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An analytical study of a model for the actin-based movement of bacteria

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Abstract: *In this work we studied a system of two partial differential type modeling the motion of some bacteria, which use an Actin-based propulsion mechanism. The resulting system is composed by two equation (a porous-media equation and a linear hyperbolic one) coupled with two moving boundaries. We proved the local (in time) well posedness for the Cauchy problem.*

1 – Introduction

In recent years the interest in mathematical and computational study of biological phenomena has rapidly increased. Models for this kind of processes are spreading in many branches of biology like oncology, epidemiology, natural pattern formation, ecology and morphogenesis. Between many challenging problems proposed by the biological modeling, cell and microorganism movement and direction process is a very interesting subject: cells and microorganisms direct their movement through chemotaxis processes, i.e. according to the presence and the concentration of specific chemical agents in their environment. Obviously it is impossible to give an exhaustive list of references, anyway we refer to [10, 14, 20] and [9] (and the references therein) for an introduction in the topic and in its applications.

The definition of models for the description of cell motility is still at the beginning. Essentially cell motion is divided into three different phases and relevant processes: *protrusion*, *adhesion* and *contraction*. More in detail, firstly the cell pulls out the front, then it adheres at the surface tightly by the leading edge and weakly by the rear one and finally it develops a contraction that pulls up the rear, completing the motile cycle (for further information see [17] and [18]). In particular, cell motion is mostly due to the use of the so called *lamellipodia* as motile appendages.

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These processes are generally described by evolution and transport equations. In particular we suggest to the reader the models introduced in [1] [9], [20] and [21].

Lamellipodia are structures very similar to the tails of a peculiar protein, the actin, that bacteria like *Lysteria Monicytogenes*, *Shigella Flexenari* and *Rickettsia Rickettsii* use for their motion inside host cells. The characterization of the movement of these microorganisms is easier than the one of the cells; for this reason, modeling the motion of these organisms can be a first step in the modeling of the protrusion at the leading edge.

Actin tail is composed by a large number of cross-linked actin filaments. As every protein the actin can polymerize and the process of polymerization drives protrusive forces generation in the motion. The mechanism by which the movement occurs is still not well understood (see for example [6, 12] and [17]). In literature there are two different ways of modeling the actin based movement. The first approach [6] considers all the actin filaments attached to the surface of the bacterium or of the cell and it assumes that the polymerization process produces a compression in the filaments. So, there are some filaments under compression and some other under tension. The opposition between these two classes of filaments makes the bacterium moving.

The other approach is based on the observation that, after a short period of time, the filaments detach from the bacterium and their place is taken from new ones. The detachment of filaments causes a compression in the tail which pushes the bacterium in the opposite direction. In this kind of model, (see for example [19] and [7]), the motion is due to the opposition between the force of pushing filaments and a frictional force resulting from the hydrodynamic drag and/or the resisting force that is necessary to break the link between the filaments and the surface. Models obtained by this approach are the so called *Brownian and tethered ratched* models.

In this work we have studied a 1-dimensional brownian and tethered ratched model from an analytical and a numerical point of view. The model describes the spatio-temporal evolution of two physical quantities of the actin tail in bacteria movement. In particular we have studied the evolution of the density and of the number of filaments per unit volume of the tail, which we denote respectively by ρ and u. The model is a modification of the one introduced in [2] and it is a system of two partial differential equations with two moving boundaries. The equation for u is a porous media parabolic equation, while ρ evolves following a linear transport equation with the flux velocity and the source that both depend from spatial and time variables. The main purpose of this work is to prove the local well posedness of the Cauchy problem for the system composed by the two coupled partial differential equations for u and ρ and the two ordinary differential equations related to the moving boundaries (the extrema of the actin tail).

The paper is organized as follows. In the first section we describe the model introduced in [2] and a generalization we have introduced for better describe some physical properties of the actin tail. The second section is devoted to prove the local existence and uniqueness theorem for the model.

In order to prove the theorem we have transformed the moving domain in a fixed one. In the new domain the equation for the evolution of u is transformed from a porous media to a nonlinear degenerate parabolic equation with coefficients depending on the spatial and temporal variables. We show that, if the initial data verify some conditions (see hypotheses (3.2)-(3.10) of Theorem 3.2) then a solution for the problem exists and is unique. Moreover, we show that $u \in C^{2,1}$ and that ρ is a continuous function.

The strategy we will use is to freeze the moving boundaries and solve the evolution system for the functions u and ρ, from which obtain new boundaries for the actin tail. So the local existence theorem for our system becomes a fixed point problem for the functions describing the extrema of the actin tail. Our principal inspiration has been the work [23].

2 – The model

Listeria Monocytogenes, Shigella Flexenari and Rickettsia Rickettsii, are particular bacteria that may cause a lot of serious diseases, such as meningitis, typhus and Rocky Mountain fever. Their virulence is strictly connected to the high speed of their movement inside a single host cell and their ability to spread out, infecting many other cells. Unlike many bacteria, they don't move using flagella, but they exploit a cytosol protein: the actin.

As a protein, actin can polymerize, and as a consequence of this process, several actin monomers aggregate in a chain. In particular, actin polymers look like gelatinous and elastic filaments. Their peculiarity is their polar structure; monomers can only attach to one end of a filament while they can only detach from the other. Polymerization and depolymerization processes are the cause of the movement of Listeria moncytogenes, Shigella Flexenari and Rickettsia Rickettsii.

On the outer membranes of these bacteria, there is an enzyme, which attracts actin causing its polymerization. Polymerization ends of actin filaments tie at the bacterium enzyme site. Monomers addiction compresses the filament, until it leaves the bacterium. This makes the bacterium and the filament move in opposite directions. New filaments now tie to the bacterium while the detached filament completely depolymerizes after several chemical reactions until it vanishes. During these processes, detached actin monomers are free to polymerize again, creating new filaments. Our model is a generalization of the one introduced by Bazaliy, Bazaliy and Friedman in [2]. For the reader's convenience we first describe their model, then we will explain our changes.

The original model. In [2] it is assumed that at any point of the tail, the physical quantities like density (that is number of filaments for unit volume), velocity, filaments length and so on, depend only on the distance of that point from the bacterium. Let x denote the spatial variable and t the temporal one. We assume that the motion happens in the x-axis negative direction. So, since the tail and the bacterium movement directions are opposite, the actin tail moves in the direction of increasing x. Moreover we assume that at time $t = 0$, the bacterium position is $x = 0$.

In [2], the following variables are introduced:

- $w(x, t)$ velocity of the tail;
- $u(x, t)$ numerical filaments density of the tail;
- $l_f(x, t)$ length of filaments;
- $\rho(x, t)$ actin density of the tail, and

$$\rho(x, t) = C_\rho l_f(x, t) u(x, t) \tag{2.1}$$

  with C_ρ positive real constant;
- $p(x, t)$ pressure of the tail;
- $l(t)$ left end of the tail;
- $r(t)$ right end of the tail;
- $V(t)$ velocity of the bacterium;

we will define our problem in the region occupied by the actin gel, $\Omega(t) = (l(t), r(t))$.

So, we have the bacterium which moves with velocity $V(t)$ together with the attached filaments, and the tail, composed by the detached filaments that moves in the opposite direction with velocity $w(x, t)$. We remark that, in our notation, $l(t)$ is the end of the tail in touch with the bacterium, so $V(t)$ is the speed of the front $l(t)$ or easily $\frac{dl}{dt} = V(t)$. It must be noted that $V(t)$ is different from $w(l(t), t)$; in fact, $V(t)$ denotes the velocity at which the front is moving, while $w(l(t), t)$ denotes the velocity at which the part of the tail in $l(t)$ moves. In the above the tail is considered as the set of detached filaments, so $w(l(t), t)$ and $V(t)$ identify, respectively, the velocity of detached and of attached filaments.

From a physical point of view this problem can be regarded as a motion in a viscous fluid. In particular, in [2] the authors assume that:

- the motion happens for low Reynolds number, so that it is laminar;
- each filament motion doesn't affect the motion of the other filaments;
- a deformation of a piece of gel is only due to a change in the numerical filaments density. As a consequence, the pressure of the tail verifies the following constitutive law:

$$p(x, t) = p_0(t) + \overline{E}(u - u_0) \tag{2.2}$$

  where u_0 is the numerical filaments density at the bacterium surface, $\overline{E}$ is a positive constant and $p_0(t)$ is the pressure at the bacterium surface which is defined as follows:

$$p_0(t) = \beta - \alpha\left(w(l(t),t) + V(t)\right) \tag{2.3}$$

with α and β positive constants;

- the viscosity coefficients for the bacterium and the actin tail, denoted respectively by μ and b, are both positive.

Note that under these assumptions, the balance between the force that pushes the gel and the drag force can be written as follows:

$$bw = -\frac{\partial p}{\partial x} = -\overline{E}\frac{\partial u}{\partial x}$$

and so:

$$w(x,t) = -\frac{\overline{E}}{b}\frac{\partial u}{\partial x}. \tag{2.4}$$

As we said previously:

$$\frac{dl}{dt} = V(t)$$

and since the motion is laminar:

$$V(t) = -\frac{p_0(t)}{\mu}. \tag{2.5}$$

So, from (2.3):

$$V(t) = -\frac{1}{\mu}\left[\beta - \alpha\left(w(l(t),t) + V(t)\right)\right]$$

where we assume that

$$\mu > \alpha$$

and then:

$$V(t) = \frac{\alpha}{\mu - \alpha}w(l(t),t) - \frac{\beta}{\mu - \alpha}.$$

Replacing $V(t)$ by dl/dt and using (2.4) we obtain the following evolution equation for $l(t)$:

$$\frac{dl}{dt} = -E\frac{\alpha}{\mu - \alpha}u_x(l(t),t) - \frac{\beta}{\mu - \alpha}. \tag{2.6}$$

With regard to the evolution of $r(t)$ we assume, as in [2], that at $x = r(t)$ the average distance between the tail filaments is proportional both to $u(x,t)^{-\frac{1}{3}}$ and to $l_f(x,t)$. So there exists a constant $\overline{C}$ such that:

$$u(r(t),t)^{-\frac{1}{3}} = \overline{C}l_f(r(t),t)$$

and from (2.1):

$$u(r(t),t)^{\frac{2}{3}} = C\overline{C}\rho(r(t),t).$$

Set:

$$\rho_d(t) = du(r(t),t)^{\frac{2}{3}}$$

with $d > 0$.

Following [2]:

$$\frac{dr}{dt} = w(r(t),t) - \frac{\nu}{\rho(r(t),t) - \rho_d(t)}$$

and so:

$$\frac{dr}{dt} = -Eu_x(r(t),t) - \frac{\nu}{\rho(r(t),t) - \rho_d(t)}. \tag{2.7}$$

As far as the boundary data is concerned

$$u(l(t),t) = u_0 > 0. \tag{2.8}$$

Due to the disintegration of the tail at $x = r(t)$, in [2] it is assumed that $p(r(t),t) = 0$. So, from equation (2.2):

$$p(r(t),t) = p_0(t) + \overline{E}(u(r(t),t) - u_0) = -\mu\frac{dl}{dt} + \overline{E}(u(r(t),t) - u_0) = 0$$

where last equation is derived using (2.5).

Then:

$$u(r(t),t) = u_0 + \frac{\mu}{\overline{E}}\frac{dl}{dt}. \tag{2.9}$$

Also for ρ in [2] a boundary condition is given:

$$\rho(l(t),t) = \rho_0 > 0. \tag{2.10}$$

Let's now turn to the derivation of the system of partial differential equations for u and ρ. In order to study the evolution of u we write down the conservation law for the numerical density of the filaments in the following way:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(wu) = 0 \quad \forall x \in \Omega(t), t > 0 \tag{2.11}$$

and replacing w by (2.4), we find:

$$\frac{\partial u}{\partial t} - \frac{\overline{E}}{b}\frac{\partial}{\partial x}(u_x u) = 0 \quad \forall x \in \Omega(t), t > 0$$

that is:

$$\frac{\partial u}{\partial t} - \frac{\overline{E}}{2b}\frac{\partial^2}{\partial x^2}(u^2) = 0 \quad \forall x \in \Omega(t), t > 0.$$

[2] assumes that the filaments length decreases with a constant rate $\overline{K}$ as the filaments distance from the bacterium increases. So they have the following equation for $l_f(x,t)$

$$\frac{\partial l_f}{\partial t} + w\frac{\partial l_f}{\partial x} = -\overline{K} \quad \forall x \in \Omega(t), t > 0. \tag{2.12}$$

Using (2.1), they obtain the following conservation law for ρ:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(w\rho) = -C_\rho \overline{K} u. \tag{2.13}$$

In fact, from (2.1):

$$\frac{\partial \rho}{\partial t} = C\left(\frac{\partial l_f}{\partial t}u + \frac{\partial u}{\partial t}l_f\right) \quad \text{and} \quad \frac{\partial \rho}{\partial x} = C\left(\frac{\partial l_f}{\partial x}u + \frac{\partial u}{\partial x}l_f\right).$$

So

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial (w\rho)}{\partial x} &= C_\rho\left[\left(\frac{\partial l_f}{\partial t} + w\frac{\partial l_f}{\partial x}\right)u + \left(\frac{\partial u}{\partial t} + \frac{\partial (wu)}{\partial x}\right)l_f\right] \\ &= C_\rho\left(\frac{\partial l_f}{\partial t} + w\frac{\partial l_f}{\partial x}\right)u = -C\overline{K}u.\end{aligned}$$

Finally they define an initial data for the problem. That is:

$$\begin{cases} u(x,0) = \tilde{u}_0(x) & x \in (l(t), r(t)) \\ \rho(x,0) = \tilde{\rho}_0(x) & x \in (l(t), r(t)) \end{cases} \tag{2.14}$$

with $\tilde{u}_0(l(t)) = u_0$ and $\tilde{\rho}_0(l(t)) = \rho_0$

From (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.13) and (2.14) the original model could be summarized as follows:

$$\begin{cases} u_t - \dfrac{E}{2}(u^2)_{xx} = 0 & x \in (l(t), r(t)), t > 0 \\ \rho_t - E(u_x\rho)_x = -K_2 u & x \in (l(t), r(t)), t > 0 \end{cases} \tag{M_o}$$

subject to the following initial conditions:

$$\begin{cases} u(x,0) = \tilde{u}_0(x) & x \in (0, r_0) \\ \rho(x,0) = \tilde{\rho}_0(x) & x \in (0, r_0) \end{cases} \tag{I}$$

and to the boundaries conditions:

$$\begin{cases} u(l(t),t) = u_0 \\ u(r(t),t) = u_0 + \dfrac{\mu}{bE}\dfrac{dl}{dt} \\ \rho(l(t),t) = \rho_0 \end{cases} \tag{B}$$

with the following equations for the evolution of the two moving boundaries:

$$\begin{cases} \dfrac{dl}{dt} = -E\dfrac{\alpha}{\mu-\alpha}u_x(l(t),t) - \dfrac{\beta}{\mu-\alpha} & l(0)=0 \\ \dfrac{dr}{dt} = -Eu_x(r(t),t) - \dfrac{\nu}{\rho(r(t),t)-du(r(t),t)^{\frac{2}{3}}} & r(0)=r_0>0. \end{cases} \tag{LR}$$

Moreover, in order to ensure that the problem is well-posed and consistent from a physical point of view, u and ρ have to verify the following conditions:

$$u(x,t),\rho(x,t)>0 \quad \forall x\in(l(t),r(t)), \quad t>0 \tag{2.15}$$

$$\rho(r(t),t)-du(r(t),t)^{\frac{2}{3}}>0 \quad \forall t>0 \tag{2.16}$$

$$r(t)-l(t)>0 \quad \forall t>0 \tag{2.17}$$

where $E=\overline{E}/b$ and $K_2=CK$.

Changes to the model. Let's describe how we have changed the model. Roughy speaking, in [2], the filament length decreases at a constant rate as the distance between the rear side of the bacterium and the filament increases. So the filament elongation due to the polymerization process is not taken into account. In order to consider also this aspect, we assume that the variation of the filaments length is controlled by a nonincreasing function, that is positive in a small interval near the bacterium and negative far from it.

This means that the filaments length increases for $x=l(t)$ while it decreases for all x sufficiently far from $l(t)$. In fact, the limit of the model introduced in [2], is that the filaments length can only decrease. We introduced the following C^2 function, $\overline{K}(x,t)$ such that

$$\overline{K}(x,t)=\begin{cases} \overline{K}_1 & x\in(l(t),l(t)+\delta_l), t>0 \\ \text{decreasing} & x\in(l(t)+\delta_l,l(t)+\delta_r), t>0 \\ -\overline{K}_2 & x\in(l(t)+\delta_r,r(t)) \end{cases}$$

with $\overline{K}_1$ and $\overline{K}_2$ positive constants and $0\le\delta_l\le\delta_r$. Moreover, since the filaments length increases only for $x=l(t)$, we assume that δ_l and δ_r are small enough and in particular we assume that:

$$0<\delta_l\ll\delta_r.$$

Then we derive the following evolution equation for $l_f(x,t)$:

$$\frac{\partial l_f}{\partial t} + w\frac{\partial l_f}{\partial x} = -\overline{K}(x,t) \quad \forall x \in \Omega(t), t > 0. \tag{2.18}$$

As in the original method we derive the equation for the evolution of ρ using (2.1). So, in our case the equation for ρ becomes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (w\rho)}{\partial x} = -\overline{K}(x,t)u. \tag{2.19}$$

From (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.19) and (2.14) we can write the new model as:

$$\begin{cases} u_t - \dfrac{E}{2}(u^2)_{xx} = 0 & x \in (l(t), r(t)), t > 0 \\ \rho_t - E(u_x\rho)_x = K(x,t)u & x \in (l(t), r(t)), t > 0 \end{cases} \tag{M}$$

$$K(x,t) = C\overline{K}(x,t) = \begin{cases} K_1 & x \in (l(t), l(t)+\delta_l), t > 0 \\ C^2 \text{ and decreasing} & x \in (l(t)+\delta_l, l(t)+\delta_r), t > 0 \\ -K_2 & x \in (l(t)+\delta_r, r(t)) \end{cases} \tag{2.20}$$

subject (I), (B), (LR) and also conditions (2.15)-(2.17).

As we said previously, the model is a system of two parabolic/hyperbolic differential equations with two moving boundaries.

Remark 2.1. We want to emphasize that model (M_o) can be obtained from model (M) simply setting $K_1 = -K_2$, $\delta_l = \delta_r = 0$. In fact, with these choices of the parameters the two intervals $(l(t), l(t)+\delta_l)$ and $(l(t)+\delta_l, l(t)+\delta_r)$ are empty and on $x = l(t)$, $K(x,t) = -K_2$. So $K(x,t) \equiv -K_2$ for all $x \in \Omega(t), t > 0$. So, all the results we show in next sections for (M) also hold for (M_o).

3 – Theoretical results

This chapter is focused on the study of system (M) with (I), (B), (LR) and (2.15)-(2.17), from a theoretical point of view.

In particular we will show the following local existence theorem:

Theorem 3.1. *Let r_0 be a positive constant and let $\tilde{u}_0(x) \in C^{4+\alpha}[0, r_0]$ and $\tilde{\rho}_0(x) \in C[0, r_0]$ such that*:

$$\tilde{u}_0(x) \quad \text{and} \quad \tilde{\rho}_0(x) > 0 \quad \forall x \in [0, r_0] \tag{3.1}$$

$$\tilde{\rho}_0(r_0) - d\tilde{u}_0(r_0)^{\frac{2}{3}} > 0 \tag{3.2}$$

$$(\tilde{u}_0)_x < 0 \quad \forall x \in [0, r_0] \tag{3.3}$$

$$(\tilde{u}_0)_x(0) > -\frac{1}{E}\frac{\beta}{\alpha} \tag{3.4}$$

$$(\tilde{u}_0)_x(r_0) > -\frac{1}{E}\frac{\nu}{\tilde{\rho}_0(r_0) - d\tilde{u}_0(r_0)^{\frac{2}{3}}} \tag{3.5}$$

$$\tilde{u}_0(l(t)) = u_0 \tag{3.6}$$

$$\tilde{u}_0(r(t)) = u_0 + \frac{\mu}{bE} l'(0) \tag{3.7}$$

$$0 = \Big[(E\tilde{u}_0(\tilde{u}_0)_x)_x + l'(0)(\tilde{u}_0)_x\Big]_{x=0} \tag{3.8}$$

$$\frac{\mu}{b}\frac{\alpha}{\mu-\alpha}[E(\tilde{u}_0(\tilde{u}_0)_x)_{xx} + l'(0)(\tilde{u}_0)_{xx}] = -\Big[E(\tilde{u}_0(\tilde{u}_0)_y)_y + r'(0)(\tilde{u}_0)_y\Big]_{y=r_0} \tag{3.9}$$

$$0 = E^2(\tilde{u}_0(\tilde{u}_0(\tilde{u}_0)_x)_x)_{xx} + 2El'(0)(\tilde{u}_0(\tilde{u}_0)_x)_{xx} + l'(0)^2(\tilde{u}_0)_{xx} + l''(0)(\tilde{u}_0)_x \tag{3.10}$$

where $l'(t)$ and $r'(t)$ are defined in (LR) *and*

$$l''(t) = -E\frac{\alpha}{\mu-\alpha}\left[\frac{r_0}{r(t)-l(t)}u_{yt}(0,t) - r_0\frac{r'(t)-l'(t)}{(r(t)-l(t))^2}u_y(0,t)\right].$$

Then there exists $T > 0$ such that system (M) with (I), (B), (LR) *and* (2.15)*)*-(2.17) *has a unique solution for all $(x,t) \in \Omega_T(t) = \cup_{t\in[0,T]}((l(t), r(t)) \times \{t\})$.*

Moreover it results $u \in C^{2,1}(\Omega_T(t))$ and $\rho \in C(\Omega_T(t))$.

The hypotheses of this theorem guarantee the solvability of system (M), with (B), (I), (LR) and (2.15)-(2.17) for all $(x,t) \in [0,T]$. In fact:

- hypotheses (3.1) and (3.2) together with the positivity of r_0 imply that the initial data verify conditions (2.15)-(2.17);
- hypotheses (3.3)-(3.5) guarantee, as we will show in the proof of the theorem, that the equation for ρ together with its boundary value at $x = l(t)$ is well posed;
- hypotheses (3.6)-(3.10) are the compatibility conditions for the parabolic equation for u. They are needed to make $u \in C^{2,1}$ and $\rho \in C$.

The scheme of the proof of Theorem 3.1 follows the one in [2] for the local existence and uniqueness of the solution for the original model, (M_o).

In the first part of the proof, we fix two functions $l(t), r(t) : \mathbb{R} \to \mathbb{R}$ such that:

$$\begin{array}{llll} l(t) \in C^2 & \text{such that} & l(0) = 0 \quad \text{and} & l'(0) < 0 \\ r(t) \in C^1 & \text{such that} & r(0) = r_0 > 0 \quad \text{and} & r'(0) < 0. \end{array} \tag{3.11}$$

These two functions are used to define a change of the spatial variable x into y, which transform the moving domain $\Omega(t) = (l(t), r(t))$ in the fixed domain $Q = (0, r_0)$.

System (M) is transformed in a new system in the variable y and for this new system the existence and the uniqueness of the solution is proved. We denote by (u_{lr}, ρ_{lr}) this solution, where the subscript lr indicates that the solution is strictly dependent on $l(t)$ and $r(t)$.

Theorem 3.1 is then proved using a fixed point argument. More in detail, two functions, $\hat{l}(t)$ and $\hat{r}(t)$ verifying equations (LR) with (u, ρ) substituted by (u_{lr}, ρ_{lr}). So, $\hat{l}$ and $\hat{r}$ depend on l, r and t. Finally we define a map Γ such that:

$$\Gamma(l, r)(t) = (\hat{l}(t), \hat{r}(t))$$

and for this map we will show that it exists a unique fixed point.

In the following sections we will describe our proof of Theorem 3.1. In particular we will first show the following theorem:

THEOREM 3.2. *Let r_0 be a positive constant and let $\tilde{u}_0(x) \in C^{4+\alpha}[0, r_0]$ and $\tilde{\rho}_0(x) \in C[0, r_0]$ such that hypotheses* (3.1)-(3.10) *are verified.*

Assume, moreover, that $l(t) \in C^2$ and $r(t) \in C^1$ such that $l(0) = 0$ and $r(0) = r_0 > 0$.

Then there exists $T > 0$ such that system (M) with (I), (B), (LR) *and* (2.15)-(2.17) *has a unique solution for all $(x, t) \in \Omega_T(t)$.*

Moreover $u \in C^{2,1}(\Omega_T(t))$ and $\rho \in C(\Omega_T(t))$.

The Theorem 3.1 depends on an subtile use of the Contraction theorem, the strategy we adopt is inspired by [23]. The proof relies on several lemmas we will prove in the following subsections.

3.1 – The model in a fixed domain

As we have said at the end of the previous section, in this first part of the proof we fix two functions $l(t)$ and $r(t)$ verifying (3.11).

Note that this assumption implies that:

- there exists $T_l > 0$ such that

$$\forall t \in [0, T_l] \quad l'(t) < 0 \quad \text{and} \quad l(t) < 0$$

- there exists $T_r > 0$ such that

$$\forall t \in [0, T_r] \quad r'(t) < 0 \quad \text{and} \quad r(t) < r_0 \tag{3.12}$$

- there exists $T_{r-l} > 0$ such that

$$\forall t \in [0, T_{r-l}] \quad r(t) - l(t) > 0. \tag{3.13}$$

We define $T = \min\{T_l, T_r, T_{r-l}\}$ and we study the existence of the solution of system (M) with (I), (B), (LR) and (2.15)-(2.17) in the interval $t \in (0, T)$.

In order to transform the moving domain $\Omega(t) = (l(t), r(t))$ in the fixed one $Q = (0, r_0)$, one can choose among several change of the spatial variable.

In particular, in [2] the following transformation is used:

$$x = y + R(y, t) \tag{3.14}$$

with

$$R(y, t) = l(t)\chi(y) + (r(t) - r_0)\chi(y - r_0) \tag{3.15}$$

and $\chi(z) \in C^\infty$ such that:

$$\chi(z) = \begin{cases} 0 & |z| > \delta_0/4 \\ 1 & |z| < \delta_0/8 \end{cases}$$

with $\delta_0 < r_0/2$. Moreover $\chi(z)$ increases for $z \in (-\delta_0/4, -\delta_0/8)$ and it decreases in $(\delta_0/8, \delta_0/4)$.

From (3.14):

$$\frac{dx}{dy} = \begin{cases} 1 + l(t)\chi'(y) & y \in (\delta_0/8, \delta_0/4) \\ 1 + (r(t) - r_0)\chi'(y - r_0) & y \in (r_0 - \delta_0/4, r_0 - \delta_0/8) \\ 1 & \text{otherwise} \end{cases}$$

$x(y)$ is invertible if and only if it is monotone and we have that it is monotone if:

$$l(t)\chi'(y) > 0 \quad \forall y \in (\delta_0/8, \delta_0/4) \tag{3.16}$$

and

$$(r(t) - r_0)\chi'(y - r_0) > 0 \quad \forall y \in (r_0 - \delta_0/4, r_0 - \delta_0/8). \tag{3.17}$$

Our choice of T ensures that $l(t) < 0$ and since $\chi'(y)$ is decreasing for $y \in (\delta_0/8, \delta_0/4)$ then:

$$l(t)\chi'(y) > 0 \quad \forall y \in (\delta_0/8, \delta_0/4)$$

and so (3.16) holds.

On the other hand, $r(t) < r_0$ and $\chi(y-r_0)$ is increasing for all $y \in (r_0-\delta_0/4, r_0-\delta_0/8)$. So

$$(r(t)-r_0)\chi'(y-r_0) < 0 \quad \forall y \in (r_0-\delta_0/4, r_0-\delta_0/8)$$

and in order to ensure (3.17) we should add some hypotheses on $\chi(z)$.

For this reason we have chosen the following change of the spatial variable x:

$$y = r_0 \frac{x-l(t)}{r(t)-l(t)}. \tag{3.18}$$

That is linear and invertible for all t such that $r(t)-l(t) > 0$, and so for all $t \in (0,T)$.

The equations of the system in the moving domain are easier than in the fixed one. In fact, since:

$$\frac{\partial y}{\partial x} = \frac{r_0}{(r(t)-l(t))}$$

and

$$\frac{\partial y}{\partial t} = -\frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)}$$

system (M) in the fixed domain becomes:

$$\begin{cases} u_t - \dfrac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)} u_y - \dfrac{E}{2}\dfrac{r_0^2}{(r(t)-l(t))^2}(u^2)_{yy} = 0 \quad (y,t) \in Q_T \\ \rho_t - \dfrac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)} \rho_y - E\dfrac{r_0^2}{(r(t)-l(t))^2}(u_y\rho)_y = K(y,t)u(y,t) \end{cases} \tag{M_F}$$

where $Q_T = Q \times [0,T]$. Moreover, $K(y,t)$ is the C^2 function corresponding to $K(x,t)$. That is:

$$K(y,t) = \begin{cases} K_1 & y \in \left(0, \dfrac{r_0}{r(t)-l(t)}\delta_l\right) \\ \text{is decreasing} & y \in \left(\dfrac{r_0}{r(t)-l(t)}\delta_l, \dfrac{r_0}{r(t)-l(t)}\delta_r\right) \\ K_2 & y \in \left(\dfrac{r_0}{r(t)-l(t)}\delta_r, r_0\right) \end{cases} \tag{3.19}$$

(M_F) is subject to the same initial and boundary conditions of system (M):

$$\begin{cases} u(y,0) = \tilde{u}_0(y) & y \in (0, r_0) \\ \rho(y,0) = \tilde{\rho}_0(y) & y \in (0, r_0) \end{cases} \tag{I_F}$$

$$\begin{cases} u(0,t) = u_0 \\ u(r_0,t) = u_0 + \dfrac{\mu}{bE}\dfrac{dl}{dt} \\ \rho(0,t) = \rho_0 \end{cases} \tag{B_F}$$

and to the following evolution equations for $l(t)$ and $r(t)$:

$$\begin{cases} \dfrac{dl}{dt} = -E\dfrac{\alpha}{\mu-\alpha}\dfrac{r_0}{r(t)-l(t)}u_y(0,t) - \dfrac{\beta}{\mu-\alpha} & l(0) = 0 \\ \dfrac{dr}{dt} = -E\dfrac{r_0}{r(t)-l(t)}u_y(r_0,t) - \dfrac{\nu}{\rho(r_0,t) - du(r_0,t)^{\frac{2}{3}}} & r(0) = r_0. \end{cases} \tag{LR_F}$$

Note that $l(t)$ and $r(t)$ lose their meaning of moving boundary positions. However their computation at every time is needed to define the coefficients of system (M_F).

Moreover we want the solution to satisfy:

$$u, \rho > 0 \quad \forall (y,t) \in Q_T. \tag{3.20}$$

3.2 – Local existence and uniqueness of the solution

Since $\Omega_T(0) = \Omega(0)$ the hypotheses (3.1)-(3.10) of Theorem 3.2 are not affected by the change of variable (3.18), provided they are related to the variable y instead of x. For the sake of simplicity, in the following we will refer to them without specifying at which variable they are related, except when it will not be clear from the context.

So, with this change of variables, Theorem 3.2 is equivalent to the following one:

THEOREM 3.3. *Let r_0 be a positive constant, $\tilde{u}_0 \in C^{4+\alpha}([0, r_0])$ and $\tilde{\rho}_0 \in C([0, r_0])$ functions such that the hypotheses* (3.1)-(3.10) *of Theorem* 3.2 *are verified. Moreover, let suppose that there exist $l(t)$ and $r(t)$ defined as in the hypotheses of Theorem* 3.1. *Then there exists $T > 0$ such that system (M_F) with (I_F), (B_F) and (LR_F) has a unique solution for all $(y,t) \in Q_T = (0, r_0) \times [0, T]$.*

Moreover $u \in C^{2,1}(Q_T)$ and $\rho \in C(Q_T)$ are positive functions.

In order to study the existence and the uniqueness of the solution for (M_F) with (I_F), (B_F) and (LR_F) we split system (M_F) in two easier coupled systems and then we will show the existence and the uniqueness of the solution of both systems.

In particular, we split the system in the two following ones:

$$\begin{cases} u_t - (a(t,u)u_y)_y - b(y,t)u_y = 0 & (y,t) \in Q_T \\ u(y,0) = \tilde{u}_0(y) & y \in (0, r_0) \\ u(0,t) = u_0 & t \in (0,T] \\ u(r_0,t) = u_0 + \dfrac{\mu}{bE} l'(t) & t \in (0,T] \end{cases} \tag{S_u}$$

$$\begin{cases} \rho_t - (a(t,u)u_y\rho)_y - b(y,t)\rho_y = K(y,t)u & (y,t) \in Q_T \\ \rho(y,0) = \tilde{\rho}_0(y) & y \in [0, r_0] \\ \rho(0,t) = \rho_0 & t \in (0,T] \end{cases} \tag{S_ρ}$$

where:

$$a(t,u) = E\frac{r_0^2}{(r(t)-l(t))^2}u = a_1(t)a_2(u) \tag{3.21}$$

with

$$a_1(t) = E\frac{r_0^2}{(r(t)-l(t))^2} \qquad \text{and} \qquad a_2(u) = u \tag{3.22}$$

$$b(y,t) = \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)} \tag{3.23}$$

and $K(y,t)$ is defined in (3.19).

Existence and uniqueness for (S_ρ). The partial differential equation of system (S_ρ) is of transport type with a flux velocity and a source term depending on y, t and u.

$$\rho_t - a(t)(u_y\rho)_y - b(y,t)\rho_y = K(y,t)u \quad \forall (y,t) \in Q_T$$

Following [11], we can determine its solution using the method of characteristics. In particular, the characteristic curves are defined in the following way:

$$\begin{cases} s = t \\ \dfrac{dy}{dt} = -a(t)u_y - b(y,t) \quad y(0) = y_0. \end{cases} \tag{3.24}$$

Since the boundary data for ρ is defined on $y = 0$, (S_ρ) is well posed if:

$$\frac{dy}{dt} = -a(t)u_y - b(y,t) > 0$$

and for the definition of $a(t)$, (3.21), and $b(y,t)$, (3.23), at $t = 0$:

$$\frac{dy}{dt}(0) = -a(0)u_y - b(y,0) = -E(\tilde{u}_0)_y - \frac{l'(0)(r_0 - y) + r'(0)y}{r_0}$$

and this quantity is positive. In fact from (3.3) $(\tilde{u}_0)_y < 0$ and, from (LR_F), hypotheses (3.4) and (3.5) imply that $l'(0)$ and $r'(0)$ are also negative. So, for $t = 0$ (S_ρ) is well posed and for the assumptions on the regularity of $l(t)$ and $r(t)$, there exists T_{u_x}, such that it is well posed for all $t \in [0, T_{u_x}]$. So, setting $T = \min\{T_{lr}, T_{u_x}\}$ the problem for ρ is well posed and $r(t) - l(t) > 0$ for all $t \in [0, T]$.

Along the characteristics ρ verifies:

$$\frac{d\rho}{dt} = a(t)u_{yy}(y(t), t)\rho(y(t), t) + K(y(t))u(y(t), t) \quad t \in (0, T]. \tag{3.25}$$

Then, along that curves:

$$\rho(t) = e^{A(t)} \left(\overline{\rho}_0(y_0) + B(t)\right) \tag{3.26}$$

where:

$$A(s) = \int_0^s a(s')u_{yy}(y(s'), s')ds' \tag{3.27}$$

and

$$B(s) = \int_0^s e^{-A(s')} K(y(s'), s')u(y(s'), s')ds'. \tag{3.28}$$

This prove the existence of the solution for (S_ρ) for all $(y, t) \in Q_T$.

Moreover, this solution is a positive and continuous function.

In fact, let $\delta_0(t) : [0, T] \to [0, r_0]$ such that:

$$K(\delta_0(t), t) = 0 \quad \forall t \in [0, T].$$

Then, since $\rho_t - (a(t, u)\rho)_y - b(y, t)\rho_y = K(y, t)u$, for every fixed $t \in [0, T]$, $\rho(y, t)$ is increasing where $K(y, t)$ is positive and decreasing otherwise. So:

$$\rho(y, t) \geq \begin{cases} \rho_0 & y \in [0, \delta_0(t)] \\ \rho(r_0, t) & y \in [\delta_0(t), r_0] \end{cases}$$

that is, for all $t \in [0, T]$, $\rho(y, t) \geq \min\{\rho_0, \rho(r_0, t)\}$.

As a consequence $\rho_0 > 0$ and $\rho(r_0) > 0$ are sufficient conditions for the positivity of ρ. From hypothesis (3.1), $\rho_0 > 0$. Moreover, from the regularity of u, there exists a $T' > 0$ such that for all $t \in [0, T']$, $\rho(r_0) - du(r_0)^{\frac{2}{3}} > 0$. Then, $\rho(r_0) > du(r_0)^{\frac{2}{3}} > 0$.

Setting $T = \min\{T, T'\}$ we have that for all $t \in [0, T]$ the problem for ρ is well posed, $r(t) - l(t) > 0$ and $\rho(r_0, t) - du(r_0)^{\frac{2}{3}} > 0$ hold.

As far as the regularity of ρ is concerned, we note that from Theorem 3.2, u is a $C^{2,1}$ function and this is a sufficient condition for the continuity of ρ. We summarize the result obtained in this section as follows:

THEOREM 3.4. *Let r_0 be a positive constant, $\tilde{u}_0 \in C^{4+\alpha}([0, r_0])$ and $\tilde{\rho}_0 \in C([0, r_0])$ functions such that the hypotheses* (3.1)-(3.10) *of Theorem* 3.2 *are verified. Let $l(t)$ and $r(t)$ be defined as in the hypotheses of Theorem* 3.2*. Then there exists $T > 0$ such that (S_ρ) has a unique solution which is positive and continuous.*

Existence and uniqueness for (S_u). The study of the existence and the uniqueness of system (S_u) is more difficult. This is due on one hand to the degeneracy of the equation and on the other to the time dependent coefficients of the equation.

There is a wide literature about porous media equations (see for example [22] and the references therein) and quasilinear parabolic degenerate equations ([4, 5] or [8]). But the general case for coefficients depending on y, t, u and u_y is not treated. A masterpiece in the literature about quasilinear parabolic equation is [13] and it will be our main reference in the following.

In order to prove the existence and the uniqueness of the solution for (S_u), we will study the following more general system:

$$\begin{cases} u_t - \left(g(t,u)u_y\right)_y - b(y,t)u_y = 0 & y \in (\eta_1, \eta_2), t > 0 \\ u(y,0) = \tilde{u}_0(y) \geq 0 & y \in (\eta_1, \eta_2) \\ u(\eta_1, t) = \psi_1(t) & t > 0 \\ u(\eta_2, t) = \psi_2(t) & t > 0 \\ (\psi_1)_t = a(t)(uu_y)_y + b(y,t)(u)_y & \text{for } (y,t) = (\eta_1, 0) \\ (\psi_2)_t = a(t)(uu_y)_y + b(y,t)(u)_y & \text{for } (y,t) = (\eta_2, 0) \end{cases} \tag{3.29}$$

where, $\psi_1(t)$ and $\psi_2(t)$ are C^1 functions and

$$g(t,u) = g_1(t)g_2(u).$$

Moreover, we will assume that the following conditions hold:

$$0 < \lambda_1 \leq g_1(t) \leq \Lambda_1 \quad \forall t \in (0,T] \tag{3.30}$$

$$\begin{cases} g_2(u) \in C^1 \\ g_2(0) = 0 \quad \text{and } g_2'(s) > 0 \quad \forall s > 0 \\ ug_2(u) \text{ is a locally Holder continue function} \end{cases} \tag{3.31}$$

and

$$b(y,t), b_y(y,t) \in L^\infty(Q_T). \tag{3.32}$$

In particular, we will prove the following local existence theorem:

THEOREM 3.5. *Let* $\tilde{u}_0 \in C^2[0, r_0]$ *a positive function, let* $g_1 : [0,T] \to \mathbb{R}^+$ *and* $g_2 : \mathbb{R}^+ \to \mathbb{R}$ *be* C^1 *functions and let* λ, Λ *be positive constants such that conditions* (3.30)-(3.32) *are verified. Then* (3.29) *has at least one weak solution.*

We remark that in (3.29) conditions

$$\begin{cases} u(\eta_1, t) = \psi_1(t) & t > 0 \\ u(\eta_2, t) = \psi_2(t) & t > 0 \end{cases} \tag{3.33}$$

are the so called, *compatibility conditions* of zero order and

$$\begin{cases} (\psi_1)_t = g(t)(uu_y)_y + b(y,t)(u)_y & \text{for } (y,t) = (\eta_1, 0) \\ (\psi_2)_t = g(t)(uu_y)_y + b(y,t)(u)_y & \text{for } (y,t) = (\eta_2, 0) \end{cases} \tag{3.34}$$

are those of the first order. From classical theory about parabolic equation, see [13], these conditions are needed to have $u \in C^{2,1}$.

First of all we note if $\tilde{u}_0(y)$ verifies the hypotheses (3.1)-(3.10) of Theorem 3.2, then (S_u) with (B_F) and (LR_F) is the special case of (3.29) obtained setting:

$$\begin{cases} \eta_1 = 0 \\ \eta_2 = r_0 \\ g(t,u) = E\dfrac{r_0^2}{(r(t) - l(t))}u \\ \psi_1(t) = u_0 \\ \psi_2(t) = u_0 + \dfrac{\mu\varepsilon}{b} l'(t) \end{cases}$$

where $a(t,u)$ is defined in (3.21).

Setting $g(t,u) = a(t,u)$ conditions (3.31)-(3.32) are also satisfied. In fact, since for all $t \in [0,T]$, $r(t) - l(t)$ is a positive and continuous function, setting:

$$m = \min_{t \in [0,T]} r(t) - l(t) \quad \text{and} \quad M = \max_{t \in [0,T]} r(t) - l(t)$$

and

$$\lambda = E\frac{r_0^2}{M} \quad \text{and} \quad \Lambda = E\frac{r_0^2}{m}$$

we have that $\lambda \leq a_1(t) \leq \Lambda$ where $a_1(t)$ is defined in (3.22). Moreover $a_2(u) = u$ verifies (3.31). Finally:

$$b(y,t) = \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)} \quad \text{and} \quad b_y(y,t) = \frac{-l'(t) + r'(t)}{r(t) - l(t)}$$

are continuous functions and so they are bounded on Q_T. Then, hypothesis (3.32) is also satisfied.

Hypotheses (3.6)-(3.9) correspond to the compatibility conditions of zero and first order for (S_u) and they are verified, since $\tilde{u}_0(0) = u_0$ and $\tilde{u}_0(r_0) = u_0 + \mu\varepsilon l'(0)/b$. So

$$\begin{cases} \psi_1(0) = \overline{u}_0(0) \\ \psi_2(0) = \overline{u}_0(r_0). \end{cases}$$

With regard to the first order compatibility condition in $y = 0$, from (3.34):

$$\psi_{1t}(0) = |(a(t,u)(uu_y)_y + b(y,t)u_y)|_{y=0,t=0}\,.$$

Since $\psi_1(t) = u_0$ then $\psi_{1t} = 0$ and

$$\begin{aligned} \Big|(a(t,u)(uu_y)_y + b(y,t)u_y)\Big|_{y=0,t=0} &= a(0,\overline{u}_0)(\overline{u}_0(\overline{u}_0)_y)_y + b(0,0)(\overline{u}_0)_y \\ &= E(\overline{u}_0(\overline{u}_0)_y)_y + l'(0)(\overline{u}_0)_y. \end{aligned}$$

So hypothesis (3.8) implies that this compatibility condition holds.

In $y = r_0$, the first compatibility condition is:

$$\psi_{2t}(0) = |(a(t,u)(uu_y)_y + b(y,t)u_y)|_{y=r_0,t=0}\,.$$

From the definition of $\psi_2(t)$ we obtain that:

$$\psi_{2t}(t) = \frac{\mu}{bE} l''(t)$$

and deriving $l'(t)$, defined in (LR_F), we can write $l''(t)$ as follows:

$$\begin{aligned} l''(t) &= E\frac{\alpha}{\mu-\alpha}\left[\frac{r_0(r'(t)-l'(t))}{(r(t)-l(t))^2}u_y - \frac{r_0}{r(t)-l(t)}u_{yt}\right]_{y=0} \\ &= E\frac{\alpha}{\mu-\alpha}\frac{r_0}{r(t)-l(t)}\left[\frac{r'(t)-l'(t)}{r(t)-l(t)}u_y - \frac{r_0}{r(t)-l(t)}(u_t)_y\right]_{y=0} \end{aligned}$$

and

$$\begin{aligned} (u_t)_y &= \left[E\frac{r_0^2}{(r(t)-l(t))^2}(uu_y)_y + \frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)}u_y\right]_y \\ &= \left[E\frac{r_0^2}{(r(t)-l(t))^2}(uu_y)_{yy} + \frac{r'(t)-l'(t)}{r(t)-l(t)}u_y + \frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)}u_{yy}\right]. \end{aligned}$$

Then

$$\begin{aligned} l''(t) &= E\frac{\alpha}{\mu-\alpha}\frac{r_0}{r(t)-l(t)}\left[\frac{r'(t)-l'(t)}{r(t)-l(t)}\left(1-\frac{r_0}{r(t)-l(t)}\right)u_y\right. \\ &\quad \left. -E\frac{r_0^3}{(r(t)-l(t))^3}(uu_y)_{yy} - r_0\frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)}u_{yy}\right]_{y=0} \\ &= E\frac{\alpha}{\mu-\alpha}\frac{r_0}{r(t)-l(t)}\left[\frac{r'(t)-l'(t)}{r(t)-l(t)}\left(1-\frac{r_0}{r(t)-l(t)}\right)u_y(0,t)\right. \\ &\quad \left. -E\frac{r_0^3}{(r(t)-l(t))^3}(uu_y)_{yy}(0,t) - \frac{r_0^2}{(r(t)-l(t))^2}l'(t)u_{yy}(0,t)\right]. \end{aligned}$$

So

$$l''(0) = -E\frac{\alpha}{\mu-\alpha}\left[E(\tilde{u}_0(\tilde{u}_0)_y)_{yy} + l'(0)(\tilde{u}_0)_{yy}\right] \tag{3.35}$$

and replacing l'' in the first order compatibility condition we obtain:

$$\begin{aligned}
&-\frac{\mu}{b}\frac{\alpha}{\mu-\alpha}\left[E(\overline{u}_0(\overline{u}_0)_y)_{yy}+l'(0)(\overline{u}_0)_{yy}\right]\\
&=\left[E\frac{r_0^2}{(r(t)-l(t))^2}(\overline{u}_0(\overline{u}_0)_y)_y+\frac{l'(t)(r_0-y)+r'(t)y}{r(t)-l(t)}(\overline{u}_0)_y\right]_{y=r_0,t=0}\\
&=\left[E(\tilde{u}_0(\tilde{u}_0)_y)_y+r'(0)(\tilde{u}_0)_y\right]
\end{aligned}$$

that is hypothesis (3.9).

This proves that (S_u) is a special case of system (3.29) and it verifies hypotheses of Theorem 3.2. So Theorem 3.5 is a generalization of Theorem 3.2.

Proof of Existence of solutions to (3.29). First of all we have to give the following definition to explain what we mean for a weak solution of (3.29).

Definition 3.6. *A function u defined in $[\eta_1,\eta_2]\times[0,T]$ is a weak solution for (3.29), if:*

1. *u is real, non negative and continuous;*
2. $\begin{cases} u(y,0)=\tilde{u}_0(y) & x\in[\eta_1,\eta_2]\\ u(\eta_1,t)=\psi_1(t) & t\in[0,T]\\ u(\eta_2,t)=\psi_2(t) & t\in[0,T]\end{cases}$
3. *$G_2(u(y,t))=\left(\int_0^{u(y,t)}g_2(s)ds\right)$ is of class $C^{2,1}$ and its derivative with respect to y is a square integrable function;*
4. *u verifies the following identity:*

$$\begin{aligned}
&\int_0^T\int_{\eta_1}^{\eta_2}[u_t\varphi-g_1(t)(g_2(u)u)_y\varphi_y-(b(y,t)\varphi)_y u]dydt\\
&=-\int_{\eta_1}^{\eta_2}\tilde{u}_0(y)\varphi(y,0)dydt
\end{aligned}\tag{3.36}$$

for all $\varphi\in C^2$ such that $\varphi(\eta_1,t)=\varphi(\eta_2,t)=0$ for any $t\in(0,T]$ and $\varphi(y,T)=0$ for any $y\in(\eta_1,\eta_2)$.

Remark 3.7. In the literature about first boundary problems for degenerate parabolic equations, this special case is not treated in detail. Among others we point out the article of Gilding [8] and the one of Bertsch and Kamin [3]. In particular, in [8] the first boundary problem for equations of the following type are treated:

$$u_t=(a(u)u_y)_y+b(u)u_y \tag{3.37}$$

while in [3]:

$$u_t=(a(t,u)u_y)_y+b(y,t,u). \tag{3.38}$$

Our equation doesn't belong neither to (3.37) nor to (3.38).

The independence of $b(y,t,u)$ from u_y in the equation (3.38) as well as the independence of the coefficients from y, t and u_y in the equations (3.37) nor to (3.38), makes the study of said equations easier then the study of our model. Indeed, since in our equation for u $a(t,u) = a_1(t)a_2(u)$, we can follow the proof of the existence for (3.37).

The idea of the proof of Theorem 3.5 is to show that if, for all $y \in [0, r_0]$, $\tilde{u}(y) > 0$, then there exists a sequence of functions $\{u_{0,k}\}$ which uniformly converges to $\tilde{u}_0$ for $k \to \infty$. Then we show that for all k, the system (3.29) with initial datum $u_{0,k}$ has a unique solution, u_k. Finally we prove that setting

$$u(y,t) = \lim_{k\to\infty} u_k(y,t)$$

then u is a weak solution for system 3.29. In the next section we will show that this solution is also unique.

For the proof we need the following four lemma.

LEMMA 3.8. *Let $f \in C^1(0,\infty)$. Then given any M positive real constant there exists a function $\vartheta \in C^2[0,M]$ and a positive constant C such that for $s \in (0,M]$:*

1. $C \geq |\vartheta(s)| \geq \frac{1}{C}$;
2. $\vartheta''(s)\vartheta(s) < 0$;
3. $|f'(s)\vartheta(s) + 2f(s)\vartheta'(s)| \leq -C\vartheta''(s)\vartheta(s)$;
4. $f^2(s) \leq -C\vartheta''\vartheta$;

if and only if $F(s) = s|f'(s)| \in L^1(0,M)$.

The proof of this lemma follows from [8, Lemma 3], simply replacing $b(s)$ with a constant.

LEMMA 3.9. *Let $\varepsilon, \alpha \in (0,1]$ and $M > 0$ be fixed arbitrary constants.*

Let be $Q_T = (\eta_1, \eta_2) \times (0,T]$, with $-\infty < \eta_1 < \eta_2 < \infty$.

Suppose that $\tilde{u}_0(y) \in C^{2+\alpha}[0,r_0]$ and that $\psi_1(t)$, $\psi_2(t)$ are $C^{1+\alpha}[0,T]$ functions such that:

$$\begin{cases} \varepsilon \leq \tilde{u}_0(y) \leq M & y \in [\eta_1,\eta_2] \\ \varepsilon \leq \psi_1(t), \psi_2(t) \leq M & t \in [0,T] \\ \psi_i(0) = \tilde{u}_0(\eta_i) & i = 1,2 \\ \psi_i'(0) = \left(g(0,\tilde{u}_0)(\tilde{u}_0)_y\right)_y + b(y,0)\tilde{u}_0(y)_y & i = 1,2. \end{cases} \tag{3.39}$$

Then, if $g_1(t), g_2(u)$ *and* $b(y,t)$ *verify respectively conditions* (3.30), (3.31) *and* (3.32) *there exists a unique function* $u(y,t)$ *such that*:

1. $u(y,t) \in C^{2,1}(\overline{Q}_T)$;
2. $G_2(u) = \displaystyle\int_0^{u(y,t)} g_2(r)dr$ *is such that* $G_2(u) \in C^{2,1}(Q)$ *and its derivative with respect to* y *is a square integrable function*;
3. $\varepsilon \le u \le M$, $\forall (y,t) \in Q_T$;
4. $u_t = (g(t,u)u_y)_y + b(y,t)u_y$, $\forall (y,t) \in Q_T$;
5. $\begin{cases} u(y,0) = \tilde{u}_0(y) \\ u(\eta_i,t) = \psi_i(t) \quad \forall t \in [0,T], i = 1,2. \end{cases}$

The proof of this Lemma uses Lemma 3.8 and some properties of the non degenerate parabolic equations. In particular, since $g(t,u)$ has a continuous derivative with respect to u and verifies conditions (3.30) and (3.31), then there exists $h(u)$ such that:

$$\begin{cases} h(s) = g_2(s) & \varepsilon \le u \le M \\ h'(s) = 0 & \text{otherwise.} \end{cases}$$

Then there exist $\gamma \in (0,1)$ and a function $u \in C^{2+\gamma,1}(\overline{Q}_T)$ which satisfies (B_F) and equation

$$u_t - g_1(t)(h(u)u_y)_y - b(y,t)u_y = 0 \quad \text{for all} \quad (y,t) \in Q_T \tag{3.40}$$

moreover u is the unique solution of the problem above.

In fact, $g_1(t)h(u)$ and $b(y,t)$ verifies the hypotheses of the existence Theorem 6.1 (of [13, pag. 452]), for parabolic non degenerate equation. In particular, writing equation (3.40) as follows:

$$u_t - (g_1(t)h(u))u_{yy} - b(y,t)u_y = 0 \quad \text{for all} \quad (y,t) \in Q_T$$

we have that:

$$g_1(t)h(u)\xi^2 = E\frac{r_0^2}{(r(t) - l(t))^2}u\xi^2 \ge \lambda\varepsilon\xi^2 > 0$$

and

$$(g_1(t)h(u) + b(y,t))u_y\big|_{u_y=0} = 0 > -c_1u^2 - c_2$$

for all c_1 and c_2 positive constants.

Then to prove the Lemma, we have only to show that $G_2(u)$ has a generalized square integrable derivative. This can be shown setting $v = G_2(u)$ and noting that v verifies:

$$v_t = g_1(t)h(u)v_{yy} + b(y,t)v_y \quad \text{in} \quad Q_T.$$

The required regularity is then obtained noting that $u \in C^{2,1}(Q_T)$ and using standard theory on parabolic equations, for further detail see [13].

LEMMA 3.10. *Let the assumptions of Lemma* 3.9 *hold and let* $u(y,t)$ *be the function exhibited in the same Lemma. Suppose that*

$$|G_2'(\tilde{u}_0)| \leq K_0 \quad \forall y \in \left[\eta_1 + \frac{\delta}{2}, \eta_2 - \frac{\delta}{2}\right]$$

with K_0 *and* δ *positive constants.*

Then if

$$s g_2'(s) \in L^1(0,1)$$

there exists a constant $K = K(K_0, \delta, M)$ *such that:*

$$|G_2(u(y_1,t_1)) - G_2(u(y_2,t_2))| \leq K\left[|y_1 - y_2|^2 + |t_1 - t_2|\right]^{\frac{1}{2}}$$

$\forall (y_1,t_1),(y_2,t_2) \in \overline{Q_\delta} = \left[\eta_1 + \frac{1}{2}\delta, \eta_2 - \frac{1}{2}\delta\right] \times [0,T]$.

PROOF. $s g_2'(s) \in L^1(0,1)$, so

$$F(s) = s|g_2'(s)| \in L^1(0,M).$$

Then, for Lemma 3.8, there exists a function $\vartheta(s) \in C^2[0,M]$ such that:

1. $C \geq |\vartheta(s)| \geq \frac{1}{C}$;
2. $\vartheta''(s)\vartheta(s) < 0$;
3. $|g_2'(s)\vartheta(s) + 2g_2(s)\vartheta'(s)| \leq -C\vartheta''(s)\vartheta(s)$;
4. $g_2^2(s) \leq -C\vartheta''\vartheta$.

Set

$$w(y,t) = \int_0^{u(y,t)} g_2(s)\vartheta^{-1}(s)ds.$$

The proof of the lemma needs some properties of $w(y,t)$.

$$w_t = \frac{g_2(u)}{\vartheta(u)}u_t \quad \text{and} \quad w_y = \frac{g_2(u)}{\vartheta(u)}u_y$$

and

$$w_{yy} = \frac{g_2(u)}{\vartheta(u)}u_{yy} + \left[\frac{g_2'(u)}{\vartheta(u)} - g_2(u)\frac{\vartheta'(u)}{\vartheta^2(u)}\right]u_y^2. \tag{3.41}$$

Since $u_t = (g(t,u)u_y)_y + b(y,t)u_y$, then:

$$\begin{aligned}
w_t &= \frac{g_2(u)}{\vartheta(u)}\left[g_1(t)g_2(u)u_{yy} + g_1(t)g_2'(u)u_y^2 + b(y,t)u_y\right] \\
&= g_1(t)g_2(u)\left[w_{yy} + \left(g_2(u)\frac{\vartheta'(u)}{\vartheta^2(u)} - \frac{g_2'(u)}{\vartheta(u)}\right)u_y^2\right] + g_1(t)g_2(u)\frac{g_2'(u)}{\vartheta(u)}u_y^2 + b(y,t)w_y \\
&= g(t,u)w_{yy} + g_1(t)\vartheta'(u)w_y^2 + b(y,t)w_y.
\end{aligned}$$

We differentiate this equation with respect to y and multiply it for w_y. So, we obtain:

$$w_y w_{yt} = g_1(t)g_2(u)w_y w_{yyy} + \left[\frac{g_1(t)}{g_2(u)}g_2'(u)\vartheta(u) + 2g_1(t)\vartheta'(u)\right] w_y^2 w_{yy} + \frac{g_1(t)}{g_2(u)}\vartheta(u)\vartheta''(u)(w_y)^4 + b_y(y,t)w_y^2 + b(y,t)w_y w_{yy}.$$

Setting $p = w_y$ the previous equation becomes:

$$\frac{1}{2}(p^2)_t = g_1(t)g_2(u)pp_{yy} + \left[\frac{g_1(t)}{g_2(u)}g_2'(u)\vartheta(u) + 2g_1(t)\vartheta'(u)\right] p^2 p_y + \frac{g_1(t)}{g_2(u)}\vartheta(u)\vartheta''(u)p^4 + b_y(y,t)p^2 + b(y,t)pp_y. \tag{3.42}$$

Let's define $z(y,t)$ in the following way:

$$z(y,t) = \zeta^2(y)p^2(y,t) \tag{3.43}$$

where $\zeta \in C^2[\eta_1, \eta_2]$ is a *cut-off* function such that:

$$\zeta(y) = \begin{cases} 1 & y \in \left[\eta_1 + \frac{3}{4}\delta, \eta_2 - \frac{3}{4}\delta\right] \\ 0 & y \in \left[\eta_1, \eta_1 + \frac{1}{2}\delta\right] \cup \left[\eta_2 - \frac{1}{2}\delta, \eta_2\right] \end{cases} \tag{3.44}$$

and $\forall y \in [\eta_1, \eta_2] \quad 0 \le \zeta(y) \le 1$.

If $z(y,t)$ has a maximum point in Q_T, then at this point:

$$z_y = z_t = 0 \quad \text{and} \quad z_{yy} < 0$$

or, since $g(t,u)$ is positive,

$$z_y = 0 \quad \text{and} \quad g(t,u)z_{yy} - z_t = 0.$$

From (3.43):

$$z_t = 2\zeta^2(y)pp_t \tag{3.45}$$

$$z_y = 2\zeta p(\zeta' p + \zeta p_y) \tag{3.46}$$

and

$$z_{yy} = 2[\zeta' p + \zeta p_y]^2 + 2\zeta p[\zeta'' p + 2\zeta' p_y + \zeta p_{yy}] \tag{3.47}$$

$z_y = 0$ implies:

$$\zeta' p = -\zeta p_y \tag{3.48}$$

and at this point $g(t,u)z_{yy} - z_t$ becomes:

$$\begin{aligned} g(t,u)z_{yy} - z_t =& 2g(t,u)\zeta p[\zeta'' p + 2\zeta' p_y + \zeta p_{yy}] - 2\zeta^2 p p_t \\ =& 2[\zeta^2(g(t,u)pp_{yy} - pp_t) + g(t,u)\zeta\zeta'' p^2 - 2g(t,u)p^2(\zeta')^2] \end{aligned}$$

and so $g(t,u)z_{yy} - z_t < 0$ if and only if:

$$\zeta^2 \left(\frac{1}{2}(p^2)_t - g(t,u)pp_{yy} \right) \geq g(t,u)p^2(\zeta\zeta'' - 2(\zeta')^2) \tag{3.49}$$

and using equation (3.42):

$$\begin{aligned} &\zeta^2 \left(\frac{1}{2}(p^2)_t - g(t,u)pp_{yy} \right) \\ &= \zeta^2 \left[\frac{g_1(t)}{g_2(u)} \vartheta\vartheta'' p^4 + \left(\frac{g_1(t)}{g_2(u)} g_2' \vartheta + 2g_1(t)\vartheta' \right) p^2 p_y + b_y p^2 + bpp_y \right] \\ &= \zeta^2 p^2 \left[\frac{g_1(t)}{g_2(u)} \vartheta\vartheta'' p^2 + \left(\frac{g_1(t)}{g_2(u)} g_2' \vartheta + 2g_1(t)\vartheta' \right) p_y + b_y \right] - b\zeta\zeta' p^2 \end{aligned}$$

where in the last equation we have used (3.48). So equation (3.49) is equivalent to:

$$\zeta^2 \left[\frac{g_1(t)}{g_2(u)} \vartheta\vartheta'' p^2 + \left(\frac{g_1(t)}{g_2(u)} g_2' \vartheta + 2g_1(t)\vartheta' \right) p_y + b_y \right] - b\zeta\zeta' \geq g(t,u)(\zeta\zeta'' - 2(\zeta')^2).$$

That is:

$$\begin{aligned} -\frac{g_1(t)}{g_2(u)} \vartheta\vartheta'' \zeta^2 p^2 &\leq \zeta^2 p_y \left(\frac{g_1}{g_2} g_2' \vartheta + 2g_1\vartheta' \right) + b_y\zeta^2 - b\zeta\zeta'' + g(t,u)(2\zeta' - \zeta\zeta'') \\ &= -\zeta\zeta' p \left(\frac{g_1}{g_2} g_2' \vartheta + 2g_1\vartheta' \right) + b_y\zeta^2 - b\zeta\zeta'' + g(t,u)(2\zeta' - \zeta\zeta''). \end{aligned}$$

So, since for Lemma 3.8, $\vartheta\vartheta'' < 0$ and $g_1(t), g_2(u) > 0$:

$$\begin{aligned}\zeta^2 p^2 \leq &\frac{\zeta\zeta'}{\vartheta\vartheta''}p(g_2'\vartheta + 2g_2\vartheta) + \frac{g_2^2}{\vartheta\vartheta''}(\zeta\zeta'' - 2\zeta') - \frac{g_2}{g_1}\frac{b_y}{\vartheta\vartheta''}\zeta^2 + \frac{g_2}{g_1}\frac{b}{\vartheta\vartheta''}\zeta\zeta' \\ \leq &\left|\frac{\zeta\zeta'}{\vartheta\vartheta''}p(g_2'\vartheta + 2g_2\vartheta)\right| + \left|\frac{g_2^2}{\vartheta\vartheta''}(\zeta\zeta'' - 2\zeta')\right| + \left|\frac{g_2}{g_1}\frac{b_y}{\vartheta\vartheta''}\zeta^2\right| + \left|\frac{g_2}{g_1}\frac{b}{\vartheta\vartheta''}\zeta\zeta'\right|\end{aligned}$$

and again from Lemma 3.8

$$\left|\frac{g_2'\vartheta + 2g_2\vartheta'}{\vartheta\vartheta''}\right| \leq C \quad \text{and} \quad \left|\frac{g_2^2}{\vartheta\vartheta''}\right| \leq C.$$

Moreover, since $g_2' \geq 0$ in $[0, M]$ and $\vartheta > 0$ we have

$$\left|\frac{2g_2\vartheta'}{\vartheta\vartheta''}\right| \leq \left|\frac{g_2'\vartheta + 2g_2\vartheta'}{\vartheta\vartheta''}\right| \leq C$$

and then

$$\left|\frac{g_2}{\vartheta\vartheta''}\right| \leq \frac{1}{2}\frac{C}{\vartheta'}.$$

From Lemma 3.8, $\vartheta'' < 0$ and so ϑ' is a decreasing function. Then

$$\left|\frac{g_2}{\vartheta\vartheta''}\right| \leq \frac{1}{2}\frac{C}{\vartheta'(M)} = C'.$$

Using these inequalities we obtain the following upper bound for $\zeta^2 p^2$:

$$\zeta^2 p^2 \leq C\left[|\zeta\zeta' p| + |\zeta\zeta'' - 2\zeta'| + \frac{1}{|\vartheta'(M)|}\left|\frac{b}{g_1}\zeta\zeta'\right| + \frac{1}{|\vartheta'(M)|}\left|\frac{b_y}{g_1}\zeta^2\right|\right]. \tag{3.50}$$

Since b and b_y are bounded functions and g_1 is positive and continuous in Q_T, b/g_1 and b_y/g_1 are also bounded. Let C_b and C_{b_y} their respectively upper bounds.

Set

$$C' = \frac{1}{|\vartheta'(M)|}\max\{C_b, C_{b_y}\}$$

then equation (3.50) becomes:

$$\zeta^2 p^2 \leq C\left[|\zeta\zeta' p| + |\zeta\zeta'' - 2\zeta'| + C'|\zeta\zeta'| + C'|\zeta^2|\right].$$

Using Young's inequality:

$$C\zeta\zeta' p \leq \frac{\zeta^2 p^2}{2} + \frac{C^2(\zeta')^2}{2} \leq \frac{\zeta^2 p^2}{2} + C^2(\zeta')^2$$

and so

$$\frac{1}{2}z = \frac{1}{2}\zeta^2 p^2 \leq C\left[C(\zeta')^2 + |\zeta\zeta'' - 2\zeta'| + C'|\zeta\zeta'| + C'|\zeta^2|\right]. \tag{3.51}$$

On the other hand if the maximum value of z is not an interior point of Q_T it has to be on the lower bound of $\overline{Q}_T$ and by definition:

$$\begin{aligned} z(y,0) =& \zeta^2(y)p^2(y,0) = \zeta^2(y)(w_y(y,0))^2 = \zeta^2(y)\left[\left(\int_0^u (y,0)g_2(s)\vartheta^{-1}(s)ds\right)_y\right]^2 \\ =& \zeta^2(y)(u_y(y,0))^2 g_2^2(u(y,0))\vartheta^{-2}(u(y,0)) = \zeta^2(y)\left((G_2)'(u_0)\right)^2 \vartheta^{-2} \end{aligned}$$

$|\zeta| \leq 1$ for its definition, (3.44), and since for Lemma 3.8:

$$\vartheta^{-1}(s) \geq \frac{1}{C}$$

and for the hypothesis of the lemma:

$$|(G_2)'(\tilde{u}_0)| \leq K_0 \quad \text{for} \quad y \in \left[\eta_1 + \frac{1}{2}\delta, \eta_2 - \frac{1}{2}\delta\right]$$

then

$$z(y,0) \leq C^2 K_0^2. \tag{3.52}$$

From equations (3.51) and (3.52) we have that

$$\sup_{\overline{Q}_\delta} |w_y| \leq C_1 = C_1(K_0, M, \delta)$$

and since $w_y = \frac{1}{\vartheta}(G_2(u))_y$ it follows $\sup_{\overline{Q}_\delta} |(G_2(u))_y| \leq CC_1$.

Now setting $v(y,t) = G_2(y,t)$, then:

$$v_t = g(t,u)v_{yy} + b(y,t)v_y$$

and

$$|v(y_1,t) - v(y_2,t)| \leq CC_1|y_1 - y_2| \quad \forall (y_1,t),(y_2,t) \in \overline{Q}_\delta. \tag{3.53}$$

Moreover v has a bound which depends only on M.

Then there exists C_2 such that:

$$|v(y,t_1) - v(y,t_2)| \leq C_2|t_1 - t_2|^{\frac{1}{2}}. \tag{3.54}$$

From equations (3.53) and (3.54) the theorem is proved. In fact:

$$\begin{aligned}|v(y_1,t_1)-v(y_2,t_2)| &\le |v(y_1,t_1)-v(y_1,t_2)|+|v(y_1,t_2)-v(y_2,t_2)|\\ &\le C_2|t_1-t_2|^{\frac{1}{2}}+CC_1|y_1-y_2|\le K\left(|y_1-y_2|^2+|t_1-t_2|\right)^{\frac{1}{2}}.\end{aligned}$$

Then:

$$|G_2(y_1,t_1)-G_2(y_2,t_2)|\le K(|y_1-y_2|^2+|t_1-t_2|)^{\frac{1}{2}}\quad \forall (y_1,t_1),(y_2,t_2)\in \overline{Q}_\delta.\ \square$$

Lemma 3.11. *Let the assumptions of Lemma* 3.9 *hold and let* $u(y,t)$ *the function exhibited in the same Lemma. Suppose that there exist two positive constants,* K_0 *and* K_0' *such that:*

$$|(G_2(\tilde{u}_0))'|\le K_0\quad \forall y\in\left[\eta_1+\frac{1}{2}\delta,\eta_2-\frac{1}{2}\delta\right]$$

and

$$\int_0^T|(G_2(\psi_i))'|dot\le K_0'\quad i=1,2.$$

Then, if $sag'(s)\in L^1(0,M)$*, there exists a positive constant* $L=L(K_0,K_0',M,\delta,T)$ *such that:*

$$\iint_{Q\setminus Q_\delta}\left((G_2(u)_y)\right)^2dydt\le L.$$

Proof. We will only prove that

$$\int_0^T\int_{\eta_1}^{\eta_1+\frac{1}{2}\delta}(G_2(u)_y)^2dydt\le\frac{1}{2}L.$$

The proof is the same for

$$\int_0^T\int_{\eta_2-\frac{1}{2}\delta}^{\eta_2}(G_2(u)_y)^2dydt\le\frac{1}{2}L.$$

Set:

$$\chi(y,t)=\frac{1}{g_1(t)}[G_2(u(y,t))-G_2(\psi_1(t))]$$

then, since $u_t-(g(t,u)u_y)_y-b(y,t)u_y=0$:

$$\int_0^T\int_{\eta_1}^{\eta_1+\frac{1}{2}\delta}[u_t-(g_1(t)g_2(u)u_y)_y-b(y,t)u_y]\chi(y,t)dydt=0. \tag{3.55}$$

Noting that:

$$(g_2(u)u_y)_y = (G_2(u))_{yy}.$$

Equation (3.55) is equivalent to:

$$\begin{aligned}
0 &= \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} [u_t - (g_1(t)g_2(u)u_y)_y - b(y,t)u_y]\chi(y,t)dydt \\
&= \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(u(y,t))}{g_1(t)} dydt - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(\psi_1(t))}{g_1(t)} dydt \\
&\quad - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} (G_2(u))_{yy}[G_2(u) - G_2(\psi_1)]dydt - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u\chi(y,t)dydt.
\end{aligned}$$

But

$$\begin{aligned}
&\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} (G_2(u))_{yy}[G_2(u) - G_2(\psi_1)]dydt \\
&= \int_0^T [G_2(u)_y(G_2(u) - G_2(\psi_1))]_{\eta_1}^{\eta_1+\frac{1}{2}\delta} dot - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \left(G_2(u)_y\right)^2 dydt \\
&= \int_0^T \left[G_2\left(u\left(\eta_1 + \frac{1}{2}\delta, t\right)\right)_y (G_2(u) - G_2(\psi_1)) \right] \left(\eta_1 + \frac{1}{2}\delta, t\right) dot \\
&\qquad - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \left(G_2(u)_y\right)^2 dydt \\
&\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u_y\chi(y,t)dydt \\
&= \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} [b(y,t)u]_y\chi(y,t) - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b_y(y,t)u\chi(y,t)dydt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} [b(y,t)u]_y\chi(y,t)dydt \\
&= \int_0^T [b(y,t)u(y,t)\chi(y,t)]_{\eta_1}^{\eta_1+\frac{1}{2}\delta} dot - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u(y,t)\frac{G_2(u)_x}{g_1(t)}dydt.
\end{aligned}$$

So, equation (3.55) implies:

$$\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} (G_2(u)_y)^2\, dydt = -\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(u)-G_2(\psi_1)}{g_1(t)} dydt$$

$$-\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u(y,t)\frac{G_2(u)_y}{g_1(t)} dydt - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b_y(y,t)u\chi(y,t)dydt \quad (3.56)$$

$$+\int_0^T \left[\left(G_2(u)_y - \frac{b(y,t)}{g_1(t)}u(y,t)\right)(G_2(u)-G_2(\psi_1))\right](\eta_1+\frac{1}{2}\delta,t)dot.$$

Denote the four integrals on the right hand side by I_1, I_2, I_3, I_4 respectively. We estimate them in turn.

Let C_1 and C_{g_1} be positive constants such that:

$$0 < C_{g_1} \leq \inf_{t\in[0,T]} g_1(t)$$

and

$$C_1 \geq \sup_{s\in(0,M]} G_2(s), \qquad \sup_{s\in(0,M]} \int_0^s G_2(r)dr$$

$$C_1 \geq \sup_{y,t\in[\eta_1,\eta_1+\frac{1}{2}\delta]\times[0,T]} \left|\frac{b(y,t)}{g_1(t)}u(y,t)\right|, \qquad \sup_{y,t\in[\eta_1,\eta_1+\frac{1}{2}\delta]\times[0,T]} \left|\frac{b_y(y,t)}{g_1(t)}u(y,t)\right|.$$

Since for Lemma 3.10 there exists $K = K(K_0, M, \delta)$ such that $\forall t \in (0,T]$ and $\forall u \in (0,M]$:

$$\left|G_2(u)_y\left(\eta_1+\frac{1}{2}\delta,t\right)\right| \leq K \quad \forall t\in[0,T]$$

then:

$$I_1 = \int_0^T \left[\left(G_2(u)_y - \frac{b(y,t)}{g_1(t)}u(y,t)\right)(G_2(u)-G_2(\psi_1))\right]\left(\eta_1+\frac{1}{2}\delta,t\right) dot$$

$$\leq \int_0^T \left(\left|\left(G_2\left(u\left(\eta_1+\frac{1}{2}\delta\right),t\right)\right)_y\right| + C_1\right) 2C_1 dt$$

$$= 2C_1\left[\int_0^T \left|\left(G_2\left(u\left(\eta_1+\frac{1}{2}\delta\right),t\right)\right)_y\right| + C_1 T\right]$$

and so $I_1 \leq 2C_1T(K+C_1)$.

$$I_2 = \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b(y,t)u(y,t)\frac{G_2(u)_y}{g_1(t)}dydt$$

$$\leq C_1 \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} G_2(u)_y dydt = C_1 \int_0^T [|G_2(u)|]_{\eta_1}^{\eta_1+\frac{1}{2}\delta} dot \leq 2C_1^2 T$$

$$I_3 = \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} b_y(y,t)u\chi(y,t)dydt \leq \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} C_1|G_2(u) - G_2(\psi_1)|dydt \leq 2C_1^2 T\delta$$

$$I_4 = \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(u) - G_2(\psi_1)}{g_1(t)}dydt$$

$$= \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(u)}{g_1(t)}dydt - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t \frac{G_2(\psi_1)}{g_1(t)}dydt$$

$$\leq \frac{1}{C_{g_1}} \left[\left| \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t G_2(u)dydt \right| + \left| \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u_t G_2(\psi_1)dydt \right| \right]$$

$$= \frac{1}{C_{g_1}} \left[\left| \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \frac{\partial}{\partial t} \left(\int_0^u G_2(s)d.s. \right) dydt \right| + \right.$$

$$\left. + \left| \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} [uG_2(\psi_1)]_0^T \, dx - \int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} u(G_2(\psi_1(t))')dydt \right| \right]$$

then from the hypotheses of the lemma $\int_0^T |G(\psi_1(t))|dot \leq K_0'$

$$I_4 \leq \frac{1}{C_{g_1}} \left[\int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \left| \left(\int_0^u G_2(s)d.s. \right)_0^T \right| + \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} 2MC_1 + \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} MK_0' \right]$$

$$\leq \frac{1}{C_{g_1}} \left[2C_1\delta + 2MC_1\delta + MK_0'\delta \right].$$

These estimates of I_1, I_2, I_3 and I_4 make $\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} (A_2(u)_x)^2 dxdt$ bounded.

In fact, setting:

$$\frac{1}{2}L = 2C_1T(K+C_1) + 2C_1^2T + 2C_1^2T\delta + \frac{\delta}{C_{a_1}}[2C_1 + 2MC_1 + MK_0']$$

and substituting this estimate in equation (3.56), we obtain:

$$\int_0^T \int_{\eta_1}^{\eta_1+\frac{1}{2}\delta} \left(G_2(u)_y\right)^2 dydt \leq \frac{1}{2}L \tag{3.57}$$

and $L = L(K, K_0', M, \delta)$ but it doesn't depend on ε. □

Proof of Theorem 3.5. As in [8], we can choose positive constants M and K_0', sequences of positive constants $\{\varepsilon_k\}_{k=1}^{\infty}$, $\{\alpha_k\}_{k=1}^{\infty}$ and sequences of functions $\{u_{0,k}\}_{k=0}^{\infty}$, $\{\psi_{1,k}\}_{k=1}^{\infty}$ and $\{\psi_{2,k}\}_{k=1}^{\infty}$ such that:

1. $\varepsilon_k, \alpha_k \in (0,1] \quad \forall k$;
2. $u_{0,k} \in C^{2+\alpha_k}[\eta_1,\eta_2] \quad \forall k$;
3. $\psi_{1,k}, \psi_{2,k} \in C^{1+\alpha_k}[0,T] \quad \forall k$;
4. $\varepsilon_k \leq u_{0,k}(y) \leq M \qquad \forall y \in [\eta_1,\eta_2], \forall k$
   $\varepsilon_k \leq \psi_{1,k}(t), \psi_{2,k}(t) \leq M \quad \forall t \in [0,T], \forall k$;
5. $u_{0,k+1}(y) \leq u_{0,k}(y) \quad \forall x \in [\eta_1,\eta_2], \forall k$
   $\psi_{1,k+1}(t) \leq \psi_{1,k}(t) \quad \forall t \in [0,T], \forall k$
   $\psi_{2,k+1}(t) \leq \psi_{2,k}(t) \quad \forall t \in [0,T], \forall k$;
6. $\psi_{1,k}(0) = u_{0,k}(\eta_1)$ and $\psi_{2,k}(0) = u_{0,k}(\eta_2)$.
   Moreover:
7. $(\psi_{i,k})'(0) = (g(t,u_{0,k})(u_{0,k})_y)_y(\eta_i) + b(y,t)(u_{0,k})_y(\eta_i)$ for $i = 1,2$;
8. $\forall \delta \in (0,1)$ there exists a constant $K_0(\delta)$ such that:
$$|(G_2(u_{0,k}))'(y)| \leq K_0(\delta) \quad \forall y \in (\eta_1+\delta, \eta_2-\delta) \quad \forall k;$$
9. $\int_0^T G_2(\psi_{1,k}(t))dt, \int_0^T G_2(\psi_{2,k}(t))dt \leq K_0' \quad \forall k$;
10. $u_{0,k} \to u_0(y)$ for $k \to \infty$ uniformly $\forall y \in [\eta_1,\eta_2]$;
11. $\psi_{1,k} \to \psi_1$ and $\psi_{2,k} \to \psi_2$ for $k \to \infty$ uniformly $\forall t \in [0,T]$;
12. $\varepsilon_k \to 0$ as $k \to \infty$.

So, for Lemma 3.9 there exists a unique function $u_k(y,t)$ such that:

1. $(A_2(u_y))_y \in C^{2,1}(R)$;
2. $\varepsilon_k \leq u_k(y,t) \leq M \quad \forall (y,t) \in \overline{Q}_T$;
3. $(u_k)_t - (g(t,u_k)(u_k)_y)_y - b(y,t)(u_k)_y = 0 \quad \forall (y,t) \in \overline{Q}_T$;
4. $u_k(y,0) = \tilde{u}_0(y) \quad \forall y \in [\eta_1,\eta_2]$;
5. $u_k(\eta_1,t) = \psi_{1,k} \quad \forall t \in [0,T]$;
6. $u_k(\eta_2,t) = \psi_{2,k} \quad \forall t \in [0,T]$.

In view of the monotonicity conditions on $\{u_{0,k}\}_{k=1}^{\infty}, \{\psi_{1,k}\}_{k=1}^{\infty}$ and on $\{\psi_{2,k}\}_{k=1}^{\infty}$ we can define a real non negative bounded function

$$u(y,t) = \lim_{k\to\infty} u_k(y,t).$$

As in [8] we can prove that this is a weak solution for system (3.29).

Moreover, as in [8], thanks to Lemma 3.10 and Lemma 3.11, we can prove the continuity of u in the interior of Q and that the operator $G_2(u)$ has a generalized square integrable derivative.

So, we have only to prove the continuity of $u(y,t)$ for $y = \eta_1, \eta_2$. We will show the continuity only in $y = \eta_1$. For $y = \eta_2$ it can be shown in a similar way.

In order to prove the continuity of $u(y,t)$ in $y=\eta_1$, it's enough to prove that $\forall t_0 \in [0,T]$:

$$\limsup_{(y,t)\to(\eta_1,t_0)} u(y,t) \le \psi_1(t_0) \tag{3.58}$$

and

$$\liminf_{(y,t)\to(\eta_1,t_0)} u(y,t) \ge \psi_1(t_0). \tag{3.59}$$

Equation (3.58) can be proven as follows:

$$u(y,t) \le u_k(y,t) \quad \forall (y,t) \in \overline{Q}_T \quad \forall k.$$

Then:

$$\limsup_{(y,t)\to(\eta_1,t_0)} u(y,t) \le \limsup_{(y,t)\to(\eta_1,t_0)} u_k(y,t) = \psi_{1.k}(t_0)$$

and equation (3.58) is obtained letting $k\to\infty$.

In order to prove equation (3.59) we will show that for any $\varepsilon \in (0,\psi_1(t_0))$, we can define a function $w(y,t)$ such that:

$$\liminf_{(y,t)\to(\eta_1,t_0)} w(y,t) = \psi_1(t_0) - \varepsilon$$

and

$$u_k(y,t) \ge w(y,t) \qquad \text{for } k \gg 1 \text{ and } \forall (y,t) \in \overline{Q}_T. \tag{3.60}$$

If $\psi_1(t_0)=0$ then trivially equation (3.59) is verified. In fact:

$$\liminf_{(y,t)\to(\eta_1,t_0)} u(y,t) \ge 0$$

then we focused on $\psi(t_0) > 0$.

Let ε be a constant such that $\varepsilon \in (0,\psi_1(t_0))$. Set

$$\beta = 1 + M \sup_{(y,t)\in\overline{Q}_T} b(y,t).$$

Then we can define the following functions:

$$\rho(c) = \int_0^M a(t,r)(cr + b(y,t)r + \beta)^{-1} dr \tag{3.61}$$

and

$$\lambda(c) = t_0 - \frac{1}{c}\int_0^{\psi_1(t_0)-\varepsilon} a(t,r)(cr + b(y,t)r + \beta)^{-1} dr. \tag{3.62}$$

We note that:

$$\lim_{c\to\infty} c(t_0 - \lambda(c)) = 0^+.$$

If $t_0 > 0$ then we choose and fix c so large that:

$$\begin{cases} \lambda(c) > 0 \\ c(t_0 - \lambda(c)) \le \eta_2 - \eta_1 \\ \psi_1(t) \ge \psi_1(t_0) - \dfrac{\varepsilon}{2} \qquad \forall t \in [\lambda(c), t_0] \end{cases} \tag{3.63}$$

and set $t_1 = \lambda(c)$.

We choose and fix c so large that:

$$\begin{cases} -c\lambda(c) = c(t_0 - \lambda(c)) < \eta_2 - \eta_1 \\ u_0(y) \ge u_0(0) - \dfrac{\varepsilon}{2} = \psi_1(t_0) - \dfrac{\varepsilon}{2} \quad \forall y \in [\eta_1, \eta_1 - c\lambda(c)] \end{cases} \tag{3.64}$$

and set $t_1 = t_0$. Now we define an increasing function $h : [0, \rho(c)] \to [0, M]$ as follows:

$$\eta = \int_0^{h(\eta)} g(t,r)(cr + b(y,t)r + \beta)^{-1} dr.$$

This expression identify a bijection between $[0, \rho(c)]$ and $[0, M]$. In particular, if we define $G(t,u)$ as follows:

$$G(t,u) = \int_0^u g(t,r) dr$$

we find that:

$$(G(t, h(\eta)))_\eta = g(t,h)h' = ch + b(y,t)h + \beta \text{ on } [0, \rho(c)]$$

and

$$(G(t, h(\eta)))_{\eta\eta} = h'(c + b(y,t)) \text{ on } [0, \rho(c)]. \tag{3.65}$$

We remark that, by definition

$$h(c(t_0 - \lambda(c))) = \psi_1(t_0) - \varepsilon. \tag{3.66}$$

If $t_0 < T$ then from (3.63), (3.64) and (3.66) we can choose t_2 such that:

$$\begin{gathered} t_0 < t_2 \le T \\ c(t_2 - \lambda(c)) < \eta_2 - \eta_1 \\ h(c(t_2 - \lambda(c))) < \psi_1(t_0) - \frac{\varepsilon}{2} \\ \psi_1(t) \ge \psi_1(t_0) - \frac{\varepsilon}{2} \qquad \forall t \in [t_0, t_2] \end{gathered}$$

else we set $t_2 = T = t_0$.

Let m so large that $\varepsilon_k < \psi_1(t_0) - \varepsilon$ for all $k \geq m$ and for each m we define a point σ_k such that $h(\sigma_k) = \varepsilon_k$. We set:

$$\Omega_k = \{(y,t) : t_1 < t \leq t_2, \eta_1 < y < \eta_1 - \sigma_k - c(t - \lambda(c))\}$$

and

$$\Gamma_k = \{(y,t) : t_1 < t \leq t_2, y = \eta_1 - \sigma_k - c(t - \lambda(c))\}.$$

Set $\Omega = \cup_k \Omega_k$. Since σ_k tends to 0 as ε_k tends to 0, then

$$\Omega = \{(y,t) : t_1 < t \leq t_2, \eta_1 < y < \eta_1 + c(t - \lambda(c))\}$$

and since, by definition $c(t - \lambda(c)) < \eta_2 - \eta_1$, then $\Omega \subset Q$. Now, we define the function $w(y,t)$ in the following way:

$$w(y,t) = \begin{cases} h(\eta_1 - y + c(t - \lambda(c))) & (y,t) \in \overline{\Omega} \\ 0 & (y,t) \in \overline{Q} \setminus \overline{\Omega}. \end{cases}$$

In particular:

$$\liminf_{(y,t)\to(\eta_1,t_0),(y,t)\in R} w(y,t) = \liminf_{(y,t)\to(\eta_1,t_0),(y,t)\in\Omega} w(y,t) = w(\eta_1,t_0) = \psi_1(t_0) - \varepsilon.$$

So, in order to prove the theorem we had only to show that equation (3.60) holds. That is for any $(y,t) \in \overline{\Omega}_k \setminus \Omega_k$ and for any $k \geq m$ it holds $u_k(y,t) > w(y,t)$. In fact, if $t \in [t_1, t_2]$ then:

$$u_k(\eta_1,t) = \psi_{1,k}(t) \geq \psi_1(t) \geq \psi_1(t) - \frac{\varepsilon}{2} > h(c(t_2 - \lambda(c))) > h(c(t - \lambda(c))) = w(\eta_1,t)$$

and if $(y,t) \in \Gamma_k$ then:

$$u_k(y,t) \geq \varepsilon_k = h(\sigma_k) = w(y,t).$$

Moreover, if $t_1 = 0$ then for $y \in [\eta_1, \eta_1 - c\lambda(c)]$:

$$\begin{aligned} u_k(y,t) = u_{0,k}(y) &\geq u_0(y) \geq u_0(0) - \frac{\varepsilon}{2} = \psi_1(t_0) - \frac{\varepsilon}{2} \\ &> h(c(t_2 - \lambda(c))) > h(-c\lambda(c)) > h(\eta_1 - x - c\lambda(c)) = w(y,0). \end{aligned}$$

Now, we use the maximum principle to prove that inequality (3.60) holds in $\overline{\Omega}_k$.

From (3.65), we observe that $w(y,t)$ is a classical solution of (3.29) in Ω_k. So, $w(y,t)$ is bounded away from 0 in $\overline{\Omega}_k$, by ε_k.

Then, $\forall (y,t) \in \overline{\Omega}_k$:

$$u_k(y,t) \geq w(y,t).$$

Moreover, for any $(y,t) \in \overline{Q} \setminus \overline{\Omega}_k$ it holds $u_k(y,t) \geq \varepsilon_k = h(\sigma_k) \geq w(y,t)$, and so it follows $u_k(y,t) \geq w(y,t) \quad \forall (y,t) \in \overline{Q}$ and this proves the theorem.

Uniqueness of the solution of (3.29). In order to prove the uniqueness of the solution of (3.29), we follow the proof of a similar result in [24].

THEOREM 3.12. *Assume $u_0(y) \in L^\infty(\eta_1, \eta_2)$ and $g_1(t)$, $g_2(u)$ and $b(y,t)$ verify conditions* (3.30), (3.31) *and* (3.32). *Then problem* (3.29) *has at most one weak solution.*

PROOF. Suppose that there exists two separate weak solutions of (3.29), $u_1(y,t)$ and $u_2(y,t)$. By the definition of weak solution, u_1 and u_2 verify:

$$\begin{aligned}\int_0^T \int_{\eta_1}^{\eta_2} &[(u_1 - u_2)\varphi_t - g_1(t)(G_2(u_1) - G_2(u_2))_y \varphi_y] dydt \\ &- \int_0^T \int_{\eta_1}^{\eta_2} [b(y,t)(u_1 - u_2)\varphi_y - b_y(y,t)(u_1 - u_2)\varphi] dydt = 0\end{aligned} \tag{3.67}$$

$\forall \varphi \in C^\infty$ such that $\varphi(\eta_1, t) = \varphi(\eta_2, t) = 0$ for any $t \in [0,T]$ and $\varphi(y,T) = 0$ for any $y \in [\eta_1, \eta_2]$. Equation (3.67) can be written as follows:

$$\begin{aligned}&\int_0^T \int_{\eta_1}^{\eta_2} (u_1 - u_2)[\varphi_t - b(y,t)\varphi_y - b_y(y,t)\varphi] dydt \\ &= \int_0^T \int_{\eta_1}^{\eta_2} g_1(t)(G_2(u_1) - G_2(u_2))_y \varphi_y dydt\end{aligned} \tag{3.68}$$

and integrating by parts with respect to y, the second integral of the previous equation, we obtain:

$$\int_0^T\int_{\eta_1}^{\eta_2} g_1(t)(G_2(u_1) - G_2(u_2))_g \varphi_g dydt = - \int_0^T\int_{\eta_1}^{\eta_2} g_1(t)(G_2(u_1) - G_2(u_2))\varphi_{yy} dydt.$$

Set:

$$\overline{G}(u_1, u_2) = \int_0^1 g_1(t) G_2(\vartheta u_1 + (1-\vartheta)u_2) d\vartheta$$

then

$$G_2(u_1) - G_2(u_2) = (u_1 - u_2)\overline{G}(u_1, u_2).$$

So equation (3.68) becomes:

$$\int_0^T \int_{\eta_1}^{\eta_2} (u_1 - u_2)[\varphi_t + \overline{G}\varphi_{yy} - b(y,t)\varphi_y - b_y(y,t)\varphi] dydt = 0.$$

So, if we show that $\forall f \in C_0^\infty$ there exists a solution of the following problem

$$\begin{cases} \varphi_t + \overline{G}\varphi_y - b(y,t)\varphi_y - b_y(y,t)\varphi = f & y \in [\eta_1, \eta_2], t \in (0,T] \\ \varphi(\eta_1, t) = \varphi(\eta_2, t) = 0 & \forall t \in (0,T] \\ \varphi(y,T) = 0 & \forall y \in [\eta_1, \eta_2] \end{cases} \tag{3.69}$$

then $u_1 - u_2 = 0$ and the theorem is shown.

Since $\overline{G}$ is merely bounded, it's not easy to study the solvability of (3.69). So, we approximate the operator $\overline{G}$ as follows, for sufficiently small $\eta, \delta > 0$ we define:

$$\lambda_\eta^\delta = \begin{cases} \dfrac{b(y,t)}{\eta + \overline{G}} & |u_1 - u_2| > \delta \\ 0 & \text{otherwise.} \end{cases}$$

Since $G_2(s)$ is a strictly increasing function and $u_1, u_2 \in L^\infty(Q)$ there exists $L = L(\delta, T)$ and $K = K(\delta, T)$ such that:

$$\overline{G} = g_1(t)\frac{G_2(u_1) - G_2(u_2)}{u_1 - u_2} \geq L(\delta), \quad |\lambda_\eta^\delta| \leq K(\delta) \qquad \text{if } |u_1 - u_2| > \delta.$$

Then there exists a sequence $\{\overline{G}_\varepsilon\}$ such that:

$$\lim_{\varepsilon \to 0} \overline{G}_\varepsilon = \overline{G} \quad \text{ and } \quad |G_\varepsilon| \leq C$$

where C is a positive constant.

Then for given $f \in C_0^\infty(Q)$ the approximated system:

$$\begin{cases} \varphi_t + (\eta + \overline{G}_\varepsilon)\varphi_{yy} - b(y,t)\varphi_y - b_y(y,t)\varphi = f & y \in [\eta_1, \eta_2], t \in (0,T] \\ \varphi(\eta_1, t) = \varphi(\eta_2, t) = 0 & \forall t \in (0,T] \\ \varphi(y, T) = 0 & \forall y \in [\eta_1, \eta_2] \end{cases} \tag{3.70}$$

has a unique solution following the standard theory of parabolic linear equations.

From [24, Lemma 13.3.1], the solution of system (3.70) satisfies the following inequalities:

$$\sup_Q |\varphi(y,t)| \leq C \quad \iint_Q (\eta + \overline{G}_\varepsilon)\left(\frac{\partial^2 \varphi}{\partial y^2}\right)^2 dydt \leq \frac{K(\delta)}{\eta} \quad \iint_Q \left(\frac{\partial \varphi}{\partial y}\right)^2 dydt \leq \frac{K(\delta)}{\eta}.$$

From [24, Theorem 13.3.1] we know that:

$$\lim_{\varepsilon \to 0} \iint_Q \left(\frac{\partial \varphi}{\partial y}\right)^2 dydt \leq \frac{K(\delta)}{\eta} = 0$$

and it's enough to prove our theorem. $\square$

Proof of Theorem 3.1. The proof of Theorem 3.1 uses Theorem 3.2 and the Contraction fixed point theorem.

In particular, Theorem 3.2 is shown by Lemma 3.9 and Theorem 3.4. More in detail, we are looking for a solution of system (M) with (I), (B), (LR) and (2.15)-(2.17), with positive initial data.

In order to apply Lemma 3.9 at (S_u), we have to verify that $\psi_1(t)$ and $\psi_2(t)$ are C^1 functions. This is trivial for $\psi_1(t) = u_0$, since it is a constant function. While, since $\psi_2(t) = u_0 + \dfrac{\mu}{bE} l'(t)$, then it is in C^1 if and only if $l'(t) \in C^1$. From (LR), the regularity of l' is the same of $u_y(0,t)$. So, $\psi_2(t) \in C^1$ if and only if $u_y(0,t)$ is in C^1 and this condition is equivalent to the second order compatibility condition at $y = 0$. That is:

$$\psi_{1tt} = ((a(t,u)u_y)_y + b(y,t)u_y)_t.$$

But, since ψ_1 is a constant, we can write the previous equation as:

$$0 = ((a(t,u)u_y)_y + b(y,t)u_y)_t \qquad \text{for } y = 0, t = 0 \tag{3.71}$$

$$\begin{aligned}
((a(t,u)u_y)_y + b(y,t)u_y)_t = {} & -2Er_0^2 \frac{r'(t) - l'(t)}{(r(t) - l(t))^3}(uu_y)_y + E\frac{r_0^2}{(r(t) - l(t))^2}(uu_y)_{yt} \\
& + \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)} u_{yt} + \frac{(l''(t)(r_0 - y) + r''(t)y)(r(t) - l(t))}{(r(t) - l(t))^2} u_{yt} \\
& - \frac{(r'(t) - l'(t))(l'(t)(r_0 - y) + r'(t)y)}{(r(t) - l(t))^2} u_{yt}
\end{aligned}$$

and

$$\begin{aligned}
u_{yt} &= ((a(t,u)u_y)_y + b(y,t)u_y)_y \\
&= E\frac{r_0^2}{(r(t) - l(t))^2}(uu_y)_{yy} + \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)} u_{yy} + \frac{r'(t) - l'(t)}{r(t) - l(t)} u_y
\end{aligned}$$

while

$$(uu_y)_{yt} = \frac{Er_0^2\,(u(uu_y)_y)_{yy}}{(r(t) - l(t))^2} + \frac{l'(t)(r_0 - y) + r'(t)y}{r(t) - l(t)}(uu_y)_{yy} + \frac{r'(t) + 2l'(t)}{r(t) - l(t)}(uu_y)_y.$$

Equation (3.71) becomes:

$$0 = E^2(\tilde{u}_0(\tilde{u}_0(\tilde{u}_0)_y)_y)_{yy} + 2El'(0)(\tilde{u}_0(\tilde{u}_0)_y)_{yy} + l'(0)^2(\tilde{u}_0)_{yy} + l''(0)(\tilde{u}_0)_y$$

that is hypotesis (3.10) of the Theorem. So, the hypoteses of the Theorem guarantee that ψ_1 and ψ_2 are both C^1 functions and as a consequence Lemma 3.9 holds. So, (S_u) has a solution that is unique, positive and in $C^{2,1}$.

Moreover, from Theorem 3.4, there exists ρ which solves (S_ρ) and since $u \in C^{2,1}$ then it is a continuous and positive function.

In this way Theorem 3.2 is proved.

We denote by (u_{lr}, ρ_{lr}) the solution, dependent on $l(t)$ and $r(t)$, which existence and uniqueness is shown in Theorem 3.2. From (LR_F) we define:

$$\frac{d\tilde{l}}{dt} = -E\frac{\alpha}{\mu-\alpha}\frac{r_0}{r(t)-l(t)}(u_{lr})_y(0,t) - \frac{\beta}{\mu-\alpha}$$
$$\frac{d\tilde{r}}{dt} = -E\frac{r_0}{r(t)-l(t)}(u_{lr})_y(r_0,t) - \frac{\nu}{\rho_{lr}(r_0,t)-du_{lr}(r_0,t)^{\frac{2}{3}}}$$

and we define the transformation $\Gamma(l,t) = (\tilde{l},\tilde{r})$. We would like to show that this transformation admits a fixed point and so we could end the proof.

As we said at the end of the introduction of this chapter, in this part of the proof we follow the one in [2].

So, in the Banach space $C^{2+\alpha/2}([0,T])$ we introduce the closed set:

$$L = \{g(t) \in C^{2+\alpha/2}([0,T]) : g(0) = 0, g'(0) = l'(0), g''(0) = l''(0)\}$$

and in the Banach space $C^{1+\alpha/2}$

$$R = \left\{g(t) \in C^{1+\alpha/2}([0,T]) : g(0) = 0, g'(0) = r'(0)\right\} \tag{3.72}$$

and we define

$$||g||_L = \max_{t\in[0,T]}|g(t)| + \max_{t\in[0,T]}|g'(t)| + \max_{t\in[0,T]}|g''(t)| + < g'' >_t^{\alpha/2}$$

and

$$||g||_L = \max_{t\in[0,T]}|g(t)| + \max_{t\in[0,T]}|g'(t)| + < g' >_t^{\alpha/2}$$

where

$$< g >_t^{\alpha/2} = \sup_{0\le s<t\le T}\frac{|g(t)-g(s)|}{|t-s|^\alpha}.$$

Using the Schauder estimates it can be shown that:

$$(u_{lr})_y(0,t;l,r) \in C^{(3+\alpha)/2}[0,T] \quad \text{and} \quad (u_{lr})_y(r_0,t;l,r) \in C^{(1+\alpha)/2}[0,T].$$

From the definition of $\tilde{l}$ and $\tilde{r}$:

$$\frac{d\tilde{l}}{dt} = -E\frac{r_0}{r(t)-l(t)}(u_{lr})_y(0,t;l,r) - \frac{\beta}{\mu-\alpha} \quad \tilde{l}(0) = 0$$

and

$$\frac{d\tilde{r}}{dt} = -E\frac{r_0}{r(t)-l(t)}(u_{lr})_y(r_0,t;l,r) - \frac{\nu}{\rho_{lr}(r_0,t;l,r) - du_{lr}^{\frac{2}{3}}(r_0,t;l,r)} \quad \tilde{r}(0) = r_0.$$

Then:

$$\begin{cases} \tilde{l}(t) = \displaystyle\int_0^t q_1(0,\tau,l,r)d\tau \\ \tilde{r}(t) = r_0 + \displaystyle\int_0^t q_2(r_0,\tau,l,r)d\tau \end{cases}$$

with $q_1(0,t,l,r) \in C^{1+(1+\alpha)/2}$ and $q_2(r_0,t,l,r) \in C^{(1+\alpha)/2}$ for what we said about the functions $(u_{lr})_y(0,t,l,r)$ and $(u_{lr})_y(r_0,t,l,r)$. Finally we define:

$$F = L \times R$$

and

$$||z||_F = ||z||_L + ||z||_R$$

for $z(t) = (l(t), r(t))$.

So, denoting by B_1 the closed set of F such that $||z||_F \leq 1$, then we have that we can choose T small enough to have that Γ maps B_1 in B_1.

Finally using again the Schauder estimates we can show also that Γ is a contraction; that is:

$$||\Gamma(z_1) - \Gamma(z_2)||_F \leq q(T)||z_1 - z_2||_F$$

with $q(T) \to 0$ for $T \to 0$. So, Γ has a unique fixed point and the theorem is proved.

4 – Some open problems

First of all we want to emphasize that we expect to obtain a global existence result for our system. But, as for every parabolic degenerate equation, the global existence of the solution is harder to prove than the local one. In fact, uniform estimates for the coefficients of the equations for all time $t > 0$ are needed and, up to now, we are not able to give these estimates.

Moreover, in the forthcoming paper [15], we describe and discuss two numerical schemes that we used for some computer simulations. The first one is for the numerical characterization of the traveling wave solution (see also [16]), the second one for the approximation of the solution of the general problem. In this case, in order to verify the efficiency of the method, we set the traveling wave solution as the initial datum of the problem and we use the numerical scheme to find the corresponding approximated solution. Then we compare it and the approximated traveling wave solution at every time $t > 0$. We obtain that in a finite time the approximated solution becomes a traveling wave. This result suggests that the traveling wave is asymptotically stable with respect to the dynamics of the system (M). In fact, starting from a perturbed traveling wave solution, the system reaches in a finite time the steady state configuration of the non perturbed traveling wave.

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