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On the analogy between L-functions and Atiyah-Bott-Lefschetz trace formulas for foliated spaces

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Tribute to Gianni Rivera for his 70th birthday

Abstract: *Christopher Deninger has developed an infinite dimensional cohomological formalism which would allow to prove the expected properties of the motivic L-functions (including the Dirichlet L-functions). These cohomologies are (in general) not yet constructed. Deninger has argued that they might be constructed as leafwise cohomologies associated to ramified leafwise flat vector bundles on suitable foliated spaces. In the case of number fields we propose a set of axioms allowing to make this more precise and to motivate new theorems. We also check the coherency of these axioms and from them we derive "formally" an Atiyah-Bott-Lefschetz trace formula which would imply Artin conjecture for a Galois extension of* $\mathbb{Q}$.

1 – Introduction

Christopher Deninger's approach to the study of arithmetic zeta and motivic L-functions proceeds in two steps (see for instance [8, 9]).

In the **first step**, he postulates the existence of infinite dimensional cohomology groups satisfying some "natural properties". From these data, he has elaborated a formalism which would allow him to prove the expected properties for the arithmetic zeta functions: functional equation, conjectures of Artin, Beilinson, Riemann...etc. There it is crucial to interpret the so called explicit formulae for the zeta and motivic L-functions as Lefschetz trace formulae.

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The **second step** consists in constructing these cohomologies. Deninger has given some hope that for zeta functions these cohomologies might be constructed as leafwise cohomologies of suitable foliated spaces. Moreover, in the case of motivic L-functions, one should consider flat (ramified) vector bundles on the corresponding foliated space, see Deninger [11]. Very little is known in this direction at the moment, but this second step seems to be a good motivation to develop interesting mathematics even if they remain far from the ultimate goal.

In Section 2 we recall Deninger's cohomological formalism in the case of the Riemann zeta and (primitive) Dirichlet L-functions $\Lambda(\chi, s)$. We point out a dissymmetry in the explicit formulae (1) and (4) between the coefficients of $\delta_{k \log p}$ and $\delta_{-k \log p}$, see Comment 1.

In Section 3 is devoted to the description of the Lefschetz trace formula for a flow acting on a codimension one foliated space. In Section 3.1 we recall the Guillemin-Sternberg trace formula which is indeed an important computational tool for this goal.

In Section 3.2 we recall the theorem of Alvarez-Lopez and Kordyukov.

They consider a flow $(\phi^t)_{t\in\mathbb{R}}$ acting on $(X, \mathcal{F})$ where the compact three dimensonal manifold X is foliated by Riemann surfaces. They assume that $(\phi^t)_{t\in\mathbb{R}}$ preserves globally the foliation and is transverse to the foliation. Then Alvarez-Lopez and Kordyukov define a suitable leafwise Hodge cohomology on which ϕ^t acts and they prove an Atiyah-Bott-Lefschetz trace formula (Theorem 3.2) which has some similarities with (1) for t real positive. The dissymmetry mentioned above for (1) does not hold here. Nevertheless, by comparison with (1), it suggests that there should exist a flow $(\phi^t)_{t\in\mathbb{R}}$ acting on a certain space $S_{\mathbb{Q}}$ with the following property. To each prime number p [resp. the archimedean place of $\mathbb{Q}$] there should correspond a closed orbit with length $\log p$ [resp. a stationary point] of the flow ϕ^t. But since the dissymmetry mentioned above for (1) does not hold in Theorem 3.2, the space $S_{\mathbb{Q}}$ cannot be a foliated manifold but rather a so called laminated foliated space. Its transverse structure might involve the $p-$adic integers or even the adeles (see [16] and [24] for a simple example), but we shall not dwell here on this important point.

In Section 3.3 we introduce the concept of a ramified flat line bundle $\mathcal{L}_\rho \to X$ around a finite number of closed orbits, and define the associated leafwise Hodge cohomology groups $H^j_\tau(X; \mathcal{L}_\rho)$, $0 \leq j \leq 2$. We then prove a Lefschetz trace formula where the ramified closed orbits of (X, ϕ^t) do not appear: see Theorem 3.4. This trace formula has some similarities with the explicit formula (4) for Dirichlet L-functions, the ramified closed orbits corresponding to the (ramified) prime numbers which divide the conductor of the Dirichlet character.

In Section 4, we consider more generally the Dedekind zeta function $\widehat{\zeta}_K(s)$ of a number field K and recall the associated explicit formula (12). Then, making a synthesis of Deninger's work, we state several assumptions for a laminated foliated space $(S_K, \mathcal{F}, g, \phi^t)$ which (if satisfied) would allow to construct the required (leafwise) cohomology groups for the Dedekind zeta function. In particular, the explicit formula (12) should be interpreted as a (leafwise) Atiyah-Bott-Lefschetz trace formula. We compare carefully the contributions of the archimedean places of K in (12) with the contribution of a stationary point in the Guillemin-Sternberg formula: we explain an apparent incompatibility in the case of real places. Next, we come back to the case of a primitive Dirichlet character χ mod m and consider the cyclotomic field $K = \mathbb{Q}[e^{\frac{2i\pi}{m}}]$ associated to χ with Galois group $G = (\mathbb{Z}/m\mathbb{Z})^*$. So χ defines a group homomorphism $\chi : G \to S^1$. We consider the ramified flat complex line bundle over $S_{\mathbb{Q}}$:

$$\mathcal{L}_\chi = \frac{S_K \times \mathbb{C}}{G} \to S_{\mathbb{Q}}\,,$$

and define leafwise cohomology groups $\overline{H}^j(\mathcal{L}_\chi)$, $0 \leq j \leq 2$.

Then, imitating the proof of our Theorem 3.4 and using the assumptions of Section 4, we interpret "formally" the explicit formula (4) for the Dirichlet L-function $\Lambda(\chi, s)$ as a leafwise Lefschetz trace formula for the vectors spaces $\overline{H}^j(\mathcal{L}_\chi)$, $0 \leq j \leq 2$: see Theorem 4.5. We insist on the fact that it is not known whether or not the assumptions of Section 4 are satisfied.

In Section 5, we assume that K is a (finite) Galois extension of $\mathbb{Q}$ with Galois group G. We review the definition and properties of the Artin L-function $\Lambda(K, \chi, s)$ associated to an irreducible representation $\rho : G \to GL_{\mathbb{C}}(V)$. We recall the standard explicit formula (29) for $\Lambda(K, \chi, s)$: its spectral side involves the zeroes (with sign $-$) and the poles (with sign $+$). These poles are not controlled because the Γ-factor introduced in the definition (27) of $\Lambda(K, \chi, s)$ is not associated to a mathematical structure, this Γ-factor seems to come as a parachute. This situation is in sharp contrast with the case of the Dirichlet L-functions recalled in Section 2: there the Γ-factor is introduced in the Dirichlet L-function in order to express it as the Mellin transform of a suitable theta function.

Next we consider the ramified flat vector bundle over $S_{\mathbb{Q}}$:

$$\mathcal{E}_\rho = \frac{S_K \times V}{G} \to S_{\mathbb{Q}}\,,$$

and define leafwise cohomology groups $\overline{H}^j(\mathcal{E}_\rho)$, $0 \leq j \leq 2$.

Then, imitating the proof of our Theorem 3.4 and using the axioms of Section 4, we "prove formally" a leafwise Atiyah-Bott-Lefschetz trace formula for the vectors spaces $\overline{H}^j(\mathcal{E}_\rho)$ ($0 \leq j \leq 2$) which provides an explicit formula (31). In this formula, the Γ-factor of $\Lambda(K, \chi, s)$ *appears naturally* in the computation of the contribution of the fixed points. In the spectral side of (31), the numbers only appear with a sign $-$. Since the geometric sides of (31) and (29) coincide, one then would get formally that the Artin L-function has no poles!!

Thus, the $\overline{H}^j(\mathcal{E}_\rho)$ $(0 \leq j \leq 2)$ seem "to provide" a construction of the cohomology groups that Deninger's cohomological formalism attributes to $\Lambda(K, \chi, s)$. In [9, Section 3], working in the general cohomological formalism that he elaborated and using regularized determinants, Deninger has reduced the validity of Artin conjecture for simple motives to the vanishing of H^0 and H^2. Our approach via the trace formula is a bit different and possibly simpler in the sense that it seems to need less foundational results.

It would be interesting to confront certain ideas from automorphic theory (see *e.g.* [19, 22, 31]) with the axioms of Sections 4.3. Indeed, one of the goals of Langlands programme is to identify $\Lambda(K, \chi, s)$ with the L-function L_π of an automorphic cuspidal representation π. In the automorphic world, the Γ-factor of L_π *appears naturally* in a mathematical structure. Therefore, one may ask if the axioms of Section 4.3 and the data of $\rho : G \to GL_{\mathbb{C}}(V)$ could allow to construct "formally" the desired automorphic cuspidal representation π associated to $\Lambda(K, \chi, s)$.

Ralf Meyer [26] has provided a nice and new spectral interpretation of the explicit formula (12) (actually Meyer considers all Hecke L-functions at the same time). Unfortunately, the action of G on Meyer's cohomology groups is trivial. The fact that this action is not trivial in our (conjectural) setting is guaranted by the axioms of Section 4.3.

In another direction, it should also be interesting to confront the properties of the hypothetic foliated space $(S_K, \mathcal{F}, g, \phi^t)$ with ideas from Topos theory. For instance see Morin ([27] and [18]), Caramello [5], written talks by Laurent Lafforgue [21] and, Connes's lecture 2013 in Collège de France. In any case, we hope that this paper will be useful to somebody else.

2 – Deninger's Cohomological formalism in the case of the Dirichlet L- functions

2.1 – Dirichlet L- functions $\Lambda(\chi, s)$

The (completed) Riemann zeta function is given by:

$$\widehat{\zeta}(s) = \pi^{-s/2}\Gamma(s/2) \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}$$

where $\mathbb{P} = \{2, 3, 5, \ldots\}$ denotes the usual set of prime numbers. The following well known explicit formulas express a connection between $\mathbb{P} \cup \{\infty\}$ and the zeroes of $\widehat{\zeta}$. Let $\alpha \in C^\infty_{\text{compact}}(\mathbb{R}, \mathbb{R})$ and for $s \in \mathbb{R}$, set $\Phi(s) = \int_{\mathbb{R}} e^{st}\alpha(t)dt$; Φ belongs to the

Schwartz class $\mathcal{S}(\mathbb{R})$. Then one can prove the following formula:

$$\begin{aligned}\Phi(0) - \sum_{\rho\in\widehat{\zeta}^{-1}\{0\},\, \Re\rho\geq 0} \Phi(\rho) + \Phi(1) \\ = \sum_{p\in\mathbb{P}} \log p \left(\sum_{k\geq 1} \alpha(k\log p) \ + \sum_{k\leq -1} p^k \alpha(k\log p)\right) + W_\infty(\alpha),\end{aligned} \tag{1}$$

where

$$W_\infty(\alpha) = \alpha(0)\log\pi + \int_0^{+\infty} \left(\frac{\alpha(t) + e^{-t}\alpha(-t)}{1-e^{-2t}} - \alpha(0)\frac{e^{-2t}}{t}\right) dt.$$

Let $m \in \mathbb{N}\cap[3,+\infty[$, a Dirichlet character χ mod m is a group homomorphism

$$\chi : (\mathbb{Z}/m\mathbb{Z})^* \to S^1\,.$$

Such a character is called primitive if there exists no non trivial divisor m' of m such that $\chi = \chi'\circ\pi$ where χ' is a Dirichlet character mod m' and $\pi : (\mathbb{Z}/m\mathbb{Z})^* \to (\mathbb{Z}/m'\mathbb{Z})^*$ denotes the projection. The great commun divisor f of all such divisors m' is called the conductor of χ, one can check that χ is induced by a primitive Dirichlet character mod f.

A Dirichlet character χ mod m induces a multiplicative map, still denoted χ, from $\mathbb{Z}$ to $S^1\cup\{0\}$ by the rules:

$$\forall n\in\mathbb{Z},\ \chi(n) = \chi(n+m\mathbb{Z}) \text{ if } n\wedge m = 1,\ \chi(n) = 0 \text{ if } n\wedge m \neq 1\,,\ \chi(0) = 0\,.$$

A Dirichlet character χ mod m is primitive if and only if for any non trivial divisor m' of m,

$$\exists a\in\mathbb{Z},\ a\wedge m = 1,\ a = 1 \bmod m',\ \chi(a)\neq 1\,. \tag{2}$$

Consider a (non trivial) primitive Dirichlet character χ mod m, define $q\in\{0,1\}$ by $\chi(-1) = (-1)^q$. Then the following function, first defined on the half plane $\Re s > 1$, extends as an entire holomorphic function on $\mathbb{C}$:

$$\Lambda(\chi, s) = \left(\frac{m}{\pi}\right)^{s/2} \Gamma\left(\frac{s+q}{2}\right) \prod_{p\in\mathbb{P},\, p\wedge m = 1} \frac{1}{1-p^{-s}}\,.$$

It satisfies the functional equation:

$$\forall s\in\mathbb{C},\ \Lambda(\chi,s) = \frac{\sum\limits_{a=0}^{m-1} \chi(a) e^{\frac{2i\pi a}{m}}}{i^q\sqrt{m}} \Lambda(\overline{\chi}, 1-s)\,. \tag{3}$$

The proof uses first the fact that $\Lambda(\chi, s)$ is the Mellin transform at $\frac{s+q}{2}$ of the theta function

$$\theta(\chi, y) = \frac{1}{2}\left(\frac{\pi}{m}\right)^{q/2} \sum_{n\in\mathbb{Z}} \chi(n) n^q e^{\frac{-n^2\pi y}{m}}$$

and then a certain relation between $\theta(\chi, 1/y)$ and $\theta(\overline{\chi}, y)$ which is established with the help of the Gauss sums $\sum_{a=0}^{m-1} \chi(a) e^{\frac{2i\pi a n}{m}}$ $(n \in \mathbb{Z})$.

Let $\alpha \in C^\infty_{\text{compact}}(\mathbb{R}, \mathbb{R})$ such that $\alpha(0) = 0$ and for $s \in \mathbb{R}$, set $\Phi(s) = \int_{\mathbb{R}} e^{st}\alpha(t)dt$. One then has:

$$\begin{aligned} - \sum_{\rho\in\Lambda(\chi,\cdot)^{-1}\{0\}} \Phi(\rho) = \sum_{p\in\mathbb{P},\, p\wedge m=1} \log p \sum_{n\geq 1} \big(\chi(p)^n \alpha(n\log p) + p^{-n}\chi(p)^{-n}\alpha(-n\log p)\big) \\ + \int_0^{+\infty} \frac{\alpha(x)e^{-qx} + \alpha(-x)e^{-x(1+q)}}{1-e^{-2x}}\, dx\,. \end{aligned} \tag{4}$$

By comparison with (1), "$\Phi(0) + \Phi(1)$ has disappeared", which means that $\Lambda(\chi, \cdot)$ has no poles.

The idea of the proof of (4) is the following: apply the residue theorem to the integral of the function

$$s \mapsto \left(\int_0^{+\infty} \alpha(\log t)\, t^s \frac{dt}{t}\right) \frac{\Lambda'(\chi, s)}{\Lambda(\chi, s)}$$

along the boundary of the rectangle of $\mathbb{C}$ defined by the four points:

$$1+\epsilon+iT,\ -\epsilon+iT,\ -\epsilon-iT,\ 1+\epsilon-iT,$$

then use the functional equation (3) and the formula:

$$\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) = \int_0^{+\infty} \left(\frac{e^{-u}}{u} - \frac{e^{-u\frac{s}{2}}}{1-e^{-u}}\right) du,$$

lastly let T goes to $+\infty$.

2.2 – Deninger's cohomological formalism

Deninger's philosophy is motivated by the fact that the left hand side of (1)

$$\Phi(0) - \sum_{\rho\in\widehat{\zeta}^{-1}\{0\},\, \Re\rho\geq 0} \Phi(\rho) + \Phi(1)$$

is reminiscent of a Lefschetz trace formula of the form

$$\mathrm{TR} \int_{\mathbb{R}} \alpha(t) e^{t\,\Theta_0}\, dt - \mathrm{TR} \int_{\mathbb{R}} \alpha(t) e^{t\,\Theta_1}\, dt + \mathrm{TR} \int_{\mathbb{R}} \alpha(t) e^{t\,\Theta_2}\, dt,$$

where the following two assumptions should be satisfied.

- $\Theta_0 = 0$ acts on $H^0 = \mathbb{R}$, $\Theta_2 = \mathrm{Id}$ acts on $H^2 = \mathbb{R}$.
- The closed unbounded operator, Θ_1 acts on an infinite dimensional real vector (pre-Hilbert) space H^1 and has discrete spectrum. For any $\alpha \in C^\infty_{\mathrm{compact}}(\mathbb{R}, \mathbb{R})$ the operator $\int_{\mathbb{R}} \alpha(t) e^{t\,\Theta_1}\, dt$ is trace class. The eigenvalues of $\Theta_1 \otimes Id_{\mathbb{C}}$ acting on $H^1 \otimes_{\mathbb{R}} \mathbb{C}$ coincide with the non trivial zeroes of $\widehat{\zeta}$.

For each primitive (non trivial) Dirichlet character χ, there should exist an infinite dimensional complex (pre-Hilbert) space H^1_χ endowed with a closed unbounded operator $\Theta_{1,\chi} : H^1_\chi \to H^1_\chi$ such that for any $\alpha \in C^\infty_{\mathrm{compact}}(\mathbb{R}, \mathbb{R})$ the operator $\int_{\mathbb{R}} \alpha(t) e^{t\,\Theta_{1,\chi}}\, dt$ is trace class. Moreover, the eigenvalues of $\Theta_{1,\chi}$ coincide with the non trivial zeroes of L_χ.

In Deninger's approach one first assumes the existence of a Poincaré duality pairing:

$$H^1_\chi \times H^1_{\overline{\chi}} \to H^2$$
$$(\alpha, \beta) \to \alpha \cup \beta$$

satisfying

$$\forall (\alpha, \beta) \in H^1_\chi \times H^1_{\overline{\chi}},\ e^{t\,\Theta_{1,\chi}} \alpha \cup e^{t\,\Theta_{1,\overline{\chi}}} \beta = e^t (\alpha \cup \beta), \tag{5}$$

where the e^t is dictated by the fact that $\Theta_2 = Id$ on H^2.

In order to (re)prove the functional equation (3), one combines the property (5) and the following identity (which Deninger assumes to hold true):

$$\Lambda(\chi, s) = C\, \frac{\det_\infty \left(\dfrac{s - \theta_{1,\chi}}{2\pi} : H^1_\chi \right)}{\det_\infty \left(\dfrac{s - \theta_{1,\chi}}{2\pi} : H^0_\chi \right) \det_\infty \left(\dfrac{s - \theta_{1,\chi}}{2\pi} : H^2_\chi \right)},$$

where C is a constant depending on a choice of conventions. (Actually in this particular case, the denominator is identically equal to 1).

Second, one assumes the existence of an anti-linear Hodge star $\star$ operator:

$$\star : H^1_\chi \to H^1_{\overline{\chi}},\ \star : H^1_{\overline{\chi}} \to H^1_\chi$$

such that $\star^2 = Id$ and $\star e^{t\,\Theta_{1,\chi}} = e^{t\,\Theta_{1,\overline{\chi}}} \star$ and $\langle \alpha; \beta \rangle = \alpha \cup \star \beta$ defines a scalar product (anti-linear on the right) on the vector space H^1_χ.

These data imply easily the following:

$$\forall \alpha, \alpha' \in H^1_\chi,\ \langle e^{t\,\Theta_{1,\chi}} \alpha; e^{t\,\Theta_{1,\chi}} \alpha' \rangle = e^t \langle \alpha; \alpha' \rangle. \tag{6}$$

Therefore,

$$\frac{d}{dt} \langle e^{t\,\Theta_{1,\chi}} \alpha; e^{t\,\Theta_{1,\chi}} \alpha \rangle_{t=0} = \langle \Theta_{1,\chi}(\alpha); \alpha \rangle + \langle \alpha; \Theta_{1,\chi}(\alpha) \rangle = \langle \alpha; \alpha \rangle,$$

and
$$\langle(\Theta_{1,\chi}-1/2)(\alpha);\alpha\rangle+\langle\alpha;(\Theta_{1,\chi}-1/2)(\alpha)\rangle=0.$$
Therefore, the eigenvalues s of $\Theta_{1,\chi}$ (which coincide by (4) with the non trivial zeroes of $\Lambda(\chi,s)$) satisfy $s-1/2+\overline{s}-1/2=0$ or equivalently: $\Re s=\frac{1}{2}$. Therefore Deninger's formalism should imply the Riemann hypothesis for $\Lambda(\chi,s)$! This argument comes from an idea of Serre [30] and has been formalized in the foliation case in [13]. Of course, we have described only a very small part of Deninger's formalism which deals also with L-functions of motives, Artin conjecture, Beilinson conjectures....etc.

Comment 1. There is a dissymmetry in (1) and in (4) between the coefficients of $\alpha(k\log p)$ and $\alpha(-k\log p)$ for $k\in\mathbb{N}^*$. In the framework of Deninger's formalism the explanation is the following. Equation (5) implies "formally" that the transpose of $e^{t\Theta_{1,\chi}}$ is $e^te^{-t\Theta_{1,\overline{\chi}}}$. Therefore, if we have a Lefschetz cohomological interpretation of (1) in Deninger's formalism for a test function α with support in $]0,+\infty[$ then we have also a cohomological proof of (1) for α with support in $]-\infty,0[$. In this formalism, (5) (and the above dissymmetry) is quite connected to the Riemann hypothesis.

Recall that Alain Connes [7] has reduced the validity of the Riemann hypothesis (for the L-functions of the Hecke characters) to a trace formula.

3 – Analogy with the foliation case

3.1 – The Guillemin-Sternberg trace formula

Consider a smooth compact manifold X with a smooth action:
$$\phi:X\times\mathbb{R}\to X,\ (x,t)\to\phi^t(x),$$
so that $\phi^{t+s}=\phi^t\circ\phi^s$ for any $t,s\in\mathbb{R}$. Let $D_y\phi^t$ denote (for fixed $t\in\mathbb{R}$) the differential of the map : $y\in X\to\phi^t(y)$. One has: $\partial_s\phi^{t+s}_{|s=0}(y)=D_y\phi^t(\partial_s\phi^s_{|s=0}(y))$.

Consider also a smooth vector bundle $E\to X$. Assume that E is endowed with a smooth family of maps
$$\psi^t:(\phi^t)^*E\to E,\,t\in\mathbb{R},$$
satisfying the following cocycle condition:
$$\forall u\in C^\infty(X;E),\,\forall t,s\in\mathbb{R},\ \psi^s(\psi^t(u\circ\phi^t)\circ\phi^s)=\psi^{t+s}(u\circ\phi^{t+s}).$$
In other words, we require that the maps $K^t:u\to\psi^t(u\circ\phi^t)=K^t(u)$ define an action of the additive group $\mathbb{R}$ on $C^\infty(X;E)$. Notice that in the case of $E=\wedge^*T^*X$ and $\psi^s={}^tD\phi^s$ (the transpose of the differential $D\phi^s$ of ϕ^s), this condition is satisfied.

We shall assume that the graph of ϕ (*i.e.* $\{(x, \phi^t(x), t)\}$) meets transversally the "diagonal" $\{(x, x, t),\ x \in X,\ t \in \mathbb{R}\}$. Guillemin-Sternberg have checked ([20]) that the trace $Tr(K^t | C^\infty(X; E))$ is defined as a distribution of $t \in \mathbb{R} \setminus \{0\}$ by the formula:

$$Tr(K^t | C^\infty(X; E)) = \int_X K^t(x, x)$$

where $K^t(x, y)$ denote Schwartz (density) kernel of K^t. We warn the reader that, in general, for $\alpha \in C^\infty_{\text{compact}}(\mathbb{R}) \setminus \{0\}$, the operator $\int_{\mathbb{R}} \alpha(t) K^t dt$ is not trace class.

Now, we give the name $T^0_x = \partial_t \phi^t(x)_{t=0}\, \mathbb{R}$ to the real line generated by the vector field $\partial_t \phi^t(x)_{t=0}$ of ϕ^t at a point x where $\partial_t \phi^t(x)_{t=0} \neq 0$.

Proposition 3.1 ([20], Guillemin-Sternberg). *The following formula holds in* $\mathcal{D}'(\mathbb{R} \setminus \{0\})$.

$$Tr(K^t \,|\, C^\infty(X; E)) = \sum_\gamma l(\gamma) \sum_{k \in \mathbb{Z}^*} \frac{\mathrm{Tr}(\psi^{kl(\gamma)}_{x_\gamma}\,;\, E_{x_\gamma})}{|\det(1 - D_y \phi^{kl(\gamma)}(x_\gamma)\,;\, T_{x_\gamma}X / T^0_{x_\gamma})\,|} \delta_{kl(\gamma)} + \sum_x \frac{\mathrm{Tr}(\psi^t_x\,;\, E_x)}{|\det(1 - D_y \phi^t(x)\,;\, T_x X)|}.$$

In the first sum, γ runs over the periodic primitive orbits of ϕ^t, x_γ denotes any point of γ, $l(\gamma)$ is the length of γ, $\phi^{l(\gamma)}(x_\gamma) = x_\gamma$. In the second sum, x runs over the fixed points of the flow: $\phi^t(x) = x$ for any $t \in \mathbb{R}$.

Comment 2. Recall that $D_y \phi^t$ denotes, for fixed t, the differential of the map $y (\in X) \to \phi^t(y)$. The non vanishing of the two determinants in Proposition 3.1 is equivalent to the fact that the graph of ϕ meets transversally the "diagonal" $\{(x, x, t),\ x \in X,\ t \in \mathbb{R}\}$.

Note that the following elementary observation is the main ingredient of the proof the Proposition 3.1. Let $A \in GL_n(\mathbb{R})$ and $\delta_0(\cdot)$ denote the Dirac mass at $0 \in \mathbb{R}^n$. Then one computes the distribution $\delta_0(A\cdot)$ in the following way. For any $f \in C^\infty_{\text{comp}}(\mathbb{R}^n)$, one has:

$$\langle \delta_0(A\cdot); f(\cdot) \rangle = \int_{\mathbb{R}^n} \delta_0(Ax) f(x) dx = \int_{\mathbb{R}^n} \delta_0(y) f(A^{-1}y) \frac{1}{\mathrm{Jac}(A)} dy = \frac{1}{\mathrm{Jac}(A)} f(0)$$

where dy denotes the Lebesgue measure. Therefore: $\delta_0(A\cdot) = \frac{1}{Jac(A)} \delta_0(\cdot)$.

3.2 – The Lesfchetz trace formula of Alvarez-Lopez and Kordyukov

Now we shall assume that X is a compact three dimensional oriented manifold and endowed with a codimension one foliation $(X, \mathcal{F})$. We shall also assume that the flow ϕ^t preserves the foliation $(X, \mathcal{F})$, is transverse to it and thus has no fixed point. Therefore $(X, \mathcal{F})$ is a compact Riemannian foliation whose leaves are oriented. We shall apply later Proposition 3.1 with $E = \wedge^* T^* \mathcal{F} \to X$.

Comment 3. A typical example is of the form $X = \frac{L \times \mathbb{R}^{+*}}{\Lambda}$, where Λ a subgroup of $(\mathbb{R}^{+*}, \times)$ and $\phi^t(l, x) = (l, xe^{-t})$.

Now, we get a so called bundle like metric g_X on $(X, \mathcal{F})$ in the following way. We require that $g_X(\partial_t \phi^t(z)) = 1$, $\partial_t \phi^t(z) \perp T\mathcal{F}$ for any $(t, z) \in \mathbb{R} \times X$, and that $(g_X)_{|T\mathcal{F}}$ is a given leafwise metric. By construction, with respect to g_X, the foliation $(X, \mathcal{F})$ is defined locally by riemannian submersions.

In this setting, Alvarez-Lopez and Kordyukov [2] have proved the following Hodge decomposition theorem ($0 \leq j \leq 2$):

$$C^\infty(X, \wedge^j T^* \mathcal{F}) = \ker \Delta_\tau^j \oplus \overline{\operatorname{Im} \Delta_\tau^j} \tag{7}$$

where Δ_τ^j denotes the leafwise Laplacian. Since we have $\dfrac{\ker d_\mathcal{F}}{\overline{\operatorname{Im} d_\mathcal{F}}} = \ker \Delta_\tau^j$, we call the vector space $H_\tau^j(X) = \ker \Delta_\tau^j$ a reduced leafwise cohomology group.

Let π_τ^j denote the projection of the vector space of leafwise differential forms $C^\infty(X, \wedge^j T^* \mathcal{F})$ onto $H_\tau^j(X) = \ker \Delta_\tau^j$ according to (7) with $0 \leq j \leq 2$. Then Alvarez-Lopez and Kordyukov [1] have proved the following Atiyah-Bott-Lefschetz trace formula.

Theorem 3.2 ([1]). *Let $\alpha \in C^\infty_{\text{compact}}(\mathbb{R})$. Then the operators*

$$\int_\mathbb{R} \alpha(s)\, \pi_\tau^j \circ (\phi^s)^* \circ \pi_\tau^j \, ds$$

are trace class for $0 \leq j \leq 2$. Let χ_Λ denote the leafwise measured Connes Euler characteristic of $(X, \mathcal{F})$ ([6]). Then one has:

$$\begin{aligned} &\sum_{j=0}^{2} (-1)^j \mathrm{TR} \int_\mathbb{R} \alpha(s)\, \pi_\tau^j \circ (\phi^s)^* \circ \pi_\tau^j \, ds \\ &= \chi_\Lambda \alpha(0) + \sum_\gamma \sum_{k \geq 1} l(\gamma)\, (\epsilon_{-k\gamma} \alpha(-kl(\gamma)) + \epsilon_{k\gamma} \alpha(kl(\gamma))) \end{aligned} \tag{8}$$

where γ runs over the primitive closed orbits of ϕ^t, $l(\gamma)$ is the length of γ, $x_\gamma \in \gamma$ and $\epsilon_{\pm k\gamma} = \operatorname{sign} \det(\mathrm{id} - D\phi^{\pm kl(\gamma)}_{|\mathrm{T}_{x_\gamma} \mathcal{F}})$.

Proof. (Sketch of the idea). The case where the support of α is included in a suitably small interval $[-\epsilon, +\epsilon]$ is treated separately. So let us assume that the (compact) support of α is included in $\mathbb{R}\setminus\{0\}$. The authors show that the following quantity:

$$H(t)=\sum_{j=0}^{2}(-1)^j\mathrm{TR}\int_{\mathbb{R}}\alpha(s)\,e^{-t\Delta_\tau^j}\circ(\phi^s)^*\,ds$$

does not depend on the real $t>0$. Using non trivial arguments based on (7), the authors then prove that

$$\lim_{t\to+\infty}H(t)=\sum_{j=0}^{2}(-1)^j\mathrm{TR}\int_{\mathbb{R}}\alpha(s)\,\pi_\tau^j\circ(\phi^s)^*\circ\pi_\tau^j\,ds\,.$$

On the other hand, they show that:

$$\lim_{t\to 0^+}H(t)=\sum_{j=0}^{2}(-1)^j\mathrm{Tr}\int_{\mathbb{R}}\alpha(s)(\phi^s)^*\,ds.$$

But Proposition 3.1 (with $E=\wedge^jT^*\mathcal{F}$ and $\psi^s={}^tD\phi^s$) shows that the right handside is equal to:

$$\sum_\gamma l(\gamma)\sum_{k\in\mathbb{Z}^*}\sum_{j=0}^{2}(-1)^j\,\frac{\mathrm{Tr}\big({}^t(D_y\phi^{kl(\gamma)}(x_\gamma)):\wedge^j\mathrm{T}^*_{x_\gamma}\mathcal{F}\mapsto\wedge^j\mathrm{T}^*_{x_\gamma}\mathcal{F}\big)}{|\det(\mathrm{id}-D_y\phi^{kl(\gamma)}_{|\mathrm{T}_{x_\gamma}\mathcal{F}})|}\alpha(kl(\gamma)).$$

Then, using the equality $\lim_{t\to+\infty}H(t)=\lim_{t\to0^+}H(t)$, one then gets immediately the result. $\square$

Comment 4. If there exists a real $h_\gamma>0$ such that $h_\gamma D\phi^{l(\gamma)}_{|\mathrm{T}_{x_\gamma}\mathcal{F}}$ belongs to $S0_2(\mathrm{T}_{x_\gamma}\mathcal{F})$ (*i.e.* $D\phi^{l(\gamma)}_{|\mathrm{T}_{x_\gamma}\mathcal{F}}$ is a direct similitude) then $\epsilon_{\pm k\gamma}=1$ for any integer k.

Comment 5. The extension of Theorem 3.2 to the case where the flow has fixed points is the subject of a work in progress [4].

3.3 – The case of a ramified flat line bundle on $(X,\mathcal{F},(\phi^t)_{t\in\mathbb{R}})$

We consider now another compact (three dimensional) oriented riemannian foliation $(\tilde{X},\mathcal{F},\phi^t)$ of codimension 1 which defines a Galois ramified covering $\pi:\tilde{X}\to X$ with *finite* automorphism group G such that the action of G commutes with $\phi^t, t\in\mathbb{R}$ and permutes the leaves. We assume that for any real t, $\phi^t\circ\pi=\pi\circ\phi^t$ and that π sends leaves onto leaves. We can also assume that G preserves a bundlelike metric g' of $(\tilde{X},\mathcal{F},\phi^t)$ and we fix such one.

Consider now a non trivial character $\rho : G \to S^1$. Define an action of G on $\tilde{X} \times \mathbb{C}$ by setting

$$\forall (h, m, \lambda) \in G \times \tilde{X} \times \mathbb{C},\ h \cdot (m, \lambda) = (h \cdot m, \rho^{-1}(h)\lambda)\,.$$

To this action we associate the ramified flat complex line bundle $\mathcal{L}_\rho = \frac{\tilde{X} \times \mathbb{C}}{G} \to X$ over X, where any $(m, \lambda) \in \tilde{X} \times \mathbb{C}$ is identified to $(h \cdot m, \rho^{-1}(h)\lambda)$ for any $h \in G$.

One defines a projection P_j acting on $H^j_\tau(\tilde{X}) \otimes_{\mathbb{R}} \mathbb{C}$ $(0 \leq j \leq 2)$ by setting:

$$P_j = \frac{1}{|G|} \sum_{h \in G} h^*\,,\ \operatorname{card} G = |G|\,.$$

Definition 3.3. One then defines the leafwise cohomology group $H^j_\tau(X; \mathcal{L}_\rho)$ with coefficient in $\mathcal{L}_\rho$ by:

$$H^j_\tau(X; \mathcal{L}_\rho) = \operatorname{Im} P_j\,, 0 \leq j \leq 2\,.$$

Since the flow (ϕ^t) commutes with G, it induces an action denoted $(\phi^t)^*_j$ $(= \pi^j_\tau(\phi^t)^*)$ on each $H^j_\tau(X; \mathcal{L}_\rho)$.

The duality between $\mathcal{L}_\rho$ and $\mathcal{L}_{\overline{\rho}}$ induces a map:

$$\begin{gathered} H^1_\tau(X; \mathcal{L}_\rho) \times H^1_\tau(X; \mathcal{L}_{\overline{\rho}}) \to H^2_\tau(X) \\ (\alpha, \beta) \to \alpha \wedge \beta\,. \end{gathered}$$

The leaves of $(\tilde{X}, \mathcal{F})$ (and of $(X, \mathcal{F})$) are oriented. The restriction of the $G-$invariant metric g' along the leaves then induces the Hodge star operator:

$$\begin{gathered} H^j_\tau(X; \mathcal{L}_\rho) \to H^{2-j}_\tau(X; \mathcal{L}_{\overline{\rho}}) \\ \omega \to \overline{\star\omega}\,. \end{gathered}$$

Consider a closed orbit γ in X defined by $t \in [0, T] \to \phi^t(x_0)$ where $\phi^T(x_0) = x_0$. We shall say that this closed orbit is ramified if the cardinal of $\pi^{-1}(x_0)$ is strictly smaller than the cardinal of G. Since the action of G commutes with one of $\phi^t (t \in \mathbb{R})$, this definition does not depend on the choice of $x_0 \in \gamma$. We shall assume that there are only a finite number of closed ramified orbits. Moreover, for any such ramified closed orbit, we shall make the following two assumptions:

- First, if $\tilde{x_0} \in \pi^{-1}(\{x_0\})$ then the restriction of ρ to $G_{\tilde{x_0}} = \{r \in G / r \cdot \tilde{x_0} = \tilde{x_0}\}$ is not trivial.
- Second, if $T > 0$ and $h \in G$ are such that $\phi^T(\tilde{x_0}) = h \cdot \tilde{x_0}$, then

$$\begin{aligned} &\forall r \in G_{\tilde{x_0}},\ \operatorname{sign}\det\big(Id - D(h^{-1}\, r \circ \phi^T)\big)_{|T_{\tilde{x_0}}\mathcal{F}} \\ &= \operatorname{sign}\det\big(Id - D(h^{-1} \circ \phi^T)\big)_{|T_{\tilde{x_0}}\mathcal{F}}\,. \end{aligned} \tag{9}$$

Consider now the case where γ is an unramified closed orbit on X. Let $\tilde{x_0} \in \pi^{-1}(\{x_0\})$, there exists $h \in G$ such that $\phi^T(\tilde{x_0}) = h \cdot \tilde{x_0}$. Then, for any $\lambda \in \mathbb{C}$,

$$h^{-1} \cdot (\phi^T(\tilde{x_0}), \lambda) = (\tilde{x_0}, \rho(h)\lambda)\,.$$

Then the complex number $\rho(h)$ defines the monodromy action along γ on the flat line bundle $\mathcal{L}_\rho$ and we denote it by $\rho(\gamma)$.

We then may state a leafwise Lefschetz trace formula (with coefficients in $\mathcal{L}_\rho$) where, in analogy with the ramified primes of a Dirichlet character, the ramified closed orbits do not contribute at all.

Theorem 3.4. *Assume that for any $h \in G$, the graph of $h \circ \phi^t$ intersects transversally the "diagonal" $\{(\tilde{x}, \tilde{x}, t)/\ \tilde{x} \in \tilde{X},\ t \in \mathbb{R}\}$. Let $\alpha \in C^\infty_{\text{compact}}(\mathbb{R})$ be such that $\alpha(0) = 0$. Then for each $0 \leq j \leq 2$, the operator $\int_{\mathbb{R}} \alpha(s)(\phi^s)^*_j\, ds$ acting on $H^j_\tau(X; \mathcal{L}_\rho)$ is trace class. Moreover, one has*:

$$\begin{aligned}
&\sum_{j=0}^{2} (-1)^j \mathrm{TR} \int_{\mathbb{R}} \alpha(s)\,(\phi^s)^*_j\, ds \\
&= \sum_{\gamma} \sum_{k \geq 1} l(\gamma) \big(\, \epsilon_{-k\gamma}\, \rho(-k\gamma)\, \alpha(-kl(\gamma)) + \epsilon_{k\gamma}\, \rho(k\gamma)\, \alpha(kl(\gamma))\big)
\end{aligned}$$

where γ runs over the primitive unramified closed orbits of ϕ^t, $l(\gamma)$ is the length of γ, $x_\gamma \in \gamma$ and $\epsilon_{\pm k\gamma} = \operatorname{sign}\det(\mathrm{id} - D\phi^{\pm kl(\gamma)}_{|\mathrm{T}_{x_\gamma}\mathcal{F}})$.

Proof. We use Theorem 3.2 or rather its proof to compute, the alternate sum of traces:

$$\sum_{j=0}^{2} (-1)^j \frac{1}{|G|} \sum_{h \in G} \mathrm{TR} \int_{\mathbb{R}} \alpha(s)\, \pi^j_\tau\, (h^{-1} \circ \phi^s)^*\, ds : H^j_\tau(\tilde{X}) \otimes_{\mathbb{R}} \mathbb{C} \to H^j_\tau(\tilde{X}) \otimes_{\mathbb{R}} \mathbb{C}\,. \quad (10)$$

So we have to consider the reals $T \neq 0$ and the points $\tilde{x}_0 \in \tilde{X}$ such that $h^{-1} \circ \phi^T(\tilde{x}_0) = \tilde{x}_0$. We shall assume $T > 0$, the case $T < 0$ being similar. By considering $\pi(h^{-1} \circ \phi^t(\tilde{x}_0))$, one obtains a closed orbit on X, $\gamma_{\pi(\tilde{x}_0)} : t \mapsto \phi^t(\pi(\tilde{x}_0))$ $(0 \leq t \leq T)$ of length T. There exists $k \in \mathbb{N}^*$, such that $\gamma_{\pi(\tilde{x}_0)}$ is the $k-$iteration of the *primitive* closed orbit of length $T_0 = \frac{T}{k}$ determined by $\phi^{T_0}(\pi(\tilde{x}_0)) = \pi(\tilde{x}_0)$. Upstairs on $\tilde{X}$, this means that there exists $h_0 \in G$ such that $h_0^{-1}\phi^{T_0}(\tilde{x}_0) = \tilde{x}_0$. So $\phi^{kT_0}(\tilde{x}_0) = h_0^k \cdot \tilde{x}_0 = h \cdot \tilde{x}_0$ and hence $h^{-1}h_0^k \in G_{\tilde{x_0}}$. We then distinguish two cases.

A) The case where $\gamma_{\pi(\tilde{x}_0)}$ is unramified.

Then by considering the translates on the left of $h^{-1} \circ \phi^t(\tilde{x}_0)$, one obtains exactly the $|G|$ curves (of the flow) on $\tilde{X}$ which correspond (via π) to $\gamma_{\pi(\tilde{x}_0)}$. We write them in the following way, $t \mapsto lh^{-1}l^{-1} \circ \phi^t(l \cdot \tilde{x}_0)$ ($0 \leq t \leq T$), where $l \in G$, the corresponding monodromy being $\rho(lhl^{-1})$. Observe that $l \cdot \tilde{x}_0$ is a fixed point of $lh^{-1}l^{-1} \circ \phi^T$. Then the proof of Theorem 3.2 ([1]) shows that the geometric contribution of $\gamma_{\pi(\tilde{x}_0)}$ to (10) is computed according to Proposition 3.1 and is equal to:

$$\frac{T_0}{|G|} \sum_{l \in G} \sum_{j=0}^{2} (-1)^j \frac{\operatorname{Tr}\left({}^t D(lh^{-1}l^{-1} \circ \phi^T)(l \cdot \tilde{x_0})\,;\, \wedge^j T^*_{l \cdot \tilde{x_0}} \mathcal{F}\right)}{|\det\left(\mathrm{id} - D(lh^{-1}l^{-1} \circ \phi^T)\right)_{|T_{l \cdot \tilde{x_0}} \mathcal{F}}|}\ \rho(lhl^{-1})\, \alpha(kT_0)\,.$$

We observe that all the following reals, where l runs over G, have the same sign:

$$\det\left(\mathrm{id} - D(lh^{-1}l^{-1} \circ \phi^T)\right)_{|T_{l \cdot \tilde{x_0}} \mathcal{F}},\ \det(\mathrm{id} - D\phi^T_{|\mathrm{T}_{\pi(\tilde{x}_0)} \mathcal{F}})\,.$$

Therefore, the previous expression is clearly equal to $T_0\, \epsilon_{k \gamma_{\pi(\tilde{x}_0)}}\, \rho(h)\, \alpha(kT_0)$ which yields the desired contribution.

B) The case where $\gamma_{\pi(\tilde{x}_0)}$ is ramified.

So $G_{\tilde{x_0}} = \{u \in G /\, u \cdot \tilde{x_0} = \tilde{x_0}\}$ is not trivial. Then there are exactly $|G/G_{\tilde{x_0}}|$ curves (of the flow) upstairs on $\tilde{X}$ which correspond (via π) to $\gamma_{\pi(\tilde{x}_0)}$. They are given by $t \mapsto l_j h^{-1} l_j^{-1} \circ \phi^t(l_j \cdot \tilde{x}_0)$ ($0 \leq t \leq T$), where the l_j ($1 \leq j \leq m$) run over a system of representatives of cosets of $G/G_{\tilde{x_0}}$. Since the restriction of ρ to $G_{\tilde{x_0}}$ is *not* trivial, we have to count each such curve $|G_{\tilde{x_0}}|$ times but with (possibly) different monodromies (*i.e.* action on the line factor $\mathbb{C}$). More precisely, for each representative l_j the curve labeled

$$t \mapsto l_j h^{-1} l_j^{-1} u \circ \phi^t(l_j \cdot \tilde{x}_0), \quad \text{with} \quad u \in l_j G_{\tilde{x_0}} l_j^{-1} \tag{11}$$

has monodromy $\rho(u^{-1} l_j h l_j^{-1})$.

Thanks to the sign assumption (9), the proof given above in the unramified case shows that the sum of the contributions to (10) of the $|l_j G_{\tilde{x_0}} l_j^{-1}|$ curves in (11) is then equal to:

$$C \sum_{u \in l_j G_{\tilde{x_0}} l_j^{-1}} \rho(u^{-1} l_j h l_j^{-1}) = C\big(\sum_{s \in G_{\tilde{x_0}}} \rho(s) \big)\, \rho(h)\,,$$

where C is a suitable constant. But since the restriction of ρ to $G_{\tilde{x_0}}$ is assumed to be not trivial, this contribution is zero. $\square$

3.4 – An explicit example

Now we describe an explicit example (communicated to us by Jesus Alvarez-Lopez) of a ramified covering satisfying the conditions of Theorem 3.4. Denote by $\mathcal{S}$ the Jacob ladder, a certain noncompact surface embedded in $\mathbb{R}^3$, and by L the real line of symmetry of $\mathcal{S}$. There is a group $\mathcal{T}$ of translations isomorphic to $(\mathbb{Z}, +)$, whose vectors belong to L and which acts on $\mathcal{S}$ such that the quotient $\mathcal{S}/\mathcal{T}$ is a smooth compact Riemann surface. Let $G = \{Id, R\}$ denote the group generated by the rotation R of $\mathbb{R}^3$ whose axis is L and angle is π. Observe that we have a ramified covering $\mathcal{S} \to \mathcal{S}/G$ where the set of ramification points is $L \cap \mathcal{S}$. Moreover, there exists a vector field U on $\mathcal{S}$ whose fixed points are exactly the ones of the $G-$action and which is invariant by G and $\mathcal{T}$. Consider an action of $\mathcal{T} \simeq (\mathbb{Z}, +)$ on the circle S^1 defined by a rotation of angle $2\pi\alpha$ ($\alpha \notin \mathbb{Q}$). Now, set $\tilde{X} = \frac{\mathcal{S}\times S^1}{\mathcal{T}}$, it is foliated by the leaves induced by the sets $\mathcal{S} \times \{e^{i\theta}\}$. Consider ϕ^t the flow of the vector field $U \times \frac{\partial}{\partial_x}$ of $\tilde{X}$. Then we can choose U such that the hypothesis of Theorem 3.4 are satisfied by $\tilde{X}, (\phi^t)_{t\in\mathbb{R}}, X = \tilde{X}/G$.

3.5 – The more general case of a flat ramified complex vector bundle

More generally, one can consider a unitary representation $\rho : G \to U(E)$ where E is a complex hermitian vector space. One then gets the ramified flat complex hermitian vector bundle

$$\mathcal{E}_\rho = \frac{\tilde{X} \times E}{G} \to X$$

over X where (m, v) is identified with $(h \cdot m, \rho(h)^{-1}v)$ for any $h \in G$. Similarly, the dual representation ${}^t\rho^{-1} : G \to U(E^*)$ allows to consider the dual flat complex hermitian vector space $\mathcal{E}^*_{{}^t\rho^{-1}} \to X$.

The duality between $\mathcal{E}_\rho$ and $\mathcal{E}^*_{{}^t\rho^{-1}}$ induces a map:

$$\begin{gathered} H^1_\tau(X; \mathcal{E}_\rho) \times H^1_\tau(X; \mathcal{E}^*_{{}^t\rho^{-1}}) \to H^2_\tau(X) \\ (\alpha, \beta) \to \alpha \wedge \beta\,. \end{gathered}$$

Denote by $J : \mathcal{E}_\rho \to \mathcal{E}^*_{{}^t\rho^{-1}}$ the antilinear vector bundle isomorphism provided by the hermitian scalar product. Then, using leafwise Hodge star of the metric g', one gets the following Hodge star operator acting on the cohomology groups:

$$\begin{gathered} H^j_\tau(X; \mathcal{E}_\rho) \to H^{2-j}_\tau(X; \mathcal{E}^*_{{}^t\rho^{-1}}) \\ \omega \to J \star \omega\,. \end{gathered}$$

4 – Remarks about a conjectural dynamical laminated foliated space $(S_K, \mathcal{F}, g, \phi^t)$ associated to the Dedekind zeta function $\widehat{\zeta}_K$

Much of the following Section is speculative in nature. It should be viewed as a working programme or a motivation for developing interesting mathematics.

Let K be a number field and let $\mathcal{O}_K$ denote its ring of integers. Let r_1 (resp. $2r_2$) denote the number of real (resp. complex) embeddings of K so that the dimension of K as a $\mathbb{Q}$-vector space is equal to $r_1 + 2r_2$. If $\sigma : K \to \mathbb{C}$ is a complex embedding then of course $|\sigma(z)|$ and $|\overline{\sigma}(z)|$ define the same archimedean absolute value on K. Therefore the set S_∞ of all the archimedean absolute values of K has exactly $r_1 + r_2$ elements.

We now set:

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),\ \Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)\,.$$

The Dedekind zeta function $\widehat{\zeta}_K$ is defined for $\Re s > 1$ by:

$$\widehat{\zeta}_K(s) = |d_K|^{s/2}\,\Gamma_{\mathbb{R}}^{r_1}(s)\,\Gamma_{\mathbb{C}}^{r_2}(s)\,\prod \frac{1}{1-(N\mathcal{P})^{-s}}\,,$$

where d_K denotes the discriminant of K over $\mathbb{Q}$, $\mathcal{P}$ runs over the set of non zero prime ideals of $\mathcal{O}_K$ and $N\mathcal{P}$ ($=$ card $\mathcal{O}_K/\mathcal{P}$) denotes the norm of $\mathcal{P}$.

The function $\widehat{\zeta}_K$ extends as a meromorphic function on $\mathbb{C}$ and admits a simple pole at 0 and 1. It satisfies the functional equation $\widehat{\zeta}_K(s) = \widehat{\zeta}_K(1-s)$.

We recall the explicit formula for the zeta function $\widehat{\zeta}_K$ as an equality between two distributions in $\mathcal{D}'(\mathbb{R}\setminus\{0\})$ (t being the real variable).

$$\begin{aligned} 1 - \sum_{\rho\in\widehat{\zeta}_K^{-1}\{0\},\ \Re\rho\geq 0} e^{t\rho} + e^t = &\sum_{\mathcal{P}} \log N\mathcal{P} \sum_{k\geq 1}(\delta_{k\log N\mathcal{P}} + (N\mathcal{P})^{-k}\delta_{-k\log N\mathcal{P}}) \\ &+ r_1\left(\frac{1}{1-e^{-2t}}1_{\{t>0\}} + \frac{e^t}{1-e^{2t}}\,1_{\{t<0\}}\right) + r_2\left(\frac{1}{1-e^{-t}}\,1_{\{t>0\}} + \frac{e^t}{1-e^{t}}\,1_{\{t<0\}}\right), \end{aligned} \tag{12}$$

where $\mathcal{P}$ runs over the set of prime ideals of $\mathcal{O}_K$ and $N\mathcal{P}$ denotes the norm of $\mathcal{P}$.

4.1 – Structural assumptions and their consequences

We assume, following Deninger (*e.g.* [15, 14]), that to $\operatorname{Spec}\mathcal{O}_K \cup S_\infty$, one can associate a (laminated) foliated space $(S_K, \mathcal{F}, g, \phi^t)$ satisfying the following assumptions.

1) The leaves are Riemann surfaces, the path connected components of S_K are three dimensional and, g denotes a leafwise riemannian metric. The flow $(\phi^t)_{t\in\mathbb{R}}$ acts on $(S_K, \mathcal{F})$, it sends leaves into (other) leaves and its graph intersects transversally the "diagonal" $\{(\tilde{x}, \tilde{x}, t)/\ \tilde{x}\in S_K,\ t\in\mathbb{R}\}$.

2) To each prime ideal $\mathcal{P}$ of $\mathcal{O}_K$ there corresponds a unique primitive closed orbit $\gamma_{\mathcal{P}}$ of ϕ^t of length $\log N\mathcal{P}$. There is a bijection between the set S_∞ of archimedean absolute values and the set of fixed point $y_\infty = \phi^t(y_\infty), \forall t \in \mathbb{R}$, of the flow. Each leaf contains at most one fixed point and the flow is transverse to all the leaves different from the ones containing the $r_1 + r_2$ fixed points.
3a) We assume that for any fixed point y_∞:

$$\forall t \in \mathbb{R},\ e^{-t/2} D_y \phi^t(y_\infty)_{|T_{y_\infty}\mathcal{F}} \in SO_2(T_{y_\infty}\mathcal{F})\,. \tag{13}$$

3b) For any prime $\mathcal{P}$ of $\mathcal{O}_K$ and any $\tilde{x} \in \gamma_{\mathcal{P}}$:

$$e^{-\frac{\log N\mathcal{P}}{2}} D_y \phi^{\log N\mathcal{P}}(\tilde{x})_{|T_{\tilde{x}}\mathcal{F}} \in SO_2(T_{\tilde{x}}\mathcal{F})\,. \tag{14}$$

4) We have (Frechet) reduced real leafwise cohomology groups $\overline{H}^j_{\mathcal{F},K}$ $(0 \le j \le 2)$, on which $(\phi^t)_{t\in\mathbb{R}}$ acts naturally, with the following properties. One has $\overline{H}^0_{\mathcal{F},K} \simeq \mathbb{R}$ (the space of constant functions) and $\overline{H}^2_{\mathcal{F},K} \simeq \mathbb{R}[\lambda_g]$ where $[\lambda_g]$ denote the class in $\overline{H}^2_{\mathcal{F},K}$ of the leafwise kaehler metric λ_g associated to g. Moreover, we assume that

$$\forall t \in \mathbb{R},\ (\phi^t)^*([\lambda_g]) = e^t[\lambda_g]\,, \tag{15}$$

and that $\overline{H}^1_{\mathcal{F},K}$ is infinite dimensional.
5) The action of ϕ^t on $\overline{H}^1_{\mathcal{F},K}$ commutes with the Hodge star $\star$ induced by g. Moreover there exists a transverse measure μ on $(S_K, \mathcal{F})$ such that $\int_{S_K}(\alpha \wedge \star\beta)\mu$ defines a scalar product on $\overline{H}^1_{\mathcal{F},K}$
6) For any $\alpha \in C^\infty_{\text{compact}}(\mathbb{R} \setminus \{0\}; \mathbb{R})$, $\int_{\mathbb{R}} \alpha(t)(\phi^t)^* dt$ acting on $\overline{H}^1_{\mathcal{F},K}$ is trace class (possibly in some generalized sense, *cf.* [4]). The explicit formula (12) is interpreted as an Atiyah-Bott-Lefschetz trace formula for the foliated space $(S_K, \mathcal{F}, g, \phi^t)$ with respect to the leafwise cohomology groups $\overline{H}^j_{\mathcal{F},K}$ $(0 \le j \le 2)$. In particular, the infinitesimal generator θ_1 of $(\phi^t)_{t\in\mathbb{R}}$ acting on $\overline{H}^1_{\mathcal{F},K} \otimes \mathbb{C}$ has discrete spectrum, its set of eigenvalues coincide with the set of zeroes of $\widehat{\zeta}_K(s)$ (with the same multiplicities on each side). Moreover:

$$\overline{H}^1_{\mathcal{F},K} \otimes_{\mathbb{R}} \mathbb{C} = \overline{\sum_{z_q \in \widehat{\zeta}_K^{-1}\{0\}} \ker(\theta_1 - z_q Id)^{n(z_q)}}\,.$$

7) Let $x_\infty \in S_K$ be any fixed point corresponding (according to 2)) to a real archimedan absolute value. Then x_∞ should be a limit point of a trajectory γ_∞: $\lim_{t\to+\infty} \phi^t(y) = x_\infty$ for any $y \in \gamma_\infty$. Moreover, γ_∞ should have the following orbifold structure. Define an orbifold structure on $\mathbb{R}^{\geq 0}$ by requiring the following map to be an orbifold isomorphism:

$$Sq : \frac{\mathbb{R}}{\{1,-1\}} \to \mathbb{R}^{\geq 0},\ Sq(z) = z^2.$$

Notice that Sq transforms the flow $\phi^t_{\frac{\mathbb{R}}{\{1,-1\}}}(z) = ze^{-t}$ into the flow $\phi^t_{\mathbb{R}^{\geq 0}}(v) = ve^{-2t}$. Then we require that there exists an embedding $\Psi : \mathbb{R}^{\geq 0} \to \gamma_\infty$ such that $\Psi(0) = x_\infty$ and

$$\forall (t,v) \in \mathbb{R} \times \mathbb{R}^{\geq 0},\ \Psi(\phi^t_{\mathbb{R}^{\geq 0}}(v) = ve^{-2t}) = \phi^t(\Psi(v)). \tag{16}$$

Lastly we require that γ_∞ is transverse at x_∞ to $T_{x_\infty}\mathcal{F}$.
8) Let $z_\infty \in S_K$ be any fixed point corresponding (according to 2)) to a complex archimedean absolute value. Then there exist two trajectories $\gamma_\pm$ of the flow ϕ^t with end point z_∞. For any $z_\pm \in \gamma_\pm$, $\lim_{t\to+\infty} \phi^t(z_\pm) = z_\infty$. These two trajectories $\gamma_\pm$ are transverse to $\mathcal{F}$ at z_∞. Moreover there exists an embedding:

$$\Psi : \mathbb{R} \to \gamma_- \cup \gamma_+,$$

such that $\Psi(0) = z_\infty$, $\gamma_\pm\setminus\{0\} = \Psi(\mathbb{R}^\pm\setminus\{0\})$. Lastly, $\forall v, t \in \mathbb{R}$, $\Psi(ve^{-t}) = \phi^t(\Psi(v))$.

Comment 6. The stronger assumption $\forall t \in \mathbb{R}$, $(\phi^t)^*(g) = e^t g$ implies (13) (because $\phi^0 = Id$), (15) and the fact that ϕ^t commutes with the Hodge star not only on $\overline{H}^1_{\mathcal{F},K}$ but also on the vector space of leafwise differential 1-forms. Deninger told us privately that this assumption $(\phi^t)^*(g) = e^t g$ might be too strong. Assumption 5. and (15) implies the analogue of Equation (6) for $\widehat{\zeta}_K$ in Deninger's formalism. Therefore, the first six Assumptions imply the Riemann hypothesis for $\widehat{\zeta}_K$ as explained in Section 2!

Comment 7. The disymmetry mentionned in Comment 1 might be explained in the following way. For each prime ideal $\mathcal{P}$ of $\mathcal{O}_K$ with norm p^f, $(S_K, \mathcal{F})$ should exhibit a transversal of the type $]0,1[\times\mathbb{Z}_p$ and possibly the ring of finite Adeles $\mathbb{A}_\mathbb{Q}$ might enter into the picture. See [24] for a simple case.

4.2 – Remarks about the contribution of the archimedean places in (12)

Now we apply formally the Guillemin-Sternberg trace formula for the distribution of the real variable t:

$$\sum_{j=0}^{2} (-1)^j \mathrm{Tr}((\phi^t)^* \,;\, \Gamma(S_K \,;\, \wedge^j T^*\mathcal{F})) \tag{17}$$

where $\Gamma(S_K \,;\, \wedge^j T^*\mathcal{F})$ denotes the set of "smooth" sections.

Lemma 4.1 (Deninger [14]).

1) *The contribution of a fixed point* y_∞ *corresponding to an archimedean place of* K *in the Guillemin-Sternberg trace formula for* (17) *is*:

$$\frac{1}{|\det(1 - D_y\phi^t(y_\infty)\,;\,T_{y_\infty}S_K/T_{y_\infty}\mathcal{F})|}.$$

2) *In the case of a fixed point* x_∞ *corresponding to a real archimedean place of* K *one has*:

$$\forall t \in \mathbb{R}\setminus\{0\},\ \frac{1}{|\det(1 - D_y\phi^t(x_\infty)\,;\,T_{x_\infty}S_K/T_{x_\infty}\mathcal{F})|} = \frac{1}{|1-e^{-2t}|}.$$

3) *In the case of a fixed point* z_∞ *corresponding to a complex archimedean place of* K *one has*:

$$\forall t \in \mathbb{R}\setminus\{0\},\ \frac{1}{|\det(1 - D_y\phi^t(z_\infty)\,;\,T_{z_\infty}S_K/T_{z_\infty}\mathcal{F})|} = \frac{1}{|1-e^{-t}|}.$$

Proof. 1. Using Proposition 3.1, one sees that the contribution of the fixed point y_∞ is equal to:

$$\frac{\sum_{j=0}^{2}(-1)^j \operatorname{Tr}((D_y\phi^t)^*(y_\infty)\,;\,\wedge^j T^*_{y_\infty}\mathcal{F})}{|\det(1 - D_y\phi^t(y_\infty)\,;\,T_{y_\infty}S_K)|}$$
$$= \frac{\det(1 - D_y\phi^t(y_\infty)\,;\,T_{y_\infty}\mathcal{F})}{|\det(1 - D_y\phi^t(y_\infty)\,;\,T_{y_\infty}\mathcal{F})|}\,\frac{1}{|\det(1 - D_y\phi^t(y_\infty)\,;\,T_{y_\infty}S_K/T_{y_\infty}\mathcal{F})|}.$$

Using property (13) one checks easily that

$$\frac{\det(1 - D_y\phi^t(y_\infty)\,;\,T_{y_\infty}\mathcal{F})}{|\det(1 - D_y\phi^t(y_\infty)\,;\,T_{y_\infty}\mathcal{F})|} = 1.$$

One then gets immediately 1).
2) Since $T_{x_\infty}S_K/T_{x_\infty}\mathcal{F}$ is a real line, there exists $\kappa \in \mathbb{R}$ such that:

$$\forall t \in \mathbb{R},\ |\det(1 - D_y\phi^t(x_\infty)\,;\,T_{x_\infty}S_K/T_{x_\infty}\mathcal{F})| = |1-e^{\kappa t}|.$$

By Assumption 7), γ_∞ is transverse at x_∞ to $T_{x_\infty}\mathcal{F}$ and (16) shows that $D_y\phi^t(x_\infty)$ acts as $e^{-2t}\,Id$ on the real line $T_{x_\infty}S_K/T_{x_\infty}\mathcal{F}$. One then gets 2) immediately. One proves 3) in the same way, using Assumption 8). □

Recall that we wish to test the interpretation of (12) as a Lefschetz trace formula via the Guillemin-Sternberg formula. Part 2) of the next Proposition is a priori embarrassing...

Proposition 4.2 (Deninger [14]).

1) *The contribution of a real archimedean absolute value in* (12) *coincides* for any real positive t *with the contribution of the corresponding fixed point* x_∞ *in the Guillemin-Sternberg formula for* (17).
2) *The contributions of the fixed point* x_∞ for t real negative *in the Guillemin-Sternberg formula for* (17) *and of the corresponding real archimedean absolute value in* (12) *do* not *coincide.* (*This riddle will be resolved in Section* 4.4).

Proof. 1) This is part 2) of the previous Lemma.
2) Indeed, the Guillemin-Sternberg formula gives

$$\frac{1}{|1-e^{-2t}|} = \frac{e^{2t}}{1-e^{2t}},$$

whereas (12) gives $\dfrac{e^{t}}{1-e^{2t}}$ for $t<0$. □

Comment 8. It was in order to explain the factor -2 (instead of -1) in $\frac{1}{1-e^{-2t}}$ for $t>0$ for a real archimedean place in (12) that Deninger has proposed in [15, Section 3] the Assumption 7).

The following Proposition shows that the contribution of a *complex* archimedean place in the explicit formula (12) is better understood than the one of a *real* archimedean place (*cf.* Proposition 4.2. 2)).

Proposition 4.3 ([14], Section 5). *Let* z_∞ *be a fixed point corresponding to a complex archimedean place of* K. *The contribution of* z_∞ *in the Guillemin-Sternberg trace formula* (*for* (17)) *coincides, for* $t \in \mathbb{R}\setminus\{0\}$, *with the contribution of the corresponding complex archimedean place in* (12).

Proof. This is an easy consequence of Lemma 4.1. 3). □

4.3 – More precise assumptions when K is a Galois extension of $\mathbb{Q}$ with Galois group G

In this subsection we assume that K is a Galois extension of $\mathbb{Q}$ of degree n with Galois group G. We then require the existence of a ramified Galois covering map $\pi_K : S_K \to S_{\mathbb{Q}}$ whose automorphism group coincides with G and which *satisfies the following properties.*

For any real t, $\phi^t \circ \pi_K = \pi_K \circ \phi^t$, the map π_K [resp. G] sends leaves to leaves. The leafwise metric g is assumed to be $G-$invariant and, the actions of G and ϕ^t ($t \in \mathbb{R}$) on S_K and $\overline{H}^1_{\mathcal{F},K}$ commute. Moreover the action of G on the set of curves $\gamma_{\mathcal{P}}$ coincides with the action of G on the set of (non zero) prime ideals $\mathcal{P}$. More precisely, consider a prime number p and the decomposition of $p\mathcal{O}_K$ in prime ideals:

$$p\mathcal{O}_K = \mathcal{P}_1^e \dots \mathcal{P}_r^e .$$

Therefore, $n = e\,r\,f$ where $N\mathcal{P}_j = p^f$ for $j \in \{1, \dots, r\}$.

Consider $D_{\mathcal{P}_j} = \{h \in G/\, h \cdot \mathcal{P}_j = \mathcal{P}_j\}$ [1] , we then have a natural surjective homomorphism:

$$\begin{aligned} \Theta_j : D_{\mathcal{P}_j} &\to \operatorname{Aut} \frac{\mathcal{O}_K}{\mathcal{P}_j} \\ h &\mapsto \Theta_j(h) = (\mathrm{Fr}\,)^{a_j(h)} \end{aligned} \tag{18}$$

where Fr denote the Frobenius automorphim $z \mapsto z^p$ of $\frac{\mathcal{O}_K}{\mathcal{P}_j}$ and $a_j(h)$ is a suitable integer (modulo f). One has $|D_{\mathcal{P}_j}| = ef$. The fact that Θ_j is natural means that

$$\forall (h, v) \in G \times \mathcal{O}_K, \;\; h \cdot v - v^{p a_j(h)} \in \mathcal{P}_j \,. \tag{19}$$

Recall that e is the common cardinal of the inertia groups $I_{\mathcal{P}_j} = \ker \Theta_j$ and that $e \geq 2$ if and only if p divides the discriminant of K. We then require that each point of $\gamma_{\mathcal{P}_j}$ is fixed by $I_{\mathcal{P}_j}$ and that π_K induces the covering map

$$\begin{aligned} \pi_K : \gamma_{\mathcal{P}_j} &\simeq \frac{\mathbb{R}}{f \log p\, \mathbb{Z}} \to \gamma_p \simeq \frac{\mathbb{R}}{\log p\, \mathbb{Z}} \\ x &\mapsto x \,. \end{aligned} \tag{20}$$

Moreover, we require the following three properties:

$$\forall h \in D_{\mathcal{P}_j}, \;\forall x \in \gamma_{\mathcal{P}_j} \simeq \frac{\mathbb{R}}{f \log p\, \mathbb{Z}}, \;\; h \cdot x = x + a_j(h) \log p \,. \tag{21}$$

The restriction of ϕ^t to $\gamma_{\mathcal{P}_j} \simeq \frac{\mathbb{R}}{f \log p\, \mathbb{Z}}$ (resp. γ_p) is assumed to be the translation by t: $\phi^t(x) = x + t$. Lastly we state a strengthening of Assumption 3a):

$$\text{if } h^{-1}\phi^T(\tilde{x}) = \tilde{x} \text{ for } \tilde{x} \in S_K,\, h \in G, \text{ then } D(h^{-1}\phi^T)(\tilde{x})_{|T_{\tilde{x}}\mathcal{F}} \in \mathbb{R}^{+*} SO_2(T_{\tilde{x}}\mathcal{F}) \,. \tag{22}$$

Now we state the required conditions for the archimedean places of K. Observe that if $|\,|$ is such a place then all the other archimedean places of K are of the form

[1]When G is abelian, the $D_{\mathcal{P}_j}$ are all equal for $1 \leq j \leq r$.

$z \to |h(z)|$ where h runs over G. Therefore either they are all real or all complex. If they are all real, then $r_1 = n$ and the group G acts freely and transitively on the set S_∞ of archimedean places of K. We then require that the action of G on S_∞ coincides with the one of G on the set of corresponding fixed points $\{x_{1,\infty}, \ldots, x_{n,\infty}\}$.

If the archimedean places are all complex, then $n = 2r_2$. We require that the transitive action of G on S_∞ coincides with the (transitive) action of G on the set of corresponding fixed points $\{z_{1,\infty}, \ldots, z_{r_2,\infty}\}$. For each $j \in \{1, \ldots, r_2\}$ there exists a unique element $h_j \in G \setminus \{1\}$ such that $h_j(z_{j,\infty}) = z_{j,\infty}$. The h_j are all conjugate to each other and satisfy $h_j^2 = 1$, where $j \in \{1, \ldots, r_2\}$. In some sense each h_j represents a non canonical model of the complex conjugation. Moreover we assume that for any $j \in \{1, \ldots, r_2\}$

$$Dh_j(z_{j,\infty})_{|T_{z_{j,\infty}}\mathcal{F}} = Id, \quad Dh_j(z_{j,\infty})_{|\frac{T_{z_{j,\infty}}S_K}{T_{z_{j,\infty}}\mathcal{F}}} = -Id. \tag{23}$$

4.4 – Explanation of the incompatibility at the archimedean place of $\mathbb{Q}$ between the Guillemin-Sternberg trace formula and the explicit formula (1).

A priori, Proposition 4.2. 2) seems to raise an "objection" in Deninger's approach. We are going to explain that the Guillemin-Sternberg trace formula and the explicit formula (1) *are actually compatible.* Proposition 4.2. 2) is due to the fact that $S_{\mathbb{Q}}$ has a mild singularity at x_∞, whereas $S_{\mathbb{Q}[i]}$ is "smooth" at z_∞.

We apply the previous subsection with $K = \mathbb{Q}[i]$. Thus we require the existence of a degree two ramified covering $\pi_{\mathbb{Q}[i]} : S_{\mathbb{Q}[i]} \to S_{\mathbb{Q}}$ with structural group $G = \{Id, h_1\}$ such that $h_1(z_\infty) = z_\infty$, $Dh_1(z_\infty)_{|T_{z_\infty}\mathcal{F}} = Id_{\mathrm{T}_{z_\infty}\mathcal{F}}$, $Dh_1(z_\infty)$ induces $-Id$ on $T_{z_\infty}S_{\mathbb{Q}[i]}/T_{z_\infty}\mathcal{F}$ and $h_1 \circ \phi^t = \phi^t \circ h_1$ for any real t.

Then we have (at least formally) the following equality between distributions of the real variable t:

$$\begin{aligned}
&\sum_{j=0}^{2}(-1)^j \mathrm{Tr}\Big((\phi^t)^* \,;\, \Gamma(S_{\mathbb{Q}} \,;\, \wedge^j T^*\mathcal{F})\Big) \\
&= \sum_{j=0}^{2}(-1)^j \mathrm{Tr}\left(\frac{(\phi^t)^* + h_1^*(\phi^t)^*}{2} \,;\, \Gamma(S_{\mathbb{Q}[i]} \,;\, \wedge^j T^*\mathcal{F})\right) = B(t)\,.
\end{aligned}$$

We observe that z_∞ is also a fixed point for $h_1 \circ \phi^t = \phi^t \circ h_1$, $t \in \mathbb{R}$.

PROPOSITION 4.4. *The contribution of z_∞ in the Guillemin-Sternberg trace formula for the term $B(t)$ above is equal to*:

$$\frac{1}{1 - e^{-2t}} \quad \text{if} \quad t > 0\,,$$

and to

$$\frac{e^t}{1-e^{2t}} \quad \text{if} \quad t<0\,.$$

This contribution matches perfectly with the contribution of the real archimedean place in the explicit formula (1)*.*

Proof. The axioms of Section 4.3 (recalled in the beginning of this subsection) and the proof of Lemma 4.1 allow to check formally that the contribution of z_∞ (for $t \in \mathbb{R} \setminus \{0\}$) in

$$\sum_{j=0}^{2} (-1)^j \mathrm{Tr}\Big(h_1^*(\phi^t)^* \,;\, \Gamma(S_{\mathbb{Q}[i]}\,;\, \wedge^j T^*\mathcal{F})\Big)$$

is equal to:

$$\frac{1}{|\det(1 - D_y(h_1 \circ \phi^t(z_\infty))\,;\, T_{z_\infty}S_K/T_{z_\infty}\mathcal{F})|} = \frac{1}{1+e^{-t}}\,.$$

Now we can compute the contribution of the fixed point $x_\infty \in S_{\mathbb{Q}}$ in $B(t)$.

For $t>0$ we find:

$$\frac{1}{2}\left(\frac{1}{1-e^{-t}} + \frac{1}{1+e^{-t}}\right) = \frac{1}{1-e^{-2t}}\,.$$

For $t<0$ we find:

$$\frac{1}{2}\left(\frac{e^t}{1-e^{t}} + \frac{e^t}{1+e^{t}}\right) = \frac{e^t}{1-e^{2t}}\,.$$

The proposition is proved. □

4.5 – Primitive Dirichlet characters and leafwise flat ramified lines bundles

Let χ be a primitive Dirichlet character mod m (≥ 3), so χ is defined by a group homomorphism:

$$\chi : G = (\mathbb{Z}/m\mathbb{Z})^* \to S^1\,.$$

Consider the cyclotomic field $K = \mathbb{Q}[e^{\frac{2i\pi}{m}}]$, it is a Galois extension of $\mathbb{Q}$ whose Galois group is equal to G and has cardinal $\phi(m)$ (ϕ being the Euler function). Then define an action of G on $S_K \times \mathbb{C}$ by $h \cdot (z,\lambda) = (h\cdot z, \chi(h)^{-1}\lambda)$ for any $(h,z,\lambda) \in G \times S_K \times \mathbb{C}$.

Using the axioms of Section 4.1, we are going to argue that the following leafwise ramified flat line bundle over $S_{\mathbb{Q}}$:

$$\mathcal{L}_\chi = \frac{S_K \times \mathbb{C}}{G} \to S_{\mathbb{Q}}$$

should provide the relevant cohomology allowing to interpret the explicit formula (4) as a Lefschetz trace formula. We set for $j \in \{0,1,2\}$:

$$\overline{H}^j(\mathcal{L}_\chi) \;=\; \frac{1}{|G|}\sum_{h\in G} h^*(\overline{H}^j_{\mathcal{F},K}\otimes_{\mathbb{R}}\mathbb{C}) = (\overline{H}^j_{\mathcal{F},K}\otimes_{\mathbb{R}}\mathbb{C})^G\,.$$

Since χ is a non trivial (primitive) character, G acts trivially on $\overline{H}^0_{\mathcal{F},K}\simeq\mathbb{R}$, $\overline{H}^2_{\mathcal{F},K}\simeq\mathbb{R}[\lambda_g]$, and we get that $\overline{H}^j(\mathcal{L}_\chi)=0$ for $j=0$ or $j=2$. Recall that the actions of $(\phi^t)_{t\in\mathbb{R}}$ and G on S_K commute, so the flow ϕ^t induces an action (still) denoted $(\phi^t)^*$ on $\overline{H}^1(\mathcal{L}_\chi)$. Now we are going to give a formal proof an Atiyah-Bott-Lefschetz trace formula whose geometric side coincides with the one of the explicit formula (4) associated to $\Lambda(\chi,s)$.

Theorem 4.5. (*"informal" theorem*) *Consider $\alpha\in C^\infty_{\text{compact}}(\mathbb{R}^{+*})$. We then have*:

$$\begin{aligned}&-\operatorname{TR}\left(\int_{\mathbb{R}}\alpha(t)(\phi^t)^*dt:\overline{H}^1(\mathcal{L}_\chi)\to\overline{H}^1(\mathcal{L}_\chi)\right)\\&=\sum_{p\in\mathbb{P},\,p\wedge m=1}\log p\left(\sum_{k\geq1}\chi(p)^k\alpha(k\log p)\right)+\int_0^{+\infty}\frac{\alpha(x)e^{-qx}}{1-e^{-2x}}\,dx\,,\end{aligned}\tag{24}$$

where $q\in\{0,1\}$ is such that $\chi(-1)=(-1)^q$.

Proof. We proceed as in the proof of Theorem 3.4 and write the left handside of (24) as:

$$-\frac{1}{|G|}\sum_{h\in G}\operatorname{TR}\int_{\mathbb{R}}\alpha(s)\,\pi^1_\tau\,(h^{-1}\circ\phi^s)^*\,ds:\overline{H}^1_{\mathcal{F},K}\otimes_{\mathbb{R}}\mathbb{C}\to\overline{H}^1_{\mathcal{F},K}\otimes_{\mathbb{R}}\mathbb{C}\,.\tag{25}$$

First we compute (formally) the contributions of the closed orbits according to Guillemin-Sternberg trace formula. Let p be a prime number such that $p\wedge m=1$. Then p is unramified in $K=\mathbb{Q}[e^{\frac{2i\pi}{m}}]$ $(e=1)$ and, with the notations of (18), the residue class $[p]\in(\mathbb{Z}/m\mathbb{Z})^*$ belongs to $D_{\mathcal{P}_j}$ and is such that $\Theta_j[p]=\mathrm{Fr}$. See [29, page 109]. Notice that since here G is abelian, the $D_{\mathcal{P}_j}$ are all equal for $1\leq j\leq r$. We then consider the closed orbit $k\gamma_p$ for $k\in\mathbb{N}^*$, γ_p being iterated k times. Pick up a point $x\in\gamma_p$, for each $j\in\{1,\dots,r\}$ we select a point $\tilde{x}_j\in\gamma_{\mathcal{P}_j}$ such that $\pi_K(\tilde{x}_j)=x$. Then using (18) and (21) one immediately gets:

$$\forall(j,t)\in\{1,\dots,r\}\times\mathbb{R},\ [p]^{-k}\phi^t(\tilde{x}_j)=(\tilde{x}_j+t-k\log p)\,.$$

Now we recall Assumption 6 in Section 4.1. Then, proceeding as in the proof of Theorem 3.4 and using the equality $rf=|G|$ $(=\phi(m))$ one checks easily that the contribution of $k\gamma_p$ to the expression (25) is equal to

$$\log p\,\chi(p)^k\,\alpha(k\log p)\,\epsilon_{k\gamma_p}\,.$$

But thanks to Assumption 3a) in Section 4.1 (or (22)), the sign $\epsilon_{k\gamma_p}$ is equal to 1 (the determinant of a direct similitude being positive). Therefore the contribution of $k\gamma_p$ is the one expected in (24).

In order to deal with the ramified closed orbits, we use the following:

Lemma 4.6. *Assume that the prime number p divides m so that p is ramified in K and the inertia groups $I_{\mathcal{P}_j}$ (introduced near (18)) are not trivial. Then the restriction of χ to any of the $I_{\mathcal{P}_j}$ $(1 \leq j \leq r)$ is not trivial.*

Proof. We follow [28, Proposition 10.3, page 61]. Let $m = \prod_l l^{\nu_l}$ be the prime factorization of m and let f_p be smallest positive integer such that

$$p^{f_p} \equiv 1 \operatorname{mod}(m/p^{\nu_p}).$$

Then one has in $K = \mathbb{Q}[e^{\frac{2i\pi}{m}}]$ the factorization:

$$p\,\mathcal{O}_K = (\mathcal{P}_1 \cdots \mathcal{P}_r)^{\phi(p^{\nu_p})},$$

where $\mathcal{P}_1, \ldots, \mathcal{P}_r$ are distinct prime ideals, all of norm p^{f_p}. Using the Chinese remainder isomorphism:

$$\left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^* \simeq \left(\frac{\mathbb{Z}}{\frac{m}{p^{\nu_p}}\mathbb{Z}}\right)^* \times \left(\frac{\mathbb{Z}}{p^{\nu_p}\mathbb{Z}}\right)^*,$$

an inspection of the proof of [28, Proposition 10.3, page 61] allows to see that for any $j \in \{1, \ldots, r\}$

$$D_{\mathcal{P}_j} = D_{\mathcal{P}_1} \simeq < p > \times \left(\frac{\mathbb{Z}}{p^{\nu_p}\mathbb{Z}}\right)^*, \; I_{\mathcal{P}_j} = I_{\mathcal{P}_1} \simeq \{1\} \times \left(\frac{\mathbb{Z}}{p^{\nu_p}\mathbb{Z}}\right)^*.$$

Now, the fact that χ is primitive implies clearly that the restriction of χ to $I_{\mathcal{P}_j}$ is not trivial (use (2) with $m' = p^{\frac{m}{\nu_p}}$). □

Suppose now that the prime number p divides m so that γ_p is a ramified closed orbit. Observe that Condition (22) implies that the analogue of (9) is satisfied with all the signs being positive. The previous Lemma, Assumption 6 of Section 4.1 and the proof of Theorem 3.4 then show formally that the geometric contribution of the ramified γ_p in (25) is zero as expected in (24).

Consider now the archimedean places. Assume first that they are all complex, then $\phi(m) = 2r_2$. Recall the associated fixed points $z_{1,\infty}, \ldots, z_{r_2,\infty}$ of ϕ^t in Section 4.1 and the elements $h_1, \ldots, h_{r_2}$ of the end of Section 4.3; they satisfy $h_j(z_{j,\infty}) = z_{j,\infty}$, $(1 \leq j \leq r_2)$. Since G is abelian, the elements h_j $(1 \leq j \leq r_2)$ all equal and actually they are equal to -1 (see [29, Page 109]). Then for any

$h \in G \setminus \{1, -1\}$, $h(z_{j,\infty}) \neq z_{j,\infty}$ $(1 \leq j \leq r_2)$. The $z_{1,\infty}, \dots, z_{r_2,\infty}$ are fixed points of $h_1\phi^t$, ϕ^t for all $t \in \mathbb{R}$. So we are reduced to analyze the contributions of $z_{1,\infty}, \dots, z_{r_2,\infty}$ to:

$$-\frac{1}{|G|} \sum_{h \in \{1,-1\}} \mathrm{TR} \int_{\mathbb{R}} \alpha(s)\, \pi^1_\tau\, (h^{-1} \circ \phi^s)^*\, ds : \overline{H}^1_{\mathcal{F},K} \otimes_{\mathbb{R}} \mathbb{C} \to \overline{H}^1_{\mathcal{F},K} \otimes_{\mathbb{R}} \mathbb{C}\,. \quad (26)$$

We recall the axioms of Section 4.1 and 4.3 (for $h_1 = -1$). Then, using the arguments of the proof of Proposition 4.4 and the Guillemin-Sternberg trace formula, one shows formally that the geometric contributions (for $t > 0$) of the fixed points of ϕ^t in (25) (or (26)) is given by:

$$\int_0^{+\infty} \frac{1}{2r_2} \sum_{j=1}^{r_2} \left(\frac{1}{1 - e^{-t}} + \frac{\chi(-1)}{1 + e^{-t}} \right) \alpha(t) dt\,.$$

But this is exactly the contribution of the archimedean place of $S_{\mathbb{Q}}$ in (24).

Assume now that the archimedean places are all real so that $r_1 = \phi(m)$. Then $e^{\frac{2i\pi}{m}}$ is real which implies that $m = 2$. But this case is excluded by assumption. $\square$

We have reproved incidentally the fact that a primitive Dirichlet L-function is a special case of an Artin L-function. The general Artin L-functions will be the topic of the next Section.

5 – Artin conjecture as a consequence of an hypothetic Atiyah-Bott-Lefschetz proof of the explicit formula for $\widehat{\zeta}_K$

The following Section should be viewed as a working programme or a motivation for developing interesting mathematics. We shall perform computations using the various Assumptions of Section 4. But, notice that we shall not use here Assumption 5) of Section 4.1 (the one which would imply the Riemann Hypothesis).

5.1 – The Artin L-function $\Lambda(K, \chi, s)$

Let K be a finite Galois extension of $\mathbb{Q}$ with Galois group G. Consider a complex representation $\rho : G \to GL(V)$ where V is a complex vector space of dimension N. Its character $\chi : G \to GL(V)$ is defined by $\chi(h) = \mathrm{Tr}\, \rho(h)$, $h \in G$. We are going to recall the definition of the Artin L-function $\Lambda(K, \chi, s)$ associated to ρ (ρ is determined up to isomorphism by χ).

Let $p \in \mathbb{P}$ be a prime number, we use the notations of Section 4.3 and (18). Let $\mathcal{P}_j, \mathcal{P}$ be two prime ideals of $\mathcal{O}_K$ lying over p, there exists $h \in G$ such that $h\mathcal{P}_j = \mathcal{P}$. Choose $\Phi_{\mathcal{P}_j} \in D_{\mathcal{P}_j}$ (modulo $I_{\mathcal{P}_j}$) and $\Phi_{\mathcal{P}} \in D_{\mathcal{P}}$ such that $\Theta_j(\Phi_{\mathcal{P}_j}) =$ Fr and $\Theta(\Phi_{\mathcal{P}}) =$ Fr. Denote now by $V^{I_{\mathcal{P}_j}}$ the subset of vectors V which are invariant under the action of $I_{\mathcal{P}_j}$. It is then clear that $\det(Id - p^{-s}\rho(\Phi_{\mathcal{P}_j}); V^{I_{\mathcal{P}_j}})$ and $\det(Id - p^{-s}\rho(\Phi_{\mathcal{P}}); V^{I_{\mathcal{P}}})$ do not depend on the choice of respectively $\Phi_{\mathcal{P}_j}$, $\Phi_{\mathcal{P}}$. Therefore, we can assume that $\Phi_{\mathcal{P}} = h\Phi_{\mathcal{P}_j}h^{-1}$. Then since, $hD_{\mathcal{P}_j}h^{-1} = D_{\mathcal{P}}$ and $hI_{\mathcal{P}_j}h^{-1} = I_{\mathcal{P}}$, we obtain:

$$\forall s \in \mathbb{C},\ \det(Id - p^{-s}\rho(\Phi_{\mathcal{P}_j}); V^{I_{\mathcal{P}_j}}) = \det(Id - p^{-s}\rho(\Phi_{\mathcal{P}}); V^{I_{\mathcal{P}}}) .$$

If all the archimedean absolute values of K are real we set $n_\sigma^+ = N = \dim V$, $n_\sigma^- = 0$. If all the archimedean absolute values of K are complex, we set:

$$n_\sigma^+ = \dim \ker(\rho(h_j) - Id),\ n_\sigma^- = \dim \ker(\rho(h_j) + Id) ,$$

where h_j is any of the elements $h_1, \dots, h_{r_2}$ of G introduced at the end of Section 4.3 (they are all conjugate to each other and satisfy $h_j^2 = 1$).

Now, for $s \in \mathbb{C}$ such that $\Re s > 1$, we define the Artin L-function as:

$$\Lambda(K,\chi,s) = (\mathcal{N}(K,\chi))^{s/2}\Gamma_{\mathbb{R}}(s)^{n_\sigma^+}\Gamma_{\mathbb{R}}(s+1)^{n_\sigma^-} \prod_{p\in\mathbb{P}} \frac{1}{\det(Id - p^{-s}\rho(\Phi_{\mathcal{P}}); V^{I_{\mathcal{P}}})} , \quad (27)$$

where the positive integer $\mathcal{N}(K,\chi)$ denotes the norm of the Artin conductor of χ. Recall that $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$.

Now we recall Brauer's theorem in order to explain the meromorphic continuation of $\Lambda(K,\chi,s)$. The character χ is an integral linear combination $\chi = \sum_{i=1}^k n_i\chi_{i*}$ where the $n_i \in \mathbb{Z}$ and the χ_{i*} are induced from characters χ_i of degree 1 on subgroups $H_i = \mathrm{Gal}(K : L_i)$, L_i a suitable subfield of K. From this, one can deduce that:

$$\Lambda(K,\chi,s) = \prod_{i=1}^k \Lambda(K,\chi_{i*},s)^{n_i} = \prod_{i=1}^k \Lambda(\widetilde{\chi_i},s)^{n_i} , \quad (28)$$

where $\Lambda(\widetilde{\chi_i},s)$ is the L-function attached to the Groessencharakter $\widetilde{\chi_i}$ associated to χ_i. From these identities, one deduces that $\Lambda(K,\chi,s)$ admits a meromorphic continuation to $\mathbb{C}$ with zeroes and poles all belonging to the critical strip $0 \le \Re s \le 1$. Moreover, it satisfies the functional equation:

$$\Lambda(K,\chi,s) = W(\chi)\Lambda(K,\overline{\chi},1-s) ,$$

where $W(\chi)$ is a complex constant of modulus 1.

Artin Conjecture. If the representation ρ is irreducible then $\Lambda(K,\chi,s)$ is entire (without any pole).

So Artin conjecture means that each zero of $\Lambda(\widetilde{\chi_i},s)$ for negative n_i in (28) is compensated by a zero of another $\Lambda(\widetilde{\chi_j},s)$ for positive n_j. This conjecture is proved when G is abelian. In the non abelian case, it is proved in several particular cases using deep methods (see Taylor's survey [31]), but it remains widely open in the general case.

5.2 – Explicit Formulas and Trace Formulas

THEOREM 5.1. *Let $\{\lambda_k, k \in I\}$ (resp. $\{\mu_j, j \in J\}$) be the set of zeroes (resp. poles) of $\Lambda(K,\chi,s)$. Let $\alpha \in C^\infty_{\text{compact}}(\mathbb{R}^{+*})$. Then one has:*

$$\begin{aligned} &-\sum_{k\in I}\int_0^{+\infty}\alpha(s)e^{s\lambda_k}\,ds+\sum_{j\in J}\int_0^{+\infty}\alpha(s)e^{s\mu_j}\,ds \\ &=\sum_{p\in\mathbb{P}}\log p\sum_{n\geq 1}\alpha(n\log p)\mathrm{Tr}\,(\Phi^n_{\mathcal{P}}:V^{I_{\mathcal{P}}})+\int_0^{+\infty}\frac{\alpha(x)}{1-e^{-2x}}(n^+_\sigma+n^-_\sigma e^{-x})\,dx\,. \end{aligned} \tag{29}$$

PROOF. One proceeds exactly as for the proof of the explicit formula (4), using the functional equation for $\Lambda(K,\chi,s)$ and its Eulerian product. □

Now we define an action of G on $S_K \times V$ by

$$h\cdot(z,v)=(h\cdot z,\rho^{-1}(h)\cdot v),\ \forall(h,z,v)\in G\times S_K\times V\,.$$

Then using the axioms of Section 4.1, we are going to argue that the following leafwise ramified flat vector bundle over $S_{\mathbb{Q}}$:

$$\mathcal{E}_\rho=\frac{S_K\times V}{G}\to S_{\mathbb{Q}}$$

should provide the relevant cohomology allowing to exhibit an explicit formula for $\Lambda(K,\chi,s)$ via a Lefschetz trace formula. We set for $j\in\{0,1,2\}$:

$$\overline{H}^j(\mathcal{E}_\rho)\;=\;\frac{1}{|G|}\sum_{h\in G}h^*(\overline{H}^j_{\mathcal{F},K}\otimes_{\mathbb{R}}V)=(\overline{H}^j_{\mathcal{F},K}\otimes_{\mathbb{R}}V)^G\,.$$

Recall that the actions of (ϕ^t) and G on S_K are assumed to commute. We shall still denote by $(\phi^t)^*$ the action on $\overline{H}^j(\mathcal{E}_\rho)$ induced by the flow (ϕ^t).

Next we recall and use Assumption 6) of Section 4.1. According to the hypothesis of Section 4.3, for each zero z_q of $\widehat{\zeta}_K$, G leaves each

$$\ker(\theta_1-z_qId)^{n(z_q)}\otimes_{\mathbb{C}}V=W_q$$

globally invariant and commutes with $\theta_1\otimes_{\mathbb{C}}Id_V$ (which we shall denote simply θ_1). Next, observe that θ_1-z_qId induces a nilpotent endomorphism of $\frac{1}{|G|}\sum_{h\in G}h\cdot W_q$, therefore:

$$\forall t\in\mathbb{R},\ \mathrm{Tr}\left(e^{t\theta_1}:\frac{1}{|G|}\sum_{h\in G}h\cdot W_q\right)\;=\;d_q\,e^{tz_q}\,, \tag{30}$$

where $d_q=\dim\frac{1}{|G|}\sum_{h\in G}h\cdot W_q$. Notice that a priori some of the d_q may be equal to 0.

THEOREM 5.2 ("Informal" Theorem). *Assume that the representation ρ is irreducible. Let $\alpha \in C^{\infty}_{\mathrm{compact}}(\mathbb{R}^{+*})$. Then one has*:

$$\begin{aligned} &- \sum_{z_q \in \widehat{\zeta}_K^{-1}\{0\}} d_q \int_0^{+\infty} \alpha(s) e^{s z_q}\, ds \\ &= \sum_{p \in \mathbb{P}} \log p \sum_{k \geq 1} \alpha(k \log p) \mathrm{Tr}\,(\Phi_{\mathcal{P}}^k : V^{I_{\mathcal{P}}}) + \int_0^{+\infty} \frac{\alpha(x)}{1 - e^{-2x}} (n_\sigma^+ + n_\sigma^- e^{-x})\, dx\,. \end{aligned} \tag{31}$$

COMMENT. Actually, we can only assume that $V^G = \{0\}$, which is of course weaker than ρ irreducible. The Γ factor $\Gamma_{\mathbb{R}}(s)^{n_\sigma^+} \Gamma_{\mathbb{R}}(s+1)^{n_\sigma^-}$ comes in the definition of $\Lambda(K, \chi, s)$ as a parachute and is not well motivated by a mathematical structure. It was introduced there (by Artin himself?) in order to get (28). That is why in the spectral side of (29) we have "uncontroled poles". This Γ factor appears naturally in the computation of the contribution of the fixed points in (31) and that is why the spectral side of (31) is better "controled".

PROOF. We shall use the arguments of the proofs of Theorems 3.4 and 4.5. Since G acts trivially on $\overline{H}^0_{\mathcal{F},K} \simeq \mathbb{R}$ and $\overline{H}^2_{\mathcal{F},K} \simeq \mathbb{R}[\lambda_g]$, one gets that $\overline{H}^j(\mathcal{E}_\rho) = \{0\}$ for $j = 0, 2$. Therefore, using (30), one gets:

$$\sum_{j=0}^{2} (-1)^j \mathrm{TR} \left(\int_{\mathbb{R}} \alpha(t) (\phi^t)^* dt : \overline{H}^j(\mathcal{E}_\rho) \right) = - \sum_{z_q \in \widehat{\zeta}_K^{-1}\{0\}} d_q \int_0^{+\infty} \alpha(s) e^{s z_q}\, ds\,.$$

We now write the left hand side of this equality under the form:

$$\begin{aligned} &\frac{1}{|G|} \sum_{h \in G} \sum_{j=0}^{2} (-1)^j\, \mathrm{TR} \left(\int_{\mathbb{R}} \alpha(s)\, \pi_\tau^j\, (h^{-1} \circ \phi^s)^*\, ds : \overline{H}^j_{\mathcal{F},K} \otimes_{\mathbb{R}} V \right) \\ &\quad = - \frac{1}{|G|} \sum_{h \in G} \mathrm{TR} \left(\int_{\mathbb{R}} \alpha(s)\, \pi_\tau^1\, (h^{-1} \circ \phi^s)^*\, ds : \overline{H}^1_{\mathcal{F},K} \otimes_{\mathbb{R}} V \right) \end{aligned} \tag{32}$$

where π_τ^j (written for $\pi_\tau^j \otimes Id_V$) denotes the Hodge projection onto leafwise harmonic forms. We now proceed to compute formally the geometric contributions in (32) of the closed orbits and of the fixed points of (ϕ^t) according to the Guillemin-Sternberg trace formula.

Thus, we consider a prime number $p \in \mathbb{P}$ and the contribution of the closed orbit $k\gamma_p$, $k \in \mathbb{N}^*$. Let $\mathcal{P}$ be any of the prime ideals $\mathcal{P}_1, \ldots, \mathcal{P}_r$ of $\mathcal{O}_K$ such that $p\mathcal{O}_K = \mathcal{P}_1^e \ldots \mathcal{P}_r^e$. Fix $x_0 \in \gamma_p$ and $\tilde{x}_0 \in \gamma_{\mathcal{P}}$ such that $\pi_K(\tilde{x}_0) = x_0$. The axioms of Section 4.3 and (21) imply that:

$$\forall t \in \mathbb{R},\ \Phi_{\mathcal{P}}^{-k} \phi^t(\tilde{x}_0) = (\tilde{x}_0 + t - k \log p)\,.$$

First assume that p is not ramified in K ($e=1$). Then, there are exactly $|G|$ curves (of the flow) in S_K lying (via π_K) over $k\gamma_p$. They are given by

$$t \to l \cdot \Phi_{\mathcal{P}}^{-k} l^{-1} \cdot \phi^t (l \cdot \tilde{x_0}),\ 0 \le t \le k \log p,\ l \in G\,.$$

Then the proof of Theorem 3.2 ([1]) shows that the geometric contribution of $k\gamma_p$ to (32) is computed according to Proposition 3.1 and is equal to:

$$\frac{\log p}{|G|} \sum_{l \in G} \sum_{j=0}^{2} (-1)^j \frac{\mathrm{Tr}\left({}^t D(l \Phi_{\mathcal{P}}^{-k} l^{-1} \circ \phi^{k \log p})(l \cdot \tilde{x_0}); \wedge^j T^*_{l \cdot \tilde{x_0}} \mathcal{F}\right)}{|\det\big(\mathrm{id} - D(l \Phi_{\mathcal{P}}^{-k} l^{-1} \circ \phi^{k \log p})\big)_{|T_{l\cdot \tilde{x_0}} \mathcal{F}}|} \chi(l \Phi_{\mathcal{P}}^{k} l^{-1})\, \alpha(k \log p)\,.$$

The sign assumption (22) shows that all these determinants are positive. Therefore, the geometric contribution of $k\gamma_p$ to (32) is equal to $\log p\, \chi(\Phi_{\mathcal{P}}^k)\, \alpha(k \log p)$ as expected in (31).

Next assume that p is ramified in K ($e>1$). So $I_{\mathcal{P}} = G_{\tilde{x_0}} = \{u \in G / \, u \cdot \tilde{x_0} = \tilde{x_0}\}$ is not trivial. Then there are exactly $|G/G_{\tilde{x_0}}|$ curves (of the flow) upstairs on $\tilde{X}$ which correspond (via π) to $k\gamma_{\log p}$. They are given by $t \mapsto l_j \Phi_{\mathcal{P}}^{-k} l_j^{-1} \circ \phi^t (l_j \cdot \tilde{x}_0)$ ($0 \le t \le k \log p$), where the l_j ($1 \le j \le rf$) run over a system of representatives of cosets of $G/G_{\tilde{x_0}}$. We have to count each such curve $|G_{\tilde{x_0}}|$ times but with (possibly) different monodromies (*i.e.* action on the vector space factor V). More precisely, for each representative l_j the monodromy of the curve labeled

$$t \mapsto l_j \Phi_{\mathcal{P}}^{-k} l_j^{-1} u \circ \phi^t (l_j \cdot \tilde{x}_0),\ \text{with}\, u \in l_j G_{\tilde{x_0}} l_j^{-1}\,, \tag{33}$$

is equal to $\rho(u^{-1} l_j \Phi_{\mathcal{P}}^k l_j^{-1})$. The proof given above in the unramified case, with a more precise use of the sign assumption (22), allows to check that the geometric contribution of $k\gamma_p$ to (32) is equal to

$$\begin{aligned}&\frac{\log p\, \alpha(k \log p)}{|G|} \sum_{j=1}^{fr} \sum_{u' \in I_{\mathcal{P}}} \mathrm{Tr}\,(\rho(l_j \Phi_{\mathcal{P}}^k l_j^{-1} l_j u' l_j^{-1})\\ &= \log p\, \alpha(k \log p)\, \mathrm{Tr} \left(\rho(\Phi_{\mathcal{P}}^k)\, \frac{1}{e} \sum_{u' \in I_{\mathcal{P}}} \rho(u') \right)\,.\end{aligned}$$

Since $\frac{1}{e} \sum_{u' \in I_{\mathcal{P}}} \rho(u')$ is a projection of V onto $V^{I_{\mathcal{P}}}$ which commutes with $\rho(\Phi_{\mathcal{P}}^k)$, this contribution coincides with

$$\log p\, \alpha(k \log p)\, \mathrm{Tr}\,(\rho(\Phi_{\mathcal{P}}^k) : V^{I_{\mathcal{P}}})$$

as expected in (31).

Now we come to the contribution to (32) of the fixed points (*i.e.* the archimedean places of K). We proceed as in the proof of Theorem 4.5 and give only a sketch.

First assume that all the archimedean places are complex. Then $|G| = 2r_2$, recall the associated fixed points $z_{1,\infty}, , \dots, z_{r_2,\infty}$ of ϕ^t in Section 4.1 and the associated elements h_j of G in Section 4.3. Then for each $j \in \{1, \dots, r_2\}$, $z_{j,\infty}$ is a fixed point of $h_j \phi^t$, ϕ^t for all $t \in \mathbb{R}$, and for any $h \in G \setminus \{1, h_j\}$, $h(z_{j,\infty}) \neq z_{j,\infty}$. Thus we have to determine the contributions of $z_{1,\infty}, \dots, z_{r_2,\infty}$ to:

$$-\frac{1}{|G|} \sum_{h \in \{1, h_1, \dots, h_{r_2}\}} \mathrm{TR} \left(\int_{\mathbb{R}} \alpha(s) \pi_\tau^1 \, (h^{-1} \circ \phi^s)^* ds : \overline{H}^1_{\mathcal{F},K} \otimes_{\mathbb{R}} V \to \overline{H}^1_{\mathcal{F},K} \otimes_{\mathbb{R}} V \right). \quad (34)$$

We recall the axioms of Sections 4.1 and 4.3 (for the h_j). Then, using the arguments of the proof of Proposition 4.4 and the Guillemin-Sternberg trace formula, one shows formally that the geometric contributions (for $t > 0$) of the fixed points of ϕ^t in (32) (or (34)) is given by:

$$\int_0^{+\infty} \frac{1}{2r_2} \sum_{j=1}^{r_2} \left(\frac{\mathrm{Tr}\, \rho(Id_V)}{1 - e^{-t}} + \frac{\mathrm{Tr}\, \rho(h_j)}{1 + e^{-t}} \right) \alpha(t) dt \, .$$

Recall that $\rho(h_j)$ is a symmetry of V such that $\mathrm{Tr}\, \rho(h_j) = n_\sigma^+ - n_\sigma^-$. One then computes that for any real $x > 0$:

$$\frac{\mathrm{Tr}\, \rho(Id_V)}{1 - e^{-x}} + \frac{\mathrm{Tr}\, \rho(h_j)}{1 + e^{-x}} = \frac{n_\sigma^+ + n_\sigma^-}{1 - e^{-x}} + \frac{n_\sigma^+ - n_\sigma^-}{1 + e^{-x}} = \frac{2n_\sigma^+}{1 - e^{-2x}} + \frac{2n_\sigma^- e^{-x}}{1 - e^{-2x}} \, .$$

One then proves easily that the contribution to (32) of the fixed points coincides with

$$\int_0^{+\infty} \frac{\alpha(x)}{1 - e^{-2x}} (n_\sigma^+ + n_\sigma^- e^{-x}) \, dx$$

as expected in (31).

When all the archimedean places are real, the situation is much simpler. Recall the set of fixed points $\{x_{1,\infty}, \dots, x_{r_1,\infty}\}$ associated to the real archimedean places, then for any $h \in G \setminus \{1\}$, $h(x_{j,\infty}) \neq x_{j,\infty}$ $(1 \leq j \leq r_1)$. So we are reduced to analyze the contributions of $x_{1,\infty}, \dots, x_{r_1,\infty}$ to:

$$-\frac{1}{|G|} \mathrm{TR} \left(\int_{\mathbb{R}} \alpha(s) \, \pi_\tau^1 \, (\phi^s)^* \, ds : \overline{H}^1_{\mathcal{F},K} \otimes_{\mathbb{R}} V \to \overline{H}^1_{\mathcal{F},K} \otimes_{\mathbb{R}} V \right) . \quad (35)$$

Therefore, the axioms of Section 4.1 and the Guillemin-Sternberg trace formula show formally that the geometric contributions (for $t > 0$) of the fixed points of ϕ^t in (32) (or (35)) is given by:

$$\int_0^{+\infty} \frac{1}{r_1} \left(\sum_{j=1}^{r_1} \frac{\mathrm{Tr}\, \rho(Id_V)}{1 - e^{-2t}} \right) \alpha(t) dt \, .$$

But, since here $\mathrm{Tr}\, \rho(Id_V) = N = n_\sigma^+$ and $n_\sigma^- = 0$, this is exactly the expected contribution in (31). □

THEOREM 5.3. *Let $(u_k)_{k\in A}$ and $(v_l)_{l\in B}$ be two sequences of points (with possible multiplicity) of the critical strip $\{z\in\mathbb{C},\ 0\le \Re z\le 1\}$, A and B being two subsets of $\mathbb{N}$. Assume that $\sum_{k\in A}\frac{1}{1+|u_k|^2}+\sum_{l\in B}\frac{1}{1+|v_l|^2}<+\infty$ and that for any $\alpha\in C^\infty_{\mathrm{compact}}(\mathbb{R}^{+*})$:*

$$\sum_{k\in A}\int_0^{+\infty}\alpha(s)e^{su_k}\,ds=\sum_{l\in B}\int_0^{+\infty}\alpha(s)e^{sv_l}\,ds\,. \tag{36}$$

Then, there exists a bijection $\xi: A\to B$ such that for any $k\in A, u_k=v_{\xi(k)}$ with the same multiplicity.

PROOF. We give only a sketch. First, in (36) we can replace $\alpha(s)$ by $\alpha(s)e^{-2s}$. Then, two successive integration by parts allow to see that:

$$\int_0^{+\infty}\alpha''(s)\sum_{k\in A}\frac{e^{s(u_k-2)}}{(u_k-2)^2}\,ds=\int_0^{+\infty}\alpha''(s)\sum_{l\in B}\frac{e^{s(v_l-2)}}{(v_l-2)^2}\,ds\,.$$

Therefore there exists two constants M_1, M_2 such that:

$$\forall s\in[0,+\infty[,\ \sum_{k\in A}\frac{e^{s(u_k-2)}}{(u_k-2)^2}=\sum_{l\in B}\frac{e^{s(v_l-2)}}{(v_l-2)^2}+M_1+sM_2\,.$$

Letting $s\to+\infty$, one gets $M_1=M_2=0$. Now applying inductively $\int_{+\infty}^s$ to the previous identity, one obtains for every $r\in\mathbb{N}$:

$$\sum_{k\in A}\frac{1}{(u_k-2)^{2+r}}=\sum_{l\in B}\frac{1}{(v_l-2)^{2+r}}\,.$$

In order to finish the proof, we use an elegant argument pointed out to us by Vincent Lafforgue. The previous equality implies that all the derivatives at 0 of the following meromorphic function vanish:

$$z\mapsto\sum_{k\in A}\frac{1}{(z+u_k-2)^2}-\sum_{l\in B}\frac{1}{(z+v_l-2)^2}\,.$$

Hence this function is identically zero, which proves the theorem. □

The conjunction of the three last Theorems (formally) would imply Artin conjecture and that the zeroes of $\Lambda(K,\chi,s)$ are the zeroes z_q of $\widehat{\zeta}_K$ with multiciplity d_q. Of course it is understood that if $d_q = 0$ then z_q is not a zero of $\Lambda(K,\chi,s)$.

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