Rendiconti di Matematica, Serie VII
Volume 33, Roma (2013), 1 – 16

Riemannian geodesics of semi Riemannian warped product metrics

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ABSTRACT: *Let (M_1, g_1) and (M_2, g_2) be two C^∞–differentiable connected, complete Riemannian manifolds, $k : M_1 \to \mathbb{R}$ a C^∞-differentiable function, having $0 < k_0 < k(x) \leq K_0$, for any $x \in M_1$ and $g := g_1 - kg_2$ the semi Riemannian metric on the product manifold $M := M_1 \times M_2$.*
We associate to g a suitable family of Riemannian metrics $G_r + g_2$, with $r > -K_0^{-1}$, on M and we call Riemannian *geodesics of g the geodesics of g which are geodesics of a metric of the previous family, via a suitable reparametrization.*
Among the properties of these geodesics, we quote:
For any $z_0 = (x_0, y_0) \in M$ and for any $y_1 \in M_2$ there exists a subset $A \neq \emptyset$ of M_1, such that all the geodesics of g joining z_0 with a point (x_1, y_1), with $x_1 \in A$, are Riemannian. The Riemannian geodesics of g determine a "partial" property of geodesic connection on M. Finally, we determine two new classes of semi Riemannian metrics (one of which includes some FLRM-metrics), geodesically connected by Riemannian geodesics of g.

1 – Introduction

Let (M_1, g_1) and (M_2, g_2) be two connected, complete, Riemannian manifolds.

For the greater part of the paper, we shall use the assumption of the completeness of the two manifolds only to avoid to write a long and trivial series of inequalities.

Let $k : M_1 \to \mathbb{R}$ be a C^∞-differentiable function, bounded from below away from zero.

We consider the semi Riemannian warped product metric $g : g_1 - kg_2$ and the family of Riemannian metrics $G_r + g_2$ on the manifold $M := M_1 \times M_2$, where $G_r := (k^{-1} + r)g_1$ and $r > -K_0^{-1} := k_1$, being $K_0 := \sup_{x \in M_1}\{k(x)\}$, if k is bounded from above and $r > 0 := k_1$ in the other case.

KEY WORDS AND PHRASES: *Semi Riemannian warped product – Geodesics*
A.M.S. CLASSIFICATION: Primary 53C22, Secondary 53C50.

Then we prove that M is complete with respect to the metric $G_r + g_2$ and that the geodesics of $G_r + g_2$, belonging to a suitable subset, determine geodesics of g, via a suitable reparametrization, for any $r > k_1$.

We call them *Riemannian geodesics of* g.

We prove some properties of these geodesics and here we quote some of them as examples.

Let us consider $z_0 = (x_0, y_0) \in M$ and a geodesic $\zeta = (\gamma, \tau) : [0,1] \to M$ of g, with $\gamma(0) = x_0$, $\tau(0) = y_0$, $\dot\gamma(0) = \widetilde{X}$ and $\dot\tau(0) = \widetilde{Y}$. If k is bounded and

$$g_1(\widetilde{X}, \widetilde{X}) > k(x_0) g_2(\widetilde{Y}, \widetilde{Y}) \frac{K_0 - k(x_0)}{K_0};$$

then ζ is a Riemannian geodesic of g.

An analogous statement holds, if k is unbounded from above.

A surprising property, being the Morse theory of Riemannian and semi Riemannian metrics quite different, is the following.

Since M_1 and M_2 are connected and complete with respect to the respective Riemannian metrics g_1 and g_2, the manifold M_1 is *positive and negative geodesically connected with respect to* g; *i.e.*, for any real number $r > k_1$, for any $z_0 = (x_0, y_0) \in M$, for any $x_1 \in M_1$ and for any geodesic $\nu : \mathbb{R} \to M_2$ of g_2, having $\nu(0) = y_0$, there exists $t_0 \in \mathbb{R}$ such that the point z_0 and the point $(x_1, \nu(t_0))$ (and the point $(x_1, \nu(-t_0))$) can be joined by a Riemannian geodesic of g, obtained by reparametrizing a suitable geodesic of $G_r + g_2$.

Analogously, the manifold M_2 is *positive and negative geodesically connected with respect to* g, too.

Hence, we shall say that M *is partially Riemannian connected with respect to* g.

More surprising are the following two results.

If M_1 *and* M_2 *are connected and complete with respect to the respective Riemannian metrics* g_1 *and* g_2*, if the dimension of* M_1 *is greater than one and* M_1 *is simply connected, if* g_1 *has a negative sectional curvature, if* k *is bounded from below away from zero and if the Hessian of* k *verifies a suitable inequality* (*see* (4.2), *below*), *then* M *is geodesically connected by means Riemannian geodesic of* g.

If $M_1 = \mathbb{R}$*, then* g *is an FLRW-metric* (*with speed of light* $c = 1$) *and* M *is geodesically connected by Riemannian geodesic of* g*, provided* M_2 *connected and complete with respect to* g_2 *and* k *bounded from below away from zero.*

The FLRW-metrics are used in cosmology to study the early universe (see, *e.g.*, [9]).

The paper ends with an Appendix in which we determine a sufficient condition such that G_r has negative sectional curvature, for any $r \in (k_1, +\infty)$.

We conclude by noticing that the Levi-Civita connection of g is not used in this paper, because it hides all the relations between the metric tensor g and the Riemannian metric $G_r + g_2$.

In this case, the Levi-Civita connection of $g_1 + g_2$ allows us to use these relations.

Hence, we consider this paper as a first application of the results obtained in [1, 2] and [3].

2 – Preliminaries

This section contains the main geometric objects, which are needed in the following.

We also state some straightforward results.

Let (M_1, g_1), (M_2, g_2) be two connected, complete, Riemannian manifolds and $\overset{1}{\nabla}$, $\overset{2}{\nabla}$ the Levi-Civita connections determined by the metrics g_1 and g_2, respectively.

Let $k : M_1 \to \mathbb{R}$ be a smooth map.

We suppose

$$0 < k_0 := \inf_{x \in M_1} \{k(x)\} \ . \tag{2.1}$$

On the manifold $M := M_1 \times M_2$, we consider the tensor $g := g_1 - k \cdot g_2$, which defines a *semi Riemannian warped product metric*, having the signature equal to the dimension of M_1.

The geometry of warped product metrics is described in details in [7].

We shall set

$$G_r := \left(\frac{1}{k} + r \right) \cdot g_1$$

and G_r is a Riemannian metric on M_1, for any $r > k_1$, being $k_1 := -K_0^{-1}$ if k is bounded and $K_0 := \sup_{x \in M_1} \{k(x)\}$, and $k_1 := 0$ in the other case.

Finally, we set $I := [0, 1]$.

From [3], it follows.

Lemma 2.1. *A differentiable curve $\zeta = (\gamma, \tau) : I \to M$ is a geodesic of g, if and only if it satisfies the following system of ordinary differential equations*

$$\overset{1}{\nabla}_{\dot\gamma} \dot\gamma = -\frac{1}{2} g_2(\dot\tau, \dot\tau) \cdot g_1^\sharp(dk) \circ \gamma \tag{2.2}$$

$$\overset{2}{\nabla}_{\dot\tau} \dot\tau = -\frac{1}{k \circ \gamma} dk(\dot\gamma) \cdot \dot\tau \tag{2.3}$$

where $g_1^\sharp : T^*M_1 \to TM_1$ is the canonical isomorphism of bundles induced by g_1.

From [3], we also get:

Lemma 2.2. *The map $\mu : I \to M_1$ is a geodesic with respect to the metric G_r if and only if*

$$\overset{1}{\nabla}_{\dot\mu} \dot\mu = \frac{1}{2k \circ \mu (1 + rk \circ \mu)} \left\{ 2dk(\dot\mu) \cdot \dot\mu - g_1(\dot\mu, \dot\mu) \cdot g_1^\sharp(dk) \circ \mu \right\} . \tag{2.4}$$

We conclude this number by two lemmas needed in the following.

Lemma 2.3. *Let $\mathcal{M}$ be a topological space equipped with two distance functions d_1 and d_2. Suppose that any Cauchy sequence of d_2 is also a Cauchy sequence of d_1. Then the completeness of d_1 implies the completeness of d_2.*

A proof of the above lemma is straightforward and we omit it here.

We observe that if there exists a positive number L such that $d_1(x_1, x_2) \geq Ld_2(x_1, x_2)$, for each $x_1, x_2 \in \mathcal{M}$, then each Cauchy sequence of d_2 is also a Cauchy sequence of d_1.

Corollary 2.4. *Suppose that the inequality* (2.1) *holds.*

The manifold (M_1, g_1) is complete, if there exists an $r > k_1$ such that (M_1, G_r) is complete.

Vice versa, if (M_1, g_1) is complete, then (M_1, G_r) is complete, for any $r \in (k_1, +\infty)$.

Proof. We shall denote by d_{g_1}, d_{G_r} the distance functions associated with the Riemannian metrics g_1 and G_r, respectively.

For any $X \in T_{x_0}M_1$ and $x_0 \in M_1$, we have

$$g_1(X, X) = \frac{k(x_0)}{1 + rk(x_0)} G_r(X, X) \text{ and } G_r(X, X) = \frac{1 + rk(x_0)}{k(x_0)} g_1(X, X);$$

for any $r > k_1$.

The functions $f_1, f_2 : (k_0, +\infty) \to \mathbb{R}$ defined respectively by setting

$$f_1(t) = \frac{t}{1 + rt} \quad \text{and} \quad f_2(t) = \frac{1 + rt}{t}; \quad \forall r \in (k_1, +\infty)$$

are bounded.

Hence, there exist two positive real numbers k_2 and k_3 such that

$$d_{g_1}(x_1, x_2) \leq \sqrt{k_2} d_{G_r}(x_1, x_2) \;\text{ and }\; d_{G_r}(x_1, x_2) \leq \sqrt{k_3} d_{g_1}(x_1, x_2)$$

for all $x_1, x_2 \in M_1$.

Then our corollary follows immediately from Lemma 2.3. □

Finally, we recall that connected, complete, Riemannian manifolds are geodesically connected (see, *e.g.*, [5]).

3 – Geodesics of $(M, G_r + g_2)$ and (M, g)

In this Section we shall use the geometric objects and the notations introduced in the previous one.

Lemma 3.1. *For any $\mu : I \to M_1$ and for any $r > k_1$, there is a uniquely determined diffeomorphism $\varphi_r : I \to I$ such that*

$$\begin{aligned} \varphi_r(0) = 0, \quad \varphi_r(1) = 1 \\ \dot{\varphi}_r = a_r \frac{1+rk}{k} \circ \mu \circ \varphi_r \end{aligned} \tag{3.1}$$

where a_r is a suitable real number.

Proof. We shall determine φ_r^{-1} and then we shall obtain φ_r as the inverse of φ_r^{-1}.

Condition (3.1) is equivalent to

$$\frac{d\varphi_r^{-1}}{ds} = \frac{k(\mu(s))}{a_r(1 + rk(\mu(s)))}.$$

Hence the map φ_r^{-1} is defined by

$$\varphi_r^{-1}(s) := \frac{1}{a_r} \int_0^s \frac{k}{1+rk} \circ \mu \, d\xi, \qquad a_r := \int_0^1 \frac{k}{1+rk} \circ \mu \, d\xi; \tag{3.2}$$

for any $s \in I$.

As a consequence, φ_r^{-1} is a smooth strictly increasing diffeomorphism from I onto I. □

We need the following lemma, too.

Lemma 3.2. *For any differentiable curve $\gamma : I \to M_1$, there is a uniquely determined diffeomorphism $\psi : I \to I$, such that*

$$\begin{aligned} \psi(0) = 0, \quad \psi(1) = 1 \\ \dot{\psi} = \frac{b}{k \circ \gamma}, \end{aligned} \tag{3.3}$$

where b is a suitable positive real number.

Proof. The map ψ is defined by

$$\psi(s) := b \int_0^s \frac{1}{k \circ \gamma} d\xi, \quad b := \Big(\int_0^1 \frac{1}{k \circ \gamma} d\xi \Big)^{-1}, \tag{3.4}$$

for any $s \in I$. □

The previous lemma implies:

THEOREM 3.3. *Let $\mu : I \to M_1$ and $\nu, \tau : I \to M_2$ be smooth curves and suppose $\tau = \nu \circ \psi$, being ψ defined by the previous lemma.*
Then, τ satisfies (2.3), if and only if ν is a geodesic of g_2.
Moreover, it results $\tau(0) = \nu(0)$ and $\tau(1) = \nu(1)$.

PROOF. In fact, it results

$$\begin{aligned} \overset{2}{\nabla}_{\dot\tau}\dot\tau &= (\dot\psi)^2 \cdot (\overset{2}{\nabla}_{\dot\nu}\dot\nu) \circ \psi + \ddot\psi \cdot \dot\nu \circ \psi \\ &\overset{(3.3)}{=} (\dot\psi)^2 \cdot (\overset{2}{\nabla}_{\dot\nu}\dot\nu) \circ \psi + \frac{b}{k^2 \circ \mu}((dk)(\dot\mu)) \cdot \dot\nu \circ \psi \\ &= (\dot\psi)^2 \cdot (\overset{2}{\nabla}_{\dot\nu}\dot\nu) \circ \psi - \frac{1}{k \circ \mu} dk(\dot\mu) \cdot \dot\tau; \end{aligned}$$

and we have the assertion. □

LEMMA 3.4. *Let $\mu_r, \gamma_r : I \to M_1$ be two smooth curves, such that $\gamma_r = \mu_r \circ \varphi_r$, being φ_r the mapping defined by Lemma 2.3, with $\mu = \mu_r$.*
Then, μ_r is a geodesic with respect to the metric G_r, if and only if the curve γ_r satisfies the equation:

$$\overset{1}{\nabla}_{\dot\gamma_r}\dot\gamma_r = \frac{-1}{2k \circ \gamma_r(1 + rk_r \circ \gamma_r)} g_1(\dot\gamma_r, \dot\gamma_r) g_1^\sharp(dk) \circ \gamma_r. \tag{3.5}$$

Moreover, we have $\mu_r(0) = \gamma_r(0)$ and $\mu_r(1) = \gamma_r(1)$.

PROOF. In fact, we have

$$\begin{aligned} \overset{1}{\nabla}_{\dot\gamma_r}\dot\gamma_r &= (\dot\varphi_r)^2 \cdot (\overset{1}{\nabla}_{\dot\mu_r}\dot\mu_r) \circ \varphi_r + \ddot\varphi_r \cdot (\dot\mu_r \circ \varphi_r) \\ &\overset{(2.4)}{=} \frac{-\dot\varphi_r^2}{2k \circ \mu_r \circ \varphi_r(1 + rk \circ \mu_r \circ \varphi_r)} g_1(\dot\mu_r, \dot\mu_r) \circ \varphi_r \cdot g_1^\sharp(dk) \circ \mu_r \circ \varphi_r \\ &\quad + \frac{\dot\varphi_r^2}{k_r \circ \mu_r \circ \varphi_r(1 + rk_r \circ \mu_r \circ \varphi_r)} dk(\dot\mu_r) \circ \varphi_r \cdot \dot\mu_r \circ \varphi_r \\ &\quad + \ddot\varphi_r \cdot \dot\mu_r \circ \varphi_r \\ &\overset{(3.2)}{=} \frac{-1}{2k_r \circ \gamma_r(1 + rk_r \circ \gamma_r)} g_1(\dot\gamma_r, \dot\gamma_r) g_1^\sharp(dk) \circ \gamma_r \\ &\quad + \frac{1}{k_\circ \gamma_r(1 + rk \circ \gamma_r)} dk(\dot\gamma_r) \cdot \dot\gamma_r + \ddot\varphi_h(dk(\dot\gamma_h)) \cdot \dot\mu_h \circ \varphi_h \\ &\overset{(3.1)}{=} \frac{-1}{2k_r \circ \gamma_r(1 + rk_r \circ \gamma_r)} g_1(\dot\gamma_r, \dot\gamma_r) g_1^\sharp(dk) \circ \gamma_r\ . \end{aligned}$$

Since the vice versa can be proved in an analogous way, our lemma follows. □

Lemma 3.5. *Under the assumptions of the previous lemma, if either μ_r is a geodesic of G_r or γ_r verifies* 3.5*, we have*

$$g_1(\dot\gamma_r, \dot\gamma_r) = a_r^2 \frac{(1 + rk(x_0))(1 + rk \circ \gamma_r)}{k(x_0)k \circ \gamma_r} \cdot g_1(X_r, X_r), \tag{3.6}$$

being $\gamma_r(0) = x_0$ and $X_r = \dot\mu_r(0)$.

Proof. In fact, it results

$$\begin{aligned} g_1(\dot\gamma_r, \dot\gamma_r) &= (\dot\varphi_r)^2 \cdot g_1(\dot\mu_r \circ \varphi_r, \dot\mu_r \circ \varphi_r) \\ &\overset{(3.2)}{=} a_r^2 \left(\frac{1 + rk}{k} \circ \mu_r \circ \varphi_r \right)^2 \cdot g_1(\dot\mu_r \circ \varphi_r, \dot\mu_r \circ \varphi_r). \end{aligned}$$

Then, under the assumptions of our lemma, it follows

$$g_1(\dot\gamma_r, \dot\gamma_r) = a_r^2 \frac{1 + rk \circ \gamma_r}{k \circ \gamma_r} \cdot G_r(\dot\mu_r \circ \varphi_r, \dot\mu_r \circ \varphi_r).$$

From which (3.6) immediately follows. □

From the above lemma and Lemma 3.4, we get the following

Lemma 3.6. *Under the assumptions of the previous lemma, if $\mu_r : I \to M_1$ is a geodesic with respect to the metric G_r then*

$$\overset{1}{\nabla}_{\dot\gamma_r} \dot\gamma_r = \frac{-a_r^2(1 + rk(x_0))}{2k(x_0)k^2 \circ \gamma_r} \cdot g_1(X_0, X_0) \cdot g_1^\sharp(dk) \circ \gamma_r. \tag{3.7}$$

The next lemma characterizes the norm of the vector field $\dot\tau_r$. We skip the proof of this lemma for it is very similar to that one of Lemma 3.5.

Lemma 3.7. *Let $\mu_r : I \to M_1$ and $\tau_r, \nu : I \to M_2$ be three smooth curves such that $\tau_r = \nu \circ \psi_r$, being ψ_r defined as in Lemma 3.2, by means of μ_r. If either ν_r is a geodesic of g_2 or τ_r is a solution of Equation* 2.2*, then*

$$g_2(\dot\tau, \dot\tau) = \frac{b_r^2}{k^2 \circ \gamma_r} \cdot g_2(Y_0, Y_0); \tag{3.8}$$

with $\nu(0) = y_0$ and $\dot\nu(0) = Y_0$.

With the previous notations, we have:

Theorem 3.8. *Suppose that the curve* $(\mu_r, \nu_r) : I \to M$ *is a geodesic with respect to the metric* $G_r + g_2$ *and*

$$a_r^2 \frac{1 + rk(x_0)}{k(x_0)} \cdot g_1(X_0, X_0) = b_r^2 g_2(Y_0, Y_0); \tag{3.9}$$

with $\mu_r(0) = x_0$, $\nu_r(0) = y_0$, $\dot{\mu}_r(0) = X_0$ *and* $\dot{\nu}_r(0) = Y_0$.

Then, the curve $(\gamma_r, \tau_r) : I \to M$*, obtained as in the previous Lemmas is a geodesic with respect to the metric* g.

We have $(\mu_r(0), \nu_r(0)) = (x_0, y_0)$ *and* $(\mu_r(1), \nu_r(1)) = (\gamma_r(1), \tau_r(1))$*, too.*

Proof. Since $(\mu_r, \nu_r) : I \to M$ is a geodesic of the metric $G_r + g_2$ then $\mu_r : I \to M_1$ is a geodesic of G_r and $\nu_r : I \to M_2$ is a geodesic of g_2. Hence from Theorem 3.3 it follows that the curve (γ_r, τ_r) satisfies Equation (2.3).

As a consequence, we need only to prove that (γ_r, τ_r) satisfies Equation (2.2). In fact, we have

$$\begin{aligned} \overset{1}{\nabla}_{\dot{\gamma}_r} \dot{\gamma}_r &\overset{(3.7)}{=} \frac{-a_r^2(1 + rk(x_0))}{2k(x_0)k^2 \circ \gamma_r} \cdot g_1(X_0, X_0) \cdot g_1^\sharp(dk) \circ \gamma_r \\ &\overset{(3.9)}{=} \frac{-b_r^2}{2k^2 \circ \gamma_r} g_2(Y_0, Y_0) \cdot g_1^\sharp(dk) \circ \gamma_r \\ &\overset{(3.8)}{=} \frac{-1}{2} g_2(\dot{\tau}_r, \dot{\tau}_r) \cdot g_1^\sharp(dk) \circ \gamma_r. \end{aligned}$$

□

Hence, we put the following definition.

Definition 3.9. Let $(\mu_r, \nu_r) : I \to M$ be a geodesic of $G_r + g_2$ and let (γ_r, τ_r) be the geodesic of (M, g) obtained via the reparametrization by the functions φ_r and ψ_r from (μ_r, ν_r).

Then, (γ_r, τ_r) is called *Riemannian geodesic of* (M, g).

Remark 3.10. Under the assumptions of the previous theorem we set:

$$\mu_r(0) = x_0 = \gamma_r(0) \ , \ \dot{\mu}_r(0) = X_0 = X_r \ , \ \dot{\gamma}_r(0) = \widetilde{X}_r \tag{3.10}$$

and

$$\nu_r(0) = y_0 = \tau_r(0) \ , \ \dot{\nu}_r(0) = Y_0 = Y_r \ , \dot{\tau}_r(0) = \widetilde{Y}_r. \tag{3.11}$$

Then we have:

$$\widetilde{X}_r = a_r \frac{1 + rk(x_0)}{k(x_0)} X_r \quad \text{and} \quad \widetilde{Y}_r = \frac{b_r}{k(x_0)} Y_r. \tag{3.12}$$

With these notations, the first identity of 3.9 can be written as

$$a_r^2 \frac{1 + rk(x_0)}{k(x_0)} \cdot g_1(X_r, X_r) = b_r^2 g_2(Y_r, Y_r);$$

and it is equivalent to

$$g_1(\widetilde{X}_r, \widetilde{X}_r) = k(x_0)(1 + rk(x_0))g_2(\widetilde{Y}_r, \widetilde{Y}_r). \tag{3.13}$$

The previous equality implies that the geodesic $(\widehat{\nu}_r, \widehat{\tau}_r)$ of g, having (x_0, y_0) and $(a\widetilde{X}_r, a\widetilde{Y}_r)$ as initial conditions, is a Riemannian geodesic of g, for any $a \in \mathbb{R}$.

From Equation (3.13) we get

Remark 3.11. Let $\zeta_r = (\gamma_r, \tau_r)$ and $\zeta_s = (\gamma_s, \mu_s)$ be two Riemannian geodesics of g, with $r, s > k_1$, such that $\zeta_r(0) = \zeta_s(0)$.

Then $\zeta_r = \zeta_s$, if and only if $r = s$.

Theorem 3.12. *Suppose k bounded and let $\widetilde{\zeta} = (\gamma, \tau) : I \to M$ be a geodesic of g, such that $\dot{\gamma}(0) = \widetilde{X}_0$ and $\dot{\tau}(0) = \widetilde{Y}_0 \neq 0$.*

If

$$g_1(\widetilde{X}, \widetilde{X}) > k(x_0)g_2(\widetilde{Y}, \widetilde{Y})\frac{K_0 - k(x_0)}{K_0}; \tag{3.14}$$

the curve $\widetilde{\zeta}$ is a Riemannian geodesic of g.

Proof. We set

$$r = \frac{g_1(\widetilde{X}, \widetilde{X})}{k^2(x_0)g_2(\widetilde{Y}, \widetilde{Y})} - \frac{1}{k(x_0)}.$$

Then a symple calculation shows that $r > k_1$.

Now we consider the curve τ and we set $\nu_r = \tau \circ \psi_r^{-1} : I \to M_2$, being ψ_r defined by γ as in Lemma 3.2.

Since the curve τ verifies Equation (2.3), the curve ν_r is a geodesic of g_2.

Analogously, we set $\mu_r = \gamma \circ \varphi_r^{-1}$, with φ_r defined by Lemma 3.1, and μ_r is a geodesic of G_r, in the obvious way.

Finally, the previous contruction implies that (γ, τ) is a Riemannian geodesic of g obtained from the geodesic (μ_r, ν_r) of $G_r + g_2$. □

Remark 3.13. If k is unbounded from above and one replaces (3.14) by

$$g_1(\widetilde{X},\widetilde{X}) > k(x_0)g_2(\widetilde{Y},\widetilde{Y});$$

the previous theorem holds, again.

4 – Some properties of Riemannian geodesics

Remark 4.1. Let $\mu_r : I \to M_1$ be a geodesic of G_r, with $r > k_1$.

We recall that there exist a geodesic $\sigma_r : \mathbb{R} \to M_1$ of G_r and $t_0 \in \mathbb{R}$ such that $(\sigma_r([0,t_0]) = \mu_r(I)$, being G_r a complete Riemannian metric.

Moreover, it results $\dot{\mu}_r(0) = t_0\dot{\sigma}_r(0)$.

An analogous statement holds for g_2.

This implies that the mappings φ_r and ψ_r defined respectively by Lemmas 3.1 and 3.2 can be extended to diffeomorphims from $\mathbb{R}$ onto $\mathbb{R}$.

Theorem 4.2. *Let $(\mu_r,\nu_r) : \mathbb{R} \to M$ be a geodesic of $G_r + g_2$, with $r > k_1$.*

Then, for any $\alpha \in \mathbb{R}$, there exist two real numbers $\pm\beta \in \mathbb{R}$, such that the point $(\mu_r(0),\nu_r(0))$ and the point $(\mu_r(\alpha),\nu_r(\pm\beta))$ can be joined by Riemannian geodesics of g.

Proof. We put $\dot{\mu}_r(0) = X_r$ and $\dot{\nu}_r(0) = Y_r$ and suppose $\|X_r\|_1 = \|Y_r\|_2 = 1$, with the obvious meaning of the used symbols and without loss of generality.

Then, for any $\alpha \in \mathbb{R}$ ($\beta \in \mathbb{R}$), the point $\mu_r(\alpha)$ ($\nu_r(\beta)$) is the end point of the geodesic of G_r (g_2), determined by the vector αX_r (βY_r).

We shall denote by $a_{\alpha r}$ and $b_{\alpha r}$ the constants of Lemmas 3.1 and 3.2 determined by means of the geodesic having $(x_0, \alpha X_r)$ as initial condition, respectively.

Then, $X_{\alpha r}$ and $Y_{\beta r}$ verify Condition (3.9), if and only if

$$a^2_{\alpha r}\frac{1 + rk(x_0)}{k(x_0)}\alpha^2 = b^2_{\alpha r}\beta^2. \tag{4.1}$$

Hence, the assertion follows by computing β from (4.1). □

Theorem 4.3. *Let $(\mu_r,\nu_r) : \mathbb{R} \to M$ be a geodesic of $G_r + g_2$, with $r > k_1$.*

Then, for any $\beta \in \mathbb{R}$, there exist two real numbers $\pm\alpha \in \mathbb{R}$, such that the point $(\mu_r(0),\nu_r(0))$ and the point $(\mu_r(\pm\alpha),\nu_r(\beta))$ can be joined by Riemannian geodesics of g.

Proof. The proof is analogous to the previous one. □

Corollary 4.4. *For any $x_0, x_1 \in M_1$, for any $r > k_1$, for any geodesic $\mu_r : I \to M_2$ of G_r joining x_0 and x_1 and any geodesic $\nu_r : \mathbb{R} \to M_2$ of g_2, there exists $\beta \in \mathbb{R}$ such that the points $(x_0, \nu(0))$ and $(x_1, \nu(\pm\beta))$ can be joined by a Riemannian geodesic of g, obtained in the obvious way from the previous two geodesics.*

An analogous statement holds for any $y_0, y_1 \in M_2$.

Definition 4.5. Since Corollary 4.4 holds, we shall say that M_1 is *positively and negatively geodesically connected with respect to g.*

Analogously, we shall say that M_2 is *positively and negatively geodesically connected with respect to g.*

Finally, we shall say that M is *partially geodesically connected*, when the previous two definitions hold.

Theorem 4.6. *Let us consider $x_0, x_1 \in M_1$ and let us suppose that there exists a continuous map $X : (k_1, +\infty) \to T_{x_0}M_1$, such that for any $r \in (k_1, +\infty)$ the geodesic $\mu_r : I \to M_1$ of G_r, determined by the initial condition $(x_0, X(r))$, joins x_0 and x_1 and that μ_r is minimizing.*

Then, for any $y_0, y_1 \in M_2$, there exists a Riemannian geodesic of g joining (x_0, y_0) and (x_1, y_1).

Proof. Under the assumptions of the theorem, we consider the function $\beta : (k_1, \infty) \to \mathbb{R}$ defined by setting

$$\beta(r) = \frac{a_r}{b_r}\left(\frac{1 + rk(x_0)}{k(x_0)} \cdot g_1(X(r), X(r))\right)^{\frac{1}{2}};$$

where a_r and b_r are obtained respectively by (3.2) and (3.4) along the geodesic $\mu_r : I \to M_1$ of G_r, for any $r \in (k_1, +\infty)$.

Then, β is continuous, too.

Let $\gamma : I \to M_1$ be a minimizing geodesic of g_1 joining x_0 and x_1 and let us set $\dot\gamma(0) = X$.

Since all the involved geodesics are minimizing, we have

$$g_1(X, X)a_r^{-1} \leq \frac{1 + rk(x_0)}{k(x_0)} g_1(X(r), X(r)) \leq g_1(X, X)\int_0^1 \frac{1 + rk(\gamma(t))}{k(\gamma(t))}dt$$

and

$$g_1(X, X)\frac{a_r}{b_r^2} \leq \beta^2(r) \leq g_1(X, X)\frac{a_r^2}{b_r^2}\int_0^1 \frac{1 + rk(\gamma(t))}{k(\gamma(t))}dt.$$

The first of the previous inequalities and $k_0 > 0$ imply

$$\lim_{r\to k_1} \beta(r)^2 \geq \lim_{r\to -K_0^{-1}} \frac{k_0}{K_0^2(1 + rK_0)} = +\infty \quad \text{and} \quad \lim_{r\to+\infty} \beta(r)^2 = 0.$$

Hence, it results

$$\lim_{r\to k_1}\beta(r)=+\infty \quad \text{and} \quad \lim_{r\to+\infty}\beta(r)=0.$$

As a consequence of the well known generalization of the Weistrass β is onto.

Now, we consider two points $y_0, y_1 \in M_2$.

If $y_0 = y_1$, the point (x_0, y_0) and the point (x_1, y_1) can be joined by a Riemannian geodesic of g in a trivial way.

Suppose $y_0 \neq y_1$, then there exists a geodesic $\nu : \mathbb{R} \to M_2$ of g_2 and there exists $\beta_0 \in (0, +\infty)$, such that $\nu(0) = y_0$, $g_2(\dot\nu(0), \dot\nu(0)) = 1$ and $\nu(\beta_0) = y_1$.

Then, the geodesic of g_2 having $(y_0, \beta_0\dot\nu(0))$ joins y_0 and y_1.

Finally, we can consider $r_0 \in (k_1, +\infty)$ such that $\beta(r_0) = \beta_0$. With this choice the vectors X_{r_0} and $Y_{r_0} = \beta(r_0)\dot\nu(0)$ verify (3.9) and the assertion follows in a trivial way. □

Theorem 4.7. *Suppose that the manifold M_1 is connected, has dimension higher than one, non positive sectional curvature and that it is simply connected.*

Suppose that M_2 is connected, too.

Moreover, suppose that $k_0 > 0$ and that

$$(\overset{1}{\nabla}dk)(e,e)) < \frac{1+4rk}{2k(1+rk)}e(k)^2 + \frac{1}{4k(1+rk)}\|dk\|_1^2 - k(1+rk)\overset{1}{K}(\sigma); \tag{4.2}$$

for any $e, e_2 \in T_xM_1$, such that $g_1(e,e) = 1$, $g_1(e_2,e_2) = 1$ and $g_1(e,e_2) = 0$, and for any $x \in M_1$, being $\sigma =< e, e_2 >$ the two dimensional subspace spanned by e and e_2.

If M_1 and M_2 are geodesically connected with respect to the metrics g_1 and g_2, respectively, then for any $z_0, z_1 \in M$ there exists a Riemannian geodesic of g joining z_1 and z_2.

Proof. From the Appendix it follows that the sectional curvature of G_r is negative, for any $r > k_1$.

Since M_1 is simply connected, the exponential mapping of G_r, $exp_x^r : T_xM_1 \to M_1$, is a diffeomorphism, for any $x \in M_1$ (see, *e.g.* [5]).

Because of a theorem on the families of systems of ordinary differential equations continuously depending on a parameter, exp_x^r is continuous with respect to $r > k_1$, too.

Let us consider $x_0, x_1 \in M_1$ and the map $X : (k_1, \infty) \to T_{x_0}M_1$ defined by setting $X(r) = (exp_{x_0}^r)^{-1}(x_1)$, for any $r \in (k_1, \infty)$.

Then, X is continuous and the assertion follows from the previous theorem. □

Remark 4.8. *Obviously, under the assumption of the previous theorem, for r tending to k_1 the contribution of $k(\sigma)$ is zero, but the contribution of the second summand tends to $+\infty$.*

Suppose that $M_1 = \mathbb{R}$ and that $g_1 = dt^2$ is the standard metric on $\mathbb{R}$.

In this case the metric $g = dt^2 - k(t)g_2$ coincides with the FLRW-metric (Friedman-Lemaitre-Robertson-Walker metric), with speed of light $c = 1$, used in the Big Bang theories and we have:

Theorem 4.9. *If M_2 is complete with respect to the metric g_2 and k is bounded from above and bounded from below away from zero, then for any $z_0, z_1 \in M = \mathbb{R} \times M_2$ there exists a Riemannian geodesic of g joining z_1 and z_2.*

Proof. In this case, the metric tensor G_r on $\mathbb{R}$ is given by $G_r = (k^{-1} + r)dt^2$, for any $r > -K_0^{-1}$.

Let be $r > -K_0^{-1}$, then the Equation (2.4) of a geodesic of G_r becomes

$$\ddot{\mu}_r = \frac{1}{2(k \circ \mu_r)(1 + rk \circ \mu_r)}(k' \circ \mu_r)\dot{\mu}_r^2.$$

The previous equation admits a first integral given by

$$\dot{\mu}_r = c_r \left(\frac{k \circ \mu_r}{1 + rk \circ \mu_r} \right)^{\frac{1}{2}}.$$

Because of Corollary 2.4, $\mathbb{R}$ is complete with respect to the metric G_r.

Hence, we can determine c_r as a solution of the equation

$$c_r = (x_1 - x_0) \left(\int_0^1 \left(\frac{k(\mu_r(t))}{1 + rk(\mu_r(t))} \right)^{\frac{1}{2}} dt \right)^{-1}.$$

As a consequence, the mapping $\mu_r : I \to \mathbb{R}$ is strictly increasing, for $x_1 > x_0$ and strictly decreasing, for $x_1 > x_0$, because the function k is bounded from below by $k_0 > 0$.

This implies that $\exp^r_{x_0} : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism.

Since $\exp^r_{x_0}$ depends with continuity from $r \in (-K_0^{-1}, +\infty)$, the proof follows as in the previous case. $\square$

REMARK 4.10. *The previous theorem holds again, if one replaces the metric dt^2 on $\mathbb{R}$ by the Riemannian metric $f dt^2$, being $f : \mathbb{R} \to \mathbb{R}$ a C^∞-differentiable function such that $0 < f(t) < c$, for any $t \in \mathbb{R}$, with $c \in \mathbb{R}$.*

Now we return to the general case.

THEOREM 4.11. *Let us consider $r \in (k_1, +\infty)$, a geodesic $\mu_r : \mathbb{R} \to M_1$ of G_r and a geodesic $\nu_r : \mathbb{R} \to M_2$ of g_2.*

If $(\mu_r)_{|[0,+\infty)}$ has no auto intersections, there exists a map $\theta_r : \mu_r([0, +\infty)) \to \nu_r([0, +\infty))$, such that the points $(\mu_r(0), \nu_r(0))$ and $(\mu_r(t), \theta_r(\nu_r(t)))$, can be joined by a Riemannian geodesic of g obtained from the geodesic $(\mu_r, \nu_r) : \mathbb{R} \to M$ of $G_r + g_2$ in the obvious way and the mapping θ_r is onto.

Moreover, if ν_r has no auto intersections, the mapping θ_r is one to one, too.

PROOF. Under the assumptions of the theorem, we set $\dot{\mu}_r(0) = X_r$, $\dot{\nu}_r(0) = Y_r$ and we suppose $\|X_r\|_1 = \|Y_r\|_2 = 1$.

We recall that, for any $t \in \mathbb{R}$, the geodesic μ'_r of G_r determined by the initial conditions $(\mu_r(0), tX_r)$, has $\mu'_r(I) \subseteq \mu_r(\mathbb{R})$, joins $\mu_r(0)$ and $\mu_r(t)$ and the obvious quantities a'_r and b'_r are

$$a'_r := \int_0^1 \frac{k}{1 + rk} \circ \mu(\xi t)\ d\xi \quad \text{and} \quad (b'_r)^{-1} = \int_0^1 \frac{1}{k \circ \mu_r(\xi t)} d\xi.$$

An analogous statement holds for ν_r.

Now, we notice that, since μ_r has no autointersections, we can consider the map $\mu_r^{-1} : \mu_r([0, +\infty)) \to [0, +\infty)$.

Moreover, the Condition (3.9) determines the mapping $\beta : [0, +\infty) \to \mathbb{R}$, defined by:

$$\beta(t) = \frac{a'_r}{b'_r} \frac{1 + rk(x_0)}{k(x_0)} t\ , \quad \forall t \in [0, +\infty).$$

□

Then, we can set $\theta_r = \nu_r \circ \beta \circ \mu_r^{-1} : \mu_r([0, +\infty)) \to \nu_r([0, +\infty))$.

Let us consider $x_1 \in \mu_r([0, +\infty))$, then exists $t \in [0, +\infty)$, such that $\mu_r(t) = x_1$, hence $t = \mu_r^{-1}(x_1)$.

Then, $\beta(t)$ is such that the vectors tX_r and $\beta(t)Y_r$ verify (3.9).

As a consequence, the points $(\mu_r(0), \nu_r(0))$ and $(\mu_r(t), \nu_r(\beta(t)))$ can be joined by a Riemannian geodesic for g obtained from the geodesic (μ_r, ν_r) of $G_r + g_2$, with $\nu_r(\beta(t)) = \nu_r(\beta(\mu_r^{-1}(x_1))) = \theta_r(x_1)$.

5 – Appendix

In this Appendix we prove the following lemma:

LEMMA 5.1. *Suppose that the dimension of M_1 is higher than one and that g_1 has negative sectional curvature.*

Then, if k verifies (4.2), G_r has negative sectional curvature, for any $r \in (k_1, +\infty)$.

PROOF. Let $\Xi(M_1)$ be the Lie algebra of vector fields on M_1.

Let us consider a connection ∇^h of M_1 and let us suppose $\nabla^h = \overset{1}{\nabla} + \Pi$.

Then, the curvature tensor field R^h of ∇^h and the curvature tensor field $\overset{1}{R}$ of $\overset{1}{\nabla}$ are related by

$$\begin{aligned}&R^h(X,Y)Z = \overset{1}{R}(X,Y)Z + \Pi(X,\Pi(Y,Z)) - \Pi(Y,\Pi(X,Z)) + \\ &(\overset{1}{\nabla}_X\Pi)(Y,Z) - (\overset{1}{\nabla}_Y\Pi)(X,Z), \quad \forall X,Y,Z \in \Xi(M_1).\end{aligned}$$

Now we suppose that $h : M_1 \to \mathbb{R}$ is a C^∞-differentiable function and that $h(x) > 0$, for any $x \in M_1$.

We also suppose that ∇^h is the Levi-Civita connection of the metric tensor hg_1.

Then, we have

$$\begin{aligned}\Pi(X,Y) \quad &= \quad \frac{1}{2h}\left[X(h)Y + Y(h)X - g_1(X,Y)g_1^\sharp(d\log h)\right] \\ &\quad\quad \forall X,Y \in \Xi(M_1).\end{aligned}$$

The two previous identities imply

$$\begin{aligned}R^h(X,Y)Z = \overset{1}{R}(X,Y)Z& \\ &+ \frac{1}{2h}[(\overset{1}{\nabla}dh)(X,Z)Y - (\overset{1}{\nabla}dh)(Y,Z)X \\ &- g_1(Y,Z)g_1^\sharp(\overset{1}{\nabla}_X dh) + g_1(X,Z)g_1^\sharp(\overset{1}{\nabla}_Y dh)] \\ &- \frac{1}{4h^2}[3Y(h)Z(h)X - 3X(h)Z(h)Y \\ &- Y(h)g_1(X,Z)g_1^\sharp(dh) + X(h)g_1(Y,Z)g_1^\sharp(dh) \\ &+ g_1(Y,Z)\|d\log h\|_1^2 X - g_1(X,Z)\|d\log h\|_1^2 Y]\ , \quad \forall X,Y,Z \in \Xi(M_1).\end{aligned}$$

Let $\sigma =< \{e_1, e_2\} >$ be a two dimensional subspace of T_xM_1, with $x \in M_1$ and let us suppose $\|e_1\|_1 = \|e_2\|_1 = 1$ and $g_1(e_1,e_2) = 0$.

Then, the sectional curvature of ∇^h is

$$K(\sigma) = \frac{1}{h}\overset{1}{K}(\sigma) - \frac{1}{2h^2}[(\overset{1}{\nabla}dh)(e_1, e_1) + (\overset{1}{\nabla}dh)(e_2, e_2)] \\ - \frac{1}{4h^3}[3e_1(h)^2 + 3e_2(h)^2 - \|dh\|_1^2];$$

being $\overset{1}{K}$ the sectional curvature of $\overset{1}{\nabla}$.

Now we suppose $h = k^{-1} + r$, where k is the mapping used in the previous numbers and $r > K_0^{-1} = k_1$.

Then, $dh = -k^{-2}dk$ and $\overset{1}{\nabla}dh = 2k^{-3}dk \otimes dk - k^{-2}\overset{1}{\nabla}dk$.

Hence, the sectional curvature of G_r is

$$K_r(\sigma) = \frac{k}{1+rk}\overset{1}{K}(\sigma) + \frac{1}{2(1+rk)^2}[(\overset{1}{\nabla}dk)(e_1, e_1) + (\overset{1}{\nabla}dk)(e_2, e_2)] \\ - \frac{1+4rk}{4k(1+rk)^3}[e_1(k)^2 + e_2(k)^2] \\ - \frac{1}{4k(1+rk)^3}\|dk\|_1^2\ ;$$

for any two dimensional subspace $\sigma \subseteq T_xM_1$, for any (e_1, e_2) basis of σ such that $\|e_1\|_1 = \|e_2\|_1 = 1$ and $g_1(e_1, e_2) = 0$ and for any $x \in M_1$.

As a consequence, the sectional curvature of G_r is negative, for any $r > k_1$, if and only if

$$\begin{aligned} &(\overset{1}{\nabla}dk)(e_1, e_1) + (\overset{1}{\nabla}dk)(e_2, e_2) \\ &< \frac{1+4rk}{2k(1+rk)}[e_1(k)^2 + e_2(k)^2] \\ &\quad + \frac{1}{2k(1+rk)}\|dk\|_1^2 - 2k(1+rk)\overset{1}{K}(\sigma). \end{aligned} \tag{5.1}$$

We notice that, if the sectional curvature $\overset{1}{K}$ of g_1 is positive, then the Inequality (5.1) can not hold for any $r > k_1$.

Hence, we are forced to suppose the g_1 has either a negative or null sectional curvature.

In this case, the Inequality (5.1) holds, if and only if, it results

$$(\overset{1}{\nabla}dk)(e, e) < \frac{1+4rk}{2k(1+rk)}e(k)^2 + \frac{1}{4k(1+rk)}\|dk\|_1^2 - k(1+rk)\overset{1}{K}(\sigma); \tag{5.2}$$

for any $e \in T_xM_1$, such that $g_1(e, e) = 1$, for any $e_2 \in T_xM_1$, with $g_1(e_2, e_2) = 1$ and $g_1(e_1, e_2) = 0$, and any $x \in M_1$, being $\sigma = \langle e_1, e_2\rangle$. □

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Lavoro pervenuto alla redazione il 26 febbraio 2013
ed accettato per la pubblicazione il 4 marzo 2013

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