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Sum of the generalized harmonic series with even natural exponent

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Abstract: *In this paper we deal with real harmonic series, without considering their complex extension to the* Riemann zeta function.
It is well known that the generalized harmonic series are convergent if the exponent is greater than one, while they are divergent if the exponent is one or less than one.
Further, if the exponent is an even natural number $2k$*, there exists the sum of the series in closed form being equal to* π^{2k} *times a rational number.*
This sum was calculated for the first time by Euler (see, for example, [2]) through Taylor's expansion of the function $\sin x/x$ *and then by Fourier through the expansion of suitable periodic functions. Further, the formula* $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ *can be proved by using Cauchy's* Residue Calculus *or Weierstraß'* Product Theorem *(see, for example, the first five books in the references of [1]).*
In recent times the formula $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ *has been proved in many other ways (see [1, 3, 4, 5, 6]) through elementary goniometric arguments or simple properties of the series and product expansions. Many of these methods, however, apply only to the case* $\sum_{n=1}^{\infty} 1/n^2$.
In this paper we obtain the sum of all generalized harmonic series with an even natural exponent by calculating the eigenvalues of the differential operators derivative of order $2k$ *defined on a certain Hilbert space and then by inverting such operators, in order to obtain the sum of the series as trace of the inverse operators.*

1 – General Case

To calculate the sum of the generalized harmonic series $\sum_{n=1}^{\infty} 1/\left(n^{2k}\right)$, let us consider, for each fixed positive $k \in \mathbb{N}$, the linear differential operator

$$T_{2k} := \frac{d^{2k}}{dx^{2k}}$$

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defined on the set

$$\mathcal{D} = \left\{ u \in L^2[0,h] : \frac{d^{2i}u(0)}{dx^{2i}} = \frac{d^{2i}u(h)}{dx^{2i}} = 0\,,\, \forall i = 0,1,2,\ldots,k-1 \right\} \cap C^{2k}(0,h).$$

Since in this space the general eigenvalues equation

$$T_{2k}\psi_n(x) \;=\; \lambda_n \psi_n(x) \tag{1.1}$$

assumes the form

$$\frac{d^{2k}}{dx^{2k}}\,\left[C\,\sin\left(\frac{n\pi}{h}\,x\right)\right] \;=\; \frac{(-1)^k n^{2k}\pi^{2k}}{h^{2k}}\,\left[C\,\sin\left(\frac{n\pi}{h}\,x\right)\right]\,,$$

we recognize that the eigenvalues λ_n and the eigenfunctions $\psi_n(x)$ of the linear operators T_{2k} are

$$\lambda_n \;=\; \frac{(-1)^k n^{2k}\pi^{2k}}{h^{2k}} \qquad \text{and} \qquad \psi_n(x) \;= C\,\sin\left(\frac{n\pi}{h}\,x\right)$$

respectively.

We observe that equation (1.1) defined on the set $\mathcal{D}$ can be seen, in the frame of quantum mechanics, as a generalization of the time independent Schrödinger's equation associated to a free particle on the interval $[0,h]$.

The inverse operator of a differential operator is an integral one and by virtue of the properties of the inverse operator (in our case T^{-1}), we have

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n} = \sum_{n=1}^{\infty}\frac{(-1)^k h^{2k}}{n^{2k}\pi^{2k}} = \mathrm{Tr}(T_{2k}^{-1}) = \mathrm{Tr}\left[\left(\frac{d^{2k}}{dx^{2k}}\right)^{-1}\right] \tag{1.2}$$

where "Tr" indicates the *trace* of the operator.

To invert an operator T_{2k}, for fixed k, we have to find the *kernel* $G(x,s)$ such that

$$\int_0^h G(x,s)\,\left[\frac{d^{2k}}{dx^{2k}}\,u(x)\right]\,dx \;=\; u(s) \tag{1.3}$$

where $G(x,s)$ is the Green's function of the operator T_{2k}.

By iterating an integration by parts and using the conditions on the derivatives of even order for the functions $u \in \mathcal{D}$, the left-hand side of the equation (1.3) becomes

$$\int_0^h G(x,s)\,\left[\frac{d^{2k}}{dx^{2k}}\;u(x)\right]\,dx$$

$$= G(x,s)\,\frac{d^{2k-1}u(x)}{dx^{2k-1}}\bigg|_0^h - \frac{dG(x,s)}{dx}\,\frac{d^{2k-2}u(x)}{dx^{2k-2}}\bigg|_0^h$$

$$+\frac{d^2G(x,s)}{dx^2}\,\frac{d^{2k-3}u(x)}{dx^{2k-3}}\bigg|_0^h - \frac{d^3G(x,s)}{dx^3}\,\frac{d^{2k-4}u(x)}{dx^{2k-4}}\bigg|_0^h + \cdots - \frac{d^{2k-1}G(x,s)}{dx^{2k-1}}\,u(x)\bigg|_0^h$$

$$+\int_0^h\left[\frac{d^{2k}G(x,s)}{dx^{2k}}\right]\,u(x)\,dx$$

$$= G(x,s)\,\frac{d^{2k-1}u(x)}{dx^{2k-1}}\bigg|_0^h + \frac{d^2G(x,s)}{dx^2}\,\frac{d^{2k-3}u(x)}{dx^{2k-3}}\bigg|_0^h + \cdots + \frac{d^{2k-2}G(x,s)}{dx^{2k-2}}\,\frac{du(x)}{dx}\bigg|_0^h$$

$$+\int_0^h\left[\frac{d^{2k}G(x,s)}{dx^{2k}}\right]\,u(x)\,dx.$$

By imposing on the Green's function the $2k$ boundary conditions

$$\frac{d^{2i}G(x,s)}{dx^{2i}}\bigg|_{x=0} = \frac{d^{2i}G(x,s)}{dx^{2i}}\bigg|_{x=h} = 0 \tag{1.4}$$

for all $i = 0, 1, 2, \ldots, k-1$, the left-hand side of the equation (1.3) becomes

$$\int_0^h G(x,s)\,\left[\frac{d^{2k}}{dx^{2k}}\;u(x)\right]\,dx \;=\int_0^h\left[\frac{d^{2k}G(x,s)}{dx^{2k}}\right]\,u(x)\,dx. \tag{1.5}$$

By substituting the (1.5) into the (1.3), we obtain

$$\int_0^h\left[\frac{d^{2k}G(x,s)}{dx^{2k}}\right]u(x)\,dx = u(s) = \int_0^h\;\delta(x-s)\,u(x)\,dx \tag{1.6}$$

from which the relation

$$\frac{d^{2k}G(x,s)}{dx^{2k}} = \delta(x-s) \tag{1.7}$$

follows, where $\delta(x-s)$ is the Dirac's δ-*function*.

The solution of (1.7) is

$$G(x,s) = \begin{cases} G_-(x,s) & \text{if} \quad x\in[\,0,s\,] \\ G_+(x,s) & \text{if} \quad x\in[\,s,h\,] \end{cases}$$

where $G_-(x,s)$ and $G_+(x,s)$ are two polynomials of degree $2k-1$ with respect to the variable x, that is

$$G_-(x,s) = \mathcal{P}_{2k-1}(x) \qquad \text{and} \qquad G_+(x,s) = \mathcal{Q}_{2k-1}(x).$$

By integrating (1.7) with $\epsilon > 0$, we obtain the equation

$$\int_{s-\epsilon}^{s+\epsilon} \frac{d^{2k}G(x,s)}{dx^{2k}}\, dx = \int_{s-\epsilon}^{s+\epsilon} \delta(x-s)\, dx$$

from which, in the limit $\epsilon \to 0^+$, the *jump discontinuity* condition

$$\left.\frac{d^{2k-1}\mathcal{Q}_{2k-1}(x)}{dx^{2k-1}}\right|_s - \left.\frac{d^{2k-1}\mathcal{P}_{2k-1}(x)}{dx^{2k-1}}\right|_s = 1 \tag{1.8}$$

follows.

By substituting the solution $G(x,s)$ of the equation (1.7) into the left-hand side of the (1.6), we obtain

$$\int_0^s \left[\frac{d^{2k}\mathcal{P}_{2k-1}(x)}{dx^{2k}}\right] u(x)\, dx + \int_s^h \left[\frac{d^{2k}\mathcal{Q}_{2k-1}(x)}{dx^{2k}}\right] u(x)\, dx = u(s). \tag{1.9}$$

By iterating an integration by parts and using the $2k$ conditions (1.4) with the properties of the set $\mathcal{D}$, the equation (1.9) becomes

$$\begin{aligned} &[\mathcal{P}_{2k-1}(x) - \mathcal{Q}_{2k-1}(x)] \left.\frac{d^{2k-1}u(x)}{dx^{2k-1}}\right|_s - \left[\frac{d\mathcal{P}_{2k-1}(x)}{dx} - \frac{d\mathcal{Q}_{2k-1}(x)}{dx}\right] \left.\frac{d^{2k-2}u(x)}{dx^{2k-2}}\right|_s \\ &+ \left[\frac{d^2\mathcal{P}_{2k-1}(x)}{dx^2} - \frac{d^2\mathcal{Q}_{2k-1}(x)}{dx^2}\right] \left.\frac{d^{2k-3}u(x)}{dx^{2k-3}}\right|_s - \cdots + u(s) = u(s) \end{aligned} \tag{1.10}$$

where the last term of the left-hand side has coefficient 1 by virtue of (1.8).

The kernel $G(x,s)$ of the inverse operator given in the (1.3) is then

$$G(x,s) = \begin{cases} \mathcal{P}_{2k-1}(x) & \text{if} \quad x \in [\,0,s\,] \\ \mathcal{Q}_{2k-1}(x) & \text{if} \quad x \in [\,s,h\,] \end{cases}$$

where the $4k$ parameters ($2k$ parameters for each polynomial) are obtained as solution of the algebraic linear system consisting of the $4k$ linear equations representing the boundary conditions (1.4) in 0 and h, the continuity in $x = s$ of the derivatives up to the order $2k-2$ for the (1.10) to be an identity and the *jump discontinuity* in $x = s$ of the derivative of order $2k-1$.

This linear system is then of the form

$$\left.\frac{d^{2i}\mathcal{P}_{2k-1}(x)}{dx^{2i}}\right|_0 = \left.\frac{d^{2i}\mathcal{Q}_{2k-1}(x)}{dx^{2i}}\right|_h = 0 \tag{1.11a}$$

$$\left.\frac{d^{j}\mathcal{P}_{2k-1}(x)}{dx^{j}}\right|_s = \left.\frac{d^{j}\mathcal{Q}_{2k-1}(x)}{dx^{j}}\right|_s \tag{1.11b}$$

$$\left.\frac{d^{2k-1}\mathcal{Q}_{2k-1}(x)}{dx^{2k-1}}\right|_s - \left.\frac{d^{2k-1}\mathcal{P}_{2k-1}(x)}{dx^{2k-1}}\right|_s = 1 \tag{1.11c}$$

for all $i = 0, 1, 2, \dots, k-1$ and for all $j = 0, 1, 2, \dots, 2k-2$.

At this point, in analogy with the case of finite dimension in which the *trace* of an endomorphism $A = (a_{\rho\sigma})$ is given by the sum of its diagonal elements, that is $\mathrm{Tr}(A) = \sum_\sigma a_{\sigma\sigma}$, the *trace* of the inverse operator $\left(d^{2k}/dx^{2k}\right)^{-1}$ is then

$$\mathrm{Tr}\left[\left(\frac{d^{2k}}{dx^{2k}}\right)^{-1}\right] = \int_0^h G(s,s)\,ds. \tag{1.12}$$

By substituting the trace obtained in the right-hand side of (1.12) into the equation (1.2) and simplifying the factors $(-1)^k, \pi^{2k}, h^{2k}$, we finally obtain the sum of the generalized harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}}.$$

2 – First example: case of exponent $m = 2$

Let us consider the linear differential operator

$$T_2 := \frac{d^2}{dx^2}$$

and its eigenvalues equation having the form

$$\frac{d^2}{dx^2}\left[C\,\sin\left(\frac{n\pi}{h}\,x\right)\right] = -\,\frac{n^2\pi^2}{h^2}\left[C\,\sin\left(\frac{n\pi}{h}\,x\right)\right],$$

with the eigenvalues

$$\lambda_n = -\,\frac{n^2\pi^2}{h^2}.$$

It then follows

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = -\,\frac{\pi^2}{h^2}\left\{\mathrm{Tr}\left[\left(\frac{d^2}{dx^2}\right)^{-1}\right]\right\}. \tag{2.1}$$

In order to invert the operator T_2, we have to determine its Green's function

$$G(x,s) = \begin{cases} ax+b & \text{if} \quad x \in [\,0,s\,] \\ a'x+b' & \text{if} \quad x \in [\,s,h\,] \end{cases}$$

and have to solve the corresponding algebraic linear system given by the equations (1.11a), (1.11b) and (1.11c), whose form for this case is

$$\begin{cases} b &= 0 \\ a'h+b' &= 0 \\ as+b &= a's+b' \\ a'-a &= 1. \end{cases}$$

The solution of this algebraic linear system is

$$a = \frac{s-h}{h}, \qquad b = 0, \qquad a' = \frac{s}{h}, \qquad b' = -s.$$

Then we have

$$G(x,s) = \begin{cases} \dfrac{(s-h)x}{h} & \text{if} \quad x \in [\,0,s\,] \\[2ex] \dfrac{sx}{h} - s & \text{if} \quad x \in [\,s,h\,]. \end{cases}$$

According to (1.12), the *trace* of the inverse operator $(T_2)^{-1}$ is then

$$\operatorname{Tr}\left[(T_2)^{-1}\right] = \int_0^h G(s,s)\,ds = \int_0^h \left(\frac{s^2}{h} - s\right) ds = -\frac{h^2}{6}. \tag{2.2}$$

By substituting (2.2) into (2.1), we finally obtain the well-known result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3 – Second example: case of exponent $m = 4$

Let us consider the linear differential operator

$$T_4 := \frac{d^4}{dx^4}$$

and its eigenvalues equation having the form

$$\frac{d^4}{dx^4}\left[C \sin\left(\frac{n\pi}{h}\,x\right)\right] = \frac{n^4\pi^4}{h^4}\left[C \sin\left(\frac{n\pi}{h}\,x\right)\right],$$

with the eigenvalues

$$\lambda_n = \frac{n^4\pi^4}{h^4}.$$

It then follows

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{h^4} \left\{ \mathrm{Tr}\left[\left(\frac{d^4}{dx^4} \right)^{-1} \right] \right\}. \tag{3.1}$$

In order to invert the operator T_4, we have to determine its Green's function

$$G(x,s) = \begin{cases} ax^3 + bx^2 + cx + d & x \in [\,0, s\,] \\ a'x^3 + b'x^2 + c'x + d' & x \in [\,s, h\,] \end{cases}$$

and have to solve the corresponding algebraic linear system given by the equations (1.11a), (1.11b) and (1.11c), whose form for this case is

$$\begin{cases} d & = 0 \\ a'h^3 + b'h^2 + c'h + d' & = 0 \\ b & = 0 \\ 6a'h + 2b' & = 0 \\ as^3 + bs^2 + cs + d & = a's^3 + b's^2 + c's + d' \\ 3as^2 + 2bs + c & = 3a's^2 + 2b's + c' \\ 6as + 2b & = 6a's + 2b' \\ 6a' - 6a & = 1. \end{cases}$$

The solution of this algebraic linear system is

$$a = \frac{1}{6}\left(\frac{s}{h} - 1\right), \qquad b = 0, \qquad c = \frac{s^3}{6h} + \frac{hs}{3} - \frac{s^2}{2}, \qquad d = 0$$
$$a' = \frac{s}{6h}, \qquad b' = \frac{-s}{2}, \qquad c' = \frac{s^3}{6h} + \frac{hs}{3}, \qquad d' = \frac{-s^3}{6}.$$

Then we have

$$G(x,s) = \begin{cases} \dfrac{1}{6}\left(\dfrac{s}{h} - 1\right)x^3 + \left(\dfrac{s^3}{6h} + \dfrac{hs}{3} - \dfrac{s^2}{2}\right)x & \text{if} \quad x \in [\,0, s\,] \\[2ex] \dfrac{sx^3}{6h} - \dfrac{sx^2}{2} + \left(\dfrac{s^3}{6h} + \dfrac{hs}{3}\right)x - \dfrac{s^3}{6} & \text{if} \quad x \in [\,s, h\,]. \end{cases}$$

The *trace* of the inverse operator $(T_4)^{-1}$ is then

$$\mathrm{Tr}\left[\left(\frac{d^4}{dx^4} \right)^{-1} \right] = \int_0^h \left(\frac{s^4}{3h} - \frac{2s^3}{3} + \frac{hs^2}{3} \right) ds = \frac{h^4}{90}. \tag{3.2}$$

By substituting (3.2) into (3.1), we finally obtain the well-known result

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

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