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Hecke algebras and harmonic analysis on finite groups

FABIO SCARABOTTI – FILIPPO TOLLI

To Alessandro Figà Talamanca
on the occasion of his retirement

Abstract: *Let G be a finite group, K a subgroup and (σ, V) an irreducible representation of K. Then the Hecke algebra associated with the triple (G, K, σ) is the commutant of the induced representation $Ind_K^G\sigma$. In [3] Curtis and Fossum derived several explicit expressions for the characters of Hecke algebras. In the present paper we give an exposition of their results (see also [5], pp. 279-291) in the language of finite harmonic analysis. In particular, we show the connection with the theory of finite Gelfand pairs.*

1 – Introduction

Let $K \leq G$ be finite groups and denote by $X = G/K$ the corresponding homogeneous space. A function $f : G \longrightarrow \mathbb{C}$ is said to be bi-K-invariant if $f(k_1gk_2) = f(g)$ for all $g \in G$, $k_1, k_2 \in K$. The bi-K-invariant functions form an algebra under convolution that coincides with the commutant of the permutation representation of G on X. In other words, any G-invariant operator on X is given by the convolution with a suitable bi-K-invariant kernel.

We recall that (G, K) is a (finite) Gelfand pair when the permutation representation of G on the space G/K decomposes without multiplicity; equivalently, when the algebra of bi-K-invariant functions is commutative. In this setting it is possible to develop a harmonic analysis based on a particular basis of the space of bi-K-invariant functions constitued by the so called spherical functions. The theory of spherical functions has many applications; we refer to ([6, 13, 1]) for complete

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expositions. In [9, 10, 11] and [12] we extended the theory of spherical functions to homogeneous spaces whose associated permutation representation decomposes with multiplicity and gave several applications, mainly to probability and statistics. One of the goals of the present paper is to show how the results in [3] may be seen as generalizations of the theory of spherical functions associated with finite Gelfand pairs.

The plan of the paper is the following. In Section 2 we give some basic results on idempotents in a finite dimensional algebra focusing on the case of the commutant of a representation of a finite group. In Section 3 we show the connection between idempotents and ideals in the group algebra. In Section 4 we give the results in [3] for general Hecke algebras, showing the connections between the irreducible characters of the Hecke algebra and the irreducible characters of the group. In Section 4.1 we focus to the case of Hecke algebras that are commutant of induced representations of one-dimensional representations. In this case, the results of the general case may be formulated in a form suitable for applications. The last two sections contain new material with respect to our original sources. In Section 4.2, we show how the results in [3] are even more transparent when the Hecke algebra is the commutant of a permutation representation. In the final section we treat the case of a finite Gelfand pair and we show that the main formulas in [3] generalized the well known formulas that express a character in terms of the corresponding spherical function and the spherical function in terms of the character.

In the present paper we assume that all representations are complex and unitary. We use the notation in our books [1] and [2], where one can find introductions to representation theory of both finite groups and finite dimensional associative algebras and to finite harmonic analysis.

2 – Ideals and idempotents

We begin by recalling some basic facts on idempotents in unitary spaces (see [7]). Let V be a complex vector space endowed with a hermitian scalar product and denote by $\mathrm{End}(V)$ the vector space of all linear operators $T : V \to V$. $E \in \mathrm{End}(V)$ is called an *idempotent* when $E^2 = E$. If E is an idempotent then $V = \mathrm{Ran}E \oplus \mathrm{Ker}E$ (with direct sum) and E is the projection on $\mathrm{Ran}E$ along $\mathrm{Ker}E$. Moreover, $\mathrm{Ran}(I - E) = \mathrm{Ker}E$, $\mathrm{Ker}(I - E) = \mathrm{Ran}E$, $I - E$ is also an idempotent and it is the projection on $\mathrm{Ker}E$ along $\mathrm{Ran}E$. Conversely, if $V = V_1 \oplus V_2$ then the projection of V_1 along V_2 is an idempotent. An idempotent $E \in \mathrm{End}(V)$ is an *orthogonal projection* when $\mathrm{Ran}E \perp \mathrm{Ker}E$ and this is equivalent to E being selfadjoint. If E_1, E_2 are idempotents we have $E_1E_2 = 0$ if and only if $\mathrm{Ran}E_2 \leq \mathrm{Ker}E_1$; in particular, if $E_1E_2 = E_2E_1 = 0$ then $\mathrm{Ran}E_1 \cap \mathrm{Ran}E_2 = \{0\}$. From these observations we immediately deduce the following fact.

Lemma 2.1. *Let* E, E_1 *and* E_2 *be idempotents and suppose that* $E_1E_2 = E_2E_1 = 0$ *and* $E = E_1 + E_2$*. Then* $\mathrm{Ran}E = \mathrm{Ran}E_1 \oplus \mathrm{Ran}E_2$*.*

Let G be a finite group and denote by (σ, U) an irreducible representations of G. The commutant of (σ, U) is the algebra $\mathrm{Hom}_G(U, U)$ of all linear operators in $\mathrm{End}(U)$ that commute with the action of G:

$$\mathrm{Hom}_G(U, U) = \{T \in \mathrm{End}(U) : \sigma(g)T = T\sigma(g),\ \forall g \in G\}.$$

An idempotent $E \in \mathrm{Hom}_G(U, U)$ is *primitive* when the condition $E = E_1 + E_2$, with E_1, E_2 idempotents in $\mathrm{Hom}_G(U, U)$ and $E_1E_2 = E_2E_1 = 0$, always implies that either $E_1 = 0$ or $E_2 = 0$. If E is an idempotent in $\mathrm{Hom}_G(U, U)$ then

$$\mathcal{I}_E = \{A \in \mathrm{Hom}_G(U, U) : AE = A\}$$

is a left ideal of $\mathrm{Hom}_G(U, U)$. It is easy to see that

$$\mathcal{I}_E = \{A \in \mathrm{Hom}_G(U, U) : \mathrm{Ker}A \supseteq \mathrm{Ker}E\}$$

because $AE = A$ if and only if $\mathrm{Ker}A \supseteq \mathrm{Ker}E$.

We introduce the Hilbert-Schmidt hermitian scalar product on $\mathrm{Hom}_G(U, U)$ by setting, for $A, B \in \mathrm{Hom}_G(U, U)$,

$$\langle A, B\rangle_{HS} = \mathrm{tr}(B^*A)$$

where B^* is the adjoint of B and tr denotes the trace. Notice that if $\mathcal{I}$ is a left ideal of $\mathrm{Hom}_G(U, U)$ then also $\mathcal{I}^\perp = \{B \in \mathrm{Hom}_G(U, U) : \langle A, B\rangle_{HS} = 0, \forall A \in \mathcal{I}\}$ is a left ideal.

Lemma 2.2. *Let* $\mathcal{I}$ *be a left ideal of* $\mathrm{Hom}_G(U, U)$ *and denote by* $\mathcal{E}$ *the orthogonal projection from* $\mathrm{Hom}_G(U, U)$ *onto* $\mathcal{I}$*. Then*

$$\mathcal{E}(AB) = A\mathcal{E}(B) \tag{2.1}$$

for all $A, B \in \mathrm{Hom}_G(U, U)$*.*

Proof. It follows immediately from the following two observations:

- $A\mathcal{E}(B) \in \mathcal{I}$;
- $AB - A\mathcal{E}(B) = A[B - \mathcal{E}(B)] \in \mathcal{I}^\perp$,

for all $A, B \in \mathrm{Hom}_G(U, U)$. □

If $W \leq U$ we denote by E_W the orthogonal projection onto W. Note that E_W belongs to $\mathrm{Hom}_G(U,U)$ if and only if W is a σ-invariant subspace. In the following propositions we will formulate certain results, that connect idempotents $E \in \mathrm{Hom}_G(U,U)$ and σ-invariant subspaces in U.

PROPOSITION 2.3.

1. *The map $W \mapsto E_W$ is a linear bijection between the σ-invariant subspaces and the orthogonal projections in* $\mathrm{Hom}_G(U,U)$*. Moreover, W is irreducible if and only if E_W is primitive. This in turn is equivalent to the fact that every idempotent $E \in \mathrm{Hom}_G(U,U)$, with* $\mathrm{Ran}E = W$*, is primitive.*
2. *The map $E \mapsto \mathcal{I}_E$ is a linear bijection between the space of orthogonal idempotents and the left ideals in* $\mathrm{Hom}_G(U,U)$*. Moreover, the ideal $\mathcal{I}_E$ is minimal (i.e. it does not contain nontrivial ideals) if and only if E is primitive. This in turn is equivalent to the fact that every idempotent $F \in \mathrm{Hom}_G(U,U)$, such that $\mathcal{I}_F = \mathcal{I}_E$, is primitive.*

PROOF. (1) The first part is obvious. Suppose that W is irreducible and that $E \in \mathrm{Hom}_G(U,U)$ is an idempotent such that $\mathrm{Ran}E = W$. If $E = E_1 + E_2$ with $E_1, E_2 \in \mathrm{Hom}_G(U,U)$ idempotent and $E_1E_2 = E_2E_1 = 0$ then by Lemma 2.1 we have $W = \mathrm{Ran}E_1 \oplus \mathrm{Ran}E_2$ and $\mathrm{Ran}E_1, \mathrm{Ran}E_2$ are σ-invariant. Therefore either $\mathrm{Ran}E_1 = \{0\}$ or $\mathrm{Ran}E_2 = \{0\}$, and E is primitive. Conversely, assume that $E \in \mathrm{Hom}_G(U,U)$ is a primitive idempotent and set $W = \mathrm{Ran}E$. Suppose that $W_1 \leq W$ is σ-invariant and let $E_1 : U \to W_1$ be the orthogonal projection onto W_1. Set $E_2 = E - E_1$ so that $E_1, E_2 \in \mathrm{Hom}_G(U,U)$, they are idempotents $E = E_1 + E_2$ and $E_1E_2 = E_2E_1 = 0$. Since E is primitive either $E_1 = 0$ (and $W_1 = \{0\}$) or $E_2 = 0$ and $W_1 = W$. Therefore W is irreducible.

(2) The proof is similar to (1) and we limit ourselves to prove that for every ideal $\mathcal{I}$ there exists an idempotent E such that $\mathcal{I} = \mathcal{I}_E$. Let $\mathcal{E}$ be as in Lemma 2.2. Setting $E = \mathcal{E}(I_U)$ we have

$$A \in \mathcal{I} \Leftrightarrow A = \mathcal{E}(A) = \mathcal{E}(AI_U) =_{(*)} A\mathcal{E}(I_U) = AE$$

where $=_{(*)}$ follows from (2.1). Moreover

$$E^2 = \mathcal{E}(I_U)\mathcal{E}(I_U) = \mathcal{E}(\mathcal{E}(I_U)) = \mathcal{E}(I_U) = E$$

proving that E is an idempotent and that $\mathcal{I} = \mathcal{I}_E$. □

PROPOSITION 2.4. *Suppose that $W_1, W_2 \leq U$ are two σ-invariant subspaces and that $E_1, E_2 \in \mathrm{Hom}_G(U,U)$ are two idempotents such that $\mathrm{Ran}E_i = W_i$, $i = 1, 2$. Then, setting*

$$\mathcal{V} = \{T \in \mathrm{Hom}_G(U,U) : E_2TE_1 = T\} \equiv \{E_2TE_1 : T \in \mathrm{Hom}_G(U,U)\},$$

the map

$$\begin{array}{ccc} \mathcal{V} & \to & \mathrm{Hom}_G(W_1, W_2) \\ T & \mapsto & T|_{W_1} \end{array}$$

is a linear isomorphism. If $W_1 = W_2$ then it is also an isomorphism of algebras.

PROOF. If $T \in \mathcal{V}$ then $T|_{W_1} \in \mathrm{Hom}_G(W_1, W_2)$ and $T|_{W_1} = 0$ if and only if $T = 0$. Suppose that $T \in \mathrm{Hom}_G(W_1, W_2)$ and extend it to an operator $\tilde{T} \in \mathrm{Hom}_G(U, U)$ by setting $\tilde{T}v = 0$ for all $v \in \mathrm{Ker}E_1$. Then we have

$$\tilde{T}v = \tilde{T}E_1v = E_2\tilde{T}E_1v, \quad \forall v \in U,$$

that is $\tilde{T} \in \mathcal{V}$ and certainly $\tilde{T}|_{W_1} = T$. $\square$

PROPOSITION 2.5. *Suppose that $E \in \mathrm{Hom}_G(U, U)$ is an idempotent. Then the following facts are equivalent*:

1. $W = \mathrm{Ran}E$ *is σ-irreducible*;
2. ETE *is a multiple of E for all $T \in \mathrm{Hom}_G(U, U)$*;
3. *the algebra $\{T \in \mathrm{Hom}_G(U, U) : ETE = T\}$ is isomorphic to $\mathbb{C}$.*

PROOF. By Schur's Lemma we have $\mathrm{Ran}E$ is irreducible if and only if $\mathrm{Hom}_G(W, W) \cong \mathbb{C}$ and by Proposition 2.4 $\mathrm{Hom}_G(W, W) = \{T \in \mathrm{Hom}_G(W, W) : ETE = T\} \equiv \{ETE : T \in \mathrm{Hom}_G(W, W)\}$. $\square$

Let $\widehat{G}$ be a complete system of pairwise inequivalent irreducible representations of G. We recall [1, 2], that if $U = \bigoplus_{\theta \in J} m_\theta V_\theta$ is the decomposition of U into irreducible G-representations, where $J \subseteq \widehat{G}$ and $m_\theta = \dim \mathrm{Hom}_G(V_\theta, U)$ is the multiplicity of V_θ in U, then

$$\mathrm{Hom}_G(U, U) \cong \bigoplus_{\theta \in J} \mathrm{Hom}_G(m_\theta V_\theta, m_\theta V_\theta) \cong \bigoplus_{\theta \in J} M_{m_\theta, m_\theta}(\mathbb{C}), \tag{2.2}$$

where $M_{m_\theta, m_\theta}(\mathbb{C})$ is the full matrix algebra of $(m_\theta \times m_\theta)$-complex matrices. In particular, if $E_\theta : U \to m_\theta V_\theta$ is the orthogonal projection from U onto $m_\theta V_\theta$, then $\{E_\theta : \theta \in J\}$ is a basis for the *center* of $\mathrm{Hom}_G(U, U)$.

An idempotent $E \in \mathrm{Hom}_G(U, U)$ is *central* when it belongs to the center of $\mathrm{Hom}_G(U, U)$. A central idempotent $E \in \mathrm{Hom}_G(U, U)$ is *primitive* when every decomposition $E = E_1 + E_2$, with E_1, E_2 central idempotents and $E_1E_2 = E_2E_1 = 0$, is trivial, that is $E_1 = 0$ or $E_2 = 0$. Note that a central primitive idempotent is not necessarily a primitive idempotent. The following proposition is an immediate consequence of the above considerations.

PROPOSITION 2.6. *If E_θ is as above, then $\{E_\theta : \theta \in J\}$ are precisely the primitive idempotents in $\mathrm{Hom}_G(U, U)$.*

For the representation theory of $\mathrm{Hom}_G(U,U)$ we refer to the first chapter of [2]. Here we limit ourselves to point out that the irreducible character associated with $\theta \in J$ is given by:

$$\chi^\theta(T) = \sum_{i=1}^{m_\theta} t_{i,i}^\theta,$$

for all $T \in \mathrm{Hom}_G(U,U)$, where $(t_{i,j}^\theta)_{i,j=1,\dots,m_\theta} \in M_{m_\theta,m_\theta}(\mathbb{C})$ is given by the isomorphism (2.2).

3 – Ideals and idempotents in a group algebra

In this section we apply the previous analysis to the particular case of the regular representation of a finite group G. The *group algebra* $L(G)$ is the vector space $\{f : G \to \mathbb{C}\}$ endowed with the convolution product:

$$f * \phi(g) = \sum_{h \in G} f(gh)\phi(h^{-1})$$

for all $f, \phi \in L(G)$ and $g \in G$. For $g \in G$, the *Dirac function* δ_g centered at g is defined by setting $\delta_g(h) = \begin{cases} 0 & \text{if } h \neq g \\ 1 & \text{if } h = g. \end{cases}$ Observe that the δ_{1_G} is the unit element of $L(G)$.

We denote by λ the left regular representation of G on $L(G)$, that is,

$$[\lambda(g)f](t) = f(g^{-1}t)$$

for all $g, t \in G$ and $f \in L(G)$. It is easy to see that a subspace $V \subseteq L(G)$ is λ-invariant if and only if it is a left ideal of $L(G)$; if it is a minimal ideal if and only if λ-irreducible. In what follows, we set

$$\check{f}(g) = \overline{f(g^{-1})}$$

for all $f \in L(G)$ and $g \in G$; moreover for $\phi \in L(G)$ we denote by T_ϕ the corresponding convolution operator, that is

$$T_\phi f = f * \phi$$

for all $f \in L(G)$. We recall the following fundamental isomorphism ([1]):

Theorem 3.1. *The map*

$$\begin{array}{ccc} L(G) & \to & \mathrm{Hom}_G(L(G), L(G)) \\ \phi & \mapsto & T_\phi \end{array} \tag{3.1}$$

is an antiisomorphism of algebras.

A function $\psi \in L(G)$ is called *idempotent* if $\psi * \psi = \psi$. By the above theorem we have that ψ is an idempotent if and only if the corresponding convolution operator $E = T_\psi$ is an idempotent; moreover, it is easy to check that E is an orthogonal projection if and only if $\check{\psi} = \psi$. An idempotent is *primitive* if $\psi = \psi_1 + \psi_2$ with ψ_1, ψ_2 idempotents such that $\psi_1 * \psi_2 = \psi_2 * \psi_1$implies that $\psi_1 = 0$ or $\psi_2 = 0$.

In the next proposition we collect the main properties of left ideals in the group algebra $L(G)$. They are just a reformulation of the results of the Section 2, taking into account Theorem 3.1 (and other observations made in this section).

Proposition 3.2.

1. *If $\psi \in L(G)$ is an idempotent then $V = L(G) * \psi$ is a left ideal in $L(G)$ and every left ideal of $L(G)$ may be represented in this way;*
2. *$V = L(G) * \psi$ is minimal (that is, it is λ-irreducible) if and only if ψ is primitive;*
3. *$V = L(G) * \psi$ is minimal if and only if the algebra $\psi * L(G) * \psi$ is isomorphic to $\mathbb{C}$;*
4. *if ψ_1, ψ_2 are two idempotents and V_1, V_2 the respective ideals, then the map*

$$\begin{array}{ccc} \psi_2 * L(G) * \psi_1 & \rightarrow & \mathrm{Hom}_G(V_1, V_2) \\ \phi & \mapsto & T_\phi \end{array} \tag{3.2}$$

is a linear bijection and it is an antiisomorphism when $V_1 = V_2$.

Definition 3.3. Let (σ, U) be a representation of G and $f \in L(G)$. The *Fourier transform* of f at (σ, U) is the linear operator on U defined by setting

$$\sigma(f) = \sum_{g \in G} f(g)\sigma(g).$$

We can reconstruct the function f from its Fourier transform by the *Fourier inversion formula* ([1]):

$$f(g) = \frac{1}{|G|} \sum_{\sigma \in \widehat{G}} d_\sigma \mathrm{tr}[\sigma(g^{-1})\sigma(f)]$$

for all $g \in G$, where d_σ denotes the dimension of $\sigma \in \widehat{G}$.

Note that $\psi \in L(G)$ is an idempotent (resp. orthogonal projector) if and only if $\sigma(\psi)$ is an idempotent (resp. orthogonal projector). Moreover the Fourier transform is multiplicative: $\sigma(f_1 * f_2) = \sigma(f_1)\sigma(f_2)$ for all $f_1, f_2 \in L(G)$.

We recall the following fundamental results (see [2]).

Lemma 3.4. *Let* $L(G) = \bigoplus_{\sigma\in\widehat{G}} d_\sigma U_\sigma$ *be the isotypic decomposition of the regular representation. Then we have*:

1. *The orthogonal projection* $E_\sigma : L(G) \to d_\sigma U_\sigma$ *is given by*
$$E_\sigma f(g) = \frac{d_\sigma}{|G|}\mathrm{tr}[\sigma(g^{-1})\sigma(f)]$$
for all $f \in L(G)$ *and* $g \in G$;
2. *The map*
$$\begin{array}{ccc} d_\sigma U_\sigma & \to & \mathrm{Hom}(U_\sigma, U_\sigma) \\ f & \mapsto & \sigma(f) \end{array} \tag{3.3}$$
is an (*explicit*) *isomorphism of vector spaces*;
3. *The restriction of the map in Theorem* 3.1 *to the isotypic component* $d_\sigma U_\sigma$ *has range isomorphic to* $\mathrm{Hom}_G(d_\sigma U_\sigma, d_\sigma U_\sigma)$.

We end this section by introducing the following notation: given a G-representation σ and a function $f \in L(G)$ we set
$$\chi^\sigma(f) = \sum_{g\in G} \chi^\sigma(g) f(g) = \mathrm{tr}[\sigma(f)] \tag{3.4}$$
where χ^σ denotes the character of the representation σ.

4 – Hecke algebras

In this section we introduce the notion of the Hecke algebra associated to the induced representation from a subgroup.

Lemma 4.1. *Let* G *be a group and* $K \le G$ *a subgroup. Suppose that* V *is a left ideal in* $L(K)$ *and* $\psi \in L(K)$ *an idempotent that generates* V*. Then* $\mathrm{Ind}_K^G V \equiv \{f \in L(G) : f * \psi = f\}$*. In other words,* ψ (*which is clearly an idempotents in* $L(G)$) *generates* $\mathrm{Ind}_K^G V$ *as a left ideal in* $L(G)$.

Proof. Recall that
$$\begin{array}{ccc} L(G) & \to & \mathrm{Ind}_K^G L(K) \\ f & \mapsto & F \end{array} \tag{4.1}$$
where $F(g,k) = f(gk)$, is an explicit linear isomorphism. Indeed, $\mathrm{Ind}_K^G L(K)$ is the set of all functions $F : G \times K \to \mathbb{C}$ such that $F(gk_1, k) = F(g, k_1 k)$ and (4.1) is a

particular case of transitivity of induction (see [2, Proposition 1.6.6]). Then, in the correspondence $f \mapsto F$, we have

$$\begin{aligned} f \in \mathrm{Ind}_K^G V &\Leftrightarrow F(g,k) = \sum_{t \in K} F(g, kt^{-1})\psi(t) \\ &\Leftrightarrow f(gk) = \sum_{t\in K} f(gkt^{-1})\psi(t) \\ &\Leftrightarrow f = f * \psi. \end{aligned}$$

□

LEMMA 4.2. *Keeping the same assumptions of the previous lemma, suppose that (σ, U) is a representation of G. Then the map*

$$\begin{array}{ccc} \mathrm{Hom}_K(V, \mathrm{Res}_K^G U) & \to & \sigma(\psi)U \\ T & \mapsto & T(\psi) \end{array}$$

is a linear isomorphism of vector spaces.

PROOF. Denote by λ_K the left regular representation of K. Note that we have $T\lambda_K(k) = \sigma(k)T$, for every $T \in \mathrm{Hom}_K(V, \mathrm{Res}_K^G U)$, and therefore

$$\begin{aligned} T(\psi) &= T\left(\sum_{k\in K} \psi(k)\delta_k\right) \\ &= \sum_{k\in K} \psi(k) T(\delta_k) \\ &= \sum_{k\in K} \psi(k) T[\lambda_K(k)\delta_{1_K}] \\ &= \sum_{k \in K} \psi(k)\sigma(k)T(\delta_{1_K}) \\ &= \sigma(\psi)T(\delta_{1_K}) \in \sigma(\psi)U. \end{aligned}$$

Moreover, for every $f \in V$ we have:

$$\begin{aligned} T(f) &= T(f * \psi) \\ &= T\left(\sum_{k\in K} f(k)\lambda_K(k)\psi\right) \\ &= \sum_{k\in K} f(k)\sigma(k)T(\psi) \\ &= \sigma(f)T(\psi), \end{aligned}$$

and therefore T is determined by $T(\psi)$ and the map $T \mapsto T(\psi)$ is injective.

To prove that this map is also surjective, we show that, for $u \in U$, the map $T : V \to U$, $T(f) = \sigma(f)\sigma(\psi)u$ for all $f \in V$, belongs to $\mathrm{Hom}_K(V, \mathrm{Res}^G_K U)$. Indeed, we have

$$T(f) = \sigma(f * \psi)u = \sigma(f)u$$

and therefore, for $k \in K$, we have

$$\begin{aligned} T[\lambda_K(k)f] &= \sigma[\lambda_K(k)f]u \\ &= \sigma(\delta_k * f)u \\ &= \sigma(k)\sigma(f)u \\ &= \sigma(k)T(f). \end{aligned}$$

□

Corollary 4.3. *If (σ, U) is irreducible, then the multiplicity of U in $\mathrm{Ind}^G_K V$ is equal to*

1. $\dim \mathrm{Hom}_G(U, \mathrm{Ind}^G_K V)$;
2. $\dim[\sigma(\psi)U]$;
3. $\chi^\sigma(\psi)$.

Proof. (1) it is well known.

(2) it follows from (1), Frobenius reciprocity and the preceding lemma.

Since $\sigma(\psi) : U \to \sigma(\psi)U$ is the projection of U onto $\sigma(\psi)U$, we have also

$$\dim[\sigma(\psi)U] = \mathrm{tr}[\sigma(\psi)] = \chi^\sigma(\psi).$$

This proves (3). □

Definition 4.4. Let $K \leq G$ and ψ be an idempotent in $L(K)$. Then the *Hecke algebra* $\mathcal{H}(G, K, \psi)$ is the subalgebra $\psi * L(G) * \psi$ of $L(G)$. Clearly the antiisomorphism (3.2), the decomposition in (2.2) and Lemma 4.1 ensure us that

$$\mathcal{H}(G, K, \psi) \cong \mathrm{Hom}_G(\mathrm{Ind}^G_K V, \mathrm{Ind}^G_K V).$$

Remark 4.5. Often, it is used the notation $\mathcal{H}(G, K, \chi)$, where χ is the character of the representation of K on $L(K) * \psi$; see [4].

In the same notation of Lemma 4.1, suppose that $\mathrm{Ind}^G_K V = \bigoplus_{\sigma \in J} m_\sigma U_\sigma$ is the decomposition of $\mathrm{Ind}^G_K V$ into irreducible G-representations (*i.e.* $J \subseteq \widehat{G}$ and m_σ is the multiplicity of σ in $\mathrm{Ind}^G_K V$). Let $d_\sigma U_\sigma$ be the σ-isotypic component in $L(G)$.

LEMMA 4.6.
1. $m_\sigma U_\sigma = \{f * \psi : f \in d_\sigma U_\sigma\}$;
2. *the map*

$$\begin{array}{ccc} \{\phi \in d_\sigma U_\sigma : \psi * \phi * \psi = \phi\} & \to & \mathrm{Hom}_G(m_\sigma U_\sigma, m_\sigma U_\sigma) \\ \phi & \mapsto & T_\phi|_{m_\sigma U_\sigma} \end{array}$$

is an antiisomorphism.

PROOF. 1) Clearly, $m_\sigma U_\sigma \subseteq d_\sigma U_\sigma$ and therefore from Lemma 4.1 we deduce that

$$m_\sigma U_\sigma = \{f \in d_\sigma U_\sigma : f * \psi = f\} = \{f * \psi : f \in d_\sigma U_\sigma\}.$$

2) $d_\sigma U_\sigma$ is a two sided ideals of $L(G)$ (isomorphic to $\mathrm{End}(U_\sigma)$) and therefore by virtue of Proposition 3.4

$$\begin{array}{ccc} d_\sigma U_\sigma & \to & \mathrm{Hom}_G(d_\sigma U_\sigma, d_\sigma U_\sigma) \\ \phi & \mapsto & T_\phi \end{array}$$

is an antiisomorphism of algebras. Then we can apply (3.2) ($m_\sigma U_\sigma$ is a left ideal in $d_\sigma U_\sigma$) noting also that we can replace ψ with the orthogonal projection onto $d_\sigma U_\sigma$. □

LEMMA 4.7. *An idempotent $\xi \in \mathcal{H} := \mathcal{H}(G, K, \psi)$ is primitive in $\mathcal{H}$ if and only if it is primitive in $L(G)$.*

PROOF. Since $\xi = \psi * \xi * \psi = \xi * \psi = \psi * \xi$, we have

$$\begin{aligned} \xi * L(G) * \xi &= \xi * \psi * L(G) * \psi * \xi \\ &= \xi * \mathcal{H} * \xi. \end{aligned}$$

Then the lemma is a consequence of Proposition 3.2 and Proposition 2.5. □

LEMMA 4.8. *Let $\phi \in \mathcal{H}(G, K, \psi)$ and consider the convolution operator T_ϕ associated with ϕ. If the restriction of T_ϕ to the isotypic component $m_\sigma U_\sigma$ is the null operator, then $\sigma(\phi) = 0$.*

PROOF. Indeed, for all $f \in d_\sigma U_\sigma$ we have that $f * \psi \in m_\sigma U_\sigma$ and thus

$$\begin{aligned} \sigma(f)\sigma(\phi) &= \sigma(f)\sigma(\psi * \phi) \\ &= \sigma(f * \psi * \phi) \\ &= 0. \end{aligned}$$

Since $\{\sigma(f) : f \in d_\sigma U_\sigma\} \cong \mathrm{Hom}(U_\sigma, U_\sigma)$ (see (3.3)) we necessarily have $\sigma(\phi) = 0$. □

Corollary 4.9.

$$\mathrm{Hom}_G(m_\sigma U_\sigma, m_\sigma U_\sigma) \cong \{\phi \in d_\sigma U_\sigma : \psi * \phi * \psi = \phi\}.$$

In other words, if $\phi \in \mathcal{H}(G, K, \psi)$ and

$$T_\phi|_{m_\theta U_\theta} = 0 \tag{4.2}$$

for all $\theta \in J \setminus \{\sigma\}$, then $\phi \in d_\sigma U_\sigma$.

Proof. Using Lemma 4.8 we deduce from (4.2) that $\theta(\phi) = 0$ if $\theta \in J \setminus \{\sigma\}$. If $\theta \notin J$ then we have that $f * \psi = 0$ for all $f \in m_\theta U_\theta$ which immediately implies that $\theta(\phi) = 0$. These facts force $\phi \in d_\sigma U_\sigma$ (see (1) in Lemma 3.4). □

Let E be an idempotent on a vector space U of dimension d. Suppose that $T \in \mathrm{Hom}(U, U)$ satisfies $T = ETE$. Then $\mathrm{Ker}T \supseteq \mathrm{Ker}E$ and $\mathrm{Ran}T \subseteq \mathrm{Ran}E$. Let $\{u_1, u_2, \ldots, u_d\}$ be a basis of U such that $\{u_1, u_2, \ldots, u_m\}$ is a basis of $\mathrm{Ran}E$ and $\{u_{m+1}, u_{m+2}, \ldots, u_d\}$ is a basis of $\mathrm{Ker}E$. Let $(t_{i,j})_{i,j=1,2,\ldots,d}$ be the matrix representing T in the basis $\{u_1, u_2, \ldots, u_d\}$, that is

$$Tu_i = \sum_{j=1}^{d} t_{i,j} u_j.$$

Observe that $t_{i,j} = 0$ if $i > m$ or $j > m$ and therefore

$$\{T \in \mathrm{Hom}(U, U) : T = ETE\} \cong M_{m,m}(\mathbb{C}).$$

These considerations applied to the case $E = \sigma(\psi)$ and $T = \sigma(\phi)$ lead to the proof of the following lemma.

Lemma 4.10. *Let $\{u_1, u_2, \ldots, u_{d_\sigma}\}$ be a basis of U_σ such that $\{u_1, u_2, \ldots, u_{m_\sigma}\}$ is a basis of $\sigma(\psi)U_\sigma$ and $\{u_{m_\sigma+1}, u_{m_\sigma+2}, \ldots, u_{d_\sigma}\}$ is a basis of $\mathrm{Ker}\sigma(\psi)$. Then the map*

$$\begin{array}{ccc} \{\phi \in d_\sigma U_\sigma : \psi * \phi * \psi = \phi\} & \to & M_{m_\sigma, m_\sigma}(\mathbb{C}) \\ \phi & \mapsto & (t^\phi_{i,j})_{i,j=1,2,\ldots,m_\sigma} \end{array}$$

where $t^\phi_{i,j}$ are the coefficients of $\sigma(\phi)$ in the basis $\{u_1, u_2, \ldots, u_{d_\sigma}\}$, is an explicit isomorphism.

Corollary 4.11. *The irreducible characters of $\mathcal{H} \cong \bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C})$ are given by the restrictions of the characters of the representations in J.*

For each $\sigma \in J$, we define φ_σ by setting (recall (3.4))

$$\varphi_\sigma(\phi) = \chi^\sigma(\phi) = \sum_{g \in G} \phi(g)\chi^\sigma(g) = \mathrm{tr}[\sigma(\phi)].$$

By virtue of Corollary 4.11 $\{\varphi_\sigma : \sigma \in J\}$ are the irreducible characters of $\mathcal{H}$. We recall (see [1]) that an alternative expression of the orthogonal projection of $L(G)$ onto the isotypic component $d_\sigma U_\sigma \cong U_\sigma \otimes U_{\sigma'}$ is given by

$$E_\sigma f = \frac{d_\sigma}{|G|} f * \overline{\chi^\sigma} = \frac{d_\sigma}{|G|} f * \chi^{\sigma'},$$

where σ' is the adjoint of σ. Equivalently

$$E_\sigma f = \frac{d_\sigma}{|G|} \sum_{g \in G} \chi^\sigma(g)\lambda(g^{-1})f.$$

Finally we observe that $\psi * \overline{\chi^\sigma} = \overline{\chi^\sigma} * \psi \in \mathrm{Ind}_K^G V$ (see Lemma 4.1).

Lemma 4.12. *The orthogonal projection from* $\mathrm{Ind}_K^G V$ *onto* $m_\sigma U_\sigma$ *is given by*

$$E_\sigma^\psi f = \frac{d_\sigma}{|G|} f * \psi * \overline{\chi^\sigma}.$$

Proof. If $f \in \mathrm{Ind}_K^G V$ then $f = f * \psi$ and therefore

$$E_\sigma f = E_\sigma^\psi f.$$

If $f \in m_\sigma U_\sigma \subseteq d_\sigma U_\sigma$ then $f = E_\sigma f = E_\sigma^\psi f$, while if $\theta \not\sim \sigma$ and $f \in m_\theta U_\theta$ then $0 = E_\sigma f = E_\sigma^\psi f$. □

Corollary 4.13 ([8], Janusz). *If the multiplicity of* U_σ *in* $\mathrm{Ind}_K^G V$ *is equal to 1, then* E_σ^ψ *is a primitive idempotent in* $L(G)$ *whose range is a subspace isomorphic to* U_σ.

An idempotent ξ in $\mathcal{H}$ or in $L(G)$ is called *central* when $\xi * \phi = \phi * \xi$ for all $\phi \in \mathcal{H}$. A central idempotent ξ is called *primitive* when every decomposition $\xi = \xi_1 + \xi_2$, with ξ_1, ξ_2 central idempotents and $\xi_1 * \xi_2 = \xi_2 * \xi_1 = 0$, is trivial, *i.e.* $\xi_1 = 0$ or $\xi_2 = 0$.

Corollary 4.14. *The central primitive idempotents of* $\mathcal{H}$ *are given by*

$$\left\{ \frac{d_\sigma}{|G|} \psi * \overline{\chi^\sigma} : \sigma \in J \right\}.$$

Proof. It is an immediate consequence of Proposition 2.6 and Lemma 4.12. □

In the following theorem the characters of the representations of G contained in $\mathrm{Ind}_K^G V$ are expressed in terms of the characters of $\mathcal{H}$.

THEOREM 4.15. *For $g \in G$, let $\mathcal{C}(g)$ be the conjugacy class of g and denote by $\mathbf{1}_{\mathcal{C}(g)}$ the characteristic function of $\mathcal{C}(g)$. Then we have, $\forall \sigma \in J$,*

$$\chi^\sigma(g) = \frac{|G|}{|\mathcal{C}(g)|}\varphi_\sigma(\psi * \mathbf{1}_{\mathcal{C}(g)} * \psi)\left[\sum_{h\in G}\varphi_\sigma(\psi * \delta_{h^{-1}} * \psi)\varphi_\sigma(\psi * \delta_h * \psi)\right]^{-1}.$$

PROOF. Since the characteristic function $\mathbf{1}_{\mathcal{C}(g)}$ is a central function, we have $\psi * \mathbf{1}_{\mathcal{C}(g)} * \psi = \mathbf{1}_{\mathcal{C}(g)} * \psi$. Therefore

$$\sigma(\psi * \mathbf{1}_{\mathcal{C}(g)} * \psi) = \sigma(\mathbf{1}_{\mathcal{C}(g)})\sigma(\psi). \tag{4.3}$$

Moreover, using again the centrality of $\mathbf{1}_{\mathcal{C}(g)}$, we have $\sigma(\mathbf{1}_{\mathcal{C}(g)}) = \lambda I_\sigma$ and taking the trace of both sides we get that $\lambda = \frac{|\mathcal{C}(g)|\chi^\sigma(g)}{d_\sigma}$ and therefore

$$\sigma(\mathbf{1}_{\mathcal{C}(g)}) = \frac{|\mathcal{C}(g)|\chi^\sigma(g)}{d_\sigma}I_\sigma. \tag{4.4}$$

Plugging (4.4) in (4.3) and taking the trace of both sides, we get

$$\chi^\sigma(\psi * \mathbf{1}_{\mathcal{C}(g)} * \psi) = \frac{|\mathcal{C}(g)|\chi^\sigma(g)}{d_\sigma}\chi^\sigma(\psi). \tag{4.5}$$

From the orthogonality relations for matrix coefficients it follows that $\sigma(\overline{\chi^\sigma}) = I_\sigma \frac{|G|}{d_\sigma}$ and therefore

$$\sigma(\overline{\chi^\sigma} * \psi) = \sigma(\overline{\chi^\sigma})\sigma(\psi) = \sigma(\psi)\frac{|G|}{d_\sigma}.$$

Taking the trace of the first and the last term we get

$$\chi^\sigma(\overline{\chi^\sigma} * \psi) = \chi^\sigma(\psi)\frac{|G|}{d_\sigma}. \tag{4.6}$$

Being $\psi = \psi * \psi$ and recalling the notation δ_g for the Dirac function we have

$$\begin{aligned}
\overline{\chi^\sigma} * \psi &= \psi * \overline{\chi^\sigma} * \psi \\
&= \sum_{g\in G} \chi^\sigma(g^{-1})\psi * \delta_g * \psi \\
&= \sum_{g\in G} \chi^\sigma(g^{-1})\psi * \psi * \delta_g * \psi * \psi \\
&= \sum_{g\in G} \chi^\sigma(g^{-1}) \sum_{s,t\in G} (\psi * \delta_{sgt} * \psi)\psi(s)\psi(t) \qquad (4.7)\\
\text{(setting } h = sgt) \quad &= \sum_{h\in G} \chi^\sigma \left[\sum_{s,t\in G} \psi(s)\psi(t)\delta_{th^{-1}s}\right] \psi * \delta_h * \psi \\
&= \sum_{h\in G} \chi^\sigma(\psi * \delta_{h^{-1}} * \psi)\psi * \delta_h * \psi.
\end{aligned}$$

Taking into account (4.6), we have

$$\begin{aligned}
\chi^\sigma(\psi) &= \frac{d_\sigma}{|G|} \sum_{h\in G} \chi^\sigma(\psi * \delta_{h^{-1}} * \psi)\chi^\sigma(\psi * \delta_h * \psi) \\
\text{(by Corollary 4.11)} &= \frac{d_\sigma}{|G|} \sum_{h\in G} \varphi_\sigma(\psi * \delta_{h^{-1}} * \psi)\varphi_\sigma(\psi * \delta_h * \psi).
\end{aligned} \qquad (4.8)$$

Then the theorem follows from (4.5) and (4.8). □

4.1 – The Hecke algebra of a representation of G induced by a one-dimensional representation of K

In this section we suppose that the K-representation V is one-dimensional. Therefore, if we denote by χ its character, the idempotent generator is necessarily $\psi = \frac{1}{|K|}\overline{\chi}$. In the following lemma we give in this particular setting an explicit description of the corresponding Hecke algebra $\mathcal{H}$.

THEOREM 4.16. *In the above hypothesis, the Hecke algebra is given by*

$$\mathcal{H} = \{f \in L(G) : f(k_1 g k_2) = \overline{\chi}(k_1)\overline{\chi}(k_2)f(g),\ \forall k_1, k_2 \in K, g \in G\}.$$

PROOF. Let $f \in \mathcal{H}$, then $f = \frac{1}{|K|^2}\overline{\chi} * f * \overline{\chi}$ and therefore for $k_1, k_2 \in K, g \in G$, we have

$$\begin{aligned}
f(k_1 g k_2) &= \frac{1}{|K|^2}[\overline{\chi} * f * \overline{\chi}](k_1 g k_2) \\
&= \frac{1}{|K|^2} \sum_{\substack{t \in k_2^{-1} g^{-1} K \\ k \in K}} \overline{\chi}(k_1 g k_2 t) f(t^{-1}k)\overline{\chi}(k^{-1}) \\
\text{(setting } u = k_2 t\text{)} \quad &= \frac{1}{|K|^2} \sum_{\substack{u \in g^{-1} K \\ k \in K}} \overline{\chi}(k_1 g u) f(u^{-1}k_2 k)\overline{\chi}(k^{-1}) \\
\text{(setting } h = k_2 k\text{)} \quad &= \frac{1}{|K|^2} \sum_{\substack{u \in g^{-1} K \\ h \in K}} \overline{\chi}(k_1 g u) f(u^{-1}h)\overline{\chi}(h^{-1}k_2) \\
&= \frac{1}{|K|^2} \sum_{\substack{u \in g^{-1} K \\ h \in K}} \overline{\chi}(k_1)\overline{\chi}(g u) f(u^{-1}h)\overline{\chi}(h^{-1})\overline{\chi}(k_2) \\
&= \frac{1}{|K|^2}\overline{\chi}(k_1) \cdot [\overline{\chi} * f * \overline{\chi}](g) \cdot \overline{\chi}(k_2) \\
&= \overline{\chi}(k_1) f(g) \overline{\chi}(k_2).
\end{aligned}$$

Vice versa, if $f \in L(G)$ and $f(k_1 g k_2) = \overline{\chi}(k_1) f(g) \overline{\chi}(k_2)$ for all $g \in (G)$, and $k_1, k_2 \in K$, then

$$\begin{aligned}
[\overline{\chi} * f * \overline{\chi}](g) &= \sum_{\substack{t \in g^{-1} K \\ k_2 \in K}} \overline{\chi}(g t) f(t^{-1}k_2)\overline{\chi}(k_2^{-1}) \\
\text{(setting } g t = k_1\text{)} \quad &= \sum_{k_1, k_2 \in K} \overline{\chi}(k_1) f(k_1^{-1} g k_2)\overline{\chi}(k_2^{-1}) \\
&= \sum_{k_1, k_2 \in K} \overline{\chi}(k_1)\overline{\chi}(k_1^{-1}) f(g)\overline{\chi}(k_2)\overline{\chi}(k_2^{-1}) \\
&= |K|^2 f(g).
\end{aligned}$$

□

Let $G = \coprod_{s \in S} K s K$ be the decomposition of G into double K-cosets, with respect a system of representatives S, with $1_G \in S$. Then the theorem just proved ensures that the values of $f \in \mathcal{H}$ on a double coset $K\overline{s}K$ is determined by the value of f on $\overline{s}$: $f(k_1 \overline{s} k_2) = \overline{\chi}(k_1) f(\overline{s}) \overline{\chi}(k_2)$. We should note that on certain double cosets, f must necessarily vanish, as shown in the next lemma.

LEMMA 4.17. *For $s \in S$ and $x \in sKs^{-1} \cap K$, we set $\chi_s(x) = \chi(s^{-1}xs)$. Let $T = \{s \in S : \chi_s(x) = \chi(x), \forall x \in sKs^{-1} \cap K\}$. Then a function $f \in \mathcal{H}$ is determined* *by its values on T by the relation:*

$$f(k_1 s k_2) = \begin{cases} \overline{\chi}(k_1) f(s) \overline{\chi}(k_2) & \text{if } s \in T \\ 0 & \text{if } s \in S \setminus T \end{cases}$$

for all $k_1, k_2 \in T$.

Proof. Let $s \in S$, $x = sks^{-1} \in sKs^{-1} \cap K$ and $f \in \mathcal{H}$. We have

$$\begin{aligned} f(s)\chi(x) &= f(s)\overline{\chi}(x^{-1}) = f(x^{-1}s) \\ &= f(sk^{-1}) = \overline{\chi}(k^{-1}) f(s) = \chi(k) f(s) \\ &= f(s)\chi_s(x). \end{aligned}$$

Therefore, considering $f \in \mathcal{H}$ such that $f(s) \neq 0$, we deduce that necessarily $\chi(x) = \chi_s(s)$ for all $x \in sKs^{-1} \cap K$ and thus $s \in T$. Vice versa, let $\{\alpha_s : s \in T\}$ be an arbitrary set of complex numbers and define f by setting:

$$f(k_1 s k_2) = \begin{cases} \overline{\chi}(k_1) \alpha_s \overline{\chi}(k_2) & \text{if } s \in T \\ 0 & \text{if } s \in S \setminus T. \end{cases}$$

Let us show that f is well defined, *i.e.* if $k_1 s k_2 = h_1 s h_2$ with $k_1 k_2, h_1, h_2 \in K$ and $s \in T$, then

$$\overline{\chi}(k_1)\overline{\chi}(k_2) f(s) = \overline{\chi}(h_1)\overline{\chi}(h_2) f(s). \tag{4.9}$$

Since $k_1 s k_2 = h_1 s h_2$ implies that $s k_2 h_2^{-1} s^{-1} = k_1^{-1} h_1 \in sKs^{-1} \cap K$ and therefore

$$\chi(k_1^{-1} h_1) = \chi_s(k_1^{-1} h_1) = \chi(s^{-1} k_1^{-1} h_1 s) = \chi(k_2 h_2^{-1})$$

which gives

$$\chi(h_1)\chi(h_2) = \chi(k_1)\chi(k_2)$$

which in turn immediately gives (4.9). Since it is obvious that the function f belongs to $\mathcal{H}$, the proof is complete. □

Definition 4.18. *The Curtis and Fossum basis* of $\mathcal{H}$ is given by the elements $\{a_s : s \in T\}$ defined by setting

$$a_s(g) = \begin{cases} \overline{\chi}(k_1)\overline{\chi}(k_2) \frac{1}{|K|} & \text{if } g = k_1 s k_2 \ (k_1, k_2 \in K) \\ 0 & \text{if } g \notin KsK. \end{cases} \tag{4.10}$$

Note that changing the double coset representatives will multiply each basis element by some root of 1 (in the case V is the trivial representation of K such a root is just 1). Note also that $a_{1_G} \equiv \psi$ and, more generally, $a_s(k_1 s k_2) = |K| \psi(k_1) \psi(k_2)$.

LEMMA 4.19. *For all $s \in T$ we have*

$$a_s = \frac{1}{|sKs^{-1} \cap K| \cdot |K|} \overline{\chi} * \delta_s * \overline{\chi}.$$

PROOF. First of all we observe that

$$[\overline{\chi} * \delta_s * \overline{\chi}](g) = \sum_{t,u \in G} \overline{\chi}(gt)\delta_s(t^{-1}u)\overline{\chi}(u^{-1}) \tag{4.11}$$

with the conditions that $gt \in K$, $t^{-1}u = s$ and $u \in K$. In particular,

$$g = gt \cdot t^{-1} = gt \cdot s \cdot u^{-1} \in KsK$$

(in other words, if $g \notin KsK$ the above convolution is 0). Let $g = k_1 s k_2$ with $k_1, k_2 \in K$. Then (4.11) becomes (setting $t = us^{-1}$)

$$\begin{aligned}
[\overline{\chi} * \delta_s * \overline{\chi}](k_1 s k_2) &= \sum_{u \in K} \overline{\chi}(k_1 s k_2 u s^{-1})\overline{\chi}(u^{-1}) \\
(x = sk_2us^{-1}) \quad &= \sum_{x \in sKs^{-1} \cap K} \overline{\chi}(k_1)\overline{\chi}(x)\overline{\chi}(s^{-1}x^{-1}sk_2) \\
(\chi(x) = \chi_s(x)) \quad &= \overline{\chi}(k_1)\overline{\chi}(k_2) \sum_{x \in sKs^{-1} \cap K} \overline{\chi}(x)\chi(x) \\
&= \overline{\chi}(k_1)\overline{\chi}(k_2)|sKs^{-1} \cap K|.
\end{aligned}$$

□

Clearly, there exist complex numbers μ_{rst}, $r, s, t \in T$, such that

$$a_r * a_s = \sum_{t \in T} \mu_{rst} a_t \tag{4.12}$$

for all $r, s \in T$. These numbers are called the *structure constants* of the Hecke algebra $\mathcal{H}$ relative to the basis $\{a_s : s \in T\}$.

LEMMA 4.20. *The structure constants μ_{rst} are given by the following formula:*

$$\mu_{rst} = |K| \sum_{g \in (KrK) \cap (tKs^{-1}K)} a_r(g) a_s(g^{-1}t).$$

PROOF. On the one hand, from (4.10) and (4.12) we have

$$a_r * a_s(t) = \frac{1}{|K|}\mu_{rst} \tag{4.13}$$

for all $r, s, t \in T$. On the other hand, just computing the convolution, we get:

$$\begin{aligned} a_r * a_s(t) &= \sum_{g\in G} a_r(g)a_s(g^{-1}t) \\ &= \sum_{g\in(KrK)\cap(tKs^{-1}K)} a_r(g) * a_s(g^{-1}t). \end{aligned} \tag{4.14}$$

Comparing (4.13) and (4.14), the lemma follows. □

Lemma 4.21. *For all $\sigma \in J$ and $s \in T$, we have*:

$$\chi^\sigma[\overline{\chi} * \delta_{s^{-1}} * \overline{\chi}] = \overline{\chi^\sigma[\overline{\chi} * \delta_s * \overline{\chi}]}.$$

Proof. Since

$$[\overline{\chi} * \delta_{s^{-1}} * \overline{\chi}](g) \sum_{k_1,k_2\in K} \overline{\chi}(k_1)\delta_{s^{-1}}(k_1^{-1}gk_2^{-1})\overline{\chi}(k_2)$$

and

$$\sum_{g\in G} \chi^\sigma(g)\delta_{s^{-1}}(k_1^{-1}gk_2^{-1}) = \chi^\sigma(k_1s^{-1}k_2)$$

we have

$$\begin{aligned} \chi^\sigma[\overline{\chi} * \delta_{s^{-1}} * \overline{\chi}] &= \sum_{k_1,k_2\in K} \overline{\chi}(k_1)\overline{\chi}(k_2)\chi^\sigma(k_1s^{-1}k_2) \\ &= |K| \sum_{k\in K} \overline{\chi}(k)\chi^\sigma(s^{-1}k) \\ \text{(replacing } k \text{ with } k^{-1}) \quad &= |K| \sum_{k\in K} \chi(k)\overline{\chi^\sigma}(ks). \end{aligned}$$

Similarly,

$$\chi^\sigma[\overline{\chi} * \delta_s * \overline{\chi}] = |K| \sum_{k\in K} \overline{\chi}(k)\chi^\sigma(ks)$$

and the lemma follows. □

In the following we will use this technical result.

Lemma 4.22. *For every $s \in S$, we have*:

$$|KsK| = \frac{|K|^2}{|K \cap sKs^{-1}|}.$$

Proof. First note that the map $k_1sk_2 \mapsto k_1sk_2s^{-1}$ is a bijection between KsK and $KsKs^{-1}$. Since sKs^{-1} is a group, then

$$|KsK| = |K \cdot sKs^{-1}| = \frac{|K||sKs^{-1}|}{|K \cap sKs^{-1}|} = \frac{|K|^2}{|K \cap sKs^{-1}|}.$$ □

REMARK 4.23. Note also that, for every $g \in KsK$, we have

$$|K \cap sKs^{-1}| = |\{(k_1, k_2) \in K \times K : g = k_1 s k_2\}|.$$

THEOREM 4.24.

1. *For each $\sigma \in J$, the central primitive idempotent of $\mathcal{H}$ associated with σ is*

$$\frac{d_\sigma}{|G|} \sum_{s \in T} |sKs^{-1} \cap K| \overline{\varphi_\sigma}(a_s) a_s. \tag{4.15}$$

2. *The irreducible characters φ_σ, $\sigma \in J$, satisfy the following orthogonality relations:*

$$\sum_{s \in T} |sKs^{-1} \cap K| \varphi_{\sigma_1}(a_s) \overline{\varphi_{\sigma_2}}(a_s) = \delta_{\sigma_1, \sigma_2} \frac{|G|}{d_{\sigma_1}} \varphi_{\sigma_1}(\psi).$$

3. *The dimension d_σ of σ is also given by*

$$d_\sigma = |G| m_\sigma \cdot \left[\sum_{s \in T} |sKs^{-1} \cap K| \cdot |\varphi_\sigma(a_s)|^2 \right]^{-1},$$

where m_σ is the multiplicity of $\sigma \in \operatorname{Ind}_K^G \chi$.

PROOF. From Corollary 4.14 and (4.7), we get the following expression for the central primitive idempotent associated with $\sigma \in J$:

$$\frac{d_\sigma}{|G|} \sum_{g \in G} \chi^\sigma(\psi * \delta_{g^{-1}} * \psi) \psi * \delta_g * \psi. \tag{4.16}$$

Since, for $k \in K$, $u \in G$,

$$\begin{aligned} \psi * \delta_k(u) &= \psi(uk^{-1}) \\ &= \frac{1}{|K|} \overline{\chi}(uk^{-1}) \\ &= \psi(u)\chi(k), \end{aligned}$$

if $g = k_1 s k_2$, $s \in S$, we have

$$\begin{aligned} \chi^\sigma(\psi * \delta_{g^{-1}} * \psi) \psi * \delta_g * \psi &= \chi(k_1^{-1}) \chi(k_1) \chi(k_2^{-1}) \chi(k_2) \chi^\sigma(\psi * \delta_{s^{-1}} * \psi) \psi * \delta_s * \psi \\ &= \begin{cases} \overline{\varphi_\sigma}(a_s) a_s \dfrac{|sKs^{-1} \cap K|^2}{|K|^2} & \text{if } s \in T \\ 0 & \text{if } s \in S \setminus T \end{cases} \end{aligned} \tag{4.17}$$

where the last equality follows from Lemma 4.19 (and (4.11)) and Lemma 4.21.

From (4.16) and (4.17), we get the following expression for the central primitive idempotent associated with σ:

$$\frac{d_\sigma}{|G|}\sum_{s\in T}|KsK|\frac{|sKs^{-1}\cap K|^2}{|K|^2}\overline{\varphi_\sigma}(a_s)a_s = \frac{d_\sigma}{|G|}\sum_{s\in T}|sKs^{-1}\cap K|\overline{\varphi_\sigma}(a_s)a_s,$$

where the last equality follows from Lemma 4.22. This ends the proof of (1).

(2) On the one hand, from Corollary 4.14 and (1) we have (for $\sigma_1,\sigma_2\in J$):

$$\varphi_{\sigma_1}(\psi * \overline{\chi^{\sigma_2}}) = \sum_{s\in T}|sKs^{-1}\cap K|\overline{\varphi_{\sigma_2}}(a_s)\varphi_{\sigma_1}(a_s). \tag{4.18}$$

On the other hand

$$\begin{aligned}\varphi_{\sigma_1}(\psi * \overline{\chi^{\sigma_2}}) &= \chi^{\sigma_1}(\psi * \overline{\chi^{\sigma_2}})\\ &= \sum_{h\in G}\chi^{\sigma_1}(h)\sum_{g\in G}\psi(g)\chi^{\sigma_2}(h^{-1}g)\\ &= \sum_{g\in G}(\chi^{\sigma_1} * \chi^{\sigma_2})(g)\psi(g)\\ &= \frac{|G|}{d_{\sigma_1}}\delta_{\sigma_1,\sigma_2}\sum_{g\in G}\chi^{\sigma_1}(g)\psi(g)\\ &= \frac{|G|}{d_{\sigma_1}}\delta_{\sigma_1,\sigma_2}\chi^{\sigma_1}(\psi).\end{aligned} \tag{4.19}$$

Comparing (4.18) and (4.19) (2) follows immediately.

(3) It follows immediately from (2) and Corollary 4.3. □

4.2 – The Hecke algebra associated with the trivial character

In this section we study the Hecke algebra associated to a permutation representation, that is we consider the case $\chi = \iota_K$, where ι_K is the trivial character of K. We summarize the analysis of the previous section applied to this particular case in the following theorem. Note that now $T \equiv S$ and that, by virtue of Theorem 4.16, $\mathcal{H}$ coincides with the algebra of bi-K-invariant functions.

Theorem 4.25. *With the notation of the previous theorem suppose that* $\chi = \iota_K$. *Then*:

1. *The Curtis-Fossum basis for* $\mathcal{H}$ *is simply given by*

$$\left\{a_s = \frac{1}{|K|}\mathbf{1}_{KsK} : s\in S\right\};$$

2. *The structure constants are given by*:

$$\mu_{rst} = \frac{1}{|K|}|(KrK) \cap (tKs^{-1}K)|;$$

3. *Recalling that* $\mathcal{C}(g)$ *is the* G*-conjugacy class containing* g*, the character formula in Theorem* 4.15 *now becomes*:

$$\begin{aligned}\chi^\sigma(g) = \frac{|G|}{|\mathcal{C}(g)| \cdot |K|} &\left[\sum_{s \in S} |sKs^{-1} \cap K| \varphi_\sigma(a_s) |\mathcal{C}(g) \cap KsK|\right] \\ &\cdot \left[\sum_{s \in S} |sKs^{-1} \cap K| |\varphi_\sigma(a_s)|^2\right]^{-1}.\end{aligned} \tag{4.20}$$

Proof. (1) and (2) are left to the reader.
(3) Since $\psi = \frac{1}{|K|}\mathbf{1}_K$, if $u \in KsK$ we find that

$$\psi * \delta_u * \psi = \psi * \delta_s * \psi = \frac{|sKs^{-1} \cap K|}{|K|} a_s,$$

because, for $g \in KsK$ we have (taking into account Remark 4.23)

$$[\psi * \delta_s * \psi](g) = \frac{1}{|K|^2} \sum_{k_1, k_2 \in K} \delta_s(k_1^{-1} g k_2^{-1}) = \frac{|sks^{-1} \cap K|}{|K|^2}.$$

Therefore,

$$\chi^\sigma(\psi * \mathbf{1}_{\mathcal{C}}(g) * \psi) = \sum_{s \in S} \frac{|sKs^{-1} \cap K|}{|K|} |\mathcal{C}(g) \cap KsK| \varphi_\sigma(a_s) \tag{4.21}$$

and

$$\begin{aligned}\sum_{h \in G} \chi^\sigma(\psi * \delta_{h^{-1}} * \psi) \chi^\sigma(\psi * \delta_h * \psi) &= \sum_{s \in S} |KsK| \frac{|sKs^{-1} \cap K|^2}{|K|^2} |\varphi_\sigma(a_s)|^2 \\ &= \sum_{s \in S} |sKs^{-1} \cap K| \cdot |\varphi_\sigma(a_s)|^2\end{aligned} \tag{4.22}$$

where the first equality follows from Lemma 4.21 and the second equality from Lemma 4.22. By applying (4.21) and (4.22) to the formula in Theorem 4.15 the proof is complete. □

4.3 – Relation with Gelfand pairs

We recall that (G, K) is a Gelfand pair when $\operatorname{Ind}_K^G \iota_K$ decomposes without multiplicity; equivalently when the Hecke algebra $\mathcal{H}(G, K, \frac{1}{|K|}\iota_K)$ (which is isomorphic to the algebra of bi-K-invariant functions) is commutative. If this is the case, let (σ, V_σ) be an irreducible G-representation contained in $\operatorname{Ind}_K^G \iota_k$ and let $u_\sigma \in V_\sigma$ be a K-invariant vector with $\|u_\sigma\| = 1$. Then the spherical function associated with σ is the bi-K-invariant function defined by

$$\phi_\sigma(g) = \langle u_\sigma, \sigma(g)u_\sigma\rangle_{V_\sigma}.$$

We recall that, by Frobenious reciprocity, u_σ is unique up to a constant $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Moreover, $\{\phi_\sigma, \sigma \in J\}$, where $J \subseteq \widehat{G}$ are the representations contained in $\operatorname{Ind}_K^G \iota_K$, is a vector space basis of $\mathcal{H}$. In [1], Exercise 9.5.8 the following formulas relating ϕ_σ and χ^σ are presented:

$$\phi_\sigma(g) = \frac{1}{|K|}\sum_{k\in K} \overline{\chi^\sigma}(gk), \quad \chi^\sigma(g) = \frac{d_\sigma}{|G|}\sum_{h\in G}\overline{\phi_\sigma}(h^{-1}gh), \tag{4.23}$$

for all $g \in G$. We want to show that these are particular cases of (4.15) and (4.20).

To show the equivalence, decompose the double coset KsK into disjoint union of left cosets: $KsK = k_1 sK \coprod k_2 sK \coprod \cdots\ k_n sK$, with $k_1 = 1_G$. By Lemma 4.22 we have that $n = \frac{|K|}{|K\cap sKs^{-1}|}$. Using the G-conjugacy invariance of the character χ^σ we have that

$$\sum_{g\in k_i sK} \chi^\sigma(g) = \sum_{k\in K}\chi^\sigma(k_i sk) = \sum_{k\in K}\chi^\sigma(sk).$$

Therefore the central primitive idempotents of Theorem 4.24 evaluated at KsK equals

$$\begin{aligned}
\frac{d_\sigma}{|G|}\frac{|K\cap sKs^{-1}|}{|K|}\overline{\varphi_\sigma}(a_s) &= \frac{d_\sigma}{|G|}\frac{|K\cap sKs^{-1}|}{|K|^2}\overline{\varphi_\sigma}(\mathbf{1}_{KsK})\\
&= \frac{d_\sigma}{|G|}\frac{|K\cap sKs^{-1}|}{|K|^2}\sum_{g\in KsK}\overline{\chi^\sigma}(g)\\
&= \frac{d_\sigma}{|G|}\frac{|K\cap sKs^{-1}|}{|K|^2}\sum_{g\in\coprod_{i=1}^n k_i sK}\overline{\chi^\sigma}(g)\\
&= \frac{d_\sigma}{|G|}\frac{|K\cap sKs^{-1}|}{|K|^2}\sum_{i=1}^n\sum_{k\in K}\overline{\chi^\sigma}(k_i sk)\\
&= \frac{d_\sigma}{|G|}\frac{1}{|K|}\sum_{k\in K}\overline{\chi^\sigma}(sk)
\end{aligned}$$

which agrees with the formula of the spherical function ϕ_σ in (4.23).

Indeed, the central idempotent is equal to $\frac{d_\sigma}{|G|}\phi_\sigma$, because $\phi_\sigma * \phi_\sigma = \frac{|G|}{d_\sigma}$ (see [2, Lemma 1.5.7]). In particular,

$$\phi_\sigma(h) = \frac{|K \cap sKs^{-1}|}{|K|}\overline{\varphi}_\sigma(a_s) \tag{4.24}$$

for all $h \in KsK, s \in S$ and $\sigma \in J$.

Let now show that (4.20) reduces, in the setting of Gelfand pairs, to the second formula in (4.23). We first observe that

$$\phi_\sigma(\mathbf{1}_K) = \sum_{k \in K} \chi^\sigma(k) = |K|$$

because the multiplicity of ι_K in $\mathrm{Res}^G_K \sigma$ is equal to 1, and thus by Theorem 4.24

$$\sum_{s \in S} |sKs^{-1} \cap K||\varphi_\sigma(a_s)|^2 = \frac{|G|}{d_\sigma}.$$

Therefore

$$\begin{aligned}
\chi^\sigma(g) &= \frac{d_\sigma}{|\mathcal{C}(g)|}\left[\sum_{s \in S} \frac{|K \cap sKs^{-1}|}{|K|}\varphi_\sigma(a_s) \cdot |\mathcal{C}(g) \cap KsK|\right] \\
(\text{ by } (4.24)) &= \frac{d_\sigma}{|\mathcal{C}(g)|}\sum_{s \in S} \overline{\phi_\sigma}(s)|\mathcal{C}(g) \cap KsK| \\
&= \frac{d_\sigma}{|\mathcal{C}(g)|}\sum_{s \in S}\sum_{h \in \mathcal{C}(g) \cap KsK} \overline{\phi_\sigma}(h) \\
&= \frac{d_\sigma}{|\mathcal{C}(g)|}\sum_{h \in \mathcal{C}(g)} \overline{\phi_\sigma}(h) \\
&= \frac{d_\sigma}{|G|}\sum_{h \in G} \overline{\phi_\sigma}(hgh^{-1}).
\end{aligned}$$

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Indirizzi degli Autori:
Fabio Scarabotti – Dipartimento SBAI – Università degli Studi di Roma "La Sapienza" – via A. Scarpa 8 – 00161 Roma – Email address: scarabot@dmmm.uniroma1.it

Filippo Tolli – Dipartimento di Matematica – Università Roma TRE – L. San Leonardo Murialdo 1 – 00146 Roma – Italy – Email address: tolli@mat.uniroma3.it