k+1}(x)B_{n+k+\gamma}(y)}{(m-k+1)!(n+k+\gamma)!} \tag{4.3c}$$

$$+ m!n!\sum_{k=1}^{\gamma}(-1)^{m+n+k}\binom{-k}{m}\frac{B_{\gamma-k}(x)B_{m+n+k+1}(x-y)}{(\gamma-k)!(m+n+k+1)!} \tag{4.3d}$$

$$-\frac{B_{m+n+\gamma+1}(x)}{(n+1)(m+n+2)_\gamma}-\frac{B_{m+n+\gamma+1}(y)}{(m+1)(m+n+2)_\gamma} \tag{4.3e}$$

$$=m!n!\sum_{k=0}^{n+1}\binom{1-\gamma}{k}\frac{B_{n-k+1}(y)B_{m+k+\gamma}(x)}{(n-k+1)!(m+k+\gamma)!} \tag{4.3f}$$

$$- m!n!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{B_{\gamma-k}(y)B_{m+n+k+1}(x-y)}{(\gamma-k)!(m+n+k+1)!} \tag{4.3g}$$

$$-\frac{B_{m+n+\gamma+1}(y)}{(m+1)(m+n+2)_\gamma}-\frac{B_{m+n+\gamma+1}(x)}{(n+1)(m+n+2)_\gamma}. \tag{4.3h}$$

For $\gamma = 0,\ 1,\ 2$, it reduces respectively to the following easier formulae.

Corollary 10 (Symmetric difference).

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)B_{n+k}(y)}{m-k+1}-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)B_{m+k}(x)}{n-k+1}$$

$$=\frac{B_{m+1}(x)B_n(y)}{m+1}+\frac{B_m(x)B_{n+1}(y)}{n+1}-\frac{B_{m+n+1}(x)}{n+1}-\frac{B_{m+n+1}(y)}{m+1}.$$

As observed by Pan and Sun [19, Corollary 2.5], the last identity implies the symmetric relation on convolutions of Bernoulli numbers due to Woodcock [24].

COROLLARY 11 (PAN-SUN [19, EQ. 2.11]).

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)B_{n+k+1}(y)}{(m-k+1)(n+k+1)}-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)B_{m+k+1}(x)}{(n-k+1)(m+k+1)}$$

$$=\frac{B_{m+1}(x)B_{n+1}(y)}{(m+1)(n+1)}-\frac{(-1)^{n}}{\binom{m+n}{n}}\frac{B_{m+n+2}(x-y)}{(m+n+1)(m+n+2)}-\frac{B_{m+n+2}(x)}{(n+1)(m+n+2)}$$

$$-\frac{B_{m+n+2}(y)}{(m+1)(m+n+2)}.$$

COROLLARY 12 (SYMMETRIC DIFFERENCE).

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)B_{n+k+2}(y)}{(m-k+1)(n+k+1)_2}-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)B_{m+k+2}(x)}{(n-k+1)(m+k+1)_2}$$

$$=(-1)^{n}\frac{B_{m+n+3}(x-y)}{\binom{m+n+1}{n}(m+n+2)_2}+m!n!\sum_{k=0}^{m+1}(-1)^{k}\frac{B_{m-k+1}(x)B_{n+k+2}(y)}{(m-k+1)!(n+k+2)!}$$

$$-(-1)^{n}\frac{B_{1}(x)B_{m+n+2}(x-y)}{\binom{m+n}{n}(m+n+1)_2}-\frac{B_{m+n+3}(x)}{(n+1)(m+n+2)_2}-\frac{B_{m+n+3}(y)}{(m+1)(m+n+2)_2}$$

$$=(-1)^{n+1}\frac{B_{m+n+3}(x-y)}{\binom{m+n+1}{m}(m+n+2)_2}+m!n!\sum_{k=0}^{n+1}(-1)^{k}\frac{B_{n-k+1}(y)B_{m+k+2}(x)}{(n-k+1)!(m+k+2)!}$$

$$-(-1)^{n}\frac{B_{1}(y)B_{m+n+2}(x-y)}{\binom{m+n}{m}(m+n+1)_2}-\frac{B_{m+n+3}(y)}{(m+1)(m+n+2)_2}-\frac{B_{m+n+3}(x)}{(n+1)(m+n+2)_2}.$$

In this corollary, simplifying the last equation by combining the two sums with respect to k to a single one and then relabeling the parameters, we recover the following convolution identity (*cf.* [8, Eq. 3.2] and [14, Eq. 50.11.1]):

$$\sum_{k=0}^{\ell}(-1)^{k+\ell}\binom{\ell}{k}B_{k}(x)B_{\ell-k}(y)=\ell(x-y)B_{\ell-1}(x-y)-(\ell-1)B_{\ell}(x-y). \quad (4.4\text{a})$$

Keeping in mind the reciprocal relation

$$B_{\ell-k}(y)=(-1)^{\ell-k}B_{\ell-k}(-y)-(\ell-k)y^{\ell-k-1}$$

and then evaluating the binomial sum

$$\sum_{k=0}^{\ell}(-1)^{k+\ell}\binom{\ell}{k}(\ell-k)B_{k}(x)y^{\ell-k-1}=-\ell B_{\ell-1}(x-y)$$

we find that the last identity is equivalent to the following [14, Eq. 50.11.2])

$$\sum_{k=0}^{\ell} \binom{\ell}{k} B_k(x) B_{\ell-k}(y) = \ell(x+y-1) B_{\ell-1}(x+y) - (\ell-1) B_\ell(x+y). \qquad (4.4b)$$

Its special case $x = y = 0$ leads us to the identity

$$\sum_{k=2}^{\ell-2} \binom{\ell}{k} B_k B_{\ell-k} = -(\ell+1) B_\ell$$

which is originally due to Euler and Ramanujan (*cf.* [8, Eq. 1.2], [9, Eq. 1.2] and [11]).

4.2 – We are now going to compute the convolution with respect to $m + n = \ell$ on the equations displayed in Theorem 9.

For (4.3a), performing replacement $k \to m - k$, interchanging the summation order and then applying Lemma 5, we can compute the convolution as follows:

$$\sum_{m+n=\ell} \text{Eq. (4.3a)} = \sum_{m=0}^{\ell} \sum_{k=0}^{m} \binom{m}{k} \frac{B_{k+1}(x-y) B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}$$

$$= \sum_{k=0}^{\ell} \frac{B_{k+1}(x-y) B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma} \sum_{m=k}^{\ell} \binom{m}{k}$$

$$= \sum_{k=0}^{\ell} \binom{\ell+1}{k+1} \frac{B_{k+1}(x-y) B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}.$$

The convolution for (4.3b) can be computed in exactly the same way as (4.3a):

$$\sum_{m+n=\ell} \text{Eq. (4.3b)} = \sum_{n=0}^{\ell} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{B_{k+1}(x-y) B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}$$

$$= \sum_{k=0}^{\ell} (-1)^{k+1} \frac{B_{k+1}(x-y) B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma} \sum_{n=k}^{\ell} \binom{n}{k}$$

$$= \sum_{k=0}^{\ell} (-1)^{k+1} \binom{\ell+1}{k+1} \frac{B_{k+1}(x-y) B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}.$$

For (4.3c), recalling (3.1a) and (3.2), we can compute the convolution by means of Lemma 6 and (3.3a) as follows:

$$
\sum_{m+n=\ell} \text{Eq. (4.3c)} = \sum_{m=0}^{\ell} m!(\ell-m)! \sum_{k=-1}^{m} \binom{1-\gamma}{m-k} \frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}
$$

$$
= W'_{0,0}(\ell,\gamma-1)\frac{B_{\ell+\gamma+1}(y)}{(\ell+\gamma+1)!} + \sum_{k=0}^{\ell} \frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}W_{0,0}(k,\ell,\gamma-1)
$$

$$
= \frac{B_{\ell+\gamma+1}(y)}{(\ell+2)_\gamma}\{H_{\ell+1}-H_{\ell+\gamma}\} + \frac{1}{(\ell+2)_{\gamma-1}}\sum_{k=0}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+\gamma)}
$$

$$
+ \sum_{k=-1}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\lambda_1(k,\ell,\gamma-1).
$$

The convolution for (4.3f) can be written down directly from that for (4.3c):

$$
\sum_{m+n=\ell} \text{Eq. (4.3f)} = \sum_{n=0}^{\ell} n!(\ell-n)! \sum_{k=-1}^{n} \binom{1-\gamma}{n-k} \frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}
$$

$$
= W'_{0,0}(\ell,\gamma-1)\frac{B_{\ell+\gamma+1}(x)}{(\ell+\gamma+1)!} + \sum_{k=0}^{\ell} \frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}W_{0,0}(k,\ell,\gamma-1)
$$

$$
= \frac{B_{\ell+\gamma+1}(x)}{(\ell+2)_\gamma}\{H_{\ell+1}-H_{\ell+\gamma}\} + \frac{1}{(\ell+2)_{\gamma-1}}\sum_{k=0}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+\gamma)}
$$

$$
+ \sum_{k=-1}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\lambda_1(k,\ell,\gamma-1).
$$

For (4.3d), we can compute the convolution, by invoking Lemma 7, as follows:

$$
\sum_{m+n=\ell} \text{Eq. (4.3d)} = \sum_{m=0}^{\ell} m!(\ell-m)! \sum_{k=1}^{\gamma}(-1)^{\ell+k}\binom{-k}{m}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}
$$

$$
= \sum_{k=1}^{\gamma}(-1)^{\ell+k}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}W_{0,0}(0,\ell,k)
$$

$$
= \sum_{k=1}^{\gamma}\frac{(-1)^k}{\ell+k+1}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+2)_k}\left\{(-1)^\ell+\binom{k+\ell}{k-1}\right\}.
$$

The convolution for (4.3g) can be done in the same manner as that for (4.3d):

$$
\sum_{m+n=\ell} \text{Eq. (4.3g)} = -\sum_{n=0}^{\ell} n!(\ell-n)! \sum_{k=1}^{\gamma} \binom{-k}{n} \frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}
$$
$$
= -\sum_{k=1}^{\gamma} \frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!} W_{0,0}(0,\ell,k)
$$
$$
= -\sum_{k=1}^{\gamma} \frac{(-1)^\ell}{\ell+k+1} \frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+2)_k} \Big\{(-1)^\ell + \binom{k+\ell}{k-1}\Big\}.
$$

Finally for (4.3e) and (4.3h), it is trivial to have their convolutions as follows:

$$
\sum_{m+n=\ell} \text{Eq. (4.3e)} = -\sum_{n=0}^{\ell} \Big\{ \frac{B_{\ell+\gamma+1}(x)}{(n+1)(\ell+2)_\gamma} + \frac{B_{\ell+\gamma+1}(y)}{(\ell-n+1)(\ell+2)_\gamma}\Big\}
$$
$$
= -\frac{H_{\ell+1}}{(\ell+2)_\gamma}\Big\{ B_{\ell+\gamma+1}(x) + B_{\ell+\gamma+1}(y)\Big\}.
$$

Summarizing the computations just displayed, we get the following identities.

THEOREM 13 (MIKI-LIKE IDENTITIES).

$$
\sum_{k=0}^{\ell} \binom{\ell+1}{k+1} \frac{B_{k+1}(x-y)}{k+1} \Big\{ \frac{B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma} - \frac{(-1)^k B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}\Big\}
$$
$$
= -\frac{H_{\ell+1}}{(\ell+2)_\gamma} B_{\ell+\gamma+1}(x) + \frac{1}{(\ell+2)_{\gamma-1}} \sum_{k=0}^{\ell} \frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+\gamma)}
$$
$$
- \frac{H_{\ell+\gamma}}{(\ell+2)_\gamma} B_{\ell+\gamma+1}(y) + \sum_{k=-1}^{\ell} \frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\lambda_1(k,\ell,\gamma-1)
$$
$$
+ \sum_{k=1}^{\gamma} \frac{(-1)^k}{\ell+k+1} \frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+2)_k}\Big\{(-1)^\ell + \binom{k+\ell}{k-1}\Big\}
$$
$$
= -\frac{H_{\ell+\gamma}}{(\ell+2)_\gamma} B_{\ell+\gamma+1}(x) + \frac{1}{(\ell+2)_{\gamma-1}} \sum_{k=0}^{\ell} \frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+\gamma)}
$$
$$
- \frac{H_{\ell+1}}{(\ell+2)_\gamma} B_{\ell+\gamma+1}(y) + \sum_{k=-1}^{\ell} \frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\lambda_1(k,\ell,\gamma-1)
$$
$$
- \sum_{k=1}^{\gamma} \frac{(-1)^\ell}{\ell+k+1} \frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+2)_k}\Big\{(-1)^\ell + \binom{k+\ell}{k-1}\Big\}.
$$

For $\gamma = 1, 2$, this theorem yields the following Miki-like identities.

PROPOSITION 14 ($\gamma = 1$ IN THEOREM 13).

$$\sum_{k=0}^{\ell+1} \binom{\ell+1}{k} \frac{B_{k+1}(x-y)}{(k+1)^2} \left\{ B_{\ell-k+1}(y) - (-1)^k B_{\ell-k+1}(x) \right\}$$

$$= \sum_{k=0}^{\ell} \frac{B_{k+1}(x)B_{\ell-k+1}(y)}{(k+1)(\ell-k+1)} - \frac{H_{\ell+1}}{\ell+2} \left\{ B_{\ell+2}(x) + B_{\ell+2}(y) \right\}.$$

The special case $x = 1/2$ and $y = 0$ of this formula recovers Dunne-Schubert [9, Eq. 3.3]. We remark also that the last proposition can be derived from the convolution on the equation displayed in Corollary 10, which correspond to the case $\gamma = 0$ instead of $\gamma = 1$.

Rewrite the last formula by absorbing the alternating factor $(-1)^k$ in the first sum through the reciprocal relation

$$(-1)^{k+1} B_{k+1}(x-y) = B_{k+1}(y-x) + (k+1)(y-x)^k. \qquad (4.5)$$

Then evaluate the binomial sum corresponding to the correcting-terms by Lemma 3 with $\varepsilon = 1$ and $\gamma = 0$:

$$\sum_{k=0}^{\ell+1} \binom{\ell+1}{k} \frac{(y-x)^k}{k+1} B_{\ell-k+1}(x) = \sum_{k=0}^{\ell+1} \binom{\ell+2}{k+1} \frac{(y-x)^k}{\ell+2} B_{\ell-k+1}(x)$$

$$= \frac{B_{\ell+2}(x) - B_{\ell+2}(y)}{(x-y)(\ell+2)}.$$

This leads the last proposition equivalently to the following identity.

COROLLARY 15 (PAN-SUN [19, EQ. 2.1]).

$$\sum_{k=0}^{\ell+1} \binom{\ell+1}{k} \frac{B_{k+1}(x-y)B_{\ell-k+1}(y)}{(k+1)^2} + \sum_{k=0}^{\ell+1} \binom{\ell+1}{k} \frac{B_{k+1}(y-x)B_{\ell-k+1}(x)}{(k+1)^2}$$

$$= \sum_{k=0}^{\ell} \frac{B_{k+1}(x)B_{\ell-k+1}(y)}{(k+1)(\ell-k+1)} - \frac{H_{\ell+1}}{\ell+2} \left\{ B_{\ell+2}(x) + B_{\ell+2}(y) \right\} - \frac{B_{\ell+2}(x) - B_{\ell+2}(y)}{(x-y)(\ell+2)}.$$

Taking $x = y$ in the last equation, Pan and Sun [19, Eq. 2.3] obtained the following special case:

$$\sum_{k=0}^{\ell} \frac{B_{k+1}(x)B_{\ell-k+1}(x)}{(k+1)(\ell-k+1)} - \frac{2}{\ell+2} \sum_{k=0}^{\ell} \binom{\ell+2}{k+2} \frac{B_{k+2}B_{\ell-k}(x)}{k+2} = \frac{2}{\ell+2} H_{\ell+1} B_{\ell+2}(x).$$

For $x = 0$ and $x = 1/2$, this reduces respectively to Miki's identity [16] and an identity due to Faber-Pandharipande-Zagier [10, §4.4] (see [9, Eq. 1.8] also).

Proposition 16 ($\gamma = 2$ in Theorem 13).

$$\sum_{k=0}^{\ell+1} \binom{\ell+1}{k} \frac{B_{k+1}(x-y)}{(k+1)^2} \Big\{ B_{\ell-k+1}(y) - (-1)^k B_{\ell-k+1}(x) \Big\}$$

$$= -\frac{H_\ell}{\ell+2} B_{\ell+2}(x) - \frac{x-y}{\ell+1} B_{\ell+1}(x-y) + \sum_{k=0}^{\ell} \frac{B_{k+1}(x) B_{\ell-k+1}(y)}{(k+1)(\ell-k+1)}$$

$$\quad - \frac{H_{\ell+1}}{\ell+2} B_{\ell+2}(y) + \frac{B_{\ell+2}(x-y)}{\ell+2} - \sum_{k=-1}^{\ell} (-1)^{k+\ell} \binom{\ell+2}{k+1} \frac{B_{k+1}(x) B_{\ell-k+1}(y)}{(\ell+1)(\ell+2)}$$

$$= -\frac{H_{\ell+1}}{\ell+2} B_{\ell+2}(x) - (-1)^\ell \frac{x-y}{\ell+1} B_{\ell+1}(x-y) + \sum_{k=0}^{\ell} \frac{B_{k+1}(y) B_{\ell-k+1}(x)}{(k+1)(\ell-k+1)}$$

$$\quad - \frac{H_\ell}{\ell+2} B_{\ell+2}(y) + (-1)^\ell \frac{B_{\ell+2}(x-y)}{\ell+2} - \sum_{k=-1}^{\ell} (-1)^{k+\ell} \binom{\ell+2}{k+1} \frac{B_{k+1}(y) B_{\ell-k+1}(x)}{(\ell+1)(\ell+2)}.$$

Repeating the same process from Proposition 14 to Corollary 15, we can reformulate the last proposition to the following Miki-like identities.

Corollary 17 (Miki-like identities).

$$\sum_{k=0}^{\ell+1} \binom{\ell+1}{k} \Big\{ \frac{B_{k+1}(x-y) B_{\ell-k+1}(y)}{(k+1)^2} + \frac{B_{k+1}(y-x) B_{\ell-k+1}(x)}{(k+1)^2} \Big\}$$

$$\quad + \frac{B_{\ell+2}(x) - B_{\ell+2}(y)}{(x-y)(\ell+2)} - \sum_{k=0}^{\ell} \frac{B_{k+1}(x) B_{\ell-k+1}(y)}{(k+1)(\ell-k+1)}$$

$$= -\frac{H_\ell}{\ell+2} B_{\ell+2}(x) - \frac{x-y}{\ell+1} B_{\ell+1}(x-y) + \frac{B_{\ell+2}(x-y)}{\ell+2}$$

$$\quad - \frac{H_{\ell+1}}{\ell+2} B_{\ell+2}(y) - \sum_{k=-1}^{\ell} (-1)^{k+\ell} \binom{\ell+2}{k+1} \frac{B_{k+1}(x) B_{\ell-k+1}(y)}{(\ell+1)(\ell+2)}$$

$$= -\frac{H_{\ell+1}}{\ell+2} B_{\ell+2}(x) - (-1)^\ell \frac{x-y}{\ell+1} B_{\ell+1}(x-y) + (-1)^\ell \frac{B_{\ell+2}(x-y)}{\ell+2}$$

$$\quad - \frac{H_\ell}{\ell+2} B_{\ell+2}(y) - \sum_{k=-1}^{\ell} (-1)^{k+\ell} \binom{\ell+2}{k+1} \frac{B_{k+1}(y) B_{\ell-k+1}(x)}{(\ell+1)(\ell+2)}.$$

4.3 – Multiplying by the weight factor $(m+1)(n+1)$ the equations displayed in Theorem 9, the convolution with respect to $m+n=\ell$ can be computed similarly as follows.

For (4.3a), performing replacement $k \to m - k$, interchanging the summation order and then applying Lemma 5, we can compute the convolution as follows:

$$\sum_{m+n=\ell}(m+1)(n+1)\text{Eq.}\,(4.3a)=\sum_{m=0}^{\ell}(1+m)(1+\ell-m)\sum_{k=0}^{m}\binom{m}{k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}$$

$$=\sum_{k=0}^{\ell}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}\sum_{m=k}^{\ell}\binom{m}{k}(1+m)(1+\ell-m)$$

$$=\sum_{k=0}^{\ell}\binom{\ell+3}{k+3}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}.$$

The convolution for (4.3b) can be obtained in the same way as (4.3a):

$$\sum_{m+n=\ell}(m+1)(n+1)\text{Eq.}\,(4.3b)$$

$$=\sum_{n=0}^{\ell}(1+n)(1+\ell-n)\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}$$

$$=\sum_{k=0}^{\ell}(-1)^{k+1}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\sum_{n=k}^{\ell}\binom{n}{k}(1+n)(1+\ell-n)$$

$$=\sum_{k=0}^{\ell}(-1)^{k+1}\binom{\ell+3}{k+3}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}.$$

For (4.3c), recalling (3.1a) and (3.2), we can compute the convolution by means of Lemma 6 and (3.3b) as follows:

$$\sum_{m+n=\ell}(m+1)(n+1)\text{Eq.}\,(4.3c)$$

$$=\sum_{m=0}^{\ell}(1+m)!(1+\ell-m)!\sum_{k=-1}^{m}\binom{1-\gamma}{m-k}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}$$

$$=W'_{1,1}(\ell,\gamma-1)\frac{B_{\ell+\gamma+1}(y)}{(\ell+\gamma+1)!}+\sum_{k=0}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}W_{1,1}(k,\ell,\gamma-1)$$

$$=-\frac{\gamma-1}{(\ell+3)_\gamma}B_{\ell+\gamma+1}(y)+\sum_{k=0}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(\ell+4)_{\gamma-1}}$$

$$+\sum_{k=-1}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\lambda_2(1+k,1+\ell,\gamma-1).$$

The convolution for (4.3f) reads directly through (4.3c) as follows:

$$\sum_{m+n=\ell} (m+1)(n+1)\text{Eq. (4.3f)}$$

$$= \sum_{n=0}^{\ell}(1+n)!(1+\ell-n)!\sum_{k=-1}^{n}\binom{1-\gamma}{n-k}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}$$

$$= W'_{1,1}(\ell,\gamma-1)\frac{B_{\ell+\gamma+1}(x)}{(\ell+\gamma+1)!} + \sum_{k=0}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}W_{1,1}(k,\ell,\gamma-1)$$

$$= -\frac{\gamma-1}{(\ell+3)_\gamma}B_{\ell+\gamma+1}(x) + \sum_{k=0}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(\ell+4)_{\gamma-1}}$$

$$+ \sum_{k=-1}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\lambda_2(1+k,1+\ell,\gamma-1).$$

For (4.3d), we can compute the convolution, by invoking Lemma 7, as follows:

$$\sum_{m+n=\ell}(m+1)(n+1)\text{Eq. (4.3d)}$$

$$= \sum_{m=0}^{\ell}(1+m)!(1+\ell-m)!\sum_{k=1}^{\gamma}(-1)^{\ell+k}\binom{-k}{m}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}$$

$$= \sum_{k=1}^{\gamma}(-1)^{\ell+k}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}W_{1,1}(0,\ell,k)$$

$$= \sum_{k=1}^{\gamma}(-1)^{k}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+4)_k}\left\{(-1)^\ell + \binom{k+\ell}{k-1}\frac{\ell^2+\ell k+4\ell+k+5}{(\ell+2)(\ell+3)}\right\}.$$

The convolution for (4.3g) can be evaluated in the same way as that for (4.3d):

$$\sum_{m+n=\ell}(m+1)(n+1)\text{Eq. (4.3g)}$$

$$= -\sum_{n=0}^{\ell}(1+n)!(1+\ell-n)!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}$$

$$= -\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}W_{1,1}(0,\ell,k)$$

$$= -\sum_{k=1}^{\gamma}(-1)^\ell\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+4)_k}\left\{(-1)^\ell + \binom{k+\ell}{k-1}\frac{\ell^2+\ell k+4\ell+k+5}{(\ell+2)(\ell+3)}\right\}.$$

Finally for (4.3e) and (4.3h), it is trivial to have their convolution as follows:

$$\sum_{m+n=\ell}(m+1)(n+1)\text{Eq. (4.3e)} = -\sum_{n=0}^{\ell}\left\{\frac{1+\ell-n}{(\ell+2)_\gamma}B_{\ell+\gamma+1}(x) + \frac{n+1}{(\ell+2)_\gamma}B_{\ell+\gamma+1}(y)\right\}$$
$$= -\frac{\ell+1}{2(\ell+3)_{\gamma-1}}\left\{B_{\ell+\gamma+1}(x) + B_{\ell+\gamma+1}(y)\right\}.$$

Summarizing the computations just displayed, we get the following identities.

THEOREM 18 (MIKI-LIKE IDENTITIES).

$$\sum_{k=0}^{\ell}\binom{\ell+3}{k+3}B_{k+1}(x-y)\left\{\frac{B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma} - (-1)^k\frac{B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}\right\}$$

$$= \sum_{k=0}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(\ell+4)_{\gamma-1}} + \sum_{k=-1}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\lambda_2(1+k,1+\ell,\gamma-1)$$

$$+ \sum_{k=1}^{\gamma}(-1)^k\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+4)_k}\left\{(-1)^\ell + \binom{k+\ell}{k-1}\frac{\ell^2+\ell k+4\ell+k+5}{(\ell+2)(\ell+3)}\right\}$$

$$- \frac{\gamma-1}{(\ell+3)_\gamma}B_{\ell+\gamma+1}(y) - \frac{\ell+1}{2(\ell+3)_{\gamma-1}}\left\{B_{\ell+\gamma+1}(x) + B_{\ell+\gamma+1}(y)\right\}$$

$$= \sum_{k=0}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(\ell+4)_{\gamma-1}} + \sum_{k=-1}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\lambda_2(1+k,1+\ell,\gamma-1)$$

$$- \sum_{k=1}^{\gamma}(-1)^\ell\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+4)_k}\left\{(-1)^\ell + \binom{k+\ell}{k-1}\frac{\ell^2+\ell k+4\ell+k+5}{(\ell+2)(\ell+3)}\right\}$$

$$- \frac{\gamma-1}{(\ell+3)_\gamma}B_{\ell+\gamma+1}(x) - \frac{\ell+1}{2(\ell+3)_{\gamma-1}}\left\{B_{\ell+\gamma+1}(x) + B_{\ell+\gamma+1}(y)\right\}.$$

For $\gamma = 1, 2$, this theorem yields the following Miki-like identities.

PROPOSITION 19 ($\gamma = 1$ IN THEOREM 18).

$$\sum_{k=0}^{\ell}\binom{\ell+2}{k+2}B_k(x-y)\left\{B_{\ell-k}(y) + (-1)^kB_{\ell-k}(x)\right\} = (\ell+2)\sum_{k=0}^{\ell}B_k(x)B_{\ell-k}(y).$$

This formula can alternatively be derived from the convolution on the equation displayed in Corollary 10 corresponding to the case $\gamma = 0$ instead of $\gamma = 1$.

Rewrite the last formula by absorbing the alternating factor $(-1)^k$ in the first sum through the reciprocal relation (4.5). Then evaluate the binomial sum corresponding to the correcting-terms by Lemma 4 with $\varepsilon = 3$ and $\gamma = 0$:

$$\sum_{k=0}^{\ell} k\binom{\ell+2}{k+2}(y-x)^{k-1}B_{\ell-k}(x) = \sum_{k=0}^{\ell-1}(k+1)\binom{\ell+2}{k+3}(y-x)^k B_{\ell-k-1}(x)$$

$$= (\ell+2)\frac{B_{\ell+1}(x) + B_{\ell+1}(y)}{(x-y)^2} - 2\frac{B_{\ell+2}(x) - B_{\ell+2}(y)}{(x-y)^3}.$$

We recover from the last proposition the following Miki-like identity.

COROLLARY 20 (PAN-SUN [19, EQ. 2.2]).

$$\sum_{k=0}^{\ell}\binom{\ell+2}{k+2}\Big\{B_k(x-y)B_{\ell-k}(y) + B_k(y-x)B_{\ell-k}(x)\Big\}$$

$$= (\ell+2)\sum_{k=0}^{\ell} B_k(x)B_{\ell-k}(y)$$

$$+ 2\frac{B_{\ell+2}(x) - B_{\ell+2}(y)}{(x-y)^3} - \frac{\ell+2}{(x-y)^2}\Big\{B_{\ell+1}(x) + B_{\ell+1}(y)\Big\}.$$

Taking $x = y$ in the last equation, Pan and Sun [19, Eq. 2.4] obtained the following special case:

$$\sum_{k=0}^{\ell} B_k(x)B_{\ell-k}(x) - \frac{2}{\ell+2}\sum_{k=2}^{\ell}\binom{\ell+2}{k+2}B_k B_{\ell-k}(x) = (\ell+1)B_{\ell}(x).$$

For $x = 0$, this recovers Matiyasevich's identity [15, #0202].

PROPOSITION 21 ($\gamma = 2$ IN THEOREM 18).

$$\sum_{k=0}^{\ell}\binom{\ell+2}{k+2}B_k(x-y)\Big\{B_{\ell-k}(y) + (-1)^k B_{\ell-k}(x)\Big\}$$

$$= (\ell+2)\sum_{k=0}^{\ell} B_k(x)B_{\ell-k}(y)\Big\{1 + \frac{(-1)^{k+\ell}}{\ell}\binom{\ell}{k}\Big\}$$

$$+ \frac{(\ell+2)(\ell-1)}{\ell}B_{\ell}(x-y) - (\ell+2)(x-y)B_{\ell-1}(x-y)$$

$$= (\ell+2)\sum_{k=0}^{\ell} B_k(y)B_{\ell-k}(x)\Big\{1 + \frac{(-1)^{k+\ell}}{\ell}\binom{\ell}{k}\Big\}$$

$$+ (-1)^{\ell}\frac{(\ell+2)(\ell-1)}{\ell}B_{\ell}(x-y) - (-1)^{\ell}(\ell+2)(x-y)B_{\ell-1}(x-y).$$

Repeating the same process from Proposition 19 to Corollary 20, we can reformulate the last proposition to the following Miki-like identities.

COROLLARY 22 (MIKI-LIKE IDENTITIES).

$$\sum_{k=0}^{\ell} \binom{\ell+2}{k+2} \Big\{ B_k(x-y)B_{\ell-k}(y) + B_k(y-x)B_{\ell-k}(x) \Big\}$$

$$- 2\frac{B_{\ell+2}(x) - B_{\ell+2}(y)}{(x-y)^3} - (\ell+2)\sum_{k=0}^{\ell} B_k(y)B_{\ell-k}(x)$$

$$= -(\ell+2)\frac{B_{\ell+1}(x) + B_{\ell+1}(y)}{(x-y)^2} + \frac{\ell+2}{\ell}\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}B_k(y)B_{\ell-k}(x)$$

$$+ \frac{(\ell+2)(\ell-1)}{\ell}B_\ell(x-y) - (\ell+2)(x-y)B_{\ell-1}(x-y)$$

$$= -(\ell+2)\frac{B_{\ell+1}(x) + B_{\ell+1}(y)}{(x-y)^2} + \frac{\ell+2}{\ell}\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}B_k(x)B_{\ell-k}(y)$$

$$+ (-1)^\ell\frac{(\ell+2)(\ell-1)}{\ell}B_\ell(x-y) - (-1)^\ell(\ell+2)(x-y)B_{\ell-1}(x-y).$$

5 – $\Omega_{m,n}[{}^{0,\,\gamma}_{E,B}](x,y) - \Omega^\star_{n,m}[{}^{0,\,\gamma}_{E,B}](y,x)$

For both sequences $\Omega_{m,n}[{}^{\alpha,\gamma}_{F,G}](x,y)$ and $\Omega^\star_{m,n}[{}^{\alpha,\gamma}_{F,G}](x,y)$, we specify with $\alpha = 0$, $F_m(x) = E_m(x)$ and $G_n(y) = B_n(y)$. According to Theorem 2, the bivariate exponential generating function for the corresponding symmetric difference $\Omega_{m,n}(x,y) - \Omega^\star_{n,m}(y,x)$ reads as follows:

$$\frac{2e^{u(x-y)}}{(u+v)^\gamma(e^u+1)}\left\{ \frac{(u+v)e^{y(u+v)}}{e^{u+v}-1} - \sum_{i=0}^{\gamma-1}\frac{(u+v)^i}{i!}B_i(y) \right\}$$

$$- \frac{2e^{v(y-x)}}{(u+v)^\gamma(e^{-v}+1)}\left\{ \frac{(u+v)e^{x(u+v)}}{e^{u+v}-1} - \sum_{j=0}^{\gamma-1}\frac{(u+v)^j}{j!}B_j(x) \right\}.$$

5.1 – Reducing the main term of the last expression

$$\frac{2(u+v)^{1-\gamma}e^{ux+vy}}{(e^u+1)(e^{u+v}-1)} - \frac{2(u+v)^{1-\gamma}e^{ux+vy}}{(e^{-v}+1)(e^{u+v}-1)} = -2\frac{(u+v)^{1-\gamma}e^{ux+vy}}{(e^u+1)(e^v+1)}$$

we can reformulate the generating function as

$$-\frac{(u+v)^{1-\gamma}}{2}\frac{4e^{ux+vy}}{(e^u+1)(e^v+1)} - \frac{2e^{u(x-y)}}{e^u+1}\sum_{i=1}^{\gamma}\frac{(u+v)^{-i}}{(\gamma-i)!}B_{\gamma-i}(y) \tag{5.1a}$$

$$+\frac{2e^{v(y-x)}}{e^{-v}+1}\sum_{j=1}^{\gamma}\frac{(u+v)^{-j}}{(\gamma-j)!}B_{\gamma-j}(x). \tag{5.1b}$$

Recalling the generating functions of Euler and Bernoulli polynomials and then extracting the coefficients of $[\frac{u^m v^n}{m!n!}]$ according to the two expansions displayed in (4.2a) and (4.2b), we can respectively establish the two corresponding expressions for the symmetric difference $\Omega_{m,n}(x,y) - \Omega^\star_{n,m}(y,x)$, which are explicitly displayed as the following theorem.

THEOREM 23 (SYMMETRIC DIFFERENCE).

$$\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m-k}(x-y)B_{n+k+\gamma}(y)}{(n+k+1)_\gamma} \tag{5.2a}$$

$$-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{E_{n-k}(x-y)B_{m+k+\gamma}(x)}{(m+k+1)_\gamma} \tag{5.2b}$$

$$=-\frac{m!n!}{2}\sum_{k=0}^{m}\binom{1-\gamma}{k}\frac{E_{m-k}(x)E_{n+k+\gamma-1}(y)}{(m-k)!(n+k+\gamma-1)!} \tag{5.2c}$$

$$+m!n!\sum_{k=1}^{\gamma}(-1)^{m+n+k}\binom{-k}{m}\frac{B_{\gamma-k}(x)E_{m+n+k}(x-y)}{(\gamma-k)!(m+n+k)!} \tag{5.2d}$$

$$=-\frac{m!n!}{2}\sum_{k=0}^{n}\binom{1-\gamma}{k}\frac{E_{n-k}(y)E_{m+k+\gamma-1}(x)}{(n-k)!(m+k+\gamma-1)!} \tag{5.2e}$$

$$-m!n!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{B_{\gamma-k}(y)E_{m+n+k}(x-y)}{(\gamma-k)!(m+n+k)!}. \tag{5.2f}$$

For $\gamma = 0,\ 1,\ 2$, it reduces respectively to the following easier formulae.

COROLLARY 24 (SYMMETRIC DIFFERENCE).

$$\sum_{k=0}^{m}\binom{m}{k}E_{m-k}(x-y)B_{n+k}(y) - \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}E_{n-k}(x-y)B_{m+k}(x)$$

$$=-\frac{1}{2}\Big\{mE_{m-1}(x)E_n(y) + nE_m(x)E_{n-1}(y)\Big\}.$$

COROLLARY 25 (PAN-SUN [19, EQ. 2.12]).

$$\sum_{k=0}^{m}\binom{m}{k}E_{m-k}(x-y)\frac{B_{n+k+1}(y)}{n+k+1}-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}E_{n-k}(x-y)\frac{B_{m+k+1}(x)}{m+k+1}$$

$$=-\frac{E_m(x)E_n(y)}{2}-\frac{(-1)^n}{\binom{m+n}{n}}\frac{E_{m+n+1}(x-y)}{m+n+1}.$$

COROLLARY 26 (SYMMETRIC DIFFERENCE).

$$\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m-k}(x-y)B_{n+k+2}(y)}{(n+k+1)_2}-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{E_{n-k}(x-y)B_{m+k+2}(x)}{(m+k+1)_2}$$

$$=-\frac{m!n!}{2}\sum_{k=0}^{m}(-1)^k\frac{E_{m-k}(x)E_{n+k+1}(y)}{(m-k)!(n+k+1)!}+(-1)^n\frac{E_{m+n+2}(x-y)}{(m+2)\binom{m+n+2}{n}}$$

$$-(-1)^n\frac{B_1(x)E_{m+n+1}(x-y)}{(m+1)\binom{m+n+1}{n}}=-\frac{m!n!}{2}\sum_{k=0}^{n}(-1)^k\frac{E_{n-k}(y)E_{m+k+1}(x)}{(n-k)!(m+k+1)!}$$

$$-(-1)^n\frac{E_{m+n+2}(x-y)}{(n+2)\binom{m+n+2}{m}}-(-1)^n\frac{B_1(y)E_{m+n+1}(x-y)}{(n+1)\binom{m+n+1}{m}}.$$

In the analogous manner to the derivation of (4.4a) and (4.4b), the last corollary can futher be simplified to the following convolution identities (*cf.* [8, Eq. 4.2] and [14, Eqs. 51.6.1 and 51.6.2]):

$$\sum_{k=0}^{\ell}(-1)^{k+\ell}\binom{\ell}{k}E_k(x)E_{\ell-k}(y)=2(x-y)E_\ell(x-y)-2E_{\ell+1}(x-y); \qquad (5.3\text{a})$$

$$\sum_{k=0}^{\ell}\binom{\ell}{k}E_k(x)E_{\ell-k}(y)=2E_{\ell+1}(x+y)-2(x+y-1)E_\ell(x-y). \qquad (5.3\text{b})$$

Taking $x=y=0$, we get the following identity on Genocchi numbers

$$\sum_{k=1}^{\ell-1}\binom{\ell}{k}G_kG_{\ell-k}=2G_{\ell+1} \quad \text{with} \quad G_n=E_n(0)$$

which resembles Euler's identity on Bernoulli numbers.

5.2 – We are now going to compute the convolution with respect to $m+n=\ell$ on the equations displayed in Theorem 23.

For (5.2a), performing replacement $k \to m - k$, interchanging the summation order and then applying Lemma 5, we can compute the convolution as follows:

$$\sum_{m+n=\ell} \text{Eq. (5.2a)} = \sum_{m=0}^{\ell} \sum_{k=0}^{m} \binom{m}{k} \frac{E_k(x-y)B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}$$

$$= \sum_{k=0}^{\ell} \frac{E_k(x-y)B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma} \sum_{m=k}^{\ell} \binom{m}{k}$$

$$= \sum_{k=0}^{\ell} \binom{\ell+1}{k+1} \frac{E_k(x-y)B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}.$$

The convolution for (5.2b) reads directly through (5.2a) as follows:

$$\sum_{m+n=\ell} \text{Eq. (5.2b)} = \sum_{n=0}^{\ell} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{E_k(x-y)B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}$$

$$= \sum_{k=0}^{\ell} (-1)^{k+1} \frac{E_k(x-y)B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma} \sum_{n=k}^{\ell} \binom{n}{k}$$

$$= \sum_{k=0}^{\ell} (-1)^{k+1} \binom{\ell+1}{k+1} \frac{E_k(x-y)B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}.$$

For (5.2c), by invoking (3.1a) and Lemma 6, we can compute the convolution:

$$\sum_{m+n=\ell} \text{Eq. (5.2c)} = -\frac{1}{2} \sum_{m=0}^{\ell} m!(\ell-m)! \sum_{k=0}^{m} \binom{1-\gamma}{m-k} \frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{k!(\ell-k+\gamma-1)!}$$

$$= -\frac{1}{2} \sum_{k=0}^{\ell} \frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{k!(\ell-k+\gamma-1)!} W_{0,0}(k,\ell,\gamma-1)$$

$$= -\frac{1}{2} \sum_{k=0}^{\ell} \frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{(\ell+2)_{\gamma-1}} - \frac{1}{2} \sum_{k=0}^{\ell} \frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{k!(\ell-k+\gamma-1)!} \lambda_1(k,\ell,\gamma-1).$$

The convolution for (5.2e) can be done in the same way as (5.2c):

$$
\sum_{m+n=\ell} \text{Eq. (5.2e)} = -\frac{1}{2}\sum_{n=0}^{\ell} n!(\ell-n)! \sum_{k=0}^{n} \binom{1-\gamma}{n-k} \frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{k!(\ell-k+\gamma-1)!}
$$

$$
= -\frac{1}{2}\sum_{k=0}^{\ell} \frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{k!(\ell-k+\gamma-1)!} W_{0,0}(k,\ell,\gamma-1)
$$

$$
= -\frac{1}{2}\sum_{k=0}^{\ell} \frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{(\ell+2)_{\gamma-1}} - \frac{1}{2}\sum_{k=0}^{\ell} \frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{k!(\ell-k+\gamma-1)!}\lambda_1(k,\ell,\gamma-1).
$$

For (5.2d), by invoking (3.1a) and Lemma 7, we can compute the convolution:

$$
\sum_{m+n=\ell} \text{Eq. (5.2d)} = \sum_{m=0}^{\ell} m!(\ell-m)! \sum_{k=1}^{\gamma}(-1)^{\ell+k}\binom{-k}{m} \frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}
$$

$$
= \sum_{k=1}^{\gamma}(-1)^{\ell+k}\frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!} W_{0,0}(0,\ell,k)
$$

$$
= \sum_{k=1}^{\gamma}(-1)^{k}\frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+2)_k}\left\{(-1)^{\ell}+\binom{k+\ell}{k-1}\right\}.
$$

The convolution for (5.2f) can be obtained directly from (5.2d) as follows:

$$
\sum_{m+n=\ell} \text{Eq. (5.2f)} = -\sum_{n=0}^{\ell} n!(\ell-n)! \sum_{k=1}^{\gamma}\binom{-k}{n} \frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}
$$

$$
= -\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!} W_{0,0}(0,\ell,k)
$$

$$
= -\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+2)_k}\left\{1+(-1)^{\ell}\binom{k+\ell}{k-1}\right\}.
$$

Summarizing the computations just displayed, we get the following identities.

Theorem 27 (Miki-like identities).

$$
\sum_{k=0}^{\ell} \binom{\ell+1}{k+1} E_k(x-y) \left\{ \frac{B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma} - (-1)^k \frac{B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma} \right\}
$$

$$
= -\sum_{k=0}^{\ell} \frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{2(\ell+2)_{\gamma-1}} - \sum_{k=0}^{\ell} \frac{\lambda_1(k,\ell,\gamma-1)\,E_k(x)E_{\ell-k+\gamma-1}(y)}{2\times k!(\ell-k+\gamma-1)!}
$$

$$
+ \sum_{k=1}^{\gamma} (-1)^k \frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+2)_k} \left\{ (-1)^\ell + \binom{k+\ell}{k-1} \right\}
$$

$$
= -\sum_{k=0}^{\ell} \frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{2(\ell+2)_{\gamma-1}} - \sum_{k=0}^{\ell} \frac{\lambda_1(k,\ell,\gamma-1)\,E_k(y)E_{\ell-k+\gamma-1}(x)}{2\times k!(\ell-k+\gamma-1)!}
$$

$$
- \sum_{k=1}^{\gamma} (-1)^\ell \frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+2)_k} \left\{ (-1)^\ell + \binom{k+\ell}{k-1} \right\}.
$$

For $\gamma = 1, 2$, this theorem yields the following Miki-like identities.

Proposition 28 ($\gamma = 1$ in Theorem 27).

$$
\sum_{k=0}^{\ell+1} \binom{\ell+2}{k+1} E_k(x-y) \left\{ B_{\ell-k+1}(y) - (-1)^k B_{\ell-k+1}(x) \right\} = -\frac{\ell+2}{2} \sum_{k=0}^{\ell} E_k(x) E_{\ell-k}(y).
$$

Rewrite the last formula by absorbing the alternating factor $(-1)^k$ in the first sum through the reciprocal relation

$$
(-1)^{k+1} E_k(x-y) = E_k(y-x) - 2(y-x)^k. \tag{5.4}
$$

Then evaluate the binomial sum corresponding to the correcting-terms by Lemma 3 with $\varepsilon = 1$ and $\gamma = 0$:

$$
\sum_{k=0}^{\ell+1} \binom{\ell+2}{k+1} (y-x)^k B_{\ell-k+1}(x) = \frac{B_{\ell+2}(x) - B_{\ell+2}(y)}{x-y}.
$$

We recover the following Miki-like identity from the last proposition.

Corollary 29 (Pan-Sun [19, Eq. 2.5]).

$$
\sum_{k=0}^{\ell+1} \binom{\ell+2}{k+1} \left\{ E_k(x-y)B_{\ell-k+1}(y) + E_k(y-x)B_{\ell-k+1}(x) \right\}
$$

$$
= \frac{2}{x-y} \left\{ B_{\ell+2}(x) - B_{\ell+2}(y) \right\} - \frac{\ell+2}{2} \sum_{k=0}^{\ell} E_k(x)E_{\ell-k}(y).
$$

PROPOSITION 30 ($\gamma = 2$ IN THEOREM 27).

$$\sum_{k=0}^{\ell+1} \binom{\ell+2}{k+1} E_k(x-y)\Big\{ B_{\ell-k+1}(y) - (-1)^k B_{\ell-k+1}(x) \Big\}$$

$$= -\frac{\ell+2}{2} \sum_{k=0}^{\ell} E_k(x)E_{\ell-k}(y)\Big\{ 1 - \binom{\ell}{k}(-1)^{k+\ell} \Big\}$$

$$+ (\ell+2)E_{\ell+1}(x-y) - (\ell+2)(x-y)E_\ell(x-y)$$

$$= -\frac{\ell+2}{2} \sum_{k=0}^{\ell} E_k(y)E_{\ell-k}(x)\Big\{ 1 - \binom{\ell}{k}(-1)^{k+\ell} \Big\}$$

$$+ (-1)^\ell(\ell+2)E_{\ell+1}(x-y) - (-1)^\ell(\ell+2)(x-y)E_\ell(x-y).$$

Repeating the same process from Proposition 28 to Corollary 29, we can reformulate the last proposition to the following Miki-like identities.

COROLLARY 31 (MIKI-LIKE IDENTITIES).

$$\sum_{k=0}^{\ell+1} \binom{\ell+2}{k+1} \Big\{ E_k(x-y)B_{\ell-k+1}(y) + E_k(y-x)B_{\ell-k+1}(x) \Big\}$$

$$= (\ell+2)E_{\ell+1}(x-y) - \frac{\ell+2}{2} \sum_{k=0}^{\ell} E_k(x)E_{\ell-k}(y)\Big\{ 1 - \binom{\ell}{k}(-1)^{k+\ell} \Big\}$$

$$+ \frac{2}{x-y}\Big\{ B_{\ell+2}(x) - B_{\ell+2}(y) \Big\} - (\ell+2)(x-y)E_\ell(x-y)$$

$$= (-1)^\ell(\ell+2)E_{\ell+1}(x-y) - \frac{\ell+2}{2} \sum_{k=0}^{\ell} E_k(y)E_{\ell-k}(x)\Big\{ 1 - \binom{\ell}{k}(-1)^{k+\ell} \Big\}$$

$$+ \frac{2}{x-y}\Big\{ B_{\ell+2}(x) - B_{\ell+2}(y) \Big\} - (-1)^\ell(\ell+2)(x-y)E_\ell(x-y).$$

5.3 – Multiplying by the weight factor $(m+1)(n+1)$ the equations displayed in Theorem 23, the convolution with respect to $m+n = \ell$ can be computed similarly as follows.

For (5.2a), performing replacement $k \to m-k$, interchanging the summation order and then applying Lemma 5, we can compute the convolution as follows:

$$
\sum_{m+n=\ell} (m+1)(n+1)\text{Eq. (5.2a)} = \sum_{m=0}^{\ell} (m+1)(\ell-m+1)\sum_{k=0}^{m} \binom{m}{k} \frac{E_k(x-y)B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}
$$

$$
= \sum_{k=0}^{\ell} \frac{E_k(x-y)B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma} \sum_{m=k}^{\ell} \binom{m}{k}(m+1)(\ell-m+1)
$$

$$
= \sum_{k=0}^{\ell} \frac{k+1}{(\ell-k+1)_\gamma} \binom{\ell+3}{k+3} E_k(x-y)B_{\ell-k+\gamma}(y).
$$

The convolution for (5.2b) reads directly from (5.2a) as follows:

$$
\sum_{m+n=\ell} (m+1)(n+1)\text{Eq. (5.2b)}
$$

$$
= \sum_{n=0}^{\ell} (m+1)(\ell-m+1)\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{E_k(x-y)B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}
$$

$$
= \sum_{k=0}^{\ell} (-1)^{k+1} \frac{E_k(x-y)B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma} \sum_{n=k}^{\ell} \binom{n}{k}(m+1)(\ell-m+1)
$$

$$
= \sum_{k=0}^{\ell} (-1)^{k+1} \frac{k+1}{(\ell-k+1)_\gamma} \binom{\ell+3}{k+3} E_k(x-y)B_{\ell-k+\gamma}(x).
$$

For (5.2c), by invoking (3.1a) and Lemma 6, we can compute the convolution:

$$
\sum_{m+n=\ell} (m+1)(n+1)\text{Eq. (5.2c)}
$$

$$
= -\frac{1}{2} \sum_{m=0}^{\ell} (1+m)!(1+\ell-m)!\sum_{k=0}^{m} \binom{1-\gamma}{m-k} \frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{k!(\ell-k+\gamma-1)!}
$$

$$
= -\frac{1}{2} \sum_{k=0}^{\ell} \frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{k!(\ell-k+\gamma-1)!} W_{1,1}(k,\ell,\gamma-1)
$$

$$
= -\frac{1}{2} \sum_{k=0}^{\ell} \frac{(1+k)(\ell+\gamma-k)}{(\ell+4)_{\gamma-1}} E_k(x)E_{\ell-k+\gamma-1}(y)
$$

$$
- \frac{1}{2} \sum_{k=0}^{\ell} \frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{k!(\ell-k+\gamma-1)!} \lambda_2(1+k,1+\ell,\gamma-1).
$$

The convolution for (5.2e) can be done from that for (5.2c) as follows:

$$\sum_{m+n=\ell} (m+1)(n+1)\text{Eq. (5.2e)}$$

$$= -\frac{1}{2}\sum_{n=0}^{\ell}(1+n)!(1+\ell-n)!\sum_{k=0}^{n}\binom{1-\gamma}{n-k}\frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{k!(\ell-k+\gamma-1)!}$$

$$= -\frac{1}{2}\sum_{k=0}^{\ell}\frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{k!(\ell-k+\gamma-1)!}W_{1,1}(k,\ell,\gamma-1)$$

$$= -\frac{1}{2}\sum_{k=0}^{\ell}\frac{(1+k)(\ell+\gamma-k)}{(\ell+4)_{\gamma-1}}E_k(y)E_{\ell-k+\gamma-1}(x)$$

$$-\frac{1}{2}\sum_{k=0}^{\ell}\frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{k!(\ell-k+\gamma-1)!}\lambda_2(1+k,1+\ell,\gamma-1).$$

For (5.2d), by invoking (3.1a) and Lemma 7, we can compute the convolution:

$$\sum_{m+n=\ell} (m+1)(n+1)\text{Eq. (5.2d)}$$

$$= \sum_{m=0}^{\ell}(1+m)!(1+\ell-m)!\sum_{k=1}^{\gamma}(-1)^{\ell+k}\binom{-k}{m}\frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}$$

$$= \sum_{k=1}^{\gamma}(-1)^{\ell+k}\frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}W_{1,1}(0,\ell,k)$$

$$= \sum_{k=1}^{\gamma}(-1)^{k}\frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+4)_k}(\ell+k+1)$$

$$\times\left\{(-1)^{\ell}+\binom{k+\ell}{k-1}\frac{\ell^2+k\ell+4\ell+k+5}{(\ell+2)(\ell+3)}\right\}.$$

The convolution for (5.2f) can be computed in the same manner as that for (5.2d):

$$\sum_{m+n=\ell} (m+1)(n+1)\text{Eq. (5.2f)}$$

$$= -\sum_{n=0}^{\ell} (1+n)!(1+\ell-n)! \sum_{k=1}^{\gamma} \binom{-k}{n} \frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}$$

$$= -\sum_{k=1}^{\gamma} \frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!} W_{1,1}(0,\ell,k)$$

$$= -\sum_{k=1}^{\gamma} \frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+4)_k}(\ell+k+1) \times \left\{ 1+(-1)^{\ell} \binom{k+\ell}{k-1} \frac{\ell^2+k\ell+4\ell+k+5}{(\ell+2)(\ell+3)} \right\}.$$

Summarizing the computations just displayed, we get the following identities.

Theorem 32 (Miki-like identities).

$$\sum_{k=0}^{\ell} \frac{k+1}{(\ell-k+1)_{\gamma}} \binom{\ell+3}{k+3} E_k(x-y) \left\{ B_{\ell-k+\gamma}(y) - (-1)^k B_{\ell-k+\gamma}(x) \right\}$$

$$= -\sum_{k=0}^{\ell} \frac{(1+k)(\ell+\gamma-k)}{2(\ell+4)_{\gamma-1}} E_k(x)E_{\ell-k+\gamma-1}(y)$$

$$- \sum_{k=0}^{\ell} \frac{\lambda_2(1+k,1+\ell,\gamma-1)}{2 \times k!(\ell-k+\gamma-1)!} E_k(x)E_{\ell-k+\gamma-1}(y)$$

$$+ \sum_{k=1}^{\gamma} (-1)^k \frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+4)_k}(\ell+k+1) \left\{ (-1)^{\ell} + \binom{k+\ell}{k-1} \frac{\ell^2+k\ell+4\ell+k+5}{(\ell+2)(\ell+3)} \right\}$$

$$= -\sum_{k=0}^{\ell} \frac{(1+k)(\ell+\gamma-k)}{2(\ell+4)_{\gamma-1}} E_k(y)E_{\ell-k+\gamma-1}(x)$$

$$- \sum_{k=0}^{\ell} \frac{\lambda_2(1+k,1+\ell,\gamma-1)}{2 \times k!(\ell-k+\gamma-1)!} E_k(y)E_{\ell-k+\gamma-1}(x)$$

$$- \sum_{k=1}^{\gamma} (-1)^{\ell} \frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+4)_k}(\ell+k+1) \left\{ (-1)^{\ell} + \binom{k+\ell}{k-1} \frac{\ell^2+k\ell+4\ell+k+5}{(\ell+2)(\ell+3)} \right\}.$$

For $\gamma = 1, 2$, this theorem yields the following Miki-like identities.

Proposition 33 ($\gamma = 1$ in Theorem 32).

$$\sum_{k=0}^{\ell+1} \frac{k+1}{k+3} \binom{\ell+3}{k+2} E_k(x-y) \left\{ B_{\ell-k+1}(y) - (-1)^k B_{\ell-k+1}(x) \right\}$$

$$= -\sum_{k=0}^{\ell} \frac{(1+k)(\ell-k+1)}{2} E_k(x)E_{\ell-k}(y).$$

Rewrite the last formula by absorbing the alternating factor $(-1)^k$ in the first sum through the reciprocal relation (5.4). Then evaluate the binomial sum corresponding to the correcting-terms by Lemma 4 with $\varepsilon = 3$ and $\gamma = 0$:

$$\sum_{k=0}^{\ell+1} \frac{k+1}{k+3} \binom{\ell+3}{k+2} (y-x)^k B_{\ell-k+1}(x)$$

$$= \sum_{k=0}^{\ell+1} \frac{k+1}{\ell+4} \binom{\ell+4}{k+3} (y-x)^k B_{\ell-k+1}(x)$$

$$= \frac{B_{\ell+3}(x) + B_{\ell+3}(y)}{(x-y)^2} - 2\frac{B_{\ell+4}(x) - B_{\ell+4}(y)}{(x-y)^3(\ell+4)}.$$

We get from the last proposition the following equivalent identity.

COROLLARY 34 (MIKI-LIKE IDENTITY).

$$\sum_{k=0}^{\ell+1} \frac{k+1}{k+3} \binom{\ell+3}{k+2} \Big\{ E_k(x-y)B_{\ell-k+1}(y) + E_k(y-x)B_{\ell-k+1}(x) \Big\}$$

$$= 2\frac{B_{\ell+3}(x) + B_{\ell+3}(y)}{(x-y)^2} - 4\frac{B_{\ell+4}(x) - B_{\ell+4}(y)}{(x-y)^3(\ell+4)} - \sum_{k=0}^{\ell} \frac{(1+k)(\ell-k+1)}{2} E_k(x)E_{\ell-k}(y).$$

PROPOSITION 35 ($\gamma = 2$ IN THEOREM 32).

$$\sum_{k=0}^{\ell+1} \frac{k+1}{k+3} \binom{\ell+3}{k+2} E_k(x-y) \Big\{ B_{\ell-k+1}(y) - (-1)^k B_{\ell-k+1}(x) \Big\}$$

$$= (\ell+1)E_{\ell+1}(x-y) - \sum_{k=0}^{\ell} \frac{(1+k)(1+\ell-k)}{2} E_k(x)E_{\ell-k}(y)$$

$$- (\ell+1)(x-y)E_{\ell}(x-y) + \frac{\ell+1}{2} \sum_{k=0}^{\ell} (-1)^{k+\ell} \binom{\ell}{k} E_k(x)E_{\ell-k}(y)$$

$$= (-1)^{\ell}(\ell+1)E_{\ell+1}(x-y) - \sum_{k=0}^{\ell} \frac{(1+k)(1+\ell-k)}{2} E_k(y)E_{\ell-k}(x)$$

$$- (-1)^{\ell}(\ell+1)(x-y)E_{\ell}(x-y) + \frac{\ell+1}{2} \sum_{k=0}^{\ell} (-1)^{k+\ell} \binom{\ell}{k} E_k(y)E_{\ell-k}(x).$$

Repeating the same process from Proposition 33 to Corollary 34, we can reformulate the last proposition to the following Miki-like identities.

COROLLARY 36 (MIKI-LIKE IDENTITIES).

$$\sum_{k=0}^{\ell+1} \frac{k+1}{k+3} \binom{\ell+3}{k+2} \left\{ E_k(x-y)B_{\ell-k+1}(y) + E_k(y-x)B_{\ell-k+1}(x) \right\}$$

$$+ 4\frac{B_{\ell+4}(x) - B_{\ell+4}(y)}{(x-y)^3(\ell+4)} - 2\frac{B_{\ell+3}(x) + B_{\ell+3}(y)}{(x-y)^2}$$

$$= (\ell+1)E_{\ell+1}(x-y) - \sum_{k=0}^{\ell} \frac{(1+k)(1+\ell-k)}{2} E_k(x)E_{\ell-k}(y)$$

$$- (\ell+1)(x-y)E_\ell(x-y) + \frac{\ell+1}{2}\sum_{k=0}^{\ell}(-1)^{k+\ell}\binom{\ell}{k} E_k(x)E_{\ell-k}(y)$$

$$= (-1)^\ell(\ell+1)E_{\ell+1}(x-y) - \sum_{k=0}^{\ell} \frac{(1+k)(1+\ell-k)}{2} E_k(y)E_{\ell-k}(x)$$

$$- (-1)^\ell(\ell+1)(x-y)E_\ell(x-y) + \frac{\ell+1}{2}\sum_{k=0}^{\ell}(-1)^{k+\ell}\binom{\ell}{k} E_k(y)E_{\ell-k}(x).$$

6 $-\ 2\Omega_{m,n}\big[{1,\gamma \atop B,E}\big](x,y) + \Omega^\star_{n,m}\big[{0,\gamma \atop E,E}\big](y,x)$

For both sequences $\Omega_{m,n}\big[{\alpha,\gamma \atop F,G}\big](x,y)$ and $\Omega^\star_{m,n}\big[{\beta,\gamma \atop G,G}\big](x,y)$, we specify with $\alpha = 1$, $\beta = 0$, $F_m(x) = B_m(x)$ and $G_n(y) = E_n(y)$. According to Theorem 2, the bivariate exponential generating function for the corresponding symmetric combination $2\Omega_{m,n}(x,y) + \Omega^\star_{n,m}(y,x)$ reads as follows:

$$\frac{2/u}{(u+v)^\gamma} \left\{ \frac{ue^{u(x-y)}}{e^u - 1} - 1 \right\} \left\{ \frac{2e^{y(u+v)}}{e^{u+v} + 1} - \sum_{i=0}^{\gamma-1} \frac{(u+v)^i}{i!} E_i(y) \right\}$$

$$+ \frac{2e^{v(y-x)}}{(u+v)^\gamma(e^{-v} + 1)} \left\{ \frac{2e^{x(u+v)}}{e^{u+v} + 1} - \sum_{j=0}^{\gamma-1} \frac{(u+v)^j}{j!} E_j(x) \right\}.$$

6.1 – Reducing the main term of the last expression

$$\frac{4(u+v)^{-\gamma}e^{ux+vy}}{(e^u - 1)(e^{u+v} + 1)} + \frac{4(u+v)^{-\gamma}e^{ux+vy}}{(e^{-v} + 1)(e^{u+v} + 1)} = 4\frac{(u+v)^{-\gamma}e^{ux+vy}}{(e^u - 1)(e^v + 1)}$$

we can reformulate the generating function as

$$4\frac{(u+v)^{-\gamma}e^{ux+vy}}{(e^u-1)(e^v+1)} - \frac{4/u}{(u+v)^\gamma}\frac{e^{(u+v)y}}{(e^{u+v}+1)} + \frac{2}{u}\sum_{i=1}^{\gamma}\frac{(u+v)^{-i}}{(\gamma-i)!}E_{\gamma-i}(y) \tag{6.1a}$$

$$- \frac{2e^{u(x-y)}}{e^u-1}\sum_{j=1}^{\gamma}\frac{(u+v)^{-j}}{(\gamma-j)!}E_{\gamma-j}(y) - \frac{2e^{v(y-x)}}{e^{-v}+1}\sum_{j=1}^{\gamma}\frac{(u+v)^{-j}}{(\gamma-j)!}E_{\gamma-j}(x). \tag{6.1b}$$

Recalling the generating functions of Euler and Bernoulli polynomials and then extracting the coefficients of $[\frac{u^m v^n}{m!n!}]$ according to the two expansions displayed in (4.2a) and (4.2b), we can respectively establish the two corresponding expressions for the symmetric sum $2\Omega_{m,n}(x,y) + \Omega^\star_{n,m}(y,x)$, which are explicitly displayed as the following theorem.

THEOREM 37 (SYMMETRIC DIFFERENCE).

$$2\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)E_{n+k+\gamma}(y)}{(m-k+1)(n+k+1)_\gamma} \tag{6.2a}$$

$$+ \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{E_{n-k}(x-y)E_{m+k+\gamma}(x)}{(m+k+1)_\gamma} \tag{6.2b}$$

$$= 2m!n!\sum_{k=0}^{m+1}\binom{-\gamma}{k}\frac{B_{m-k+1}(x)E_{n+k+\gamma}(y)}{(m-k+1)!(n+k+\gamma)!} - \frac{2E_{m+n+\gamma+1}(y)}{(m+1)(m+n+2)_\gamma} \tag{6.2c}$$

$$- m!n!\sum_{k=1}^{\gamma}(-1)^{m+n+k}\binom{-k}{m}\frac{E_{\gamma-k}(x)E_{m+n+k}(x-y)}{(\gamma-k)!(m+n+k)!} \tag{6.2d}$$

$$= 2m!n!\sum_{k=0}^{n}\binom{-\gamma}{k}\frac{E_{n-k}(y)B_{m+k+\gamma+1}(x)}{(n-k)!(m+k+\gamma+1)!} - \frac{2E_{m+n+\gamma+1}(y)}{(m+1)(m+n+2)_\gamma} \tag{6.2e}$$

$$- 2m!n!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{E_{\gamma-k}(y)B_{m+n+k+1}(x-y)}{(\gamma-k)!(m+n+k+1)!}. \tag{6.2f}$$

For $\gamma = 0,\ 1,\ 2$, it reduces respectively to the following easier formulae.

COROLLARY 38 (SYMMETRIC DIFFERENCE).

$$2\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)E_{n+k}(y)}{m-k+1}$$

$$+ \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}E_{n-k}(x-y)E_{m+k}(x)$$

$$= \frac{2B_{m+1}(x)E_n(y)}{m+1} - \frac{2E_{m+n+1}(y)}{m+1}.$$

COROLLARY 39 (SYMMETRIC DIFFERENCE).

$$2\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)E_{n+k+1}(y)}{(m-k+1)(n+k+1)}$$

$$+\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{E_{n-k}(x-y)E_{m+k+1}(x)}{m+k+1}$$

$$=2m!n!\sum_{k=0}^{m+1}(-1)^{k}\frac{B_{m-k+1}(x)E_{n+k+1}(y)}{(m-k+1)!(n+k+1)!}$$

$$+(-1)^{n}\frac{m!n!}{(m+n+1)!}E_{m+n+1}(x-y)-\frac{2E_{m+n+2}(y)}{(m+1)(m+n+2)}$$

$$=2m!n!\sum_{k=0}^{n}(-1)^{k}\frac{E_{n-k}(y)B_{m+k+2}(x)}{(n-k)!(m+k+2)!}$$

$$-(-1)^{n}\frac{2m!n!}{(m+n+2)!}B_{m+n+2}(x-y)-\frac{2E_{m+n+2}(y)}{(m+1)(m+n+2)}.$$

COROLLARY 40 (SYMMETRIC DIFFERENCE).

$$2\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)E_{n+k+2}(y)}{(m-k+1)(n+k+1)_{2}}$$

$$+\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{E_{n-k}(x-y)E_{m+k+2}(x)}{(m+k+1)_{2}}$$

$$=2m!n!\sum_{k=0}^{m+1}(-1)^{k}\frac{(k+1)B_{m-k+1}(x)E_{n+k+2}(y)}{(m-k+1)!(n+k+2)!}-\frac{2E_{m+n+3}(y)}{(m+1)(m+n+2)_{2}}$$

$$+(-1)^{n}\frac{m!n!E_{1}(x)}{(m+n+1)!}E_{m+n+1}(x-y)-(-1)^{n}\frac{(m+1)!n!}{(m+n+2)!}E_{m+n+2}(x-y)$$

$$=2m!n!\sum_{k=0}^{n}(-1)^{k}\frac{(k+1)E_{n-k}(y)B_{m+k+3}(x)}{(n-k)!(m+k+3)!}-\frac{2E_{m+n+3}(y)}{(m+1)(m+n+2)_{2}}$$

$$-(-1)^{n}\frac{2m!n!E_{1}(y)}{(m+n+2)!}B_{m+n+2}(x-y)-(-1)^{n}\frac{2m!(n+1)!}{(m+n+3)!}B_{m+n+3}(x-y).$$

In addition, we can derive from the last equation the convolution identities:

$$\sum_{k=0}^{\ell}(-1)^{k+\ell}\binom{\ell}{k}B_{k}(x)E_{\ell-k}(y)=B_{\ell}(x-y)+\frac{\ell}{2}E_{\ell-1}(x-y);\qquad(6.3\text{a})$$

$$\sum_{k=0}^{\ell}\binom{\ell}{k}B_{k}(x)E_{\ell-k}(y)=B_{\ell}(x+y)-\frac{\ell}{2}E_{\ell-1}(x+y).\qquad(6.3\text{b})$$

6.2 – We are now going to compute the convolution with respect to $m + n = \ell$ on the equations displayed in Theorem 37.

For (6.2a), performing replacement $k \to m - k$, interchanging the summation order and then applying Lemma 5, we can compute the convolution as follows:

$$\sum_{m+n=\ell} \text{Eq. (6.2a)} = 2 \sum_{m=0}^{\ell} \sum_{k=0}^{m} \binom{m}{k} \frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_{\gamma}}$$

$$= 2 \sum_{k=0}^{\ell} \frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_{\gamma}} \sum_{m=k}^{\ell} \binom{m}{k}$$

$$= 2 \sum_{k=0}^{\ell} \binom{\ell+1}{k+1} \frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_{\gamma}}.$$

The convolution for (6.2b) reads symmetrically from that for (6.2a) as follows:

$$\sum_{m+n=\ell} \text{Eq. (6.2b)} = \sum_{n=0}^{\ell} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \frac{E_{k}(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_{\gamma}}$$

$$= \sum_{k=0}^{\ell} (-1)^{k} \frac{E_{k}(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_{\gamma}} \sum_{n=k}^{\ell} \binom{n}{k}$$

$$= \sum_{k=0}^{\ell} (-1)^{k} \binom{\ell+1}{k+1} \frac{E_{k}(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_{\gamma}}.$$

For (6.2c), by invoking Lemma 6 and (3.3a), we can compute the convolution:

$$\sum_{m+n=\ell} \text{Eq. (6.2c)}$$

$$= -\frac{2H_{\ell+1}}{(\ell+2)_{\gamma}} E_{\ell+\gamma+1}(y) + 2 \sum_{m=0}^{\ell} m!(\ell-m)! \sum_{k=-1}^{m} \binom{-\gamma}{m-k} \frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}$$

$$= -\frac{2H_{\ell+1}}{(\ell+2)_{\gamma}} E_{\ell+\gamma+1}(y) + 2 \frac{E_{\ell+\gamma+1}(y)}{(\ell+\gamma+1)!} W'_{0,0}(\ell,\gamma) + 2 \sum_{k=0}^{\ell} \frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!} W_{0,0}(k,\ell,\gamma)$$

$$= -\frac{2H_{\ell+\gamma+1}}{(\ell+2)_{\gamma}} E_{\ell+\gamma+1}(y) + 2 \sum_{k=0}^{\ell} \frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)(\ell+2)_{\gamma}} + 2 \sum_{k=-1}^{\ell} \frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!} \lambda_{1}(k,\ell,\gamma).$$

The convolution for (6.2e) can be computed analogously as that for (6.2c) as follows:

$$\sum_{m+n=\ell} \text{Eq. (6.2e)}$$

$$= -\frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(y) + 2\sum_{n=0}^{\ell} n!(\ell-n)! \sum_{k=0}^{n} \binom{-\gamma}{n-k} \frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!}$$

$$= -\frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(y) + 2\sum_{k=0}^{\ell} \frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!} W_{0,0}(k,\ell,\gamma)$$

$$= -\frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(y) + 2\sum_{k=0}^{\ell} \frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{(\ell-k+\gamma+1)(\ell+2)_\gamma} + 2\sum_{k=0}^{\ell} \frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!} \lambda_1(k,\ell,\gamma).$$

For (6.2d), by invoking (3.1a) and Lemma 7, we can compute the convolution:

$$\sum_{m+n=\ell} \text{Eq. (6.2d)} = \sum_{m=0}^{\ell} m!(\ell-m)! \sum_{k=1}^{\gamma} (-1)^{\ell+k-1} \binom{-k}{m} \frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}$$

$$= \sum_{k=1}^{\gamma} (-1)^{\ell+k-1} \frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!} W_{0,0}(0,\ell,k)$$

$$= \sum_{k=1}^{\gamma} (-1)^{k-1} \frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+2)_k} \left\{ (-1)^\ell + \binom{k+\ell}{k-1} \right\}.$$

The convolution for (6.2f) can be got directly from that for (6.2d) as follows:

$$\sum_{m+n=\ell} \text{Eq. (6.2f)} = -2\sum_{n=0}^{\ell} n!(\ell-n)! \sum_{k=1}^{\gamma} \binom{-k}{n} \frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}$$

$$= -2\sum_{k=1}^{\gamma} \frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!} W_{0,0}(0,\ell,k)$$

$$= -2\sum_{k=1}^{\gamma} \frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)(\ell+2)_k} \left\{ 1 + (-1)^\ell \binom{k+\ell}{k-1} \right\}.$$

Summarizing the computations just displayed, we get the following identities.

THEOREM 41 (MIKI-LIKE IDENTITIES).

$$2\sum_{k=0}^{\ell}\binom{\ell+1}{k+1}\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}+\sum_{k=0}^{\ell}(-1)^k\binom{\ell+1}{k+1}\frac{E_k(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}$$

$$=2\sum_{k=0}^{\ell}\frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)(\ell+2)_\gamma}+2\sum_{k=-1}^{\ell}\frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\lambda_1(k,\ell,\gamma)$$

$$-\frac{2H_{\ell+\gamma+1}}{(\ell+2)_\gamma}E_{\ell+\gamma+1}(y)+\sum_{k=1}^{\gamma}(-1)^{k-1}\frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+2)_k}\left\{(-1)^\ell+\binom{k+\ell}{k-1}\right\}$$

$$=2\sum_{k=0}^{\ell}\frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{(\ell-k+\gamma+1)(\ell+2)_\gamma}+2\sum_{k=0}^{\ell}\frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!}\lambda_1(k,\ell,\gamma)$$

$$-\frac{2H_{\ell+1}}{(\ell+2)_\gamma}E_{\ell+\gamma+1}(y)-2\sum_{k=1}^{\gamma}\frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)(\ell+2)_k}\left\{1+(-1)^\ell\binom{k+\ell}{k-1}\right\}.$$

For $\gamma=0,1$, we derive from this theorem the following Miki-like identities.

PROPOSITION 42 ($\gamma=0$ IN THEOREM 41).

$$\sum_{k=0}^{\ell}\binom{\ell+1}{k+1}\left\{\frac{2B_{k+1}(x-y)E_{\ell-k}(y)}{k+1}+(-1)^kE_k(x-y)E_{\ell-k}(x)\right\}$$

$$=2\sum_{k=0}^{\ell}\frac{B_{k+1}(x)E_{\ell-k}(y)}{k+1}-2H_{\ell+1}E_{\ell+1}(y).$$

Rewrite the last formula by absorbing the alternating factor $(-1)^k$ in the first sum through the reciprocal relation (5.4). Then evaluate the binomial sum corresponding to the correcting-terms by Lemma 3 with $\varepsilon=1$ and $\gamma=0$:

$$\sum_{k=0}^{\ell}\binom{\ell+1}{k+1}(y-x)^kE_{\ell-k}(x)=\frac{E_{\ell+1}(x)-E_{\ell+1}(y)}{x-y}.$$

We derive from the last proposition the following formula.

COROLLARY 43 (PAN-SUN [19, EQ. 2.6]).

$$2\sum_{k=0}^{\ell}\binom{\ell+1}{k+1}\frac{B_{k+1}(x-y)E_{\ell-k}(y)}{k+1}-\sum_{k=0}^{\ell}\binom{\ell+1}{k+1}E_k(y-x)E_{\ell-k}(x)$$

$$=2\sum_{k=0}^{\ell}\frac{B_{k+1}(x)E_{\ell-k}(y)}{k+1}-2\frac{E_{\ell+1}(x)-E_{\ell+1}(y)}{(x-y)}-2H_{\ell+1}E_{\ell+1}(y).$$

PROPOSITION 44 ($\gamma = 1$ IN THEOREM 41).

$$\sum_{k=0}^{\ell} \binom{\ell+1}{k+1} \left\{ \frac{2B_{k+1}(x-y)E_{\ell-k}(y)}{k+1} + (-1)^k E_k(x-y)E_{\ell-k}(x) \right\}$$

$$= 2\sum_{k=0}^{\ell} \frac{B_{k+1}(x)E_{\ell-k}(y)}{k+1} \left\{ 1 - (-1)^{k+\ell}\binom{\ell}{k} \right\} + E_\ell(x-y)$$

$$- 2H_{\ell+1}E_{\ell+1}(y) + \frac{2(-1)^\ell}{\ell+1}E_{\ell+1}(y) + 2\frac{B_{\ell+1}(x-y)}{\ell+1}$$

$$= 2\sum_{k=0}^{\ell} \frac{E_k(y)B_{\ell-k+1}(x)}{(\ell-k+1)} \left\{ 1 - (-1)^{k+\ell}\binom{\ell}{k} \right\} + (-1)^\ell E_\ell(x-y)$$

$$- 2H_{\ell+1}E_{\ell+1}(y) + \frac{2E_{\ell+1}(y)}{\ell+1} + 2(-1)^\ell\frac{B_{\ell+1}(x-y)}{\ell+1}.$$

Repeating the same process from Proposition 42 to Corollary 43, we can reformulate the last proposition to the following Miki-like identities.

COROLLARY 45 (MIKI-LIKE IDENTITIES).

$$\sum_{k=0}^{\ell} \binom{\ell+1}{k+1} \left\{ \frac{2B_{k+1}(x-y)E_{\ell-k}(y)}{k+1} - E_k(y-x)E_{\ell-k}(x) \right\}$$

$$= 2\sum_{k=0}^{\ell} \frac{B_{k+1}(x)E_{\ell-k}(y)}{k+1} \left\{ 1 - (-1)^{k+\ell}\binom{\ell}{k} \right\} - 2\frac{E_{\ell+1}(x) - E_{\ell+1}(y)}{x-y}$$

$$- 2H_{\ell+1}E_{\ell+1}(y) + \frac{2(-1)^\ell}{\ell+1}E_{\ell+1}(y) + 2\frac{B_{\ell+1}(x-y)}{\ell+1} + E_\ell(x-y)$$

$$= 2\sum_{k=0}^{\ell} \frac{E_k(y)B_{\ell-k+1}(x)}{(\ell-k+1)} \left\{ 1 - (-1)^{k+\ell}\binom{\ell}{k} \right\} - 2\frac{E_{\ell+1}(x) - E_{\ell+1}(y)}{x-y}$$

$$- 2H_{\ell+1}E_{\ell+1}(y) + \frac{2E_{\ell+1}(y)}{\ell+1} + 2(-1)^\ell\frac{B_{\ell+1}(x-y)}{\ell+1} + (-1)^\ell E_\ell(x-y).$$

6.3 – Multiplying by the weight factor $(m+1)$ the equations displayed in Theorem 37, the convolution with respect to $m+n = \ell$ can be computed similarly as follows.

For (6.2a), performing replacement $k \to m - k$, interchanging the summation order and then applying Lemma 5, we can compute the convolution as follows:

$$
\begin{aligned}
\sum_{m+n=\ell} (m+1)\text{Eq. (6.2a)} &= 2 \sum_{m=0}^{\ell} (m+1) \sum_{k=0}^{m} \binom{m}{k} \frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma} \\
&= 2 \sum_{k=0}^{\ell} \frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma} \sum_{m=k}^{\ell} \binom{m}{k}(m+1) \\
&= 2 \sum_{k=0}^{\ell} \binom{\ell+2}{k+2} \frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}.
\end{aligned}
$$

The convolution for (6.2b) can similarly be done as that for (6.2a) as follows:

$$
\begin{aligned}
\sum_{m+n=\ell} (m+1)\text{Eq. (6.2b)} &= \sum_{n=0}^{\ell} (1+\ell-n) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{E_k(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma} \\
&= \sum_{k=0}^{\ell} (-1)^k \frac{E_k(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma} \sum_{n=k}^{\ell} \binom{n}{k}(1+\ell-n) \\
&= \sum_{k=0}^{\ell} (-1)^k \binom{\ell+2}{k+2} \frac{E_k(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}.
\end{aligned}
$$

For (6.2c), by invoking Lemma 6 and (3.3b), we can compute the convolution:

$$
\begin{aligned}
\sum_{m+n=\ell} (m+1)\text{Eq. (6.2c)} &= -\frac{2(\ell+1)}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(y) \\
&\quad + 2 \sum_{m=0}^{\ell} (1+m)!(\ell-m)! \sum_{k=-1}^{m} \binom{-\gamma}{m-k} \frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!} \\
&= -\frac{2(\ell+1)}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(y) + 2 \frac{E_{\ell+\gamma+1}(y)}{(\ell+\gamma+1)!} W'_{1,0}(\ell,\gamma) \\
&\quad + 2 \sum_{k=0}^{\ell} \frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!} W_{1,0}(k,\ell,\gamma) \\
&= -\frac{2(\ell+\gamma+1)}{(\ell+3)_\gamma} E_{\ell+\gamma+1}(y) \Big\{ 1 + \frac{(-1)^\ell}{\ell+2} \binom{\ell+\gamma}{\gamma-1} \Big\} \\
&\quad + 2 \sum_{k=0}^{\ell} \frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(\ell+3)_\gamma} + 2 \sum_{k=0}^{\ell} \frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!} \lambda_1(1+k,1+\ell,\gamma).
\end{aligned}
$$

The convolution for (6.2e) can be computed analogously as follows:

$$
\sum_{m+n=\ell} (m+1)\text{Eq. (6.2e)}
$$

$$
= -\frac{2(\ell+1)}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(y) + 2\sum_{n=0}^{\ell} n!(1+\ell-n)! \sum_{k=0}^{n} \binom{-\gamma}{n-k} \frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!}
$$

$$
= -\frac{2(\ell+1)}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(y) + 2\sum_{k=0}^{\ell} \frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!} W_{0,1}(k,\ell,\gamma)
$$

$$
= -\frac{2(\ell+1)}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(y) + 2\sum_{k=0}^{\ell} \frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{(\ell+3)_\gamma} + 2\sum_{k=0}^{\ell} \frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!} \lambda_2(k,\ell,\gamma).
$$

For (6.2d), by invoking (3.1a) and Lemma 7, we can compute the convolution:

$$
\sum_{m+n=\ell} (m+1)\text{Eq. (6.2d)}
$$

$$
= \sum_{m=0}^{\ell} (1+m)!(\ell-m)! \sum_{k=1}^{\gamma} (-1)^{\ell+k-1} \binom{-k}{m} \frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}
$$

$$
= \sum_{k=1}^{\gamma} (-1)^{\ell+k-1} \frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!} W_{1,0}(0,\ell,k)
$$

$$
= \sum_{k=1}^{\gamma} (-1)^{k-1} \frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+3)_k} \left\{ (-1)^\ell + \binom{k+\ell}{k-1} \frac{\ell^2+k\ell+3\ell+k+3}{\ell+2} \right\}.
$$

Finally for (6.2f), the convolution can be evaluated similarly as follows:

$$
\sum_{m+n=\ell} (m+1)\text{Eq. (6.2f)} = -2\sum_{n=0}^{\ell} n!(1+\ell-n)! \sum_{k=1}^{\gamma} \binom{-k}{n} \frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}
$$

$$
= -2\sum_{k=1}^{\gamma} \frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!} W_{0,1}(0,\ell,k)
$$

$$
= -2\sum_{k=1}^{\gamma} \frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+3)_k} \left\{ 1 + \frac{(-1)^\ell}{\ell+2} \binom{k+\ell}{k-1} \right\}.
$$

Summarizing the computations just displayed, we get the following identities.

THEOREM 46 (MIKI-LIKE IDENTITIES).

$$2\sum_{k=0}^{\ell}\binom{\ell+2}{k+2}\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_{\gamma}}+\sum_{k=0}^{\ell}(-1)^k\binom{\ell+2}{k+2}\frac{E_k(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_{\gamma}}$$

$$=2\sum_{k=0}^{\ell}\frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(\ell+3)_{\gamma}}+2\sum_{k=-1}^{\ell}\frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\lambda_1(1+k,1+\ell,\gamma)$$

$$-2\frac{\ell+\gamma+1}{(\ell+3)_{\gamma}}E_{\ell+\gamma+1}(y)$$

$$-\sum_{k=1}^{\gamma}(-1)^k\frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+3)_k}\left\{(-1)^{\ell}+\binom{k+\ell}{k-1}\frac{\ell^2+k\ell+3\ell+k+3}{\ell+2}\right\}$$

$$=2\sum_{k=0}^{\ell}\frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{(\ell+3)_{\gamma}}+2\sum_{k=0}^{\ell}\frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!}\lambda_2(k,\ell,\gamma)$$

$$-\frac{2(\ell+1)}{(\ell+2)_{\gamma}}E_{\ell+\gamma+1}(y)-2\sum_{k=1}^{\gamma}\frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+3)_k}\left\{1+\frac{(-1)^{\ell}}{\ell+2}\binom{k+\ell}{k-1}\right\}.$$

For $\gamma=0,1$, we derive from this theorem the following Miki-like identities.

PROPOSITION 47 ($\gamma=0$ IN THEOREM 46).

$$\sum_{k=0}^{\ell}\binom{\ell+2}{k+2}\left\{2B_{k+1}(x-y)E_{\ell-k}(y)+(-1)^k E_k(x-y)E_{\ell-k}(x)\right\}$$

$$=2\sum_{k=0}^{\ell}B_{k+1}(x)E_{\ell-k}(y)-2(\ell+1)E_{\ell+1}(y).$$

Rewrite the last formula by absorbing the alternating factor $(-1)^k$ in the first sum through the reciprocal relation (5.4). Then evaluate the binomial sum corresponding to the correcting-terms by Lemma 3 with $\varepsilon=2$ and $\gamma=0$:

$$\sum_{k=0}^{\ell}\binom{\ell+2}{k+2}(y-x)^k E_{\ell-k}(x)=\frac{E_{\ell+2}(y)-E_{\ell+2}(x)}{(x-y)^2}+\frac{\ell+2}{x-y}E_{\ell+1}(x).$$

We obtain the following equivalent identity.

COROLLARY 48 (PAN-SUN [19, EQ. 2.7]).

$$\sum_{k=0}^{\ell}\binom{\ell+2}{k+2}\left\{2B_{k+1}(x-y)E_{\ell-k}(y)-E_k(y-x)E_{\ell-k}(x)\right\}$$

$$=2\sum_{k=0}^{\ell}B_{k+1}(x)E_{\ell-k}(y)-\frac{2(\ell+2)}{x-y}E_{\ell+1}(x)-2(\ell+1)E_{\ell+1}(y)+2\frac{E_{\ell+2}(x)-E_{\ell+2}(y)}{(x-y)^2}.$$

PROPOSITION 49 ($\gamma = 1$ IN THEOREM 46).

$$\sum_{k=0}^{\ell} \binom{\ell+2}{k+2} \left\{ 2B_{k+1}(x-y)E_{\ell-k}(y) + (-1)^k E_k(x-y)E_{\ell-k}(x) \right\}$$

$$= -2(\ell+1)E_{\ell+1}(y) - 2\sum_{k=-1}^{\ell} (-1)^{k+\ell} \binom{\ell+1}{k+1} B_{k+1}(x)E_{\ell-k}(y)$$

$$+ (\ell+1)E_{\ell}(x-y) + 2B_{\ell+1}(x-y) + 2\sum_{k=0}^{\ell} B_{k+1}(x)E_{\ell-k}(y)$$

$$= -2\frac{\ell(\ell+2)}{(\ell+1)}E_{\ell+1}(y) - \frac{2}{\ell+1}\sum_{k=0}^{\ell}(-1)^{k+\ell}\binom{\ell+1}{k}E_k(y)B_{\ell-k+1}(x)$$

$$+ \frac{2(-1)^{\ell}}{\ell+1}B_{\ell+1}(x-y) + (-1)^{\ell}E_{\ell}(x-y) + 2\sum_{k=0}^{\ell}E_k(y)B_{\ell-k+1}(x).$$

Repeating the same process from Proposition 47 to Corollary 48, we can reformulate the last proposition to the following Miki-like identities.

COROLLARY 50 (MIKI-LIKE IDENTITIES).

$$\sum_{k=0}^{\ell} \binom{\ell+2}{k+2} \left\{ 2B_{k+1}(x-y)E_{\ell-k}(y) - E_k(y-x)E_{\ell-k}(x) \right\}$$

$$+ 2\frac{\ell+2}{x-y}E_{\ell+1}(x) - 2\frac{E_{\ell+2}(x) - E_{\ell+2}(y)}{(x-y)^2}$$

$$= -2(\ell+1)E_{\ell+1}(y) - 2\sum_{k=-1}^{\ell} (-1)^{k+\ell} \binom{\ell+1}{k+1} B_{k+1}(x)E_{\ell-k}(y)$$

$$+ (\ell+1)E_{\ell}(x-y) + 2B_{\ell+1}(x-y) + 2\sum_{k=0}^{\ell} B_{k+1}(x)E_{\ell-k}(y)$$

$$= -2\frac{\ell(\ell+2)}{(\ell+1)}E_{\ell+1}(y) - \frac{2}{\ell+1}\sum_{k=0}^{\ell}(-1)^{k+\ell}\binom{\ell+1}{k}E_k(y)B_{\ell-k+1}(x)$$

$$+ \frac{2(-1)^{\ell}}{\ell+1}B_{\ell+1}(x-y) + (-1)^{\ell}E_{\ell}(x-y) + 2\sum_{k=0}^{\ell}E_k(y)B_{\ell-k+1}(x).$$

$7 - \Omega_{m,n}[{}^{0,\,\gamma}_{E,E}](x,y) - 2\Omega^\star_{n,m}[{}^{1,\,\gamma}_{B,E}](y,x)$

For both sequences $\Omega_{m,n}[{}^{\alpha,\gamma}_{F,F}](x,y)$ and $\Omega^\star_{m,n}[{}^{\beta,\gamma}_{G,F}](x,y)$, we specify with $\alpha = 0$, $\beta = 1$, $F_m(x) = E_m(x)$ and $G_n(y) = B_n(y)$. According to Theorem 2, the bivariate exponential generating function for the corresponding symmetric combination $\Omega_{m,n}(x,y) - 2\Omega^\star_{n,m}(y,x)$ reads as follows:

$$\frac{2e^{u(x-y)}}{(u+v)^\gamma(e^u+1)}\left\{\frac{2e^{y(u+v)}}{e^{u+v}+1} - \sum_{i=0}^{\gamma-1}\frac{(u+v)^i}{i!}E_i(y)\right\}$$

$$+\frac{2}{(u+v)^\gamma}\left\{\frac{e^{v(y-x)}}{e^{-v}-1} + \frac{1}{v}\right\}\left\{\frac{2e^{x(u+v)}}{e^{u+v}+1} - \sum_{j=0}^{\gamma-1}\frac{(u+v)^j}{j!}E_j(x)\right\}.$$

This generating function may be considered as the dual one to that of the last section, because their first factors have been exchanged. Therefore, we expect to derive summation formulae concerning both Bernoulli and Euler polynomials analogous to those displayed in the last section.

7.1 – Reducing the main term of the last expression

$$\frac{4(u+v)^{-\gamma}e^{ux+vy}}{(e^u+1)(e^{u+v}+1)} + \frac{4(u+v)^{-\gamma}e^{ux+vy}}{(e^{-v}-1)(e^{u+v}+1)} = -4\frac{(u+v)^{-\gamma}e^{ux+vy}}{(e^u+1)(e^v-1)}$$

we can reformulate the generating function as

$$-4\frac{(u+v)^{-\gamma}e^{ux+vy}}{(e^u+1)(e^v-1)} + \frac{4/v}{(u+v)^\gamma}\frac{e^{(u+v)x}}{(e^{u+v}+1)} - \frac{2}{v}\sum_{i=1}^{\gamma}\frac{(u+v)^{-i}}{(\gamma-i)!}E_{\gamma-i}(x) \qquad (7.1a)$$

$$-\frac{2e^{u(x-y)}}{e^u+1}\sum_{j=1}^{\gamma}\frac{(u+v)^{-j}}{(\gamma-j)!}E_{\gamma-j}(y) - \frac{2e^{v(y-x)}}{e^{-v}-1}\sum_{j=1}^{\gamma}\frac{(u+v)^{-j}}{(\gamma-j)!}E_{\gamma-j}(x). \qquad (7.1b)$$

Recalling the generating functions of Euler and Bernoulli polynomials and then extracting the coefficients of $[\frac{u^m v^n}{m!n!}]$ according to the two expansions displayed in (4.2a) and (4.2b), we can respectively establish the two corresponding expressions for the symmetric difference $\Omega_{m,n}(x,y) - 2\Omega^\star_{n,m}(y,x)$, which are explicitly displayed as the following theorem.

THEOREM 51 (SYMMETRIC DIFFERENCE).

$$\sum_{k=0}^{m} \binom{m}{k} \frac{E_{m-k}(x-y)E_{n+k+\gamma}(y)}{(n+k+1)_{\gamma}} \tag{7.2a}$$

$$+ 2\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)E_{m+k+\gamma}(x)}{(n-k+1)(m+k+1)_{\gamma}} \tag{7.2b}$$

$$= -2m!n!\sum_{k=0}^{m}\binom{-\gamma}{k}\frac{E_{m-k}(x)B_{n+k+\gamma+1}(y)}{(m-k)!(n+k+\gamma+1)!} + \frac{2E_{m+n+\gamma+1}(x)}{(n+1)(m+n+2)_{\gamma}} \tag{7.2c}$$

$$- 2m!n!\sum_{k=1}^{\gamma}(-1)^{m+n+k}\binom{-k}{m}\frac{E_{\gamma-k}(x)B_{m+n+k+1}(x-y)}{(\gamma-k)!(m+n+k+1)!} \tag{7.2d}$$

$$= -2m!n!\sum_{k=0}^{n+1}\binom{-\gamma}{k}\frac{B_{n-k+1}(y)E_{m+k+\gamma}(x)}{(n-k+1)!(m+k+\gamma)!} + \frac{2E_{m+n+\gamma+1}(x)}{(n+1)(m+n+2)_{\gamma}} \tag{7.2e}$$

$$- m!n!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{E_{\gamma-k}(y)E_{m+n+k}(x-y)}{(\gamma-k)!(m+n+k)!}. \tag{7.2f}$$

For $\gamma = 0, 1, 2$, it reduces respectively to the following easier formulae.

COROLLARY 52 (SYMMETRIC DIFFERENCE).

$$\sum_{k=0}^{m}\binom{m}{k}E_{m-k}(x-y)E_{n+k}(y) + 2\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)E_{m+k}(x)}{n-k+1}$$
$$= \frac{2E_{m+n+1}(x)}{n+1} - \frac{2E_m(x)B_{n+1}(y)}{n+1}.$$

COROLLARY 53 (PAN-SUN [19, EQ. 2.13] FOR THE FIRST EQUATION).

$$\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m-k}(x-y)E_{n+k+1}(y)}{n+k+1} + 2\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)E_{m+k+1}(x)}{(n-k+1)(m+k+1)}$$

$$= -2m!n!\sum_{k=0}^{m}(-1)^k\frac{E_{m-k}(x)B_{n+k+2}(y)}{(m-k)!(n+k+2)!}$$

$$+ \frac{2E_{m+n+2}(x)}{(n+1)(m+n+2)} + (-1)^n\frac{2m!n!}{(m+n+2)!}B_{m+n+2}(x-y)$$

$$= -2m!n!\sum_{k=0}^{n+1}(-1)^k\frac{B_{n-k+1}(y)E_{m+k+1}(x)}{(n-k+1)!(m+k+1)!}$$

$$+ \frac{2E_{m+n+2}(x)}{(n+1)(m+n+2)} - (-1)^n\frac{m!n!}{(m+n+1)!}E_{m+n+1}(x-y).$$

COROLLARY 54 (SYMMETRIC DIFFERENCE).

$$\sum_{k=0}^{m} \binom{m}{k} \frac{E_{m-k}(x-y)E_{n+k+2}(y)}{(n+k+1)_2}$$

$$+ 2\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)E_{m+k+2}(x)}{(n-k+1)(m+k+1)_2}$$

$$= \frac{2E_{m+n+3}(x)}{(n+1)(m+n+2)_2} - 2m!n!\sum_{k=0}^{m}(-1)^k\frac{(k+1)E_{m-k}(x)B_{n+k+3}(y)}{(m-k)!(n+k+3)!}$$

$$+ (-1)^n\frac{2m!n!E_1(x)}{(m+n+2)!}B_{m+n+2}(x-y) - (-1)^n\frac{2(m+1)!n!}{(m+n+3)!}B_{m+n+3}(x-y)$$

$$= \frac{2E_{m+n+3}(x)}{(n+1)(m+n+2)_2} - 2m!n!\sum_{k=0}^{n+1}(-1)^k\frac{(k+1)B_{n-k+1}(y)E_{m+k+2}(x)}{(n-k+1)!(m+k+2)!}$$

$$- (-1)^n\frac{m!n!E_1(y)}{(m+n+1)!}E_{m+n+1}(x-y) - (-1)^n\frac{m!(n+1)!}{(m+n+2)!}E_{m+n+2}(x-y).$$

7.2 – We are now going to compute the convolution with respect to $m+n=\ell$ on the equations displayed in Theorem 51.

For (7.2a), performing replacement $k \to m-k$, interchanging the summation order and then applying Lemma 5, we can compute the convolution as follows:

$$\sum_{m+n=\ell} \text{Eq. (7.2a)} = \sum_{m=0}^{\ell}\sum_{k=0}^{m}\binom{m}{k}\frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}$$

$$= \sum_{k=0}^{\ell}\frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}\sum_{m=k}^{\ell}\binom{m}{k}$$

$$= \sum_{k=0}^{\ell}\binom{\ell+1}{k+1}\frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}.$$

The convolution for (7.2b) reads similarly as the following expression:

$$\sum_{m+n=\ell} \text{Eq. (7.2b)} = 2\sum_{n=0}^{\ell}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}$$

$$= 2\sum_{k=0}^{\ell}(-1)^k\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\sum_{n=k}^{\ell}\binom{n}{k}$$

$$= 2\sum_{k=0}^{\ell}(-1)^k\binom{\ell+1}{k+1}\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}.$$

For (7.2c), by invoking (3.1a) and Lemma 6, we can compute the convolution:

$$\sum_{m+n=\ell} \text{Eq. (7.2c)}$$

$$= \frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(x) - 2\sum_{m=0}^{\ell} m!(\ell-m)! \sum_{k=0}^{m} \binom{-\gamma}{m-k} \frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!}$$

$$= \frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(x) - 2\sum_{k=0}^{\ell} \frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!} W_{0,0}(k,\ell,\gamma)$$

$$= \frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(x) - 2\sum_{k=0}^{\ell} \frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{(\ell-k+\gamma+1)(\ell+2)_\gamma}$$

$$- 2\sum_{k=0}^{\ell} \frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!} \lambda_1(k,\ell,\gamma).$$

The convolution for (7.2e) can be obtained through Lemma 6 and (3.3a) as follows:

$$\sum_{m+n=\ell} \text{Eq. (7.2e)}$$

$$= \frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(x) - 2\sum_{n=0}^{\ell} n!(\ell-n)! \sum_{k=-1}^{n} \binom{-\gamma}{n-k} \frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}$$

$$= \frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(x) - 2\frac{E_{\ell+\gamma+1}(x)}{(\ell+\gamma+1)!} W'_{0,0}(\ell,\gamma)$$

$$- 2\sum_{k=0}^{\ell} \frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!} W_{0,0}(k,\ell,\gamma) = \frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(x)$$

$$- 2\sum_{k=0}^{\ell} \frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)(\ell+2)_\gamma} - 2\sum_{k=-1}^{\ell} \frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!} \lambda_1(k,\ell,\gamma).$$

For (7.2d), by invoking (3.1a) and Lemma 7, we can compute the convolution:

$$\sum_{m+n=\ell} \text{Eq. (7.2d)} = 2\sum_{m=0}^{\ell} m!(\ell-m)! \sum_{k=1}^{\gamma} (-1)^{\ell+k-1} \binom{-k}{m} \frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}$$

$$= 2\sum_{k=1}^{\gamma} (-1)^{\ell+k-1} \frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!} W_{0,0}(0,\ell,k)$$

$$= 2\sum_{k=1}^{\gamma} (-1)^{k-1} \frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\ell+k+1)(\gamma-k)!(\ell+2)_k} \left\{ (-1)^\ell + \binom{k+\ell}{k-1} \right\}.$$

Finally for (7.2f), we write down directly the convolution from (7.2d) as follows:

$$
\begin{aligned}
\sum_{m+n=\ell} \text{Eq. (7.2f)} &= -\sum_{n=0}^{\ell} n!(\ell-n)! \sum_{k=1}^{\gamma} \binom{-k}{n} \frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!} \\
&= -\sum_{k=1}^{\gamma} \frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!} W_{0,0}(0,\ell,k) \\
&= -\sum_{k=1}^{\gamma} \frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+2)_k} \left\{ 1 + (-1)^\ell \binom{k+\ell}{k-1} \right\}.
\end{aligned}
$$

Summarizing the computations just displayed, we get the following identities.

THEOREM 55 (MIKI-LIKE IDENTITIES).

$$
\begin{aligned}
&\sum_{k=0}^{\ell} \binom{\ell+1}{k+1} \frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma} + 2\sum_{k=0}^{\ell} (-1)^k \binom{\ell+1}{k+1} \frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma} \\
&= -2\sum_{k=0}^{\ell} \frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{(\ell-k+\gamma+1)(\ell+2)_\gamma} - 2\sum_{k=0}^{\ell} \frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!} \lambda_1(k,\ell,\gamma) \\
&\quad + \frac{2H_{\ell+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(x) - 2\sum_{k=1}^{\gamma} (-1)^k \frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\ell+k+1)(\gamma-k)!(\ell+2)_k} \left\{ (-1)^\ell + \binom{k+\ell}{k-1} \right\} \\
&= -2\sum_{k=0}^{\ell} \frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)(\ell+2)_\gamma} - 2\sum_{k=-1}^{\ell} \frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!} \lambda_1(k,\ell,\gamma) \\
&\quad + \frac{2H_{\ell+\gamma+1}}{(\ell+2)_\gamma} E_{\ell+\gamma+1}(x) - \sum_{k=1}^{\gamma} \frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+2)_k} \left\{ 1 + (-1)^\ell \binom{k+\ell}{k-1} \right\}.
\end{aligned}
$$

For $\gamma = 0, 1$, we derive from this theorem the following Miki-like identities.

PROPOSITION 56 ($\gamma = 0$ IN THEOREM 55).

$$
\begin{aligned}
&\sum_{k=0}^{\ell} \binom{\ell+1}{k+1} E_k(x-y)E_{\ell-k}(y) + 2\sum_{k=0}^{\ell} (-1)^k \binom{\ell+1}{k+1} \frac{B_{k+1}(x-y)E_{\ell-k}(x)}{k+1} \\
&= 2H_{\ell+1}E_{\ell+1}(x) - 2\sum_{k=0}^{\ell} \frac{B_{k+1}(y)E_{\ell-k}(x)}{k+1}.
\end{aligned}
$$

PROPOSITION 57 ($\gamma = 1$ IN THEOREM 55).

$$\sum_{k=0}^{\ell} \binom{\ell+1}{k+1} \left\{ E_k(x-y)E_{\ell-k}(y) + 2(-1)^k \frac{B_{k+1}(x-y)E_{\ell-k}(x)}{k+1} \right\}$$

$$= 2H_\ell E_{\ell+1}(x) + \frac{2}{\ell+1} \sum_{k=0}^{\ell} (-1)^{k+\ell} \binom{\ell+1}{k} E_k(x)B_{\ell-k+1}(y)$$

$$+ E_\ell(x-y) + 2\frac{B_{\ell+1}(x-y)}{\ell+1} - 2\sum_{k=0}^{\ell} \frac{B_{k+1}(y)E_{\ell-k}(x)}{k+1}$$

$$= 2H_{\ell+1} E_{\ell+1}(x) + \frac{2}{\ell+1} \sum_{k=-1}^{\ell} (-1)^{k+\ell} \binom{\ell+1}{k+1} B_{k+1}(y)E_{\ell-k}(x)$$

$$+ (-1)^\ell E_\ell(x-y) + 2(-1)^\ell \frac{B_{\ell+1}(x-y)}{\ell+1} - 2\sum_{k=0}^{\ell} \frac{B_{k+1}(y)E_{\ell-k}(x)}{k+1}.$$

By following the same process from Proposition 42 to Corollary 43, we can recover the formulae displayed in Corollaries 43 and 45 respectively from the last two propositions.

7.3 – Multiplying by the weight factor $(n+1)$ the equations displayed in Theorem 51, the convolution with respect to $m + n = \ell$ can be computed similarly as follows.

For (7.2a), performing replacement $k \to m - k$, interchanging the summation order and then applying Lemma 5, we can compute the convolution as follows:

$$\sum_{m+n=\ell} (n+1)\text{Eq. (7.2a)} = \sum_{m=0}^{\ell} (1+\ell-m) \sum_{k=0}^{m} \binom{m}{k} \frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}$$

$$= \sum_{k=0}^{\ell} \frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma} \sum_{m=k}^{\ell} \binom{m}{k} (1+\ell-m)$$

$$= \sum_{k=0}^{\ell} \binom{\ell+2}{k+2} \frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}.$$

The convolution for (7.2b) can be evaluated by means of Lemma 5 as follows:

$$\sum_{m+n=\ell} (n+1)\text{Eq. (7.2b)} = 2\sum_{n=0}^{\ell}(n+1)\sum_{k=0}^{n}(-1)^k \binom{n}{k}\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}$$

$$= 2\sum_{k=0}^{\ell}(-1)^k\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\sum_{n=k}^{\ell}\binom{n}{k}(n+1)$$

$$= 2\sum_{k=0}^{\ell}(-1)^k\binom{\ell+2}{k+2}\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}.$$

For (7.2c), by invoking (3.1a) and Lemma 6, we can compute the convolution:

$$\sum_{m+n=\ell}(n+1)\text{Eq. (7.2c)}$$

$$= \frac{2(\ell+1)}{(\ell+2)_\gamma}E_{\ell+\gamma+1}(x) - 2\sum_{m=0}^{\ell}m!(1+\ell-m)!\sum_{k=0}^{m}\binom{-\gamma}{m-k}\frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!}$$

$$= \frac{2(\ell+1)}{(\ell+2)_\gamma}E_{\ell+\gamma+1}(x) - 2\sum_{k=0}^{\ell}\frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!}W_{0,1}(k,\ell,\gamma)$$

$$= \frac{2(\ell+1)}{(\ell+2)_\gamma}E_{\ell+\gamma+1}(x) - 2\sum_{k=0}^{\ell}\frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{(\ell+3)_\gamma} - 2\sum_{k=0}^{\ell}\frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!}\lambda_2(k,\ell,\gamma).$$

The convolution for (7.2e) can be done by invoking Lemma 6 and (3.3b) as follows:

$$\sum_{m+n=\ell}(n+1)\text{Eq. (7.2e)}$$

$$= \frac{2(\ell+1)}{(\ell+2)_\gamma}E_{\ell+\gamma+1}(x) - 2\sum_{n=0}^{\ell}(n+1)!(\ell-n)!\sum_{k=-1}^{n}\binom{-\gamma}{n-k}\frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}$$

$$= \frac{2(\ell+1)}{(\ell+2)_\gamma}E_{\ell+\gamma+1}(x) - 2\frac{E_{\ell+\gamma+1}(x)}{(\ell+\gamma+1)!}W'_{1,0}(\ell,\gamma) - 2\sum_{k=0}^{\ell}\frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}W_{1,0}(k,\ell,\gamma)$$

$$= \frac{2(\ell+\gamma+1)}{(\ell+3)_\gamma}E_{\ell+\gamma+1}(x) - 2\sum_{k=0}^{\ell}\frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(\ell+3)_\gamma}$$

$$- 2\sum_{k=-1}^{\ell}\frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\lambda_1(1+k,1+\ell,\gamma).$$

For (7.2d), by invoking (3.1a) and Lemma 7, we can compute the convolution:

$$\sum_{m+n=\ell} (n+1)\text{Eq. (7.2d)}$$

$$= 2\sum_{m=0}^{\ell} m!(1+\ell-m)!\sum_{k=1}^{\gamma}(-1)^{\ell+k-1}\binom{-k}{m}\frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}$$

$$= 2\sum_{k=1}^{\gamma}(-1)^{\ell+k-1}\frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}W_{0,1}(0,\ell,k)$$

$$= 2\sum_{k=1}^{\gamma}(-1)^{\ell+k-1}\frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+3)_k}\left\{1+\frac{(-1)^\ell}{\ell+2}\binom{k+\ell}{k-1}\right\}.$$

Finally for (7.2f), by invoking (3.1a) and Lemma 7, we get the convolution:

$$\sum_{m+n=\ell}(n+1)\text{Eq. (7.2f)} = -\sum_{n=0}^{\ell}(1+n)!(\ell-n)!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}$$

$$= -\sum_{k=1}^{\gamma}\frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}W_{1,0}(0,\ell,k)$$

$$= -\sum_{k=1}^{\gamma}\frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+3)_k}\left\{1+(-1)^\ell\binom{k+\ell}{k-1}\frac{\ell^2+k\ell+3\ell+k+3}{\ell+2}\right\}.$$

Summarizing the computations just displayed, we get the following identities.

THEOREM 58 (MIKI-LIKE IDENTITIES).

$$\sum_{k=0}^{\ell}\binom{\ell+2}{k+2}\left\{\frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}+2(-1)^k\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}\right\}$$

$$= -2\sum_{k=0}^{\ell}\frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{(\ell+3)_\gamma}-2\sum_{k=0}^{\ell}\frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!}\lambda_2(k,\ell,\gamma)$$

$$\quad -2\sum_{k=1}^{\gamma}(-1)^{\ell+k}\frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+3)_k}\left\{1+\frac{(-1)^\ell}{\ell+2}\binom{k+\ell}{k-1}\right\}+\frac{2(\ell+1)}{(\ell+2)_\gamma}E_{\ell+\gamma+1}(x)$$

$$= -2\sum_{k=0}^{\ell}\frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(\ell+3)_\gamma}-2\sum_{k=-1}^{\ell}\frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\lambda_1(1+k,1+\ell,\gamma)$$

$$\quad -\sum_{k=1}^{\gamma}\frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+3)_k}\left\{1+(-1)^\ell\binom{k+\ell}{k-1}\frac{\ell^2+k\ell+3\ell+k+3}{\ell+2}\right\}$$

$$\quad +\frac{2(\ell+\gamma+1)}{(\ell+3)_\gamma}E_{\ell+\gamma+1}(x).$$

For $\gamma = 0, 1$, we derive from this theorem the following Miki-like identities.

PROPOSITION 59 ($\gamma = 0$ IN THEOREM 58).

$$\sum_{k=0}^{\ell} \binom{\ell+2}{k+2} \Big\{ E_k(x-y)E_{\ell-k}(y) + 2(-1)^k B_{k+1}(x-y)E_{\ell-k}(x) \Big\}$$

$$= 2(\ell+1)E_{\ell+1}(x) - 2\sum_{k=0}^{\ell} E_k(x)B_{\ell-k+1}(y).$$

PROPOSITION 60 ($\gamma = 1$ IN THEOREM 58).

$$\sum_{k=0}^{\ell} \binom{\ell+2}{k+2} \Big\{ E_k(x-y)E_{\ell-k}(y) + 2(-1)^k B_{k+1}(x-y)E_{\ell-k}(x) \Big\}$$

$$= 2\frac{\ell(\ell+2)}{\ell+1}E_{\ell+1}(x) + \frac{2}{\ell+1}\sum_{k=0}^{\ell}(-1)^{k+\ell}\binom{\ell+1}{k}E_k(x)B_{\ell-k+1}(y)$$

$$+ E_\ell(x-y) + \frac{2}{\ell+1}B_{\ell+1}(x-y) - 2\sum_{k=0}^{\ell}E_k(x)B_{\ell-k+1}(y)$$

$$= 2(\ell+1)E_{\ell+1}(x) + 2\sum_{k=-1}^{\ell}(-1)^{k+\ell}\binom{\ell+1}{k+1}B_{k+1}(y)E_{\ell-k}(x)$$

$$+ (-1)^{\ell}(\ell+1)E_\ell(x-y) + 2(-1)^{\ell}B_{\ell+1}(x-y) - 2\sum_{k=0}^{\ell}B_{k+1}(y)E_{\ell-k}(x).$$

Finally, we observe that by following the same process from Proposition 47 to Corollary 48, we can recover the formulae displayed in Corollaries 48 and 50 respectively from the last two propositions.

8 – Further convolutions with different weight factors

With different weight-factors, the convolutions $m+n = \ell$ on the equations displayed in the corollaries labeled with 10, 11, 12; 24, 25, 26 and 38, 39, 40 may lead to further summation formulae on Bernoulli and Euler polynomials. This section sketches briefly the resulting identities, whose detailed proofs will not be reproduced due to space limitation.

8.1 – Weight factor $\binom{\alpha+m}{m}\binom{\gamma+n}{n}$

Here we consider only the three simplest cases from Corollaries 10, 24 and 38. The convolutions $m + n = \ell$ on the equations displayed there result, respectively, in the

following identities.

$$(\alpha + \gamma + \ell) \sum_{k=0}^{\ell} \binom{\alpha + k - 1}{k} \binom{\gamma + \ell - k - 1}{\ell - k} B_k(x) B_{\ell - k}(y) \qquad (8.1a)$$

$$= \gamma \sum_{k=0}^{\ell} \binom{\alpha + k - 1}{k} \binom{\alpha + \gamma + \ell}{\ell - k} B_k(x - y) B_{\ell - k}(y) \qquad (8.1b)$$

$$+ \alpha \sum_{k=0}^{\ell} (-1)^k \binom{\gamma + k - 1}{k} \binom{\alpha + \gamma + \ell}{\ell - k} B_k(x - y) B_{\ell - k}(x); \qquad (8.1c)$$

$$\frac{\alpha + \gamma + \ell + 1}{2} \sum_{k=1}^{\ell} \binom{\alpha + k - 1}{k - 1} \binom{\gamma + \ell - k}{\ell - k} E_{k-1}(x) E_{\ell - k}(y) \qquad (8.2a)$$

$$= \sum_{k=0}^{\ell} (-1)^k \binom{\gamma + k}{k} \binom{\alpha + \gamma + \ell + 1}{\ell - k} E_k(x - y) B_{\ell - k}(x) \qquad (8.2b)$$

$$- \sum_{k=0}^{\ell} \binom{\alpha + k}{k} \binom{\alpha + \gamma + \ell + 1}{\ell - k} E_k(x - y) B_{\ell - k}(y); \qquad (8.2c)$$

$$2 \sum_{k=0}^{\ell} \binom{\alpha + k - 1}{k} \binom{\gamma + \ell - k}{\ell - k} B_k(x) E_{\ell - k}(y) \qquad (8.3a)$$

$$= 2 \sum_{k=0}^{\ell} \binom{\alpha + k - 1}{k} \binom{\alpha + \gamma + \ell}{\ell - k} B_k(x - y) E_{\ell - k}(y) \qquad (8.3b)$$

$$- \alpha \sum_{k=1}^{\ell} (-1)^k \binom{\gamma + k - 1}{k - 1} \binom{\alpha + \gamma + \ell}{\ell - k} E_{k-1}(x - y) E_{\ell - k}(x). \qquad (8.3c)$$

We point out that the special case $\gamma = 1$ of (8.1a-8.1b-8.1c) has previously been obtained by Sun and Pan [22, Eq. 1.5].

8.2 – Weight factor $\binom{m+n}{m}$

From the convolutions $m + n = \ell$ on the equations displayed in Corollaries 10, 24 and 38, we obtain the following three identities.

$$2\sum_{k=0}^{\ell}\binom{\ell}{k}B_k(x)B_{\ell-k}(y)=\sum_{k=0}^{\ell}\binom{\ell}{k}2^{\ell-k}B_k(x-y)B_{\ell-k}(y) \tag{8.4a}$$

$$+\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}2^{\ell-k}B_k(x-y)B_{\ell-k}(x); \tag{8.4b}$$

$$\ell\sum_{k=1}^{\ell}\binom{\ell-1}{k-1}E_{k-1}(x)E_{\ell-k}(y)=\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}2^{\ell-k}E_k(x-y)B_{\ell-k}(x) \tag{8.5a}$$

$$-\sum_{k=0}^{\ell}\binom{\ell}{k}2^{\ell-k}E_k(x-y)B_{\ell-k}(y); \tag{8.5b}$$

$$\sum_{k=0}^{\ell}\binom{\ell}{k}B_k(x)E_{\ell-k}(y)=\sum_{k=0}^{\ell}\binom{\ell}{k}2^{\ell-k}B_k(x-y)E_{\ell-k}(y) \tag{8.6a}$$

$$-\frac{\ell}{2}\sum_{k=1}^{\ell}(-1)^k\binom{\ell-1}{k-1}2^{\ell-k}E_{k-1}(x-y)E_{\ell-k}(x). \tag{8.6b}$$

8.3 – Further convolution formulae

From the formulae just displayed, we can deduce further convolution identities. First replacing the Bernoulli polynomials $B_{\ell-k}(x)$ appeared in (8.4b) and (8.5a) by

$$B_k(x) = \sum_{i=0}^{k}(x-y)^i\binom{k}{i}B_{k-i}(y)$$

then manipulating the double sums there by means of the reciprocal relations

$$B_k(x) = (-1)^k B_k(-x) - kx^{k-1} \quad \text{and} \quad E_k(x) = 2x^k - (-1)^k E_k(-x)$$

and finally simplifying the resulting equations through (4.4a-4.4b) and (5.3a-5.3b), we derive the following four convolution identities, where the first two can be found

in Hansen [14, Eqs. 50.11.4 and 50.11.5].

$$\sum_{k=0}^{\ell} 2^k \binom{\ell}{k} B_k(x) B_{\ell-k}(y) = \ell(2x + y - 3/2) B_{\ell-1}(2x + y) \tag{8.7a}$$

$$- (\ell - 1) B_\ell(2x + y) + \frac{\ell(\ell - 1)}{2} E_{\ell-2}(2x + y); \tag{8.7b}$$

$$\sum_{k=0}^{\ell} (-2)^k \binom{\ell}{k} B_k(x) B_{\ell-k}(y) = \ell(y - 2x + 1/2) B_{\ell-1}(y - 2x) \tag{8.8a}$$

$$- (\ell - 1) B_\ell(y - 2x) + \frac{\ell(\ell - 1)}{2} E_{\ell-2}(y - 2x); \tag{8.8b}$$

$$\sum_{k=0}^{\ell} 2^k \binom{\ell}{k} B_k(x) E_{\ell-k}(y) = \ell(2x + y - 3/2) E_{\ell-1}(2x + y) \tag{8.9a}$$

$$- \ell E_\ell(2x + y) + B_\ell(2x + y); \tag{8.9b}$$

$$\sum_{k=0}^{\ell} (-2)^k \binom{\ell}{k} B_k(x) E_{\ell-k}(y) = \ell(y - 2x + 1/2) E_{\ell-1}(y - 2x) \tag{8.10a}$$

$$- \ell E_\ell(y - 2x) + B_\ell(y - 2x). \tag{8.10b}$$

However, we have failed to find closed forms for two similar convolution sums:

$$\sum_{k=0}^{\ell} (\pm 2)^k \binom{\ell}{k} E_k(x) B_{\ell-k}(y) \quad \text{and} \quad \sum_{k=0}^{\ell} (\pm 2)^k \binom{\ell}{k} E_k(x) E_{\ell-k}(y).$$

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