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Remark on a magnetically confined plasma with infinite charge

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Abstract: *We consider a one-species plasma moving in an infinite cylinder in* $\mathbb{R}^3$*, in which it is confined by means of a magnetic field diverging on the walls of the cylinder. In a recent paper [4] we have supposed that the plasma satisfies the Vlasov equation with a Yukawa mutual interaction (i.e. Coulomb at short distance and exponentially decreasing at infinity). Assuming that initially the particles have bounded velocities and are distributed according to a bounded density without any hypothesis on its decreasing at infinity, we have proved the global in time existence and uniqueness of the time evolution of the plasma and its confinement. In the present paper we extend this result to a Coulomb interaction, making on the initial density some assumption of slight decreasing on average at infinity, which however does not imply that the density belongs to any* L^p *space. The proof is similar, but slightly simpler.*

1 – Introduction

In the present paper we study a plasma described by the Vlasov-Poisson equation. This equation has been largely studied for the thermonuclear fusion and for some geophysics problems. Classical results on the existence and uniqueness of the solution in case of $L^\infty \cap L^1$ data can be found in many papers as [1, 12, 14, 15, 16] and [19]. For a nice review of the mathematical results on this topic see also [10].

In some recent papers ([3, 4, 5]) the Vlasov-Poisson equation has been studied in presence of a singular magnetic field that produces a confinement by means of a magnetic mirror. Actually it is well known that a concentrated and very high magnetic field reflects charged particles as a mirror. A classical effect of this phenomenon can bee seen in the aurora borealis.

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In particular in [4] we have studied a plasma contained in an unbounded cylinder and confined by an external magnetic field parallel to the symmetry axis which becomes infinite on the border. This plasma may have infinite charge and occupy the whole cylinder with an initial bounded density. The plasma particles mutually interact via a Yukawa potential, *i.e.* Coulomb-like at short distances and exponentially decreasing at large distances. This decreasing can be caused by the Debye screening effect, or by the presence of a surrounding conductor. In that paper it has been proved the existence and uniqueness of the time evolution of data belonging to L^∞ and the confinement of the solution.

In the present paper we make a stronger assumption on the initial density (see definition (2.8)), which leads to some improvements: we can extend the result to a real Coulomb interaction (without the exponential decreasing), and the proof is slightly simpler. Anyway, also in the present case, initial data which are not contained in *any* L^p are included.

For the convenience of the reader the present paper is self-contained, that is some Lemmas already proved in [4] are reported here in detail.

Finally we quote some other papers which study a Vlasov fluid with an infinite mass: [6, 2, 7, 8, 11, 13, 17], and [18].

The plan of the paper is the following: in Section 2 we state the problem and the main result; in Section 3 we give the proof, and finally in the Appendix we collect some technical tools.

2 – Position of the problem and statement of the main result

We consider a Vlasov plasma in $\mathbb{R}^3$ constituted by particles of the same charge, on which an external magnetic field B is acting, whose role is to confine the plasma in an unbounded cylinder. If $f(x, v, t)$ denotes the distribution of charge (or equivalently mass) at the point (x, v) of the phase-space at time t, the time evolution of the plasma is described by the Vlasov-Poisson system

$$\begin{cases} \partial_t f(x,v,t) + v \cdot \nabla_x f(x,v,t) + (E(x,t) + v \wedge B(x)) \cdot \nabla_v f(x,v,t) = 0 \\ E(x,t) = \displaystyle\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\, \rho(y,t)\, dy \\ \rho(x,t) = \displaystyle\int_{\mathbb{R}^3} f(x,v,t)\, dv \\ f(x,v,0) = f_0(x,v), \end{cases} \tag{2.1}$$

where $E(x, t)$ is the electric field produced by the plasma and $B(x)$ the external magnetic field.

Equation (2.1) is a conservation equation for the density f along the characteristics of the system, that is the solutions to the following problem:

$$\begin{cases} \dot{X}(t) = V(t) \\ \dot{V}(t) = E(X(t),t) + V(t) \wedge B(X(t)) \\ (X(0),V(0)) = (x,v), \end{cases} \tag{2.2}$$

where $(X(t),V(t)) = (X(x,v,t),V(x,v,t))$ denote position and velocity of a particle starting at time $t = 0$ from (x,v). Since f is time-invariant along this motion, it is:

$$\|f(t)\|_{L^\infty} = \|f_0\|_{L^\infty} = C_0. \tag{2.3}$$

As it is well known, solutions along the characteristics produce a weak solution of the Vlasov-Poisson system, which becomes a classical solution when $f(x,v,0)$ is assumed smooth. We also remark that the Lebesgue measure $dxdv$ is conserved along the motion.

In what follows we indicate by u_i, $i = 1,2,3$, the i-th component of a vector $u \in \mathbb{R}^3$. We will also put $u_\perp = (u_2,u_3)$.

We assume that an external magnetic field B is acting in order to keep the plasma inside an infinite cylinder with x_1 as symmetry axis. It is defined as

$$B = \big(h(|x_\perp|^2),0,0\big) \tag{2.4}$$

where the function h is defined in the cylinder:

$$D = \{x \in \mathbb{R}^3 : -\infty < x_1 < +\infty,\ 0 \le |x_\perp| < L\}. \tag{2.5}$$

This choice implies that the first component of a particle velocity is not affected by the presence of the magnetic field. The function h is assumed to be nonnegative, smooth in D and diverging together with its primitive as $|x_\perp| \to L$.

We introduce the cylinder $D_0 \subset D$

$$D_0 = \{x \in \mathbb{R}^3 : -\infty < x_1 < +\infty,\ 0 \le |x_\perp| \le L_0\} \tag{2.6}$$

for some $L_0 < L$ and the set

$$S_0 = \{(x,v) : x \in D_0,\ |v| \le \mathcal{V}_0\}. \tag{2.7}$$

For any $\alpha > 0$ we define now the following family of states:

$$\mathcal{F}^\alpha = \{\rho : D_0 \to \mathbb{R}^+ \text{ such that } \ M_i \le \mathcal{M}|i|^{-\alpha}\} \tag{2.8}$$

with $\mathcal{M}$ a positive constant, $i \in \mathbb{Z} - \{0\}$ and

$$M_i = \int_{\{i \le x_1 < i+1\}} \rho(x)\,dx. \tag{2.9}$$

Remark 2.1. The requirement for the density ρ to belong to $\mathcal{F}^\alpha$ implies a very weak decreasing in average and so ρ could be constant in some regions as $|x| \to \infty$; in this case these regions have to be very small in order that M_i is small for large $|i|$. We stress that these states do not necessarily belong to any L^p space.

In what follows we put $\rho_0(x) = \rho(x, 0)$. Moreover sometimes we will put $\rho(t)$ and $f(t)$ in place of $\rho(x, t)$ and $f(x, v, t)$, when needed for the sake of conciseness.

We prove the following result:

Theorem 2.2. *Let us fix an arbitrary positive time T. Let $f_0 \in L^\infty$ be supported on the set S_0 defined in* (2.7) *and $\rho_0 \in \mathcal{F}^\alpha$ with α arbitrary and positive. Then there exists a solution to system* (2.2) *in $[0, T]$ and continuous functions $\mathcal{V}(t)$, $L(t)$ and $\mathcal{M}(t)$ from $[0, T] \to \mathbb{R}^+$, satisfying $\mathcal{V}(0) = \mathcal{V}_0$, $L(0) = L_0$, $0 \leq \sup_{t\in[0,T]} L(t) < L$ and $\mathcal{M}(0) = \mathcal{M}$ such that, for all times $t \in [0, T]$, $f(t)$ is supported on the set:*

$$S_t = \{(x, v) : x \in D_t,\ |v| \leq \mathcal{V}(t)\} \tag{2.10}$$

with

$$D_t = \left\{x \in \mathbb{R}^3 : -\infty < x_1 < +\infty,\ 0 \leq |x_\perp| \leq L(t)\right\}. \tag{2.11}$$

Moreover $\rho(t) \in \mathcal{F}^\alpha_t$ where

$$\mathcal{F}^\alpha_t = \{\rho : D_t \to \mathbb{R}^+ \text{ such that } M_i \leq \mathcal{M}(t)|i|^{-\alpha}\}. \tag{2.12}$$

This solution is unique in the class of the characteristics distributed with $f(t) \in L^\infty$, supported on S_t and belonging to $\mathcal{F}^\alpha_t$ $\forall t \in [0, T]$.

3 – Proof of Theorem 1

To prove this theorem we consider a partial dynamics, obtained by putting in system (2.1) the initial condition

$$f^N_0(x, v) = f_0(x, v)\chi_{D^N_0}(x), \tag{3.1}$$

being $\chi_{\mathcal{A}}$ the characteristic function of the set $\mathcal{A}$ and

$$D^N_0 = D_0 \cap \{x \in \mathbb{R}^3 : |x_1| \leq N\}.$$

Then we have:

Theorem 3.1. *Let us fix an arbitrary positive time T and let f^N_0 be defined in* (3.1) *with f_0 satisfying the hypotheses of Theorem* 1. *Then there exists a solution $(X^N(t), V^N(t))$ of system (2.2) on the interval $[0, T]$ such that $f^N(t)$ is supported on the set*

$$S^N_t = \{(x, v) : x \in D^N_t,\ |v| \leq \mathcal{V}^N(t)\}$$

with

$$D_t^N = \{x \in \mathbb{R}^3 : |x_1| \le C^N,\ 0 \le |x_\perp| \le L^N(t)\} \tag{3.2}$$

for some constant C^N and some continuous functions $\mathcal{V}^N(t)$ and $L^N(t) < L$. Moreover $\rho^N(t) \in (\mathcal{F}_t^\alpha)^N$ with:

$$(\mathcal{F}_t^\alpha)^N = \{\rho : D_t^N \to \mathbb{R}^+ \text{ such that } M_i \le \mathcal{M}^N(t)|i|^{-\alpha}\} \tag{3.3}$$

with $\mathcal{M}^N(t)$ a continuous function. This solution is unique in the class of the characteristics distributed with $f^N(t) \in L^\infty$, supported on S_t^N and belonging to $(\mathcal{F}_t^\alpha)^N$ $\forall t \in [0,T]$.

Proof. Since f_0^N has compact support, the existence and uniqueness of the solution is contained in [3]. The property (3.3) can be deduced from the following observation:

$$\int_{i \le x_1 \le i+1} f^N(x,v,t)\ dxdv = \int_{i-B^N \le x_1 \le i+1+B^N} f_0^N(x,v)\ dxdv \le \mathcal{M}^N(t)\frac{1}{|i|^\alpha}$$

which follows from the invariance of f along the characteristics and the conservation of the measure $dxdv$, being B^N a bound on the maximal displacement of any plasma particle. □

Our aim is to obtain the solution of eq. (2.2) by a local limit (*i.e.* for fixed (x,v,t)) of $f^N(x,v,t)$ as $N \to \infty$. In the sequel of the paper we will obtain estimates independent of N, which will allow us to perform the limit (for details see [8]).

The confinement The main step in the proof is to show that the time evolved velocities of the plasma particles have a finite bound (independent of N). Once we get this bound, then the confinement follows immediately. We sketch the proof of this fact.

We put for simplicity $l(t) = |X_\perp(t)|$. Writing by components equations (2.2) with initial datum (3.1), after elementary calculation we get:

$$(V_2X_2 + V_3X_3)h(l^2) = \dot{V}_2X_3 - \dot{V}_3X_2 + X_2E_3 - X_3E_2.$$

Denoting by H a primitive of h and integrating in time we obtain,

$$\begin{aligned}&\frac{1}{2}\int_0^t \frac{d}{ds}H(l^2(s))ds = \frac{1}{2}\left[H(l^2(t)) - H(l^2(0))\right] = \\ &\int_0^t ds\left[\dot{V}_2(s)X_3(s) - \dot{V}_3(s)X_2(s) + X_2(s)E_3(s) - X_3(s)E_2(s)\right].\end{aligned} \tag{3.4}$$

By the hypotheses on f_0 we have $H(l^2(0)) \leq C$, while the assumptions on h imply that $H(l^2(t)) \to \infty$ as $l \to L$. Hence, the left hand side in eq. (3.4) goes to infinity as $l \to L$. On the other hand, by integrating by parts the right hand side we have,

$$\begin{aligned} &\int_0^t ds \left[\dot V_2(s)X_3(s) - \dot V_3(s)X_2(s) + X_2(s)E_3(s) - X_3(s)E_2(s)\right] = \\ &[V_2(s)X_3(s) - V_3(s)X_2(s)]_0^t + \int_0^t [X_2(s)E_3(X(s),s) - X_3(s)E_2(X(s),s)]\, ds. \end{aligned} \tag{3.5}$$

Since $\mathcal{V}(T)$ is bounded, also the field E does (see Proposition 3.6), so that by (3.5) the latter term in (3.4) is bounded. Hence the plasma does remain confined in a cylinder properly contained in D, as stated before.

We discuss the strategy of the further steps in the proof. In order to get a uniform bound on the particle velocities v, we observe that, as well known, the magnetic field changes only the direction, not the intensity of v, so that the magnetic force does not contribute to the variation of $|v|$. In facts, from eqns. (2.2) it follows:

$$\frac{d}{dt}v^2 = 2v \cdot \dot v = 2v \cdot (E + v \wedge B) = 2v \cdot E, \tag{3.6}$$

which shows that in order to find a bound for $|v|$ it is sufficient to find a bound for the electric field E. Unfortunately, we are not able to prove exactly this for any fixed time t, but we prove a bound on the average of the electric field over a short time interval Δ, which anyway is sufficient to control $|v|$.

We proceed in the following way: we fix N and we study this approximated system. In [2] we have proved the existence,uniqueness and confinement of the solution using an a priori bound on the total energy. Of course this bound goes to infinity as N goes to infinity. To overcome this difficulty we introduce a sort of local energy, which is our fundamental tool to deal with the infiniteness of the charge. We then study the electric field at a fixed point x, and we observe that the main contribution to its growth does not come from far away particles, nor from those nearby, but from particles in a very large region around x, whose radius is proportional to the maximum displacement that a particle can make. At this point the proof involves the local energy of this region, quantity that, by hypothesis, is initially finite. We prove that, in spite of the infinity of the system, the local energy remains finite at any later time and this allows us to achieve the result.

We go now into details and define the local energy. For $\mu \in \mathbb{R}$ and $R > 0$ we define the function,

$$\varphi^{\mu,R}(x) = \varphi\left(\frac{|x_1 - \mu|}{R}\right), \tag{3.7}$$

with φ, assumed to be smooth for technical purposes, as it will be clear in the following, such that:

$$\varphi(r) = 1 \quad \text{if} \quad r \in [0, 1] \tag{3.8}$$

$$\varphi(r) = 0 \quad \text{if} \quad r \in [2, +\infty) \tag{3.9}$$

$$-2 \leq \varphi'(r) \leq 0. \tag{3.10}$$

The local energy is defined as

$$\begin{aligned} W(\mu, R, t) = \frac{1}{2} \int dx \ \varphi^{\mu,R}(x) \int dv \ |v|^2 f(x, v, t) + \\ \frac{1}{2} \int dx \ \varphi^{\mu,R}(x) \rho(x, t) \int dy \ \frac{\rho(y, t)}{|x - y|}. \end{aligned} \tag{3.11}$$

The function W, already introduced in [8], can be seen as a sort of energy of a bounded region interacting with the rest of the system. Note that (recalling (3.6)) it does not take into account the effects of the magnetic force, nevertheless it will be the most important tool to deal with the unboundedness of the plasma. The assumptions on ρ_0 allow to give meaning to the function W at time $t = 0$, otherwise it would have been infinite, had we assumed only L^∞ data. Indeed, the following holds:

Proposition 3.2.

$$W(\mu, R, 0) \leq CR^{1-\alpha}.$$

Proof. We consider R integer for simplicity. It is easily seen that

$$\int dx \varphi^{\mu,R}(x) \rho_0(x) \leq CR^{1-\alpha}. \tag{3.12}$$

Indeed, it is:

$$\int_{|x-\mu| \leq R} \rho_0(x) \ dx = \int_{|x-\mu| \leq 1} \rho_0(x) \ dx + \sum_{|i|=1}^{R-1} \int_{\mu+i \leq x \leq \mu+i+1} \rho_0(x) \ dx.$$

Now, since $\rho_0 \in \mathcal{F}^\alpha$, if $|\mu| \leq 2R$ it is:

$$\begin{aligned} \int_{|x-\mu| \leq R} \rho_0(x) \ dx \leq C + \int_{|x| \leq 3R} \rho_0(x) \ dx \leq \\ C + \sum_{|i|=1}^{3R-1} \int_{i \leq x \leq i+1} \rho_0(x) \ dx \leq C \sum_{|i|=1}^{3R-1} \frac{1}{|i|^\alpha} \leq CR^{1-\alpha} \end{aligned}$$

while, if $|\mu| > 2R$:

$$\int_{|x-\mu|\leq R} \rho_0(x)\, dx \leq C + \sum_{|i|=1}^{R-1} \frac{1}{|\mu+i|^\alpha} \leq CR^{1-\alpha}$$

since $|\mu + i| \geq R$, which proves (3.12).

Now we estimate the potential energy of a single particle. We have:

$$\int \frac{\rho_0(y)}{|x-y|}\, dy \leq C. \tag{3.13}$$

Indeed it is:

$$\begin{aligned} \int \frac{\rho_0(y)}{|x-y|}\, dy &\leq \int_{|x-y|\leq 1} \frac{\rho_0(y)}{|x-y|}\, dy + \sum_{|i|\geq 1} \frac{1}{|i|} \int_{x_1+i\leq y_1\leq x_1+i+1} \rho_0(y)\, dy \\ &\leq C\left(1 + \sum_{|i|\geq 1:|x_1+i|\geq 1} \frac{1}{|i||x_1+i|^\alpha}\right). \end{aligned} \tag{3.14}$$

By applying the Holder inequality we get

$$\begin{aligned} &\sum_{|i|\geq 1:|x_1+i|\geq 1} \frac{1}{|i||x_1+i|^\alpha} \\ &\leq \left(\sum_{|i|\geq 1:|x_1+i|\geq 1} \frac{1}{|i|^p}\right)^{\frac{1}{p}} \left(\sum_{|i|\geq 1:|x_1+i|\geq 1} \frac{1}{|x_1+i|^{\alpha q}}\right)^{\frac{1}{q}} \leq C \end{aligned} \tag{3.15}$$

having chosen p such that $1 < p < \frac{1}{1-\alpha}$, which implies $\alpha q > 1$. Hence (3.13) is proved and this, together with (3.12), proves the proposition. $\square$

We note that we would have proved Theorem 1 also with the weaker estimate

$$W(\mu, R, 0) \leq CR.$$

However, the result of the above Proposition allows to make a shorter and more fluent demonstration.

In the following we set

$$Q(R,t) = \max\left\{1, \sup_{\mu\in\mathbb{R}} W(\mu, R, t)\right\}. \tag{3.16}$$

Moreover we introduce the maximal velocity of a plasma particle,

$$\mathcal{V}(t) = \max \left\{ \widetilde{C}, \sup_{s \in [0,t]} \sup_{(x,v)} |V(s)| \right\}, \tag{3.17}$$

where $\widetilde{C}$ is a constant that will be chosen large enough, and the maximal displacement:

$$R(t) = 1 + \int_0^t \mathcal{V}(s)\, ds. \tag{3.18}$$

We also put

$$Q(t) = \sup_{s \in [0,t]} Q(R(s), s). \tag{3.19}$$

In what follows we will prove some estimates on the partial dynamics which, we notice, is such that $|X_\perp^N(t)| < L$. It will be clear from the demonstrations that all constants, denoted generically by C and changing from line to line, depend only on $\|f_0\|_{L^\infty}$ and the arbitrary time T, but not on N. For this reason we will omit the index N. Some constants are indexed by an integer $n = 1, 2, 3 \dots$ in order to quote them elsewhere.

3.1 – Preliminary estimates

We fix arbitrarily a time T. All the subsequent estimates hold for any $t \in [0, T]$. To simplify the notation we assume that the motion of the fluid, initially arranged in D_0, remains over the time interval $[0, T]$ in a cylinder strictly contained in D. Indeed, once we find a bound on the velocities, then the observation on the confinement made at the beginning of this section ensures that such assumption is justified.

We state the most important result on the local energy, whose proof is given in the Appendix:

Proposition 3.3. *There exists a constant C independent of N such that*

$$Q(R(t), t) \leq C Q(R(t), 0).$$

As consequence of Proposition 3.2 we have:

Corollary 3.4.

$$Q(R(t), t) \leq C R(t)^{1-\alpha}. \tag{3.20}$$

Now we give a first estimate on the electric field E, which will be refined in the following Proposition 3.6.

Proposition 3.5. *There exists a constant C_1 independent of N such that:*

$$|E(x,t)| \leq C_1 \mathcal{V}(t)^{\frac{4}{3}} Q(R(t),t)^{\frac{1}{3}}. \tag{3.21}$$

Proof. We premise an estimate on the spatial density: for any $\mu \in \mathbb{R}$ and any positive number R it is:

$$\int_{|\mu - x_1| \leq R} dx\ \rho(x,t)^{\frac{5}{3}} \leq CW(\mu, R, t). \tag{3.22}$$

In facts:

$$\begin{aligned}\rho(x,t) &\leq \int_{|v|\leq a} dv f(x,v,t) + \frac{1}{a^2}\int_{|v|>a} dv\ v^2 f(x,v,t)\\ &\leq Ca^3 + \frac{1}{a^2}\int dv\ v^2 f(x,v,t).\end{aligned}$$

By minimizing over a, taking the power $\frac{5}{3}$ of both members and integrating over the set $\{x : |\mu - x_1| \leq R\}$ we get (3.22).

Now we choose a sequence of positive numbers $A_0, A_1, A_2, \ldots A_k, \ldots$ such that $A_0 = 0$, $A_1 < 1$ has to be chosen suitably in the following and $A_k = (k-1)R(t)$ for $k = 2, 3, \ldots$ Then:

$$|E(x,t)| \leq \sum_{k=0}^{+\infty} \mathcal{J}_k(x,t) \tag{3.23}$$

with

$$\mathcal{J}_k(x,t) = \int_{A_k < |x-y| \leq A_{k+1}} dy\ \frac{\rho(y,t)}{|x-y|^2}.$$

We estimate the terms in (3.23). We have:

$$\mathcal{J}_0(x,t) \leq C\|\rho(t)\|_{L^\infty} A_1 \leq C\mathcal{V}(t)^3 A_1. \tag{3.24}$$

Moreover by (3.22) we get:

$$\begin{aligned}&\mathcal{J}_1(x,t)\\ &\leq C\left(\int_{|x-y|\leq A_2} dy\ \rho(y,t)^{\frac{5}{3}}\right)^{\frac{3}{5}} \left(\int_{A_1<|x-y|\leq A_2} \frac{1}{|x-y|^5} dy\right)^{\frac{2}{5}}\\ &\leq CW(x_1, R(t), t)^{\frac{3}{5}} \left[A_1^{-\frac{4}{5}} + R(t)^{-\frac{4}{5}}\right] \leq CQ(R(t),t)^{\frac{3}{5}} A_1^{-\frac{4}{5}}.\end{aligned}$$

The minimum value of $\mathcal{J}_0(x,t) + \mathcal{J}_1(x,t)$ is attained at

$$A_1 = C\mathcal{V}(t)^{-\frac{5}{3}} Q(R(t),t)^{\frac{1}{3}}$$

so that we get

$$\mathcal{J}_0(x,t) + \mathcal{J}_1(x,t) \leq C\mathcal{V}(t)^{\frac{4}{3}} Q(R(t),t)^{\frac{1}{3}}. \tag{3.25}$$

For the remaining terms, for any $k = 2, 3, \dots$ we observe that from the definition (3.18) of maximal displacement it follows that if $A_k < |x-y| \leq A_{k+1}$, then $A_{k-1} < |x - Y(t)| \leq A_{k+2}$. Then by the invariance of f along the characteristics and the assumptions on the support S_0 of f_0, by the change of variables $(y, w) \to (Y(t), W(t))$ we get:

$$\begin{aligned} \mathcal{J}_k(x,t) &= \int_{A_k<|x-y|\leq A_{k+1}} \frac{f(y,w,t)}{|x-y|^2} dydw \\ &\leq C\frac{1}{K^2R(t)^2}\int_{A_{k-1}<|x-y|\leq A_{k+2}} \rho_0(y)dy \leq C\frac{1}{K^2R(t)} \end{aligned} \tag{3.26}$$

since the volume of the cylinder $\{y : A_{k-1} < |x-y| \leq A_{k+2}\}$ is proportional to R. Hence:

$$\sum_{k=2}^{+\infty} \mathcal{J}_k(x,t) \leq C. \tag{3.27}$$

The proof is achieved by (3.23), (3.25) and (3.27). □

3.2 – The estimate of E

We want to prove that $\mathcal{V}(T)$ is bounded. We recall, for further purposes, that $\mathcal{V}(T)$ has been chosen sufficiently large (see (3.17)). The estimate we are going to prove is the following:

Proposition 3.6.

$$\int_0^T |E(X(s),s)|ds \leq C\mathcal{V}(T)^{1-\varepsilon}$$

where $\varepsilon \in (0, \frac{\alpha}{3})$.

To prove Proposition 3.6 we need to control the time average of E over a suitable time interval. Setting

$$\langle E\rangle_{\bar{\Delta}} := \frac{1}{\bar{\Delta}} \int_t^{t+\bar{\Delta}} |E(X(s),s)|\, ds$$

we have the following result:

Proposition 3.7. *There exists a positive number $\bar{\Delta}$ such that:*

$$\langle E \rangle_{\bar{\Delta}} \leq C_2 \mathcal{V}(T)^{1-\varepsilon} \tag{3.28}$$

for any $t \in [0, T]$ such that $t \leq T - \bar{\Delta}$.

Proof. We define a time interval

$$\Delta_1 := \frac{1}{4C_1 \mathcal{V}(T)^{\frac{4}{3}} Q(T)^{\frac{1}{3}}} \tag{3.29}$$

where C_1 is the constant in (3.21). (Note the different choice of Δ_1 from [4], where it was dependent on t). For a positive integer ℓ we set:

$$\Delta_\ell = \Delta_{\ell-1} \mathcal{G} = \cdots = \Delta_1 \mathcal{G}^{\ell-1}, \tag{3.30}$$

denoting by

$$\mathcal{G} = \mathrm{Intg}\left(\mathcal{V}(T)^\delta\right) \tag{3.31}$$

where $\mathrm{Intg}(x)$ is the integer part of x and $\delta \in (0, \frac{2}{3}\alpha]$.

We claim that the following estimate holds, for any positive integer ℓ (putting for brevity $\mathcal{V} := \mathcal{V}(T)$, $Q := Q(T)$),

$$\langle E \rangle_{\Delta_\ell} \leq C \left[\mathcal{V}^{\frac{2}{3}} Q^{\frac{1}{3}} \log \mathcal{V} + \frac{\mathcal{V}^{\frac{4}{3}} Q^{\frac{1}{3}}}{\mathcal{V}^{\frac{2}{3}\alpha}\, \mathcal{V}^{\delta(\ell-1)}} \log \mathcal{V}. \right] \tag{3.32}$$

The proof of (3.32) is done in the following subsection. Note that, since $R(t) \leq C\mathcal{V}(t)$, Corollary 3.4 implies

$$\langle E \rangle_{\Delta_\ell} \leq C \left[\mathcal{V}^{1-\frac{\alpha}{3}} \log \mathcal{V} + \frac{\mathcal{V}^{\frac{5}{3}-\alpha}}{\mathcal{V}^{\delta(\ell-1)}} \log \mathcal{V} \right]. \tag{3.33}$$

Hence, defining $\bar{\ell}$ as the smallest integer such that

$$\mathcal{V}^{\delta(\ell-1)} \geq \mathcal{V}^{\frac{2}{3}}, \tag{3.34}$$

estimate (3.33) implies (3.28) with $\bar{\Delta} = \Delta_{\bar{\ell}}$. $\square$

Such time interval is of the order

$$\bar{\Delta} \approx \frac{C}{\mathcal{V}^{\frac{2}{3}} Q^{\frac{1}{3}}}. \tag{3.35}$$

3.3 – Proof of (3.32)

Before starting with the proof of (3.32) we give some preliminary results.
Let us consider two solutions of the partial dynamics, $(X(t), V(t))$ and $(Y(t), W(t))$. By Proposition 3.5 and the definition (3.30) of Δ_ℓ the following lemmas can be stated, whose proofs are given in the Appendix.

Lemma 3.8. *Let $t \in [0, T]$ such that $t + \Delta_\ell \in [0, T]$ $\forall \ell \leq \bar{\ell}$.*

$$\textit{If} \qquad |V_1(t) - W_1(t)| \leq (\log \mathcal{V})^{\frac{3}{2}}$$

then

$$\sup_{s \in [t, t+\Delta_\ell]} |V_1(s) - W_1(s)| \leq 2(\log \mathcal{V})^{\frac{3}{2}}. \tag{3.36}$$

$$\textit{If} \qquad |V_1(t) - W_1(t)| \geq (\log \mathcal{V})^{\frac{3}{2}}$$

then

$$\inf_{s \in [t, t+\Delta_\ell]} |V_1(s) - W_1(s)| \geq \frac{1}{2}(\log \mathcal{V})^{\frac{3}{2}}. \tag{3.37}$$

Lemma 3.9. *Let $t \in [0, T]$ such that $t + \Delta_\ell \in [0, T]$ $\forall \ell \leq \bar{\ell}$.*

$$\textit{If} \quad |V_\perp(t)| \leq \mathcal{V}^{\frac{1}{4}}$$

then

$$\sup_{s \in [t, t+\Delta_\ell]} |V_\perp(s)| \leq 2\mathcal{V}^{\frac{1}{4}}. \tag{3.38}$$

$$\textit{If} \quad |V_\perp(t)| \geq \mathcal{V}^{\frac{1}{4}}$$

then

$$\inf_{s \in [t, t+\Delta_\ell]} |V_\perp(s)| \geq \frac{\mathcal{V}^{\frac{1}{4}}}{2}. \tag{3.39}$$

Lemma 3.10. *Let $t \in [0, T]$ such that $t + \Delta_\ell \in [0, T]$ $\forall \ell \leq \bar{\ell}$ and assume that $|V_1(t) - W_1(t)| \geq h\big(\log \mathcal{V}\big)^{\frac{3}{2}}$ for some $h \geq 1$. Then it exists $t_0 \in [t, t + \Delta_\ell]$ such that for any $s \in [t, t + \Delta_\ell]$ it holds:*

$$|X(s) - Y(s)| \geq \frac{h(\log \mathcal{V})^{\frac{3}{2}}}{4}|s - t_0|.$$

Lemma 3.11. *There exists a positive constant C such that, for any $\mu \in \mathbb{R}$ and for any couple of positive numbers $R < R'$ we have:*

$$W(\mu, R', t) < C\frac{R'}{R}Q(R, t).$$

Now we are ready to start the proof of (3.32). It is based on an inductive procedure, whose steps are the following:

Step i) we prove (3.32) for $\ell = 1$;

Step ii) we show that if (3.32) holds for $\ell - 1$ it holds also for ℓ;

PROOF OF STEP i). We show that the following estimate holds:

$$\langle E \rangle_{\Delta_1} \leq C \left[\mathcal{V}^{\frac{2}{3}} Q^{\frac{1}{3}} \log \mathcal{V} + \frac{\mathcal{V}^{\frac{4}{3}} Q^{\frac{1}{3}}}{\mathcal{V}^{\frac{2}{3}\alpha}} \log \mathcal{V} \right]. \tag{3.40}$$

For any $t \in [0, T]$ such that $t + \Delta_1 \leq T$, we consider the time evolution of the system over the time interval $[t, t + \Delta_1]$. For any $s \in [t, t + \Delta_1]$ we set

$$(Y(s), W(s)) := (Y(s, t, y, w), W(s, t, y, w))$$

being

$$Y(t) = y, \quad W(t) = w.$$

The time-invariance of f and of the measure $dydw$ along the characteristics allows to write, by the change of variables $(y, w) \to (Y(s), W(s))$:

$$|E(X(s), s)| \leq \int dydw \, \frac{f(y, w, s)}{|X(s) - y|^2} = \int dydw \, \frac{f(y, w, t - \Delta_t)}{|X(s) - Y(s)|^2}. \tag{3.41}$$

We decompose the phase space in the following way. We define

$$T_1 = \{y : |y_1 - X_1(t)| \leq 2R(T)\} \tag{3.42}$$

$$S_1 = \{w : |v_1 - w_1| \leq (\log \mathcal{V})^{\frac{3}{2}}\} \tag{3.43}$$

$$S_2 = \{w : \; |w_\perp| \leq \mathcal{V}^{\frac{1}{4}}\} \tag{3.44}$$

$$S_3 = \{w : \; |v_1 - w_1| > (\log \mathcal{V})^{\frac{3}{2}}\} \cap \{w : |w_\perp| > \mathcal{V}^{\frac{1}{4}}\}. \tag{3.45}$$

We have

$$|E(X(s), s)| \leq \sum_{j=1}^{4} \mathcal{I}_j(X(s)) \tag{3.46}$$

where for any $s \in [t, t + \Delta_1]$

$$\mathcal{I}_j(X(s)) = \int_{T_1 \cap S_j} dydw \, \frac{f(y, w, t)}{|X(s) - Y(s)|^2}, \qquad j = 1, 2, 3$$

and

$$\mathcal{I}_4(X(s)) = \int_{T_1^c} dydw \, \frac{f(y, w, t)}{|X(s) - Y(s)|^2}.$$

Let us start by the first integral. Putting $(Y(s), W(s)) = (\bar{y}, \bar{w})$, by the invariance of f along the trajectories, Lemma 3.8 implies

$$\mathcal{I}_1(X(s)) \le \int_{T_1' \cap S_1'} d\bar{y} d\bar{w} \frac{f(\bar{y}, \bar{w}, s)}{|X(s) - \bar{y}|^2} \tag{3.47}$$

where $T_1' = \{\bar{y} : |\bar{y}_1 - X_1(s)| \le 4R(T)\}$ and $S_1' = \{\bar{w} : |V_1(s) - \bar{w}_1| \le 2\big(\log \mathcal{V}\big)^{\frac{3}{2}}\}$. Now it is:

$$\begin{aligned}\mathcal{I}_1(X(s)) \le &\int_{T_1' \cap S_1' \cap \{|X(s) - \bar{y}| \le \varepsilon\}} d\bar{y} d\bar{w} \frac{f(\bar{y}, \bar{w}, s)}{|X(s) - \bar{y}|^2} \\ &+ \int_{T_1' \cap S_1' \cap \{|X(s) - \bar{y}| > \varepsilon\}} d\bar{y} \ \frac{f(\bar{y}, \bar{w}, s)}{|X(s) - \bar{y}|^2}.\end{aligned} \tag{3.48}$$

Notice that

$$\begin{aligned}&\int_{S_1'} dw f(y, w, s) \le 2C_0 (\log \mathcal{V})^{\frac{3}{2}} \int_{|w_\perp| \le a} dw_\perp \\ &\quad + \int_{|w_\perp| > a} dw_\perp \int dw_1 \ f(y, w, s) \\ &\le Ca^2 (\log \mathcal{V})^{\frac{3}{2}} + \frac{1}{a^2} \int dw |w|^2 f(y, w, s) = Ca^2 (\log \mathcal{V})^{\frac{3}{2}} + \frac{1}{a^2} K(y, s)\end{aligned}$$

where $K(y, s) = \int dw |w|^2 f(y, w, s)$. Minimizing in a we obtain

$$\int_{S_1'} dw f(y, w, s) \le C (\log \mathcal{V})^{\frac{3}{4}} K(y, s)^{\frac{1}{2}}. \tag{3.49}$$

Hence, setting

$$\rho_1(y, s) = \int_{S_1'} dw f(y, w, s),$$

by (3.49) and Lemma 3.11 we get

$$\begin{aligned}\left(\int_{T_1'} dy \ \rho_1(y, s)^2 \right)^{\frac{1}{2}} &\le \ C (\log \mathcal{V})^{\frac{3}{4}} \left(\int_{T_1'} dy \ K(y, s) \right)^{\frac{1}{2}} \\ &\le C (\log \mathcal{V})^{\frac{3}{4}} \sqrt{W(X_1(s), 4R(s), s)} \le C \big(\log \mathcal{V} \big)^{\frac{3}{4}} \sqrt{Q}.\end{aligned} \tag{3.50}$$

Going back to (3.48), this bound implies:

$$\begin{aligned}\mathcal{I}_1(X(s)) &\le C \mathcal{V}^2 (\log \mathcal{V})^{\frac{3}{2}} \varepsilon \\ &\quad + \left(\int_{T_1'} dy \rho_1(\bar{y}, s)^2 \right)^{\frac{1}{2}} \left(\int_{T_1' \cap \{|X(s) - \bar{y}| > \varepsilon\}} d\bar{y} \ \frac{1}{|X(s) - \bar{y}|^4} \right)^{\frac{1}{2}} \\ &\le C \Big(\mathcal{V}^2 (\log \mathcal{V})^{\frac{3}{2}} \varepsilon + (\log \mathcal{V})^{\frac{3}{4}} \sqrt{\frac{Q}{\varepsilon}} \Big).\end{aligned}$$

Minimizing in ε we obtain:

$$\mathcal{I}_1(X(s)) \le CQ^{\frac{1}{3}}\mathcal{V}^{\frac{2}{3}}\log\mathcal{V}. \tag{3.51}$$

Now we pass to the term $\mathcal{I}_2$. Arguing as in (3.48), defining $S_2' = \{w: \ |w_\perp| \le 2\mathcal{V}^{\frac{1}{4}}\}$, by Lemma 3.9 we have:

$$\begin{aligned}\mathcal{I}_2(X(s)) \le & \int_{T_1'\cap S_2'} d\bar{y}d\bar{w}\frac{f(\bar{y},\bar{w},s)}{|X(s)-\bar{y}|^2}\\ \le & \int_{T_1'\cap S_2'\cap\{|X(s)-y|\le\varepsilon\}} d\bar{y}d\bar{w}\frac{f(\bar{y},\bar{w},s)}{|X(s)-\bar{y}|^2} + \int_{T_1'\cap\{|X(s)-\bar{y}|>\varepsilon\}} d\bar{y}\frac{\rho(\bar{y},s)}{|X(s)-\bar{y}|^2}\\ \le & \, C\mathcal{V}^{\frac{3}{2}}\varepsilon + \left(\int_{T_1'} d\bar{y}\ \rho(\bar{y},s)^{\frac{5}{3}}\right)^{\frac{3}{5}}\left(\int_{\{|X(s)-\bar{y}|>\varepsilon\}} d\bar{y}\ \frac{1}{|X(s)-\bar{y}|^5}\right)^{\frac{2}{5}}.\end{aligned}$$

The bound 3.22 implies

$$\mathcal{I}_2(X(s)) \le C\mathcal{V}(t)^{\frac{3}{2}}\varepsilon + CQ(t)^{\frac{3}{5}}\varepsilon^{-\frac{4}{5}}.$$

By minimizing in ε we get:

$$\mathcal{I}_2(X(s)) \le C\mathcal{V}^{\frac{2}{3}}Q^{\frac{1}{3}}. \tag{3.52}$$

For the third term we cover $S_3 \cap T_1$ by means of the sets $A_{h,k}$ and $B_{h,k}$, with $k = 0, 1, 2, \dots, m$ and $h = 1, 2, \dots, m'$, defined in the following way:

$$\begin{aligned}A_{h,k} = \Big\{(y,w,s): \ & h\big(\log\mathcal{V}\big)^{\frac{3}{2}} < |v_1 - w_1| \le (h+1)\big(\log\mathcal{V}\big)^{\frac{3}{2}},\\ & \alpha_{k+1} < |w_\perp| \le \alpha_k, \ |X(s)-Y(s)| \le l_{h,k}\Big\}\end{aligned} \tag{3.53}$$

$$\begin{aligned}B_{h,k} = \Big\{(y,w,s): \ & h(\log\mathcal{V})^{\frac{3}{2}} < |v_1 - w_1| \le (h+1)(\log\mathcal{V})^{\frac{3}{2}},\\ & \alpha_{k+1} < |w_\perp| \le \alpha_k, \ |X(s)-Y(s)| > l_{h,k}\Big\}\end{aligned} \tag{3.54}$$

where:

$$\alpha_k = \frac{\mathcal{V}}{2^k} \qquad l_{h,k} = \frac{2^{2k}Q^{\frac{1}{3}}}{h\mathcal{V}^{\frac{4}{3}}(\log\mathcal{V})^{\frac{5}{2}}} \tag{3.55}$$

Since we are in S_3, it is immediately seen that

$$m \le \frac{3}{4}\log_2\mathcal{V} \quad m' \le \frac{2\mathcal{V}}{(\log\mathcal{V})^{\frac{3}{2}}} - 1. \tag{3.56}$$

Consequently we put

$$\mathcal{I}_3(X(s)) \le \sum_{h=1}^{m'}\sum_{k=0}^{m}(\mathcal{I}_3'(h,k) + \mathcal{I}_3''(h,k)) \tag{3.57}$$

being

$$\mathcal{I}_3'(h,k) = \int_{T_1\cap A_{h,k}} \frac{f(y,w,t)}{|X(s)-Y(s)|^2}\, dydw \tag{3.58}$$

and

$$\mathcal{I}_3''(h,k) = \int_{T_1\cap B_{h,k}} \frac{f(y,w,t)}{|X(s)-Y(s)|^2}\, dydw. \tag{3.59}$$

By adapting Lemma 3.8 and Lemma 3.9 to this context it is easily seen that $\forall\ (y,w,s)\in A_{h,k}$ it holds:

$$(h-1)(\log\mathcal{V})^{\frac{3}{2}} \le |V_1(s)-W_1(s)| \le (h+2)(\log\mathcal{V})^{\frac{3}{2}}, \tag{3.60}$$

and

$$\frac{\alpha_{k+1}}{2} \le |W_\perp(s)| \le 2\alpha_k. \tag{3.61}$$

Hence, setting

$$\begin{aligned} A_{h,k}' = &\big\{(\bar y,\bar w,s):\ (h-1)(\log\mathcal{V})^{\frac{3}{2}} \le |V_1(s)-\bar w_1| \le (h+2)(\log\mathcal{V})^{\frac{3}{2}},\\ &\frac{\alpha_{k+1}}{2} \le |\bar w_\perp| \le 2\alpha_k,\ |X(s)-\bar y|\le l_{h,k}\big\}. \end{aligned} \tag{3.62}$$

we have

$$\mathcal{I}_3'(h,k) \le \int_{T_1'\cap A_{h,k}'} \frac{f(\bar y,\bar w,s)}{|X(s)-\bar y|^2}\, d\bar y d\bar w. \tag{3.63}$$

By the choice of the parameters α_k and $l_{h,k}$ made in (3.55) we have:

$$\begin{aligned} \mathcal{I}_3'(h,k) \le&\ C\, l_{h,k}\int_{A_{h,k}'} d\bar w \le C l_{h,k}\alpha_k^2 \int_{A_{h,k}'} d\bar w_1\\ \le&\ C\, l_{h,k}\alpha_k^2(\log\mathcal{V})^{\frac{3}{2}} \le C\frac{\mathcal{V}^{\frac{2}{3}}Q^{\frac{1}{3}}}{h\log\mathcal{V}}. \end{aligned} \tag{3.64}$$

Hence by (3.56)

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\mathcal{I}_3'(h,k) \le C\frac{\mathcal{V}^{\frac{2}{3}}Q^{\frac{1}{3}}}{\log\mathcal{V}}\sum_{k=0}^{m}\sum_{h=1}^{m'}\frac{1}{h} \le C\mathcal{V}^{\frac{2}{3}}Q^{\frac{1}{3}}\log\mathcal{V}. \tag{3.65}$$

Now we pass to $\mathcal{I}_3''(h,k)$, for which we need to make the time average over the interval $[t,t+\Delta_1]$. Setting

$$B_{h,k}' = \big\{(y,w):(y,w,s)\in B_{h,k}\ \text{ for some }\ s\in[t,t+\Delta_1]\big\} \tag{3.66}$$

we have:

$$\begin{aligned}\int_t^{t+\Delta_1} \mathcal{I}_3''(h,k)\ ds \leq & \int_t^{t+\Delta_1} ds \int_{T_1'\cap B_{h,k}'} dydw\ \frac{f(y,w,t)}{|X(s)-Y(s)|^2} \\ \leq & \int_{T_1'\cap B_{h,k}'} f(y,w,t) \left(\int_t^{t+\Delta_1} \frac{\chi_{B_{h,k}'}(y,w)}{|X(s)-Y(s)|^2}\, ds \right) dydw.\end{aligned} \tag{3.67}$$

By Lemma 3.10, putting $a = \frac{4\, l_{h,k}}{h(\log \mathcal{V})^{3/2}}$ we have:

$$\begin{aligned}&\int_t^{t+\Delta_1} \frac{\chi_{B_{h,k}'}(y,w)}{|X(s)-Y(s)|^2}\, ds \\ &\leq \int_{\{s:|s-t_0|\leq a\}} \frac{\chi_{B_{h,k}'}(y,w)}{|X(s)-Y(s)|^2}\, ds + \int_{\{s:|s-t_0|>a\}} \frac{\chi_{B_{h,k}'}(y,w)}{|X(s)-Y(s)|^2}\, ds \\ &\leq \frac{1}{l_k^2} \int_{\{s:|s-t_0|\leq a\}} ds + \frac{16}{h^2(\log \mathcal{V})^3} \int_{\{s:|s-t_0|>a\}} \frac{ds}{|s-t_0|^2}\, ds \\ &\leq \frac{2a}{l_{h,k}^2} + \frac{32}{h^2(\log \mathcal{V})^3} \int_a^{+\infty} \frac{1}{s^2}\, ds = \frac{16}{l_{h,k} h (\log \mathcal{V})^{\frac{3}{2}}}.\end{aligned} \tag{3.68}$$

Moreover:

$$\int_{T_1'\cap B_{h,k}'} f(y,w,t)\, dydw \leq \frac{C}{\alpha_k^2} \int_{T_1'\cap B_{h,k}'} w^2 f(y,w,t)\, dydw. \tag{3.69}$$

Now it is:

$$\int_{T_1'\cap B_{h,k}'} w^2 f(y,w,t)\, dydw \leq \int_{T_1'\cap C_{h,k}} w^2 f(y,w,t)\, dydw \tag{3.70}$$

where

$$\begin{aligned}C_{h,k} = \big\{ w:\ & (h-1)\big(\log \mathcal{V}\big)^{\frac{3}{2}} \leq |v_1 - w_1| \leq (h+2)\big(\log \mathcal{V}\big)^{\frac{3}{2}}, \\ & \alpha_{k+1} \leq |w_\perp| \leq \alpha_k \big\},\end{aligned}$$

so that:

$$\begin{aligned}\sum_{h=1}^{m'} \sum_{k=0}^{m} \int_{T_1'\cap B_{h,k}'} w^2 f(y,w,t)\, dydw \leq & C \int_{T_1'} K(y,t)\, dy \\ \leq & CW(X_1(t), 5R(t), t) \leq CQ\end{aligned} \tag{3.71}$$

by Lemma 3.11.

Taking into account (3.55), by (3.67), (3.68), (3.69) and (3.71) we get:

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_t^{t+\Delta_1}\mathcal{I}_3''(h,k)\,ds\le C\frac{Q^{\frac{2}{3}}}{\mathcal{V}^{\frac{2}{3}}}\log\mathcal{V}. \tag{3.72}$$

By multiplying and dividing by Δ_1 defined in (3.29) we obtain, recalling Corollary 3.4:

$$\sum_{h=1}^{m'}\sum_{k=0}^{m}\int_t^{t+\Delta_1}\mathcal{I}_3''(h,k)\,ds\le\ C\frac{\mathcal{V}^{\frac{4}{3}}Q^{\frac{1}{3}}}{\mathcal{V}^{\frac{2}{3}\alpha}}\log\mathcal{V}\,\Delta_1. \tag{3.73}$$

Finally the bounds (3.51), (3.52), (3.57), (3.65) and (3.73) imply:

$$\sum_{j=1}^{3}\int_t^{t+\Delta_1}\mathcal{I}_j(X(s))\,ds\le C\Delta_1\left[\mathcal{V}^{\frac{2}{3}}Q^{\frac{1}{3}}\log\mathcal{V}+\frac{\mathcal{V}^{\frac{4}{3}}Q^{\frac{1}{3}}}{\mathcal{V}^{\frac{2}{3}\alpha}}\log\mathcal{V}\right]. \tag{3.74}$$

It remains the estimate of the last term $\mathcal{I}_4(X(s))$. It can be done by the same procedure we used in Proposition 3.5 for the bound (3.27) with $k\ge 2$ to obtain

$$\mathcal{I}_4(X(s))\le C. \tag{3.75}$$

Hence by (3.41), (3.46) and (3.74), this last bound implies:

$$\int_t^{t+\Delta_1}|E(X(s),s)|\,ds\le C\,\Delta_1\left[\mathcal{V}^{\frac{2}{3}}Q^{\frac{1}{3}}\log\mathcal{V}+\frac{\mathcal{V}^{\frac{4}{3}}Q^{\frac{1}{3}}}{\mathcal{V}^{\frac{2}{3}\alpha}}\log\mathcal{V}\right],$$

so that we have proved (3.32) for $\ell=1$. □

Proof of step *ii*). In the previous step we have seen that, starting from estimate (3.21), we arrive at (3.32) on Δ_1. Let us now assume that (3.32) holds at level $\ell-1$ over an interval of size $\Delta_{\ell-1}$, $\ell>1$. Then it can be seen that it holds over an interval of size Δ_ℓ (see Remark 2 in the Appendix). In particular we get, analogously to (3.72),

$$\sum_h\sum_k\int_t^{t+\Delta_\ell}\mathcal{I}_3''(h,k)\,ds\le C\frac{\Delta_\ell}{\Delta_\ell}\frac{Q^{\frac{2}{3}}}{\mathcal{V}^{\frac{2}{3}}}\log\mathcal{V}\le C\frac{\mathcal{V}^{\frac{4}{3}}Q^{\frac{1}{3}}}{\mathcal{V}^{\frac{2}{3}\alpha}\mathcal{V}^{(\ell-1)\delta}}\log\mathcal{V}\,\Delta_\ell \tag{3.76}$$

and consequently

$$\langle E\rangle_{\Delta_\ell}\le C\left[\mathcal{V}^{\frac{2}{3}}Q^{\frac{1}{3}}\log\mathcal{V}+\frac{\mathcal{V}^{\frac{4}{3}}Q^{\frac{1}{3}}}{\mathcal{V}^{\frac{2}{3}\alpha}\mathcal{V}^{(\ell-1)\delta}}\log\mathcal{V}\right] \tag{3.77}$$

which proves the second step. Hence (3.32) is proved. □

3.4 – Proof of Proposition 3.6

We recall that Proposition 3.7 is proved by choosing $\bar{\Delta} = \Delta_{\bar{\ell}}$. We divide now the interval $[0, T]$ by n subintervals $[t_{i-1}, t_i]$, $i = 1, \dots, n$, with $t_0 = 0$, $t_n = T$ and $\frac{1}{2}\bar{\Delta} \le t_i - t_{i-1} \le \bar{\Delta}$. Hence it is:

$$\int_0^T |E(X(s), s)|\, ds \le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |E(X(s), s)|\, ds \le C \sum_{i=1}^{n} \bar{\Delta} \langle E \rangle_{\bar{\Delta}}. \tag{3.78}$$

Proposition 3.7 implies:

$$\int_0^T |E(X(s), s)|\, ds \le C \sum_{i=1}^{n} \bar{\Delta}\, \mathcal{V}^{1-\varepsilon} \le C\, T\, \mathcal{V}^{1-\varepsilon} \tag{3.79}$$

with $\varepsilon > 0$, which proves the Proposition.

3.5 – Proof of Theorem 2.2

Formula (3.6) and Proposition 3.6 easily imply that $\mathcal{V}(T)$ is bounded. As we have seen at the beginning of the section, this in turn implies that for any N the solution remains confined in the same inner cylinder $D_t \subset D$, independently of N. This argument implies also that the magnetic force is bounded uniformly in N. Moreover in the Appendix it is proved the quasi-Lipschitz property for the electric field, which is uniform in N. From these facts the existence and uniqueness of the solution to eq. (2.2) and its belonging to the class $\mathcal{F}_t^{\alpha}$ follow. For the limit $N \to \infty$ we address to [8].

4 – Appendix

Proof of Lemma 3.11. It follows from the definition of the function $\varphi^{\mu,R}$ that, for any $\mu \in \mathbb{R}$ and any couple R, R' such that $0 < R < R'$, it is:

$$\varphi^{\mu,R'}(x) = \varphi\left(\frac{|x_1 - \mu|}{R'}\right) \le \sum_{i \in \mathbb{Z}: |i| \le \frac{R'}{R}} \varphi\left(\frac{|x_1 - (\mu + iR)|}{R}\right).$$

Hence, since both terms in the function W are positive, we have:

$$W(\mu, R', t) \le \sum_{i \in \mathbb{Z}: |i| \le \frac{R'}{R}} W(\mu + iR, R, t) \le C\left(\frac{R'}{R}\right) Q(R, t). \qquad \square$$

Proof of Proposition 3.3. For any s and t such that $0 \leq s < t \leq T$ we define

$$R(t,s) = R(t) + \int_s^t \mathcal{V}(\tau)\, d\tau. \tag{4.1}$$

Then, it is:

$$R(t,t) = R(t) \quad \text{and} \quad R(t,0) = R(t) + \int_0^t \mathcal{V}(\tau) \leq 2R(t). \tag{4.2}$$

Let $(X(s), V(s))$ and $(Y(s), W(s))$ be two characteristics starting at time $s = 0$ from (x, v) and (y, w) respectively. Since the flow preserves the measure in the phase space and f is invariant along the characteristics we have:

$$\begin{aligned} W(\mu, R(t,s), s) &= \frac{1}{2}\int dx \int dv\ \varphi^{\mu,R(t,s)}(X(s))|V(s)|^2 f_0^N(x,v) \\ &+ \frac{1}{2}\iint dxdv \Big[\ \varphi^{\mu,R(t,s)}(X(s)) f_0^N(x,v) \iint dydw f_0^N(y,w)|X(s)-Y(s)|^{-1}\Big]. \end{aligned} \tag{4.3}$$

Deriving the function W with respect to the time s we get:

$$\partial_s W(\mu, R(t,s), s) = A_1(t,s) + A_2(t,s) \tag{4.4}$$

with

$$\begin{aligned} A_1(t,s) = \iint dxdv\ &\varphi^{\mu,R(t,s)}(X(s)) f_0^N(x,v)\Big[V(s)\cdot \dot{V}(s) \\ &+ \frac{1}{2}\iint dydw f_0^N(y,w)\nabla|X(s)-Y(s)|^{-1}\cdot(V(s)-W(s))\Big] \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} A_2(t,s) = \frac{1}{2}\iint dxdv\ f_0^N(x,v)\ &\partial_s\left[\varphi^{\mu,R(t,s)}(X(s))\right] \\ &\left[V^2(s) + \iint dydw\ f_0^N(y,w)\ |X(s)-Y(s)|^{-1}\right]. \end{aligned} \tag{4.6}$$

We see that the $A_2(t,s)$ is negative. Indeed the quantity in square brackets is positive. On the other hand, by the definition of the function φ it is:

$$\begin{aligned} &\partial_s\left[\varphi^{\mu,R(t,s)}(X(s))\right] \\ &= \varphi'\left(\frac{|X_1(s)-\mu|}{R(t,s)}\right)\left[\frac{X_1(s)-\mu}{|X_1(s)-\mu|}\cdot\frac{V_1(s)}{R(t,s)} - \frac{\partial_s R(t,s)}{R^2(t,s)}|X_1(s)-\mu|\right]. \end{aligned}$$

Now, $\varphi'(r) \neq 0$ only if $1 \leq r \leq 2$ and by definition $\partial_s R(t,s) = -\mathcal{V}(s)$, so that:

$$-\frac{\partial_s R(t,s)}{R^2(t,s)}|X_1(s) - \mu| \geq \frac{\mathcal{V}(s)}{R(t,s)}.$$

Hence

$$\frac{X_1(s) - \mu}{|X_1(s) - \mu|} \cdot \frac{V_1(s)}{R(t,s)} - \frac{\partial_s R(t,s)}{R^2(t,s)}|X_1(s) - \mu| \geq \frac{-|V_1(s)| + \mathcal{V}(s)}{R(t,s)} \geq 0.$$

Thus, being $\varphi' \leq 0$, we have proved that

$$A_2(t,s) \leq 0. \tag{4.7}$$

In the term A_1 we observe that, by (3.6), $V(s) \cdot \dot{V}(s) = V(s) \cdot E(X(s), s)$. Noticing that $\nabla|x - y|^{-1}$ is an odd function, by the change of variables $(x, v) \to (y, w)$ we obtain:

$$\begin{aligned} A_1(t,s) = &-\frac{1}{2} \iint dxdv \iint dydw\ f_0^N(x,v) f_0^N(y,w)\ \left[\varphi^{\mu,R(t,s)}(X(s))\right. \\ &\left.\nabla|X(s) - Y(s)|^{-1} \cdot (V(s) + W(s))\right] \\ = &-\frac{1}{2} \iint dxdv \iint dydw\ f_0^N(x,v) f_0^N(y,w) \\ &\times, \left\{\nabla|X(s) - Y(s)|^{-1} \cdot V(s)\left[\varphi^{\mu,R(t,s)}(X(s)) - \varphi^{\mu,R(t,s)}(Y(s))\right]\right\}. \end{aligned}$$

By the definition of $\varphi^{\mu,R(t,s)}$ it follows

$$|\varphi^{\mu,R(t,s)}(X(s)) - \varphi^{\mu,R(t,s)}(Y(s))| \leq 2\frac{|X(s) - Y(s)|}{R(t,s)},$$

and then:

$$\begin{aligned} |A_1(t,s)| \leq &\frac{\mathcal{V}(s)}{R(t,s)} \iint dxdv \iint dydw\ f_0^N(x,v) f_0^N(y,w) \\ &\left|\nabla|X(s) - Y(s)|^{-1}\right|\ |X(s) - Y(s)| \left[\chi_{B(s)}(x,v) + \chi_{\bar{B}(s)}(y,w)\right] \end{aligned}$$

where, by the definition of φ,

$$B(s) = \{x : |X_1(s) - \mu| \leq 2R(t,s)\} \text{ and } \bar{B}(s) = \{y : |Y_1(s) - \mu| \leq 2R(t,s)\}.$$

By symmetry we have:

$$|A_1(t,s)| \leq 2\frac{\mathcal{V}(s)}{R(t,s)} \iint dxdv \iint dydw\ f_0^N(x,v) f_0^N(y,w)$$
$$\left|\nabla |X(s)-Y(s)|^{-1}\right|\ |X(s)-Y(s)|\ \chi_{B(s)}(x,v).$$

For the obvious fact

$$r\left|\nabla |r|^{-1}\right| = \frac{1}{r},$$

we have,

$$|A_1(t,s)|\ \leq 2\frac{\mathcal{V}(s)}{R(t,s)} \iint dxdv \iint dydw\ \frac{f_0^N(x,v) f_0^N(y,w)}{|X(s)-Y(s)|}\ \chi_{B(s)}(x,v).$$

We make the change of variables $(X(s),V(s)) \to (x,v)$ and $(Y(s),W(s)) \to (y,w)$; then the conservation of the measure and the invariance of f^N along the characteristics imply, after integrating out the velocities:

$$|A_1(t,s)| \leq C\frac{\mathcal{V}(s)}{R(t,s)} \int_{B'(s)} dx \int dy\ \frac{\rho(x,s)\rho(y,s)}{|x-y|}$$

where

$$B'(s) = \{x : |x_1 - \mu| \leq 3R(t,s)\}.$$

Let us put

$$I_1(t,s) = \int_{B'(s)} dx \int dy\ \frac{\rho(x,s)\rho(y,s)}{|x-y|}.$$

Then, setting

$$B'(s) = \bigcup_{i\in\mathbb{Z}:|i|\leq 3} B_i(s)$$

and

$$B_i(s) = \{x : |x_1 - \mu_i| \leq R(t,s)\}, \qquad \mu_i = \mu + iR(t,s),$$

by the definition of φ we get:

$$\begin{aligned} I_1(t,s) &= \sum_{i\in\mathbb{Z}:|i|\leq 3} \int_{B_i(s)} dx \int dy\ \varphi^{\mu_i,R(t,s)}(x)\frac{\rho(x,s)\rho(y,s)}{|x-y|} \\ &\leq C \sum_{i\in\mathbb{Z}:|i|\leq 3} W(\mu_i, R(t,s), s) \leq CQ(R(t,s),s). \end{aligned} \tag{4.8}$$

In conclusion, being by definition

$$|A_1(t,s)| \leq C\frac{\mathcal{V}(s)}{R(t,s)}\ I_1(t,s)),$$

we have,

$$|A_1(t,s)| \leq C\frac{\mathcal{V}(s)}{R(t,s)}Q(R(t,s),s). \tag{4.9}$$

Going back to (4.4), we see that (4.7) and (4.9) imply:

$$\partial_s W(\mu, R(t,s), s) \leq C\frac{\mathcal{V}(s)}{R(t,s)}Q(R(t,s),s). \tag{4.10}$$

Notice that

$$\int_0^t \frac{\mathcal{V}(s)}{R(t,s)}ds = -\int_0^t \frac{\partial_s R(t,s)}{R(t,s)}ds = \log\frac{R(t,0)}{R(t,t)} \leq \log 2,$$

so that, by integrating in s both members and taking the supremum over μ in (4.10) we get, by the Gronwall lemma,

$$Q(R(t,s),s) \leq CQ(R(t,0),0).$$

The thesis follows by putting $s = t$, since by (4.2) $Q(R(t,t),t) = Q(R(t),t)$, while the monotonicity of the function Q and Lemma 3.11 imply $Q(R(t,0),0) \leq Q(2R(t),0) \leq CQ(R(t),0)$. $\square$

Proof of the quasi-Lipschitz property of E. We put $D := |x-y|$. Since $\|E(t)\|_{L^\infty} \leq C$, if $D \geq 1$ we have

$$|E(x,t) - E(y,t)| \leq 2\|E(t)\|_{L^\infty} \leq C \leq CD.$$

If $D < 1$, we define $\bar{z} = \frac{x+y}{2}$ and decompose the space in the following way:

$$|E(x,t) - E(y,t)| \leq C\big[I_1(x,y,t) + I_2(x,y,t) + I_3(x,y,t)\big]$$

with

$$I_1(x,y,t) = \int_{|z-\bar{z}|\leq 2D}\left(\frac{1}{|x-z|^2} + \frac{1}{|y-z|^2}\right)\rho(z,t)dz$$

$$I_2(x,y,t) = \int_{2D<|z-\bar{z}|\leq\frac{2}{D}}\left|\frac{1}{|x-z|^2} - \frac{1}{|y-z|^2}\right|\rho(z,t)dz$$

$$I_3(x,y,t) = \int_{|z_1-\bar{z}_1|\geq\frac{2}{D}}\left(\frac{1}{|x-z|^2} + \frac{1}{|y-z|^2}\right)\rho(z,t)dz.$$

Note that if $|z-\bar{z}| \le 2D$ then $|x-z| \le 3D$ and $|y-z| \le 3D$. Hence the first integral can be bounded by

$$I_1(x,y,t) \le C\|\rho(t)\|_{L^\infty} \left(\int_{|x-z|\le 3D} \frac{dz}{|x-z|^2} + \int_{|y-z|\le 3D} \frac{dz}{|y-z|^2} \right) \le CD. \tag{4.11}$$

For the term I_2 we have:

$$I_2(x,y,t) \le CD \int_{2D<|z-\bar{z}|\le \frac{2}{D}} \frac{1}{|z-\xi|^3} dz \le CD \log D, \tag{4.12}$$

with $\xi = \alpha x + (1-\alpha)y$ and $\alpha \in [0,1]$. Finally, if $|z_1 - \bar{z}_1| \ge \frac{2}{D}$, then $\min\{|x_1 - z_1|,\ |y_1 - z_1|\} \ge \frac{1}{D}$, so that

$$\begin{aligned} I_3(x,y,t) \le\ & C \int_{|x_1-z_1|\ge \frac{1}{D}} \frac{1}{|x-z|^2} \rho(z,t)\, dz \\ & + C \int_{|y_1-z_1|\ge \frac{1}{D}} \frac{1}{|y-z|^2} \rho(z,t)\, dz \\ \le\ & CD \int_{|x_1-z_1|\ge \frac{1}{D}} \frac{1}{|x-z|} \rho(z,t)\, dz \\ & + CD \int_{|y_1-z_1|\ge \frac{1}{D}} \frac{1}{|y-z|} \rho(z,t)\, dz \le CD. \end{aligned} \tag{4.13}$$

The bounds (4.11), (4.12) and (4.13) prove the estimate. □

Remark 4.1. We premise the following remark to the proofs of Lemmas 3.8, 3.9, and 3.10. We have,

$$\langle E \rangle_{\Delta_\ell} \le \langle E \rangle_{\Delta_{\ell-1}}, \qquad \forall\, \ell \le \bar{\ell}. \tag{4.14}$$

In fact, $\Delta_\ell = \mathcal{G}\Delta_{\ell-1}$, hence recalling (3.31),

$$\Big[t, t+\Delta_\ell\Big] = \bigcup_{i=1}^{\mathcal{G}} \Big[t + (i-1)\Delta_{\ell-1}, t + i\Delta_{\ell-1}\Big] \tag{4.15}$$

and so,

$$\frac{1}{\Delta_\ell}\int_t^{t+\Delta_\ell} |E(X(s),s)|\, ds \le \max_i \frac{1}{\Delta_{\ell-1}}\int_{t+(i-1)\Delta_{\ell-1}}^{t+i\Delta_{\ell-1}} |E(X(s),s)|\, ds, \tag{4.16}$$

whence we get (4.14), since the estimate (3.32) is built with the maximal time T.

Proof of Lemma 3.8. We give first the proof for $\ell = 1$, that is $\Delta_\ell = \Delta_1$.

Since the magnetic force gives no contribution to the first component of the velocity, by (3.21) and (3.29) we get, for any $s \in [t, t+\Delta_1]$,

$$\begin{aligned}|V_1(s) - W_1(s)| &\le |V_1(t) - W_1(t)| \\ &\quad + \int_t^{t+\Delta_1} \Big[|E(X(s),s)| + |E(Y(s),s)|\Big] ds \\ &\le (\log \mathcal{V})^{\frac{3}{2}} + 2C_1 \mathcal{V}^{\frac{4}{3}} Q^{\frac{1}{3}} \Delta_1 \le 2(\log \mathcal{V})^{\frac{3}{2}}.\end{aligned}$$

Analogously we prove the second statement:

$$\begin{aligned}|V_1(s) - W_1(s)| &\ge |V_1(t) - W_1(t)| \\ &\quad - \int_t^{t+\Delta_1} \Big[|E(X(s),s)| + |E(Y(s),s)|\Big] ds \\ &\ge (\log \mathcal{V})^{\frac{3}{2}} - 2C_1 \mathcal{V}^{\frac{4}{3}} Q^{\frac{1}{3}} \Delta_1 \ge \frac{1}{2}(\log \mathcal{V})^{\frac{3}{2}}.\end{aligned}$$

We show now that Lemma 3.8 holds true also over a time interval Δ_ℓ, $\ell > 1$, supposing for the electric field the estimate (3.32) at level $\ell - 1$. Proceeding as before we get by Remark 2, for any $s \in [t, t+\Delta_\ell]$,

$$\begin{aligned}&|V_1(s) - W_1(s)| \le |V_1(t) - W_1(t)| \\ &\quad + \int_t^{t+\Delta_\ell} \Big[|E(X(s),s)| + |E(Y(s),s)|\Big] ds \\ &\le (\log \mathcal{V})^{\frac{3}{2}} + C\left[\mathcal{V}^{\frac{2}{3}} Q^{\frac{1}{3}} \log \mathcal{V} + \frac{\mathcal{V}^{\frac{4}{3}} Q^{\frac{1}{3}}}{\mathcal{V}^{\frac{2}{3}\gamma}\, \mathcal{V}^{\delta(\ell-2)}} \log \mathcal{V}\right] \frac{\mathcal{V}^{\delta(\ell-1)}}{4C_1 \mathcal{V}^{\frac{4}{3}} Q^{\frac{1}{3}}} \\ &\le (\log \mathcal{V})^{\frac{3}{2}} + C\frac{\mathcal{V}^{\delta(\ell-1)} \log \mathcal{V}}{4C_1 \mathcal{V}^{\frac{2}{3}}} + C\frac{\mathcal{V}^{\delta}}{4C_1 \mathcal{V}^{\frac{2}{3}\alpha}} \log \mathcal{V} \\ &\le 2(\log \mathcal{V})^{\frac{3}{2}},\end{aligned}$$

using (3.34) and recalling that $\delta \in (0, \frac{2}{3}\alpha]$.

We proceed analogously for the lower bound. □

Proof of Lemma 3.9. We begin with the case $\ell = 1$, that is $\Delta_\ell = \Delta_1$.

By (3.6) and the definition of B it is:

$$\frac{d}{ds} V_\perp^2(s) = 2V_\perp(s) \cdot E_\perp(X(s), s). \tag{4.17}$$

We prove the thesis by contradiction. Assume that there exists a time interval $[t^*, t^{**}] \subset [t, t+\Delta_1)$, such that $|V_\perp(t^*)| = \mathcal{V}^{\frac14}$, $|V_\perp(t^{**})| = 2\mathcal{V}^{\frac14}$ and $\mathcal{V}^{\frac14} < |V_\perp(s)| < 2\mathcal{V}^{\frac14}$ $\forall s \in (t^*, t^{**})$. Then from (4.17) it follows, by (3.21):

$$\begin{aligned} |V_\perp(t^{**})|^2 \le\ & |V_\perp(t^*)|^2 + 2\int_{t^*}^{t^{**}} ds\ |V_\perp(s)|\,|E_\perp(X(s), s)| \\ & \le \mathcal{V}^{\frac12} + 4\mathcal{V}^{\frac14} \int_{t^*}^{t^{**}} ds\ |E(X(s), s)| \\ & \le \mathcal{V}^{\frac12} + 4\mathcal{V}^{\frac14}\Delta_1 C_1 \mathcal{V}^{\frac43} Q^{\frac13} < 2\mathcal{V}^{\frac12}. \end{aligned} \tag{4.18}$$

The contradiction proves the thesis.

Now we prove (3.39). As before, assume that there exists a time interval $[t^*, t^{**}] \subset [t, t+\Delta_1)$, such that $|V_\perp(t^*)| = \mathcal{V}^{\frac14}$, $|V_\perp(t^{**})| = \frac12\mathcal{V}^{\frac14}$ and $\frac12\mathcal{V}^{\frac14} < |V_\perp(s)| < \mathcal{V}^{\frac14}$ $\forall s \in (t^*, t^{**})$. Then from (4.17) it follows, by (3.21):

$$\begin{aligned} |V_\perp(t^{**})|^2 \ge |V_\perp(t^*)|^2 - 2\int_{t^*}^{t^{**}} ds\ |V_\perp(s)|\,|E_\perp(X(s), s)| \ge \\ \mathcal{V}^{\frac12} - 2\mathcal{V}^{\frac14} \int_{t^*}^{t^{**}} ds\ |E(X(s), s)| \ge \\ \mathcal{V}^{\frac12} - 2\mathcal{V}^{\frac14}\Delta_1 C_1 \mathcal{V}^{\frac43} Q^{\frac13} > \frac12 \mathcal{V}^{\frac12}. \end{aligned} \tag{4.19}$$

Hence also in this case the contradiction proves the thesis. $\square$

The same argument works also in an interval $[t, t+\Delta_\ell]$, $\ell > 1$, supposing for the electric field the estimate (3.32) at level $\ell - 1$. In fact we have the bound (see before, at the end of the proof of Lemma 3.8),

$$\langle E \rangle_{\Delta_{\ell-1}} \Delta_\ell \le C \log \mathcal{V}$$

which, used in (4.18) and (4.19), allows to achieve the proof.

Proof of Lemma 3.10. We treat first the case $\ell = 1$, that is $\Delta_\ell = \Delta_1$.

Let $t_0 \in [t, t+\Delta_1]$ be the time at which $|X_1(s) - Y_1(s)|$ has the minimum value. We put $\Gamma(s) = X_1(s) - Y_1(s)$. Moreover we define the function

$$\bar\Gamma(s) = \Gamma(t_0) + \dot\Gamma(t_0)(s - t_0).$$

Since the magnetic force does not act on the first component of the velocity it is:

$$\ddot{\Gamma}(s) - \ddot{\bar{\Gamma}}(s) = E_1(X(s), s) - E_1(Y(s), s)$$
$$\Gamma(t_0) = \bar{\Gamma}(t_0), \quad \dot{\Gamma}(t_0) = \dot{\bar{\Gamma}}(t_0)$$

from which it follows

$$\Gamma(s) = \bar{\Gamma}(s) + \int_{t_0}^{s} d\tau \int_{t_0}^{\tau} d\xi \ \big[E_1(X(\xi), \xi) - E_1(Y(\xi), \xi)\big].$$

By (3.21)

$$\begin{aligned} \int_{t_0}^{s} d\tau \int_{t_0}^{\tau} d\xi |E_1(X(\xi), \xi) - E_1(Y(\xi), \xi)| &\leq 2C_1 \mathcal{V}^{\frac{4}{3}} Q^{\frac{1}{3}} \frac{|s - t_0|^2}{2} \\ &\leq C_1 \mathcal{V}^{\frac{4}{3}} Q^{\frac{1}{3}} \Delta_1 |s - t_0| \leq \frac{|s - t_0|}{4}. \end{aligned} \tag{4.20}$$

Hence,

$$|\Gamma(s)| \geq |\bar{\Gamma}(s)| - \frac{|s - t_0|}{4}. \tag{4.21}$$

Now we have:

$$|\bar{\Gamma}(s)|^2 = |\Gamma(t_0)|^2 + 2\Gamma(t_0)\dot{\Gamma}(t_0)(s - t_0) + |\dot{\Gamma}(t_0)|^2 |s - t_0|^2.$$

We observe that $\Gamma(t_0)\dot{\Gamma}(t_0)(s - t_0) \geq 0$. Indeed, if $t_0 \in (t, t + \Delta_1)$ then $\dot{\Gamma}(t_0) = 0$ while if $t_0 = t$ or $t_0 = t + \Delta_1$ the product $\Gamma(t_0)\dot{\Gamma}(t_0)(s - t_0) \geq 0$. Hence

$$|\bar{\Gamma}(s)|^2 \geq |\dot{\Gamma}(t_0)|^2 |s - t_0|^2.$$

By Lemma 3.8 (adapted to this context with a factor $h \geq 1$), since $t_0 \in [t, t + \Delta_1]$ it is

$$|\dot{\Gamma}(t_0)| \geq h \frac{(\log \mathcal{V})^{\frac{3}{2}}}{2}$$

hence

$$|\bar{\Gamma}(s)| \geq h \frac{(\log \mathcal{V})^{\frac{3}{2}}}{2} |s - t_0|$$

and finally by (4.21),

$$|\Gamma(s)| \geq h \frac{(\log \mathcal{V})^{\frac{3}{2}}}{4} |s - t_0|.$$

From this the thesis follows, since obviously $|X(s) - Y(s)| \geq |\Gamma(s)|$. $\square$

We note that the same proof works also considering the interval $[t, t+\Delta_\ell]$, $\ell > 1$, and for the electric field the estimate (3.32) at level $\ell - 1$. In fact we have for the product (see at the end of the proof of Lemma 3.8),

$$\langle E \rangle_{\Delta_{\ell-1}} \Delta_\ell \leq C \log \mathcal{V}$$

which, used in (4.20), allows to achieve the proof.

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