Rendiconti di Matematica, Serie VII
Volume 35, Roma (2014), 47 – 68

Subharmonic functions with a Bergman type growth

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Abstract: *Subharmonic functions with a Bergman-type growth on the unit ball of* $\mathbb{R}^N$ *(*$N \in \mathbb{N}$, $N \geq 2$*) are studied jointly with their Riesz measure. Various estimations are obtained which generalize previous results due to C. Horowitz and A. A. Dolgoborodov concerning holomorphic functions in the unit disk of* $\mathbb{C}$*, belonging for instance to the Bergman space.*

1 – Introduction

Given N an integer ≥ 2, let $|\,.\,|$ denote the Euclidean norm in $\mathbb{R}^N$. The growth of a function u subharmonic on the open unit ball $B_N = \{x \in \mathbb{R}^N : |x| < 1\}$ impacts on the Riesz measure μ associated to u, as well as on its repartition function ρ given by $\rho(r) = \int_{|\zeta| \leq r} d\mu(\zeta)$ and on P_μ defined by $P_\mu(r) = \int_{|\zeta| < s_\mu(r)} h(|\zeta|)\, d\mu(\zeta)$ $\forall r \in]0,1[$ where both functions s_μ and h will be explicitly defined in Section 2.

Definition 1.1. Let $d\sigma$ denote the area element on the unit sphere $S_N = \{x \in \mathbb{R}^N : |x| = 1\}$. The area of S_N is written $\sigma_N = \int_{S_N} d\sigma$. For information $\sigma_N = \frac{2\,\pi^{N/2}}{\Gamma(N/2)}$ (see [3, page 29]). Let $\mathcal{M}_u(r) = \frac{1}{\sigma_N} \int_{S_N} u(r\eta)\, d\sigma_\eta$ for any $r \in [0,1[$ and any function u subharmonic in B_N.

This article studies subharmonic functions u under such a growth condition as for instance:

$$\int_0^1 \mathcal{M}_u(r)\,[-\varphi'(r)]\, dr < +\infty$$

with a decreasing weight function φ which will be detailed later. In this case both $\mathcal{M}_u(r)$ and $P_\mu(r)$ appear as $= \mathbf{o}(\frac{1}{\varphi(r)})$ as $r \to 1^-$, together with $\rho(r^2) = \mathbf{o}(\frac{1}{\varphi(r)\,h(r)})$.

Key Words and Phrases: *Subharmonic functions – Growth – Riesz measure – Unit ball*
A.M.S. Classification: 26B15, 28A75, 30H20, 31B05, 40A10.

Besides that, for any increasing function g such that $\int_0^{\to 1} g(t)\,dt$ diverges, it turns out that:

$$\sup_{r\in[s,1[}\left(\frac{1}{1-r}\,mes\left\{t\in[r,1[:\ \mathcal{M}_u(t)<\frac{g(t)}{-\varphi'(t)}\right\}\right)=1\qquad\forall s\in[0,1[$$

and the same holds for the sets $\left\{t\in[r,1[:\rho(t^2)<\frac{-g(t)}{\varphi'(t)\,h(t)}\right\}$ and $\left\{t\in[r,1[:P_\mu(t)<\frac{-g(t)}{\varphi'(t)}\right\}$.

Theorems 4.3, 4.9, 5.1 and 5.4 (in Sections 4 and 5) establish these results and refine them when u is subject to the stricter assumption:

$$\int_0^1 e^{\mathcal{M}_u(r)}\,[-\varphi'(r)]\,dr<+\infty \tag{1.1}$$

Corollary 4.4, Proposition 4.8, Proposition 4.10 and Example 5.5 pay a particular attention to the special weight φ defined by $\varphi(r)=(1-r^2)^{\alpha+1}$ $\forall r\in[0,1[$ with some fixed $\alpha>-1$. For instance, we obtain that $e^{P_\mu(r)}=\mathbf{o}(\frac{1}{(1-r)^{\alpha+1}})$ as $r\to 1^-$ and that

$$\sup_{r\in[s,1[}\left(\frac{1}{1-r}\,mes\left\{t\in[r,1[:\ e^{P_\mu(t)}<\frac{1}{2(\alpha+1)(1-t)^{\alpha+1}\log\left(\frac{1}{1-t}\right)}\right\}\right)=1$$
$$\forall s\in[0,1[.$$

A motivation for the study of P_μ was its link (in the case $N=2$) with products of the kind $\prod_{k=1}^{n}\frac{1}{|z_k|}$ involving zeros of some function f holomorphic in the unit disk of $\mathbb{C}$. When f belongs to the Bergman space of parameters $p>0$ and $\alpha>-1$, that is when f fulfills

$$\int_0^1\left(\int_0^{2\pi}|f(r\,e^{i\theta})|^p\,d\theta\right)(1-r^2)^\alpha\,r\,dr<+\infty \tag{1.2}$$

it was already known (see [2, page 103]) that the number of zeros z_k with moduli $\le r$ was a $\mathbf{O}\left(\frac{1}{1-r}\log\frac{1}{1-r}\right)$. Here we improve the estimate to $\mathbf{O}\left(\frac{1}{1-r}\left(\lambda(r)+\log\frac{1}{1-r}\right)\right)$ where λ is a function with limit $-\infty$ (see Example 4.7 in Section 4 for the explicit expression of λ). Corollary 4.12 evaluates when this number of zeros is $<\frac{2/p}{1-t}\left[(\alpha+1)\log(\frac{1}{1-t})-\log\log(\frac{1}{1-t})\right]$.

Several known results about Bergman spaces thus appear as particular cases when $N=2$ of the theorems established for $N\ge 2$ in Sections 4 and 5. For instance, in Example 5.3, we recover $\prod_{k=1}^{n}\frac{1}{|z_k|}=\mathbf{o}(n^{(\alpha+1)/p})$ as $n\to+\infty$ (the estimation with $\mathbf{O}$ was proved by Horowitz [4], the refinement with $\mathbf{o}$ was shown by Dolgoborodov[1]).

Theorem 5.7 and Theorem 5.8 of Section 5 are building subsequences of integers n for which $\prod_{k=1}^{n} \frac{1}{|z_k|^p}$ may be majorized by $\frac{n^{\alpha+1}}{(\log n)(\log\log n)}$. A similar result had been obtained by Dolgoborodov [1] who provided a majorant of the kind $\frac{n^{\alpha+1}}{\log n}$. In these applications to Bergman spaces, we do not make a full use of condition (1.2). Actually we only use (1.1) with $u = p \log|f|$. Hence all statements concerning Bergman spaces in Sections 4 and 5 remain valid for the set $H^{\log}_{p,\beta}$ introduced by Dolgoborodov [1] with $p > 0$ and $\beta > 0$ (containing the Bergman space of parameters p and α when $\beta = (\alpha+1)/p$):

$$f \in H^{\log}_{p,\beta} \iff \int_0^1 e^{\mathcal{M}_u(r)}\,(1-r)^{p\beta-1}\,dr < +\infty \qquad \text{(again with } u = p\,\log|f|\text{)}.$$

The paper is organized as follows:

- Section 2 gathers preparatory results about $\mathcal{M}_u$, ρ and s_μ.
- Section 3 is devoted to a technical comparison between two functions on $[0,1[$, one of which giving rise to a convergent integral and the other to a divergent one.
- Section 4 collects the results relative to the growth of $\mathcal{M}_u$ and ρ.
- Section 5 deals with the growth of P_μ.

2 – Auxiliary lemmas

LEMMA 2.1. *Let* $\tau_N = \max(1, N-2)$. *The function* $h :]0,+\infty[\to \mathbb{R}$ *given by*

$$h(s) = \begin{Bmatrix} \log\frac{1}{s} & \text{if } N = 2 \\ \frac{1}{s^{N-2}} - 1 & \text{if } N \geq 3 \end{Bmatrix} \qquad \forall s > 0$$

fulfills $\lim_{r\to 1} \frac{h(r)}{1-r} = \tau_N$, *together with* $\frac{h(r)}{1-r} \geq \tau_N$ *and* $\int_{r^2}^{r} \frac{dt}{t^{N-1}} = \frac{1}{\tau_N}\frac{h(r)}{r^{N-2}}$ $\forall r \in]0,1[$.

PROOF. In the case $N = 2$, the inequality merely follows from the well-known estimation $-\log r \geq 1 - r > 0$ $\forall r \in]0,1[$. When $N \geq 3$, we have

$$\frac{h(r)}{1-r} = \frac{1}{r^{N-2}}\frac{1-r^{N-2}}{1-r} = \frac{1}{r^{N-2}}\sum_{k=0}^{N-3} r^k = \sum_{j=1}^{N-2}\frac{1}{r^j} \geq N-2 \qquad \text{since } \frac{1}{r^j} \geq 1.$$

If $N = 2$, then $\int_{r^2}^{r} t^{1-N}\,dt = \Big[\log t\Big]_{r^2}^{r} = \log r - \log(r^2) = -\log r = h(r)$.
If $N > 2$, then $\int_{r^2}^{r} t^{1-N}dt = \left[\frac{t^{2-N}}{2-N}\right]_{r^2}^{r} = \frac{1}{N-2}\Big(\frac{1}{(r^2)^{N-2}} - \frac{1}{r^{N-2}}\Big) = \frac{1}{\tau_N}\frac{1}{r^{N-2}}\Big(\frac{1}{r^{N-2}} - 1\Big)$. □

DEFINITION 2.2. A related function $h_r : \mathbb{R}^N \setminus \{O\} \to \mathbb{R}$ will also be used outside of O the origin of $\mathbb{R}^N$. It is defined (for a given $r > 0$) by

$$h_r(\zeta) = h(|\zeta|) - h(r) = \int_{|\zeta|}^{r} \frac{\tau_N}{t^{N-1}} dt = \begin{Bmatrix} \log \frac{r}{|\zeta|} & \text{if } N = 2 \\ \frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} & \text{if } N \geq 3 \end{Bmatrix}$$

$$\forall \zeta \in \mathbb{R}^N , \ \ \zeta \neq O.$$

Throughout the paper, adjectives "increasing" and "decreasing" are meant non-strictly: there may be some flat levels.

Given a function u subharmonic in B_N, this paragraph is devoted to some technical results related to the Riesz measure μ associated to u (see [3, page 81]). It is assumed that u is harmonic in some neighborhood of O: there exists $\varepsilon > 0$ such that u is harmonic in $\varepsilon\, B_N$.

LEMMA 2.3 (see [6]). *For any $r \in]0, 1[$ and any $r' \in]0, 1[$, the following hold:*

$$\int_{|\zeta| \leq r'} h_r(\zeta)\, d\mu(\zeta) \leq \int_{|\zeta| \leq r} h_r(\zeta)\, d\mu(\zeta) \ = \ \tau_N \int_0^r \frac{\rho(t)}{t^{N-1}}\, dt$$

with the repartition function ρ defined by: $\rho(t) = \int_{|\zeta| \leq t} d\mu(\zeta)$.

REMARK 2.4. The function ρ is increasing and right-continuous on $[0, 1[$. This probabilistic name "repartition function" is inspired from the situation in the special case $\mu(B_N) = 1$: then μ may be used to describe the law of some random variable X with values in B_N, thus $\rho(t)$ appears as the probability of the event $\{|X| \leq t\}$. More precisely, ρ is the repartition function of the variable $|X|$.

JENSEN–PRIVALOV FORMULA (see [5, page 44]).

$$\mathcal{M}_u(r) = \frac{1}{\sigma_N} \int_{S_N} u(rx)\, d\sigma_x = \tau_N \int_0^r \frac{\rho(t)}{t^{N-1}}\, dt + u(O) \qquad \forall r \in [0, 1[.$$

LEMMA 2.5. *If $u(O) \geq 0$, then $\mathcal{M}_u(r) \geq \rho(r^2)\, h(r)$ for all $r \in [0, 1[$.*

PROOF. We have $\mathcal{M}_u(r) \geq \tau_N \int_{r^2}^{r} \frac{\rho(t)}{t^{N-1}}\, dt \geq \tau_N \rho(r^2) \int_{r^2}^{r} t^{1-N}\, dt$. The result proceeds from Lemma 2.1 since $\frac{1}{r^{N-2}} \geq 1$. □

Remark 2.6. For a subharmonic function v (harmonic in some neighborhood of the origin) whose Riesz measure has a repartition function ϱ satisfying: $\varrho(t) \leq \rho(t)$ $\forall t \in [0,1[$, it obviously holds that: $\mathcal{M}_v(r) \leq \mathcal{M}_u(r) - u(O) + v(O)$ $\forall r \in [0,1[$.

Lemma 2.7. *For any* $r \in]0,1[$, *let* $s_\mu(r) = \sup\{t \in]0,1[: \rho(t) \leq \frac{1}{h(r)}\}$. *The function* s_μ *is increasing and right-continuous at any point* $r_0 \in]0,1[$. *Moreover* $\varepsilon \leq s_\mu(r) \leq 1$ $\forall r \in]0,1[$. *The sup-bound is not necessarily attained but*

$$\rho(s_\mu(r)) \geq \frac{1}{h(r)} \qquad \textit{and} \qquad \rho(t) \leq \frac{1}{h(r)} \qquad \forall t < s_\mu(r) \qquad \forall r \in]0,1[.$$

Proof. The sup-bound $s_\mu(r)$ is well-defined since $\frac{1}{h} > 0$ and $\rho \equiv 0$ on $]0,\varepsilon[$. Since $\frac{1}{h}$ is increasing, it follows that s_μ increases. Since μ is a positive measure, then ρ is increasing. Thus $\rho(t) \leq \frac{1}{h(r)}$ $\forall t \in [0, s_\mu(r)[$. Besides that $\rho(t) > \frac{1}{h(r)}$ $\forall t \in]s_\mu(r), 1[$ from the definition of $s_\mu(r)$. The right-continuity of ρ ensures that $\rho(s_\mu(r)) \geq \frac{1}{h(r)}$.

If s_μ was not right-continuous at the point r_0, there would exist $\alpha > 0$ and a sequence $(r_n)_{n\in\mathbb{N}^*}$ of numbers $> r_0$, with limit r_0, such that $s_\mu(r_n) > s_\mu(r_0) + \alpha$ for any $n \in \mathbb{N}^*$. From the above observations with $t = s_\mu(r_0) + \alpha$ and $r = r_n$, we get $\rho(\, s_\mu(r_0) + \alpha\,) \leq \frac{1}{h(r_n)}$. Letting $n \to +\infty$, we would obtain $\rho(\, s_\mu(r_0) + \alpha\,) \leq \frac{1}{h(r_0)}$ since h is continuous.

Besides that, $\rho(\, s_\mu(r_0) + \alpha\,) > \frac{1}{h(r_0)}$, hence a contradiction. □

Remark 2.8. *The function* s_μ *is not left-continuous, as shown by the following counter-example: if* $\rho \equiv \frac{1}{h(r_1)}$ *on* $[a, b[$ *and* $\rho \equiv \frac{1}{h(r_2)}$ *on* $[b, c[$ *with* $0 < a < b < c < 1$ *and* $r_1 < r_2$, *then* $s_\mu(r) = b$ $\forall r \in [r_1, r_2[$ *and* $s_\mu(r_2) \geq c$, *hence* s_μ *is not left-continuous at the point* r_2.

3 – Comparison between functions under two integrals

Definition 3.1. A function φ is said to fulfill the $\mathcal{L}$-condition if:

(i) φ is $\mathcal{C}^1$ on $[0,1[$

(ii) both φ and $(-\varphi')$ are decreasing on $[0,1[$, with values in $]0,+\infty[$ and $\lim_{t\to 1^-} \varphi(t) = 0$

(iii) there exist constants $L > 0$ and $\lambda \in]0,1[$ such that

$$\frac{\varphi(1-\lambda^k) - \varphi(1-\lambda^{k+1})}{-\lambda^{k-1}\,\varphi'(1-\lambda^{k-1})} \geq L \qquad \forall k \in \mathbb{N}^*. \tag{3.1}$$

Example 3.2. Given $\gamma \geq 1$, the function φ defined by $\varphi(t) = (1-t)^\gamma$ $\forall t \in [0,1[$ fulfills the $\mathcal{L}$-condition: we have $-\varphi'(t) = \gamma(1-t)^{\gamma-1}$ thus the fraction in (3.1) equals $\frac{\lambda^{k\gamma} - \lambda^{k\gamma}\,\lambda^\gamma}{\lambda^{k-1}\gamma\,\lambda^{k\gamma}\,\lambda^{-\gamma}\,\lambda^{-(k-1)}} = \frac{1-\lambda^\gamma}{\gamma\,\lambda^{-\gamma}}$ for any $\lambda \in]0,1[$.

Remark 3.3. Conditions (i) and (ii) alone imply a majoration for the fraction in (3.1), but not necessarily a strictly positive minoration uniform relatively to k. More precisely, given $\lambda \in]0,1[$ and $k \in \mathbb{N}^*$, there exists c such that $1-\lambda^{k-1} < 1-\lambda^k < c < 1-\lambda^{k+1}$ and

$$\begin{aligned}\varphi(1-\lambda^k) - \varphi(1-\lambda^{k+1}) &= [(1-\lambda^k)-(1-\lambda^{k+1})]\,\varphi'(c) = [\lambda^{k+1}-\lambda^k]\,\varphi'(c)\\ &= -\lambda^k(1-\lambda)\,\varphi'(c).\end{aligned}$$

Now $-\varphi'(1-\lambda^{k-1}) \geq -\varphi'(c) \geq -\varphi'(1-\lambda^{k+1}) > 0$. Thus

$$\lambda(1-\lambda)\,\frac{-\varphi'(1-\lambda^{k+1})}{-\varphi'(1-\lambda^{k-1})} \leq \frac{\varphi(1-\lambda^k)-\varphi(1-\lambda^{k+1})}{-\lambda^{k-1}\,\varphi'(1-\lambda^{k-1})} \leq \lambda(1-\lambda)$$

but this minorant may tend towards 0 as $k \to +\infty$, as it occurs in the following counter-example.

Example 3.4. Given $\lambda \in]0,1[$, let ψ be the function defined by $\psi(1-\lambda^k) = \lambda^{k^2}$ $\forall k \in \mathbb{N}$, with ψ affine on $[1-\lambda^k, 1-\lambda^{k+1}]$. Thus ψ is continuous, decreasing and > 0 on $[0,1[$. The function φ defined by $\varphi(t) = \int_t^1 \psi(r)\,dr > 0$ $\forall t \in [0,1[$ satisfies $\varphi(1) = 0$ and $\varphi' = -\psi$, thus φ is $\mathcal{C}^1$ and decreasing on $[0,1]$. Both (i) and (ii) are fulfilled, but the fraction in (3.1) is majorized by $\lambda(1-\lambda)\,\frac{\lambda^{k^2}}{\lambda^{(k-1)^2}} = \lambda(1-\lambda)\,\lambda^{2k}$ which tends towards 0 as $k \to +\infty$.

The purpose of the next example is to show that the $\mathcal{L}$-condition may be fulfilled by functions other than polynomials.

Example 3.5. Given $\gamma > 1$, let $a = 1 - \exp(-\frac{2\gamma-1}{\gamma(\gamma-1)})$ and ψ the continuous function defined by: $\psi(t) = (1-t)^{\gamma-1}[\gamma \log(1-t) + 1]$ $\forall t \in [a,1[$ and $\psi(t) = \psi(a)$ $\forall t \in [0,a]$.

We have $\psi < 0$ on $[0,1[$ since $1-t \leq 1-a = \exp(-\frac{2\gamma-1}{\gamma(\gamma-1)}) < \exp(-\frac{1}{\gamma})$ $\forall t \in [a,1[$, because of $2\gamma-1 > \gamma-1 > 0$. The increasingness of ψ follows from:

$$\psi'(t) = -(1-t)^{\gamma-2}[\gamma(\gamma-1)\log(1-t) + 2\gamma - 1] > 0 \qquad \forall t \in [a,1[.$$

Let φ be the function defined on $[0,1[$ by: $\varphi(t) = \int_0^t \psi(r)\,dr - C$ where the constant C stands for: $C = a\,\psi(a) + (1-a)^\gamma \log(1-a)$. Now $\varphi'(t) = \psi(t)$ $\forall t \in [0,1[$, thus φ is decreasing and $\mathcal{C}^1$ on $[0,1[$. Moreover $\varphi(t) = a\,\psi(a) + \int_a^t \psi(r)\,dr - C = -(1-t)^\gamma \log(1-t)$ $\forall t \in [a,1[$. Thus φ fulfills (i) and (ii).

Given $\lambda \in]0,1[$, we have $1-\lambda^{k-1} > a$ for any integer $k > 1 - \frac{2\gamma-1}{\gamma(\gamma-1)\log\lambda}$, for which the integral in (3.1) equals:

$$\frac{-\lambda^{k\gamma}\log(\lambda^k) + \lambda^{(k+1)\gamma}\log(\lambda^{k+1})}{-\lambda^{k-1}(\lambda^{k-1})^{\gamma-1}[\gamma\log(\lambda^{k-1})+1]} = \frac{-(\log\lambda)\,[k - \lambda^\gamma(k+1)]}{-\lambda^{-\gamma}[\gamma(k-1)(\log\lambda)+1]}$$

which tends towards $\frac{\log\lambda}{\lambda^{-\gamma}}\,\frac{1-\lambda^\gamma}{\gamma(\log\lambda)} = \frac{(1-\lambda^\gamma)\lambda^\gamma}{\gamma} > 0$ as k tends towards $+\infty$. Moreover (3.1) never equals zero, for no integer k, otherwise we would have φ constant on $[1-\lambda^k, 1-\lambda^{k+1}]$ thus $\psi \equiv 0$ there, which is not true. Hence (iii) holds and φ satisfies the $\mathcal{L}$-condition.

The next result generalizes a statement of Dolgoborodov [1, Lemma 2] who treated the case where φ was a polynomial:

Theorem 3.6. *Let f and g be positive increasing functions on $[0,1[$ such that*

$$\int_0^{\to 1} f(t)\,[-\varphi'(t)]\,dt \quad \text{converges and} \qquad \int_0^{\to 1} g(t)\,dt \quad \text{diverges,}$$

with φ a function fulfilling the $\mathcal{L}$-condition. Let $F = \{t \in [0,1[:\ f(t)\,[-\varphi'(t)] < g(t)\}$. Then the Lebesgue measure of the set $F \cap [r,1[$ is evaluated through:

$$\sup_{r\in[s,1[} \frac{mes(F\cap[r,1[)}{1-r} = 1 \qquad \forall s \in [0,1[.$$

In other words: there exists a sequence $(a_k)_{k\in\mathbb{N}}$ of points in $[0,1[$ tending towards 1 as $k \to +\infty$, such that $\frac{mes(F\cap[a_k,1[)}{1-a_k} \to 1$ as $k \to +\infty$.

Proof. Given L and λ as in Definition 3.1, let $J_i = [1-\lambda^{i-1}, 1-\lambda^i[$ for any $i \in \mathbb{N}^*$. We introduce the sets: $I_1 = \{i \in \mathbb{N}^* : J_i \cap E = \emptyset\}$ and $I_2 = \{i \in \mathbb{N}^* : J_i \cap E \neq \emptyset\}$ with $E = F^c = \{t \in [0,1[:\ f(t)\,[-\varphi'(t)] \geq g(t)\}$.

When I_2 is a finite set, there exists an $i_0 \in \mathbb{N}^*$ such that $i \in I_1\ \forall i > i_0$ hence $f(t)\,[-\varphi'(t)] < g(t)\ \forall t \in [1-\lambda^{i_0}, 1[$ and the proposition holds trivially. In the following, I_2 will be assumed an infinite set. It will now be shown that I_1 is infinite too, proceeding as follows: since $\mathbb{N}^* = I_1 \dot{\bigcup} I_2$ (disjoint reunion), the following argument will establish both the divergence of the series $\sum_{i\in\mathbb{N}^*} u_i$ and the convergence of $\sum_{i\in I_2} u_i$ with $u_i = \lambda^{i-1} g(1-\lambda^{i-1})$. Chasles' relation provides:

$$\int_0^1 g(t)\,dt = \sum_{i\in\mathbb{N}^*} \int_{1-\lambda^{i-1}}^{1-\lambda^i} g(t)\,dt.$$

Since g is increasing, the following holds for any $i \in \mathbb{N}^*$:

$$\int_{J_i} g(t)\,dt \leq g(1-\lambda^i)\,[\lambda^{i-1}-\lambda^i] = \lambda^i\,g(1-\lambda^i)\,(\lambda^{-1}-1) = (\lambda^{-1}-1)\,u_{i+1}$$

hence the divergence of $\sum_{i\geq 2} u_i$.

Given $i \in I_2$, there exists $i' \in I_2$ with $i' \geq i+2$. Let $a \in J_i \cap E$ and $b \in J_{i'}$. Since f is increasing and $-\varphi' > 0$, it follows that

$$\int_a^b f(t)\,[-\varphi'(t)]\,dt \geq f(a) \int_a^b [-\varphi'(t)]\,dt \geq \frac{g(a)}{-\varphi'(a)}\,[\varphi(a) - \varphi(b)]$$

because $a \in E$. Now $\varphi(a) \geq \varphi(1-\lambda^i)$ and $\varphi(b) \leq \varphi(1-\lambda^{i+1})$ because φ decreases and $b \geq 1 - \lambda^{i'-1} \geq 1 - \lambda^{i+1}$. Similarly $g(a) \geq g(1-\lambda^{i-1})$ and $0 < -\varphi'(a) \leq -\varphi'(1-\lambda^{i-1})$ since g increases and $-\varphi$ decreases on J_i. Thus, for any $i \in I_2$:

$$\begin{aligned}\int_a^b f(t)\,[-\varphi'(t)]\,dt &\geq g(1-\lambda^{i-1})\,\frac{\varphi(1-\lambda^i) - \varphi(1-\lambda^{i+1})}{-\varphi'(1-\lambda^{i-1})}\\ &\geq g(1-\lambda^{i-1})\,L\,\lambda^{i-1} = L\,u_i.\end{aligned}$$

The elements of I_2 will now be sorted by reading them in increasing order and picking some of them out of I_2 into a new set I_2'', obtaining thus a splitting $I_2 = I_2' \dot{\bigcup} I_2''$ as follows:

- if an integer $i \in I_2$ but $i+1 \notin I_2$, then i stays in I_2'
- if the integers i and $i+1$ both belong to I_2, then i is kept in I_2' and $i+1$ is put in I_2''

Thus, if for instance i, $i+1$, $i+2$, $i+3$ all belong to I_2, then i and $i+2$ remain in I_2' but $i+1$ and $i+3$ go into I_2''.

Hence $|i-j| \geq 2$ for all i and j in I_2' (and the same holds in I_2''). Thus the above integral estimation holds for $a \in J_i \cap E$ and $b \in J_{i'}$ as soon as the integers i and i' belong to I_2' with $i < i'$. Note that I_2' is an infinite set, since I_2 is.

The convergence of $\sum_{i \in I_2'} u_i$ can now be proved. For any $i \in I_2'$, let $a_i \in J_i \cap E$. These points a_i form a subdivision of $[0,1[$, allowing to apply Chasles' formula.

The subinterval with lower bound a_i gives rises to an integral of the previous kind (with $a = a_i$), so that

$$\int_0^1 f(t)\,[-\varphi'(t)]\,dt \geq \sum_{i \in I_2'} L\,u_i$$

hence the convergence of this series.

- If I_2'' is a finite set, the convergence of $\sum_{i \in I_2} u_i$ is obvious
- If I_2'' is an infinite set, the same reasoning works with other points $a_i' \in J_i \cap E$ (with now $i \in I_2''$), proceeding as before in order to establish the convergence of $\sum_{i \in I_2''} u_i$.

The convergence of $\sum_{i\in I_2} u_i$ leads to the affirmations: I_1 is infinite and $\sum_{i\in I_1} u_i$ diverges.

Now I_1 can be splitted into packs of consecutive integers. The next step of the proof considers one of these packs: $]i,i'[\cap\mathbb{N}$, with $i\in I_2$, $i'\in I_2$, $i'\geq i+2$ and $j\in I_1$ for any integer $j\in]i,i'[$.

$$\sum_{i<j<i'} u_j = \sum_{i<j<i'} \lambda^{j-1}\, g(1-\lambda^{j-1}) = \sum_{i\leq j\leq i'-2} \lambda^j\, g(1-\lambda^j) \leq g(1-\lambda^{i'-1}) \sum_{i\leq j\leq i'-2} \lambda^j$$

since the function g is increasing and the sequence $(1-\lambda^j)_{j\in\mathbb{N}}$ increases too. Thus

$$\begin{aligned}\sum_{i<j<i'} u_j &\leq g(1-\lambda^{i'-1})\,\lambda^{i'-1} \sum_{i\leq j\leq i'-2} \lambda^{j-(i'-1)}\\ &= u_{i'} \sum_{i\leq j\leq i'-2} \left(\frac{1}{\lambda}\right)^{i'-1-j} = u_{i'} \sum_{1\leq m\leq i'-i-1} \left(\frac{1}{\lambda}\right)^m .\end{aligned}$$

If the packs (of consecutive integers) constituting I_1 had a bounded number of terms, there would exist a constant K, independant of i and i', such that

$$\sum_{1\leq m\leq i'-i-1} \left(\frac{1}{\lambda}\right)^m \leq K.$$

Hence $\sum_{i<j<i'} u_j \leq K\, u_{i'}$, thus $\sum_{j\in I_1} u_j \leq K \sum_{i'\in I_2} u_{i'}$ and the convergence of the second series leads to a contradiction.

Hence it is possible to extract from I_1 a sequence of packs (of consecutive integers) whose length tends towards $+\infty$. These packs will be noted $]i_k, i_k+l_k[\cap\mathbb{N}$ (with $k\in\mathbb{N}$) where $i_k\in I_2$, $i_k+l_k\in I_2$, $j\in I_1$ for any integer j such that $i_k<j<i_k+l_k$, and with moreover $\lim_{k\to+\infty} l_k=+\infty$.

For any k, let $F_k=\bigcup_{i_k<i<i_k+l_k} J_i = [1-\lambda^{i_k}, 1-\lambda^{i_k+l_k-1}[\subset[1-\lambda^{i_k},1[$. These integers i belong to I_1 hence $J_i\subset E^c$, thus $F_k\subset E^c=F$. Besides that

$$mes\, F_k = 1-\lambda^{i_k+l_k-1}-(1-\lambda^{i_k}) = \lambda^{i_k}-\lambda^{i_k+l_k-1}.$$

For $r=1-\lambda^{i_k}$ (with a given k) the following holds:

$$1\geq \frac{mes(F\cap[r,1[)}{1-r} \geq \frac{mes(F_k\cap[r,1[)}{1-r} = \frac{\lambda^{i_k}(1-\lambda^{l_k-1})}{\lambda^{i_k}} = 1-\lambda^{l_k-1}.$$

Given $s\in[0,1[$, we have $\sup_{r\geq s}\frac{mes(F\cap[r,1[)}{1-r}\geq 1-\lambda^{l_k-1}$ for all k sufficiently large. Now $\lim_{k\to+\infty}(1-\lambda^{l_k-1})=1$ and the conclusion is immediate. $\square$

4 – Behaviour of the repartition function

DEFINITION 4.1. A subharmonic function u in B_N is said to satisfy the $\mathcal{H}$-condition if u is moreover harmonic in some neighborhood of the origin, with $u(O) = 0$.

REMARK 4.2. As soon as u fulfills the $\mathcal{H}$-condition, we have $\mathcal{M}_u(r) \geq 0 \ \forall r \in [0,1[$. The increasingness of $\mathcal{M}_u$ ensues for instance from Jensen-Privalov formula, since $\rho \geq 0$.

THEOREM 4.3. *Let ρ be the repartition function associated to some subharmonic function u in B_N, satisfying the $\mathcal{H}$-condition. Let φ denote a $\mathcal{C}^1$ decreasing function on $[0,1[$ such that $\lim_{t\to 1^-} \varphi(t) = 0$.*

(i) *If $\int_0^1 \mathcal{M}_u(t)\,[-\varphi'(t)]\,dt < +\infty$ then $\mathcal{M}_u(r) = \mathbf{o}(\frac{1}{\varphi(r)})$ and $\rho(r^2) = \mathbf{o}(\frac{1}{\varphi(r)\,h(r)})$ as r tends towards 1^-.*

(ii) *If $\int_0^1 e^{\mathcal{M}_u(t)}\,[-\varphi'(t)]\,dt < +\infty$ then $e^{\mathcal{M}_u(r)} = \mathbf{o}(\frac{1}{\varphi(r)})$ as $r \to 1^-$ and*

$$\rho(r^2) \leq \frac{1}{h(r)}\left(\ell(r) + \log\frac{1}{\varphi(r)}\right) \qquad \forall r \in]0,1[,$$

where ℓ is a function defined on $[0,1[$ such that $\lim_{r\to 1^-} \ell(r) = -\infty$. More precisely, ℓ is explicited by (4.1) below.

PROOF OF (i). The estimation of $\mathcal{M}_u(r)$ follows from $0 \leq \mathcal{M}_u(r)\,\varphi(r) \leq \int_r^1 \mathcal{M}_u(t)\,[-\varphi'(t)]\,dt$ which tends towards 0 as $r \to 1^-$. The estimation of $\rho(r^2)$ follows from Lemma 2.5. □

PROOF OF (ii). Similarly we have $e^{\mathcal{M}_u(r)} \leq \frac{1}{\varphi(r)} \int_r^1 e^{\mathcal{M}_u(t)}\,[-\varphi'(t)]\,dt$ for all $r \in [0,1[$. Thus $\mathcal{M}_u(r) \leq \left(\log\frac{1}{\varphi(r)}\right) + \ell(r)$ where

$$\ell(r) = \log\left(\int_r^1 e^{\mathcal{M}_u(t)}\,[-\varphi'(t)]\,dt\right) \tag{4.1}$$

and the conclusion follows from Lemma 2.5. □

COROLLARY 4.4. *When φ is defined by $\varphi(r) = (1-r^2)^{\alpha+1} \ \forall r \in [0,1[$ for some $\alpha > -1$, then the repartition function ρ of any subharmonic function u under the assumptions of Theorem 4.3 (ii) fulfills:*

$$\rho(r) = \mathbf{O}\left(\frac{1}{1-r}\left(\lambda(r) + \log\frac{1}{1-r}\right)\right) \qquad \text{as } r \to 1^-,$$

where λ denotes a function defined on $[0,1[$ such that $\lim_{r\to 1^-} \lambda(r) = -\infty$.

Remark 4.5. In this growth estimation of $\rho(r)$, the term $\lambda(r)$ is not to be neglected: the Proposition 4.8 below will provide an example of subharmonic function u for which $\rho(r) = \mathbf{O}\left(\frac{1}{1-r}\log\log\frac{1}{1-r}\right)$ which is more accurate than a mere $\mathbf{O}\left(\frac{1}{1-r}\log\frac{1}{1-r}\right)$.

Proof of Corollary 4.4. Theorem 4.3 gives:

$$\rho(r^2) \le \frac{1}{h(r)}\left(\ell(r) + (\alpha+1)\log\frac{1}{1-r^2}\right).$$

Whence $\rho(t) \le \frac{\alpha+1}{h(\sqrt{t})}\left(\frac{\ell(\sqrt{t})}{\alpha+1} + \log\frac{1}{1-t}\right)$. Now $h(r) \ge \tau_N(1-r)$ and $1-\sqrt{t} = \frac{1-t}{1+\sqrt{t}} > \frac{1-t}{2}$, thus

$$\rho(t) < \frac{2(\alpha+1)}{\tau_N(1-t)}\left(\frac{\ell(\sqrt{t})}{\alpha+1} + \log\frac{1}{1-t}\right) \qquad \forall t \in [0,1[.$$

Remark 4.6. Later, in Proposition 4.10, we will study the size of the set of those t in $[0,1[$ for which $\rho(t) < \frac{2(\alpha+1)}{\tau_N(1-t)}\left[\log\left(\frac{1}{1-t}\right) - \frac{1}{\alpha+1}\log\log\left(\frac{1}{1-t}\right)\right]$.

Example 4.7. With $N = 2$, let $u = \log|f|$ with f a function holomorphic in the unit disk of $\mathbb{C}$, assuming that $f(0) = 1$. Then the Riesz measure μ of u is a sum of Dirac masses: $\mu = \sum_{k\in\mathbb{N}^*}\delta_{z_k}$ where the z_k denote the zeros of f, repeated according to their multiplicities and indexed by increasing moduli. Thus $\rho(r)$ is the number of points z_k located in the disk $\{z \in \mathbb{C} : |z| \le r\}$. Let $p > 0$, then

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|f(r\,e^{i\theta})|^p\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[p\,\log|f(r\,e^{i\theta})|\right]d\theta \\ &\ge \exp\left(\frac{1}{2\pi}\int_0^{2\pi}p\,\log|f(r\,e^{i\theta})|\,d\theta\right)\end{aligned}$$

through Jensen's inequality. Thus $\frac{1}{2\pi}\int_0^{2\pi}|f(r\,e^{i\theta})|^p\,d\theta \ge \exp\left(\mathcal{M}_{p\,u}(r)\right)$.

When f belongs to the Bergman space of parameters $p > 0$ and $\alpha > -1$, it means that:

$$\int_0^1\left(\frac{1}{2\pi}\int_0^{2\pi}|f(r\,e^{i\theta})|^p\,d\theta\right)(1-r^2)^\alpha\,r\,dr < +\infty,$$

thus Corollary 4.4 applies to the function $p\,u$. The Riesz measure associated to the subharmonic function $p\,u$ is $p\,\mu$ and its repartition function is $p\rho$. It leads to:

$$\rho(r) = \mathbf{O}\left(\frac{1}{1-r}\left(\lambda(r) + \log\frac{1}{1-r}\right)\right) \qquad \text{as } r \to 1^-$$

where $\lambda(r) = \frac{1}{\alpha+1} \log\left(\int_{\sqrt{r}}^1 e^{\mathcal{M}_{pu}(t)}\,[-\varphi'(t)]\,dt\right)$. We note that

$$\lambda(r) \leq \frac{1}{\alpha+1} \log\left(2(\alpha+1)\int_{\sqrt{r}\leq|z|<1} |f(z)|^p(1-|z|^2)^\alpha dA(z)\right) \qquad \forall r \in [0,1[,$$

with $dA(z)$ the normalized area element on the unit disk of $\mathbb{C}$. This growth estimation for $\rho(r)$ completes the traditional result (see [2, page 103]), according to which $\rho(r)$ is a $\mathbf{O}\left(\frac{1}{1-r}\,\log\frac{1}{1-r}\right)$.

Proposition 4.8. Let $g\,:]0,1[\to \mathbb{R}$ be defined by $g(t) = \log\log\sqrt{\frac{1}{1-t}}$. The function v defined by $v(O) = -\infty$ and $v(x) = g(|x|)$ for all $x \in B_N \setminus \{O\}$ is subharmonic on B_N. The function u defined by $u(x) = \max\{0, v(x)\}$ $\forall x \in B_N$ is subharmonic on B_N, fulfills the $\mathcal{H}$-condition, together with $\int_0^1 e^{\mathcal{M}_u(t)}\,[-\varphi'(t)]\,dt < +\infty$ for the same function φ as in Corollary 4.4. Its repartition function ρ satisfies $\rho(r) = \mathbf{O}\left(\frac{1}{1-r}\,\log\log\frac{1}{1-r}\right)$ as $r \to 1^-$.

Proof. The function g is well-defined on $]0,1[$ since $\log\frac{1}{1-t} > 0$. Moreover $g(t) \leq 0$ for all $t \leq 1-e^{-2} = 0,86466\ldots$ Since $\Delta v(x) = g''(t) + \frac{N-1}{t}g'(t)$ for $t = |x| \neq 0$ (see [3, page 26]), we need some derivatives from $g(t) = \log(\frac{1}{2}) + \log\log\frac{1}{1-t} = \log(\frac{1}{2}) + \log(-\log(1-t))$:

- $g'(t) = 1/[(1-t)\log\frac{1}{1-t}]$
- $g''(t) = -\left(-\log(\frac{1}{1-t}) + (1-t)\frac{1}{1-t}\right)[(1-t)\log\frac{1}{1-t}]^{-2} = \left(\log(\frac{1}{1-t}) - 1\right)[(1-t)\log\frac{1}{1-t}]^{-2}$

Whence $\Delta v(x) = \left(\log(\frac{1}{1-t}) - 1 + \frac{N-1}{t}(1-t)\log\frac{1}{1-t}\right)[(1-t)\log\frac{1}{1-t}]^{-2}$. Thus $\Delta v(x)$ has the same sign than

$$\begin{aligned} &t\left(\log\frac{1}{1-t}\right) - t + (N-1)(1-t)\log\frac{1}{1-t} \\ &= -t + (N-1-(N-2)t)\log\frac{1}{1-t} \geq -t + \log\frac{1}{1-t} \end{aligned}$$

since $N-1-(N-2)t > 1$ (as $(N-2)t < N-2 < N-1$) and $\log\frac{1}{1-t} > 0$ $\forall t \in]0,1[$. Now $\log(1-t) \leq -t$ $\forall t < 1$, thus $\log\frac{1}{1-t} = -\log(1-t) \geq t$. Finally $\Delta v(x) \geq 0$ and subsequently v is subharmonic on B_N.

The subharmonicity of u proceeds from [3, page 41]. The $\mathcal{H}$-condition holds since $u \equiv 0$ on some neighborhood of O. For $t \geq 1-e^{-2}$, we have $e^{\mathcal{M}_u(t)} = e^{g(t)} = \frac{1}{2}\log\frac{1}{1-t}$.

There exists $r_0 \in [1-e^{-2}, 1[$ such that $\log\frac{1}{1-t} \geq \frac{t+1}{(\alpha+1)t}$ $\forall t \in [r_0, 1[$, since $\log\frac{1}{1-t} \to +\infty$ as $t \to 1^-$, whereas the right-hand term remains bounded. Now $\frac{\varphi(t)}{\varphi'(t)} = -(1-t)\frac{1+t}{2t(\alpha+1)}$. Thus $\frac{1}{2}\log\frac{1}{1-t} \geq -\frac{\varphi(t)}{\varphi'(t)\,(1-t)}$ and

$$e^{\mathcal{M}_u(t)} = \left(\log\frac{1}{1-t}\right) - \frac{1}{2}\log\frac{1}{1-t} \leq \left(\log\frac{1}{1-t}\right) + \frac{\varphi(t)}{\varphi'(t)\,(1-t)} \qquad \forall t \in [r_0, 1[$$

hence $e^{\mathcal{M}_u(t)}\,[-\varphi'(t)] \leq [-\varphi'(t)]\left(\log\frac{1}{1-t}\right) - \frac{\varphi(t)}{1-t}$ $\forall t \in [r_0, 1[$, since $-\varphi'(t) > 0$. In other words $e^{\mathcal{M}_u(t)}\,[-\varphi'(t)] \leq -\psi'(t)$ $\forall t \in [r_0, 1[$, with ψ defined by $\psi(t) = \varphi(t)\log\frac{1}{1-t}$ $\forall t \in]0, 1[$. For any $r \in [r_0, 1[$, we obtain:

$$\int_{r_0}^{r} e^{\mathcal{M}_u(t)}\,[-\varphi'(t)]\,dt \leq -\left[\psi(t)\right]_{r_0}^{r} = \psi(r_0) - (1-r^2)^{\alpha+1}\log\frac{1}{1-r}.$$

The right-hand side tends towards $\psi(r_0)$ as $r \to 1^-$ because $\alpha + 1 > 0$. In other words: $\int_{r_0}^{1} e^{\mathcal{M}_u(t)}\,[-\varphi'(t)]\,dt < +\infty$.

Similarly $\int_{r}^{1} e^{\mathcal{M}_u(t)}\,[-\varphi'(t)]\,dt = \psi(r)$ $\forall r \in [r_0, 1[$, which leads to:

$$e^{\mathcal{M}_u(r)} \leq \frac{1}{\varphi(r)}\int_{r}^{1} e^{\mathcal{M}_u(t)}\,[-\varphi'(t)]\,dt \leq \frac{\psi(r)}{\varphi(r)} = \log\frac{1}{1-r} \qquad \forall r \in [r_0, 1[.$$

Now $\log\frac{1}{1-r} \leq \left(\log\frac{1}{1-r^2}\right)^2$ as soon as $r > 0{,}873106\ldots$ Hence there exists $r_1 \in [r_0, 1[$ such that $e^{\mathcal{M}_u(r)} \leq \left(\log\frac{1}{1-r^2}\right)^2$ $\forall r \in [r_1, 1[$, thus $\mathcal{M}_u(r) \leq 2\log\log\frac{1}{1-r^2}$. Lemma 2.5 provides $\rho(r^2) \leq \frac{2}{h(r)}\log\log\frac{1}{1-r^2}$ $\forall r \in [r_1, 1[$. Besides that $h(r) \sim \frac{\tau_N}{2}(1-r^2)$ as $r \to 1^-$ and the conclusion follows. $\square$

Theorem 4.9. *Given φ and g two positive functions on $[0, 1[$ such that $\int_0^{\to 1} g(t)dt$ diverges, with g increasing and φ fulfilling the $\mathcal{L}$-condition, let u denote a subharmonic function on B_N (satisfying the $\mathcal{H}$-condition) and ρ its repartition function. For any $r \in [0, 1[$, we define the following sets*:

$$F_1(r) = \{t \in [r, 1[\colon\ \mathcal{M}_u(t)\,[-\varphi'(t)] < g(t)\}$$

$$G_1(r) = \left\{t \in [r, 1[\colon\ \rho(t^2)\,[-\varphi'(t)] < \frac{g(t)}{h(t)}\right\}$$

$$G_2(r) = \left\{t \in [r, 1[\colon\ \rho(t^2) < \frac{\log[g(t)] - \log[-\varphi'(t)]}{h(t)}\right\}$$

$$H_1(r) = \left\{t \in [r, 1[\colon\ \int_{t^2}^{t} \mathcal{M}_u(s)\,[-\varphi'(s)]\,ds < \frac{g(t)\,\varphi(t^2)}{-\varphi'(t)}\right\}.$$

Let the set $F_2(r)$ (resp. $H_2(r)$) be defined on the same way than $F_1(r)$ (resp. $H_1(r)$), only with $\mathcal{M}_u$ replaced by $e^{\mathcal{M}_u}$.

(i) *If* $\int_0^1 \mathcal{M}_u(r)\,[-\varphi'(r)]\,dr < +\infty$ *then there exists a sequence* $(a_k)_{k\in\mathbb{N}}$ *of points in* $[0,1[$, *with limit* 1, *such that* $mes\,F_1(a_k) \sim mes\,G_1(a_k) \sim mes\,H_1(a_k) \sim 1-a_k$ *as* $k \to +\infty$.

(ii) *If* $\int_0^1 e^{\mathcal{M}_u(r)}\,[-\varphi'(r)]\,dr < +\infty$ *then there exists a sequence* $(b_k)_{k\in\mathbb{N}}$ *of points in* $[0,1[$, *with limit* 1, *such that* $mes\,F_2(b_k) \sim mes\,G_2(b_k) \sim mes\,H_2(b_k) \sim 1-b_k$ *as* $k \to +\infty$.

Proof of (i). Theorem 3.6 provides the sequence $(a_k)_k$ and the estimation of $mes\,F_1(a_k)$. If some point t belongs to $F_1(r)$, it implies that $t \in G_1(r)$ through Lemma 2.5. Thus $F_1(r) \subset G_1(r) \subset [r,1[$, hence $mes\,F_1(r) \le mes\,G_1(r) \le 1-r$ $\forall r \in [0,1[$ and the estimation of $mes\,G_1(a_k)$ follows. Remark 4.2 implies for every $t \in [0,1[$:

$$\int_{t^2}^{t} \mathcal{M}_u(s)[-\varphi'(s)]\,ds \le \mathcal{M}_u(t)\int_{t^2}^{t}[-\varphi'(s)]\,ds = \mathcal{M}_u(t)\,[\varphi(t^2)-\varphi(t)] \le \mathcal{M}_u(t)\,\varphi(t^2).$$

We deduce $F_1(r) \subset H_1(r) \subset [r,1[$ $\forall r \in [0,1[$ and the estimation of $mes\,H_1(a_k)$ follows on the same way as above.

Proof of (ii). The sequence $(b_k)_k$ and the estimation of $mes\,F_2(b_k)$ are obtained through Theorem 3.6 again. Now $t \in F_2(r)$ implies $\mathcal{M}_u(t) < \log\left(\frac{g(t)}{-\varphi'(t)}\right)$ and Lemma 2.5 leads to $t \in G_2(r)$. Similarly $F_2(r) \subset H_2(r)$. The argument ends as in (i), up to obvious adaptations. □

Proposition 4.10. *Given a real number* $\alpha \ge -1/2$, *let* φ *and* g *be defined by* $\varphi(t) = (1-t^2)^{\alpha+1}$ *and* $g(t) = \frac{-t^2}{(1-t^2)\log(1-t^2)}$ $\forall t \in]0,1[$ (*with* g *continuously extended by* $g(0)=1$).

Given ρ *the repartition function associated to some subharmonic function* u *on* B_N *under the assumptions of Theorem* 4.9 (ii), *let* $G(r)$ *denote for any* $r \in [0,1[$ *the set*:

$$G(r) = \left\{ t \in [r,1[:\ \rho(t) < \frac{2}{\tau_N(1-t)}\left[(\alpha+1)\log\left(\frac{1}{1-t}\right) - \log\log\left(\frac{1}{1-t}\right)\right]\right\}.$$

Then there exists a sequence $(b_k)_{k\in\mathbb{N}}$ *of points in* $[0,1[$, *tending towards* 1, *such that* $mes\,G(b_k) \sim 1-b_k$ *as* $k \to +\infty$.

Proof. First we make sure that the integral $\int_0^{\to 1} g(t)\,dt$ diverges. This integral has the same nature as $\int_{1/2}^{\to 1} g_1(t)\,dt$ with $g_1(t) = \frac{-2t}{(1-t^2)\log(1-t^2)}$, since $g_1(t) \sim 2\,g(t)$ as $t \to 1$. Now $g_1(t) = \frac{u'(t)}{u(t)}$ with $u(t) = -\log(1-t^2)$. hence

$$\int_{1/2}^{r} g_1(t) = \Big[\log u(t)\Big]_{1/2}^{r} = \left(\log\log\frac{1}{1-r^2}\right) - \log\log(4/3)$$

which tends towards $+\infty$ as $r \to 1^-$. Next, we have to establish that g increases on $[0,1[$. It is enough to show the increasingness of g_2 defined by: $g_2(t) = \frac{-t}{(1-t)\log(1-t)}$. For any $t \in]0,1[$, we compute $g_2'(t) = \frac{-(1-t)\log(1-t)+t[-\log(1-t)+(1-t)\frac{-1}{1-t}]}{[(1-t)\log(1-t)]^2}$. Hence $g_2'(t)$ has the same sign than $-\log(1-t)-t \geq 0$ from the well-known estimation $\log(1+x) \leq x \; \forall x > -1$.

Theorem 4.9 (ii) applies and provides the sequence $(b_k)_{k\in\mathbb{N}}$ as well as the estimation of $mes\, G_2(b_k)$. The following holds for all $t \in]0,1[$:

$$\begin{aligned}
&\log[g(t)] - \log[-\varphi'(t)] \\
&= \log(t^2) + \log\frac{1}{1-t^2} - \log\log\frac{1}{1-t^2} - \log[2t(\alpha+1)] - \alpha\log(1-t^2) \\
&= (\alpha+1)\log\frac{1}{1-t^2} - \log\log\frac{1}{1-t^2} + \log\frac{t}{2(\alpha+1)}.
\end{aligned}$$

Now $\log\frac{t}{2(\alpha+1)} \leq \log\frac{1}{2(\alpha+1)} = -\log[2(\alpha+1)] \leq 0$ since $2(\alpha+1) \geq 1$. Thus

$$\frac{\log[g(t)] - \log[-\varphi'(t)]}{h(t)} \leq \frac{2}{\tau_N(1-t^2)}\left[(\alpha+1)\log\left(\frac{1}{1-t^2}\right) - \log\log\left(\frac{1}{1-t^2}\right)\right]$$

for all $t \in]0,1[$, because of $h(t) \geq \tau_N(1-t^2)/2$.

If $t \in G_2(r)$ then $\rho(t^2) < \frac{2}{\tau_N(1-t^2)}\left[(\alpha+1)\log\left(\frac{1}{1-t^2}\right) - \log\log\left(\frac{1}{1-t^2}\right)\right]$, in other words $t \in G(r)$. Hence $G_2(r) \subset G(r) \subset [r,1[\; \forall r \in [0,1[$ and the estimation of $mes\, G(b_k)$ follows. $\square$

Remark 4.11. When $-1 < \alpha < -1/2$, Proposition 4.10 still works, the definition of $G(r)$ only requiring an additional term $\log\frac{t}{2(\alpha+1)}$ inside the square brackets. The same remark holds for the following statement: a particular situation where $N = 2$ and $u = p\log|f|$.

Corollary 4.12. *For a holomorphic function f from the Bergman space of parameters $p > 0$ and $\alpha \geq -1/2$, there exists a sequence $(b_k)_{k\in\mathbb{N}}$ of points in $[0,1[$, with limit 1, such that $mes\left\{t \in [b_k,1[\colon \rho(t) < \frac{2/p}{1-t}\left[(\alpha+1)\log\left(\frac{1}{1-t}\right) - \log\log\left(\frac{1}{1-t}\right)\right]\right\} \sim 1-b_k$ as $k \to +\infty$ where $\rho(t)$ is counting (with multiplicities) the zeros of f with modulus $\leq t$.*

Proof. The subharmonic function $p\log|f|$ fulfills the conditions of Theorem 4.9 (ii), as noticed in Example 4.7. Its repartition function is $p\,\rho$, thus Proposition 4.10 applies, with ρ replaced by $p\,\rho$. $\square$

5 – Behaviour of the Riesz measure

THEOREM 5.1. *Given μ the Riesz measure associated to a subharmonic function u in B_N, satisfying the $\mathcal{H}$-condition, let $P_\mu(r) = \int_{|\zeta|<s_\mu(r)} h(|\zeta|)\, d\mu(\zeta)$ $\forall r \in]0,1[$, with $s_\mu(r)$ from Lemma 2.7. Let φ denote a $\mathcal{C}^1$ decreasing function on $[0,1[$ such that $\lim_{t\to 1^-} \varphi(t) = 0$.*

(i) *If $\int_0^1 \mathcal{M}_u(r)\,[-\varphi'(r)]\,dr < +\infty$ then $P_\mu(r) = \mathbf{o}(\frac{1}{\varphi(r)})$ as $r \to 1^-$.*
(ii) *If $\int_0^1 e^{\mathcal{M}_u(r)}\,[-\varphi'(r)]\,dr < +\infty$ then $e^{P_\mu(r)} = \mathbf{o}(\frac{1}{\varphi(r)})$ as $r \to 1^-$. More precisely:*

$$e^{P_\mu(r)} \le \frac{e}{\varphi(r)} \int_r^1 e^{\mathcal{M}_u(t)}\,[-\varphi'(t)]\,dt \qquad \forall r \in [0,1[.$$

REMARK 5.2. In the case (ii), it obviously holds that $P_\mu(r) = \mathbf{o}(\frac{1}{\varphi(r)})$ as $e^{P_\mu(r)} \ge P_\mu(r)$ $\forall r$.

PROOF OF THEOREM 5.1. Since $h \ge 0$ on $]0,1[$ and μ is a positive measure, then P_μ is an increasing function on $]0,1[$ since s_μ is (but not necessarily strictly increasing). Given $r \in]0,1[$, it follows from Lemma 2.3 together with Jensen-Privalov formula that:

$$\mathcal{M}_u(r) \ge \int_{|\zeta|\le r'} h_r(\zeta)\,d\mu(\zeta) = \int_{|\zeta|\le r'} h(|\zeta|)\,d\mu(\zeta) - h(r)\int_{|\zeta|\le r'} d\mu(\zeta) \qquad \forall r' \in]0,1[.$$

For any $r' < s_\mu(r)$, we have: $\int_{|\zeta|\le r'} d\mu(\zeta) = \rho(r') \le \frac{1}{h(r)}$ according to Lemma 2.7, hence

$$\int_{|\zeta|\le r'} h(|\zeta|)\,d\mu(\zeta) \le \mathcal{M}_u(r) + 1.$$

The left-hand term tends towards $P_\mu(r)$ as $r' \to s_\mu(r)$, with $r' < s_\mu(r)$, since $h \ge 0$ on $]0,1[$. Hence $0 \le P_\mu(r) \le \mathcal{M}_u(r) + 1$ $\forall r \in]0,1[$.

PROOF OF (i). The above estimation leads to $\int_0^1 P_\mu(r)\,[-\varphi'(r)]\,dr < +\infty$ since $-\varphi' \ge 0$. Consequently $\int_t^1 P_\mu(r)\,[-\varphi'(r)]\,dr$ tends towards 0 as $t \to 1^-$. Besides that,

$$\int_t^1 P_\mu(r)\,[-\varphi'(r)]\,dr \ge P_\mu(t)\int_t^1 [-\varphi'(r)]\,dr = P_\mu(t)[\varphi(t) - \lim_{r\to 1^-} \varphi(r)] = P_\mu(t)\,\varphi(t) \ge 0$$

for all $t \in]0,1[$. Thus finally $P_\mu(t)\,\varphi(t) \to 0$ as $t \to 1^-$.

PROOF OF (ii). From $e^{P_\mu(r)} \le e.e^{\mathcal{M}_u(r)}$, it follows that $\int_0^1 e^{P_\mu(r)}\,[-\varphi'(r)]\,dr < +\infty$. Hence the conclusion since $\int_t^1 e^{P_\mu(r)}\,[-\varphi'(r)]\,dr \ge e^{P_\mu(t)}\,\varphi(t)$. $\square$

Example 5.3. In the case $N = 2$, let $u = \log|f|$ with f a function holomorphic in the unit disk of $\mathbb{C}$, assuming that $f(0) = 1$. Here again we make use of the notations introduced in Example 4.7. When f belongs to the Bergman space of parameters $p > 0$ and $\alpha > -1$, Theorem 5.1 (ii) applies to the function $p\,u$, with φ defined by: $\varphi(r) = (1-r^2)^{\alpha+1}$. The Riesz measure relative to the subharmonic function $p\,u$ is $p\,\mu$, thus $\exp[P_{p\mu}(r)] = \mathbf{o}\left(\frac{1}{(1-r)^{\alpha+1}}\right)$ since $\frac{1}{1+r} \leq 1$. Now $P_{p\mu}(r) = p\sum_{|z_k|<s_{p\mu}(r)} \log\frac{1}{|z_k|}$ for every $r \in [0,1[$, thus

$$\exp(P_{p\mu}(r)) = \prod_{|z_k|<s_{p\mu}(r)} \frac{1}{|z_k|^p} = \mathbf{o}\left(\frac{1}{(1-r)^{\alpha+1}}\right),$$

in other words:

$$\prod_{|z_k|<s_{p\mu}(r)} \frac{1}{|z_k|} = \mathbf{o}\left(\frac{1}{(1-r)^{(\alpha+1)/p}}\right) \qquad \text{as } r \to 1^-.$$

For all $n \in \mathbb{N}^*$ let $r_n = e^{-1/np} = 1 - \frac{1}{np}(1+\varepsilon_n)$ with $\lim_{n\to+\infty} \varepsilon_n = 0$. Hence $1 - r_n = \frac{1}{n}\frac{1+\varepsilon_n}{p}$ thus $(1-r_n)^{-(\alpha+1)/p} = n^{(\alpha+1)/p}(\frac{p}{1+\varepsilon_n})^{(\alpha+1)/p}$.

Given $n \in \mathbb{N}^*$, let m denote the largest integer such that $|z_m| < |z_n|$.

Then $\rho(|z_m|) = m < n$ thus $p\,\rho(|z_m|) < pn = -1/\log r_n = 1/h(r_n)$, hence $|z_m| < s_{p\mu}(r_n)$, the strict inequality following from Lemma 2.7, since $p\,\rho(s_{p\mu}(r_n)) \geq \frac{1}{h(r_n)}$.

Whence $\prod_{k=1}^m \frac{1}{|z_k|} \leq \prod_{|z_k|<s_{p\mu}(r_n)} \frac{1}{|z_k|}$ since $\frac{1}{|z_k|} > 1$. Finally:

$$\prod_{|z_k|<|z_n|} \frac{1}{|z_k|} = \mathbf{o}(n^{(\alpha+1)/p}) \qquad \text{as } n \to +\infty$$

which completes the result of Horowitz [4] according to whom $\prod_{k=1}^n \frac{1}{|z_k|} = \mathbf{O}(n^{(\alpha+1)/p})$ as $n \to +\infty$.

In order to refine our estimation, we introduce (for a fixed integer $n \in \mathbb{N}^*$) the holomorphic function g given by $g(z) = \prod_{k=1}^n (z - z_k)$ $\forall z \in \mathbb{C}$.

The subharmonic function $v = p\log|g| - p\log|g(0)|$ fulfills the $\mathcal{H}$-condition. Its Riesz measure is $\nu = p\sum_{k=1}^n \delta_{z_k}$ and its repartition function ϱ satisfies: $\varrho(t) = pn$ $\forall t \geq |z_n|$, together with $\varrho(t) \leq p\,\rho(t)$ $\forall t \in [0,1[$, thus $\mathcal{M}_v(r) \leq \mathcal{M}_{pu}(r)$ $\forall r \in [0,1[$ through Jensen-Privalov formula.

Besides that, $s_\nu(r) = \sup\{t \in]0,1[:\ \varrho(t) \leq \frac{1}{h(r)}\} \geq s_{p\mu}(r)$ $\forall r \in [0,1[$ since

$$\left\{t \in]0,1[:\ p\rho(t) \leq \frac{1}{h(r)}\right\} \subset \left\{t \in]0,1[:\ \varrho(t) \leq \frac{1}{h(r)}\right\}.$$

With r_n as above, we get $s_\nu(r_n) = \sup\{t \in]0,1[:\ \varrho(t) \leq np\} = 1$. Hence it turns out that $P_\nu(r_n) = p\sum_{k=1}^{n} \log \frac{1}{|z_k|}$, thus $\exp(P_\nu(r_n)) = \prod_{k=1}^{n} \frac{1}{|z_k|^p}$.

We have $\int_0^1 e^{\mathcal{M}_v(t)}\,[-\varphi'(t)]\,dt < +\infty$ hence Theorem 5.1 (ii) provides for any $r \in [0,1[$:

$$\begin{aligned} e^{P_\nu(r)} &\leq \frac{e}{\varphi(r)} \int_r^1 e^{\mathcal{M}_v(t)}\,[-\varphi'(t)]\,dt \leq \frac{e}{(1-r^2)^{\alpha+1}} \int_r^1 e^{\mathcal{M}_{pu}(t)}\,[-\varphi'(t)]\,dt \\ &\leq \frac{e}{(1-r)^{\alpha+1}} \int_r^1 \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t\,e^{i\theta})|^p\,d\theta \right) [-\varphi'(t)]\,dt \\ &\leq \frac{2e(\alpha+1)}{(1-r)^{\alpha+1}} \int_{r\leq|z|<1} |f(z)|^p (1-|z|^2)^\alpha dA(z). \end{aligned}$$

With $r := r_n$, the above integral tends towards 0 when $n \to +\infty$. We have finally recovered that $\prod_{k=1}^{n} \frac{1}{|z_k|} = \mathbf{o}(n^{(\alpha+1)/p})$ as $n \to +\infty$, stated in [1, page 257].

THEOREM 5.4. *Given* φ *and* g *two positive functions on* $[0,1[$ *such that* $\int_0^{\to 1} g(t)dt$ *diverges, with* g *increasing and* φ *fulfilling the* $\mathcal{L}$*-condition, let* u *denote a subharmonic function on* B_N *(satisfying the* $\mathcal{H}$*-condition) and* μ *its Riesz measure. For any* $r \in [0,1[$*, we define the following sets*:

$$\begin{aligned} \Phi_1(r) &= \{t \in [r,1[:\ P_\mu(t)\,[-\varphi'(t)] < g(t)\} \\ \Phi_2(r) &= \left\{t \in [r,1[:\ e^{P_\mu(t)}\,[-\varphi'(t)] < g(t)\right\}. \end{aligned}$$

(i) *If* $\int_0^1 \mathcal{M}_u(r)\,[-\varphi'(r)]\,dr < +\infty$ *then there exists a sequence* $(a'_k)_{k\in\mathbb{N}}$ *of points in* $[0,1[$*, with limit* 1*, such that* mes $\Phi_1(a'_k) \sim 1 - a'_k$ *as* $k \to +\infty$.

(ii) *If* $\int_0^1 e^{\mathcal{M}_u(r)}\,[-\varphi'(r)]\,dr < +\infty$ *then there exists a sequence* $(b'_k)_{k\in\mathbb{N}}$ *of points in* $[0,1[$*, with limit* 1*, such that* mes $\Phi_2(b'_k) \sim 1 - b'_k$ *as* $k \to +\infty$.

PROOF. During the proof of Theorem 5.1, we outlined $P_\mu \leq \mathcal{M}_u + 1$. Thus Theorem 5.4 is a straightforwardapplication of Theorem 3.6 to the integral $\int_0^1 P_\mu(r)\,[-\varphi'(r)]\,dr$ in the case (i) and $\int_0^1 e^{P_\mu(r)}\,[-\varphi'(r)]\,dr$ in the case (ii). $\square$

EXAMPLE 5.5. When φ and g are defined by $\varphi(t) = (1-t^2)^{\alpha+1}$ and $g(t) = \frac{-t}{(1-t)\log(1-t)}$ $\forall t \in [0,1[$ (for a fixed $\alpha > -1$), we notice that

$$\frac{g(t)}{-\varphi'(t)} = \frac{-t}{2t(\alpha+1)(1-t^2)^\alpha(1-t)\log(1-t)} \leq \frac{1}{2(\alpha+1)(1-t)^{\alpha+1}\log\left(\dfrac{1}{1-t}\right)}$$

since $\frac{1}{1+t} \leq 1$. For every $r \in]0,1[$, it turns out that:

$$\Phi_2(r) \subset \Phi(r) := \left\{ t \in [r,1[:\ e^{P_\mu(t)} < \frac{1}{2(\alpha+1)(1-t)^{\alpha+1} \log\left(\dfrac{1}{1-t}\right)} \right\}.$$

Thus $mes\,\Phi(b'_k) \sim 1 - b'_k$ as $k \to +\infty$ for a function u under condition (ii) of Theorem 5.4.

Lemma 5.6. *Let c denote the solution on $]1,+\infty[$ of $1 + (1-c)\log c = 0$ (for information $c = 2,23998\ldots$) and $\gamma = 1 - e^{-c} = 0,89354\ldots$*

Given $p > 0$ and $\alpha > -1$, let $r_n = \exp(\frac{-1}{np})$ $\forall n \in \mathbb{N}^*$. For functions φ and g defined on $[0,1[$ by $\varphi(t) = (1-t^2)^{\alpha+1}$ $\forall t \in [0,1[$ and $g(t) = \frac{1}{(1-t)\ \log(\frac{1}{1-t})\ \log\log(\frac{1}{1-t})}$ $\forall t \in [\gamma,1[$ (with $g(t) = g(\gamma)$ for $t \leq \gamma$), there is a constant K (depending only on α and p) such that:

$$0 < e\,\frac{g(r_n)}{-\varphi'(r_n)} \leq K\,\frac{n^{\alpha+1}}{(\log n)(\log\log n)} \qquad \forall n \in \mathbb{N}\, \textit{such that}\ n \geq \max\left\{3, \frac{-1}{p\,\log\gamma}\right\}.$$

Moreover g increases on $[0,1[$ and $\int_0^{\to 1} g(t)\,dt$ diverges.

Proof. As $1 - e^{-c} > 1 - e^{-1}$, we know that $\frac{1}{1-t} > e$ $\forall t \in [\gamma,1[$ hence $\log\log(\frac{1}{1-t}) > 0$ thus $g(t)$ is well-defined and positive. The increasingness of g will follow from the decreasingness of g_0 defined by $g_0(t) = (1-t)\,(\log\frac{1}{1-t})\,(\log\log\frac{1}{1-t})$. Its derivative is:

$$g_0'(t) = -\left(\log\frac{1}{1-t}\right)\left(\log\log\frac{1}{1-t}\right) + (1-t)\frac{+1}{1-t}\left(\log\log\frac{1}{1-t}\right)$$
$$+(1-t)\left(\log\frac{1}{1-t}\right)\frac{\dfrac{1}{1-t}}{\log\dfrac{1}{1-t}} = \left(1 - \log\frac{1}{1-t}\right)\left(\log\log\frac{1}{1-t}\right) + 1 \leq 0$$

because of $\log\frac{1}{1-t} \geq c$ $\forall t \in [\gamma,1[$ and $1 + (1-y)\log y \leq 0$ $\forall y \geq c$.

Furthermore we have to check that $\int_0^{\to 1} g(t)\,dt$ diverges: since $g(t) = \frac{u'(t)}{u(t)}$ $\forall t \in [\gamma,1[$ with $u(t) = \log\log\frac{1}{1-t}$ and $u'(t) = \frac{-\frac{-1}{1-t}}{\log\frac{1}{1-t}}$,we obtain for every $r \in [\gamma,1[$:

$$\int_\gamma^r g(t)\,dt = \Big[\log u(t)\Big]_\gamma^r = \log\Big(\log\log\frac{1}{1-r}\Big) - \log\Big(\log\log\frac{1}{1-\gamma}\Big)$$

which tends towards $+\infty$ as $r \to 1^-$. Now

$$\begin{aligned}\frac{g(t)}{-\varphi'(t)} &= \frac{1}{2t(\alpha+1)(1-t^2)^\alpha(1-t)\left(\log\frac{1}{1-t}\right)\left(\log\log\frac{1}{1-t}\right)}\\ &= \frac{1}{2t(\alpha+1)(1+t)^\alpha(1-t)^{\alpha+1}\left(\log\frac{1}{1-t}\right)\left(\log\log\frac{1}{1-t}\right)} \qquad \forall t \in [\gamma, 1[.\end{aligned}$$

We already worked with r_n in Example 5.3. We had $\frac{1}{1-r_n} = n\,\frac{p}{1+\varepsilon_n}$ where $\lim_{n\to+\infty}\varepsilon_n = 0$. Therefore $\log\frac{1}{1-r_n} \sim \log n$ and $\log\log\frac{1}{1-r_n} \sim \log\log n$ as $n \to +\infty$. Here $r_n \geq \gamma$ as soon as $\frac{-1}{pn} \geq \log\gamma$. This leads to

$$\frac{g(r_n)}{-\varphi'(r_n)} \sim \frac{(np)^{\alpha+1}}{2(\alpha+1)\,2^\alpha(\log n)\,(\log\log n)} \qquad \text{as } n \to +\infty. \qquad \square$$

THEOREM 5.7. *Given f a holomorphic function belonging to the Bergman space of parameters $p > 0$ and $\alpha > -1$ (with $f(0) = 1$), there exists a set $F \subset [0,1[$ such that*

$$\sup_{r\in[s,1[} \frac{mes(F\cap[r,1[)}{1-r} = 1 \qquad \forall s \in [0,1[$$

and such that the zeros $(z_k)_k$ of f (indexed by increasing moduli and taking multiplicities into account) satisfy $\prod_{k=1}^n \frac{1}{|z_k|^p} \leq K\,\frac{n^{\alpha+1}}{(\log n)(\log\log n)}$ for any integer $n \geq \max\{3, \frac{-1}{p\,\log\gamma}\}$ such that $r_n := \exp(\frac{-1}{np}) \in F$ (the constants K and γ both stemming from Lemma 5.6).

PROOF. In Example 4.7, we have already seen that the subharmonic function $pu = p\log|f|$ fulfills the condition (ii) of Theorem 4.3 as well as Theorem 4.9, applied with $N = 2$ together with φ and g defined as in Lemma 5.6.

Let $F = \left\{t \in [0,1[:\ e^{\mathcal{M}_{pu}(t)} < \frac{g(t)}{-\varphi'(t)}\right\}$. Thus $F \cap [r,1[= F_2(r)$ of Theorem 4.9 (ii) applied to pu, hence the estimation of $mes(F\cap[r,1[)$.

Given n a fixed integer, let v be the subharmonic function defined in Example 5.3, where we noticed that $\mathcal{M}_v(r) \leq \mathcal{M}_{pu}(r)\ \forall r \in [0,1[$. We also outlined that its Riesz measure ν satisfies $\exp(P_\nu(r_n)) = \prod_{k=1}^n \frac{1}{|z_k|^p}$ with $r_n = \exp(\frac{-1}{np})$.

Besides that $P_\nu(r) \leq \mathcal{M}_v(r) + 1\ \forall r \in [0,1[$ according to an argument performed at the beginning of the proof of Theorem 5.1. Therefore $e^{P_\nu(t)} \leq e.e^{\mathcal{M}_v(t)} < e.\frac{g(t)}{-\varphi'(t)}$ provided that $t \in F$. Lemma 5.6 provides the required majoration for $\prod_{k=1}^n \frac{1}{|z_k|^p}$ as soon as $r_n \in F$. $\square$

Theorem 5.8. *With the same notations as in Theorem* 5.7*, we also have a constant* K' (*depending only on* p *and* α) *such that* $\prod_{k=1}^{n} \frac{1}{|z_k|^p} \leq K' \frac{n^{\alpha+1}}{(\log n)(\log\log n)}$ *for any integer* $n \geq \max\{3, \frac{-1}{p \log \gamma}\}$ *such that* $]r_n, r_{n+1}] \cap F \neq \emptyset$ *and* $|z_n| < |z_{n+1}|$.

Proof. Given such an integer n for which there exists $b \in F$ such that $r_n < b \leq r_{n+1}$ and moreover $|z_n| < |z_{n+1}|$, the Riesz measure $p\,\mu$ of the subharmonic function $p\,u = p \log|f|$ (explicited in Example 4.7) gives rise to $\exp(P_{p\mu}(b)) = \prod_{|z_k|<s_{p\mu}(b)} \frac{1}{|z_k|^p}$ (see Example 5.3). The repartition function $p\,\rho$ associated to $p\,u$ leads to the computation of

$$s_{p\mu}(b) = \sup\left\{t \in]0,1[:\ p\,\rho(t) \leq \frac{1}{h(b)}\right\}.$$

On one hand $\exp(\frac{-1}{np}) < b$ means that $-\log b = h(b) < \frac{1}{np}$. Now $|z_n| < |z_{n+1}|$ implies $p\,\rho(|z_n|) = pn < \frac{1}{h(b)}$ hence $|z_n| \leq s_{p\mu}(b)$. But $|z_n| = s_{p\mu}(b)$ is impossible: otherwise Lemma 2.7 would assert $p\,\rho(|z_n|) \geq \frac{1}{h(b)}$ hence a contradiction. Finally $|z_n| < s_{p\mu}(b)$ thus $\exp(P_{p\mu}(b)) \geq \prod_{1\leq k\leq n} \frac{1}{|z_k|^p}$.

On the other hand $\exp(P_{p\mu}(b)) \leq e.\, e^{\mathcal{M}_{pu}(b)} < e.\frac{g(b)}{-\varphi'(b)}$ since b belongs to the same set F as in the previous proof. Moreover g and φ still stem from Lemma 5.6, thus $g(b) \leq g(r_{n+1})$ because g increases. Besides that

$$\begin{aligned} -\varphi'(b) &= 2b(\alpha+1)(1-b^2)^\alpha \geq 2\,r_n(\alpha+1)(1-r_{n+1}^2)^\alpha = -\varphi'(r_{n+1})\,\frac{r_n}{r_{n+1}} \\ &= -\varphi'(r_{n+1})\,\exp\left[\frac{-1}{p}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right] = -\varphi'(r_{n+1})\,\exp\left(\frac{-1}{pn(n+1)}\right). \end{aligned}$$

Consequently

$$\begin{aligned} \prod_{1\leq k\leq n} \frac{1}{|z_k|^p} &\leq e.\,\frac{g(r_{n+1})}{-\varphi'(r_{n+1})} \exp\left(\frac{1}{pn(n+1)}\right) \\ &\leq K\,e^{1/pn(n+1)}\,\frac{(n+1)^{\alpha+1}}{[\log(n+1)]\,[\log\log(n+1)]} \end{aligned}$$

and the conclusion follows. □

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Lavoro pervenuto alla redazione il 25 marzo 2014
ed accettato per la pubblicazione il 23 aprile 2014
Bozze licenziate il 15 giugno 2014

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