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An elliptic system with degenerate coercivity

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a Bernard, nostro maestro[1]

ABSTRACT: *We study the existence of solutions of a class of degererate elliptic systems.*

1 – Introduction

1.1 – Setting

In this paper we study the existence of solutions of the degererate elliptic system

$$\begin{cases} -\mathrm{div}\Big(\dfrac{a(x)\nabla u}{(b(x)+|z|)^2}\Big) + u = f(x), \\ \\ -\mathrm{div}\Big(\dfrac{A(x)\nabla z}{(B(x)+|u|)^2}\Big) + z = F(x), \end{cases} \tag{1.1}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $a(x)$ and $A(x)$ are measurable matrices such that, for $\alpha, \beta \in \mathbb{R}^+$,

$$\alpha|\xi|^2 \leq a(x)\xi\xi,\ \alpha|\xi|^2 \leq A(x)\xi\xi; \quad |a(x)| \leq \beta,\ |A(x)| \leq \beta. \tag{1.2}$$

Moreover we assume

$$0 < \lambda \leq b(x),\ B(x) \leq \gamma, \tag{1.3}$$

for some $\lambda, \gamma \in \mathbb{R}^+$ and

$$f(x),\ F(x) \in L^2(\Omega). \tag{1.4}$$

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[1] (see [14, 15, 6, 7, 13, 16, 17]).

Theorem 1.1. *Under the assumptions* (1.2), (1.3), (1.4), *there exist* $u \in W_0^{1,1}(\Omega)$ *and* $z \in W_0^{1,1}(\Omega)$, *distributional solutions of the system* (1.1).

1.2 – Comments

First of all, we note that existence of solutions belonging to the nonreflexive space $W_0^{1,1}(\Omega)$ is not so usual in the study of elliptic problems. Recently the existence of solutions in $W_0^{1,1}(\Omega)$ was proved in [3, 4, 5], for elliptic scalar problems with degenerate coercivity (so that this paper is an extension to the systems of some of those results) and in some borderline cases of the Calderon-Zygmund theory of nonlinear Dirichlet problems in [9].

The main difficulty of the problem is that even if the differential operator is well defined between $W_0^{1,2}(\Omega)$ and its dual, it is not coercive on $W_0^{1,2}(\Omega)$: degenerate coercivity means that when $|v|$ is "large", $\frac{1}{(b(x)+|v|)^2}$ goes to zero: for an explicit example see [18].

The study of problems involving degenerate equations begins with the paper [8] and it is developed in [1, 10, 11, 12, 3, 4, 5] (see also [2])

2 – Existence

2.1 – A priori estimates

The first existence result is concerned with the case of a bounded data.

We recall the following definitions.

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k; \\ k\frac{s}{|s|}, & \text{if } |s| > k; \end{cases} \qquad G_k(s) = s - T_k(s).$$

Proposition 2.1. Let $\rho > 0$, $\sigma > 0$ and g, $G \in L^\infty(\Omega)$. Then there exist weak solutions w, W belonging to $W_0^{1,2}(\Omega)$ of the system

$$\begin{cases} w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \ -\text{div}\Big(\dfrac{a(x)\nabla w}{(b(x)+|T_\rho(W)|)^2}\Big) + w = g(x), \\ W \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \ -\text{div}\Big(\dfrac{A(x)\nabla W}{(B(x)+|T_\sigma(w)|)^2}\Big) + W = G(x). \end{cases}$$

Proof. The existence is a consequence of the Leray-Lions theorem (or Schauder theorem) since the principal part is not degenerate, thanks to the presence of T_ρ and T_σ. Moreover, if we take $G_h(w)$ as test function in the first equation and $G_k(W)$

as test function in the second equation, we have, dropping two positive terms,

$$\begin{cases} \displaystyle\int_\Omega [|w| - |g(x)|]|G_h(w)| \le 0, \\ \displaystyle\int_\Omega [|W| - |G(x)|]|G_k(w)| \le 0. \end{cases}$$

Then the choice $h = \|g\|_{L^\infty(\Omega)}$, $k = \|G\|_{L^\infty(\Omega)}$ implies

$$\begin{cases} |w| \le \|g\|_{L^\infty(\Omega)} , \\ |W| \le \|G\|_{L^\infty(\Omega)} . \end{cases}$$

Thus, if we set $\rho = \|g\|_{L^\infty(\Omega)}$ and $\sigma = \|G\|_{L^\infty(\Omega)}$, we can say that w and W are bounded weak solutions of the system

$$\begin{cases} w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \ -\mathrm{div}\left(\dfrac{a(x)\nabla w}{(b(x) + |W|)^2} \right) + w = g(x), \\ W \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \ -\mathrm{div}\left(\dfrac{A(x)\nabla W}{(B(x) + |w|)^2} \right) + W = G(x). \end{cases}$$

□

Now we define

$$f_n = \frac{f}{1 + \frac{1}{n}|f|}, \qquad F_n = \frac{F}{1 + \frac{1}{n}|F|},$$

so that

$$\|f_n - f\|_{L^2(\Omega)} \to 0, \qquad \|F_n - F\|_{L^2(\Omega)} \to 0. \tag{2.1}$$

Thanks to the Proposition 2.1, there exists a solution (u_n, z_n) of the system

$$\begin{cases} u_n \in W_0^{1,2}(\Omega) : \ -\mathrm{div}\left(\dfrac{a(x)\nabla u_n}{(b(x) + |z_n|)^2} \right) + u_n = f_n(x), \\ z_n \in W_0^{1,2}(\Omega) : \ -\mathrm{div}\left(\dfrac{A(x)\nabla z_n}{(B(x) + |u_n|)^2} \right) + z_n = F_n(x), \end{cases} \tag{2.2}$$

Now we prove our first estimates.

LEMMA 2.2. *The sequences $\{u_n\}$ and $\{z_n\}$ are bounded in $L^2(\Omega)$.*

Proof. We take $G_k(u_n)$ as a test function in the first equation and we have

$$\alpha \int_\Omega \frac{|\nabla G_k(u_n)|^2}{(b(x)+|z_n|)^2} + \int_\Omega |G_k(u_n)|^2 \le \int_\Omega |f|\,|G_k(u_n)| \tag{2.3}$$

If we drop the first positive term and we use the Hölder inequality, then we have

$$\left[\int_\Omega |G_k(u_n)|^2\right]^{\frac{1}{2}} \le \left[\int_{\{k\le|u_n|\}} |f|^2\right]^{\frac{1}{2}}. \tag{2.4}$$

Similar estimates hold true for z_n. In particular, taking $k=0$, we have the boundedness of the sequences $\{u_n\}$ and $\{z_n\}$ in $L^2(\Omega)$. So we have that there exist u, z such that, up to subsequences,

$$u_n \rightharpoonup u, \quad z_n \rightharpoonup z \qquad \text{weakly in } L^2(\Omega). \tag{2.5}$$

Then if we drop the second term in (2.3), we have

$$\alpha \int_\Omega \frac{|\nabla G_k(u_n)|^2}{(b(x)+|z_n|)^2} \le \int_{\{k\le|u_n|\}} |f|^2. \tag{2.6}$$

A similar estimate for z_n comes from the second equation. □

Lemma 2.3. *The sequences $\{u_n\}$ and $\{z_n\}$ are bounded in $W_0^{1,1}(\Omega)$.*

Proof. A consequence of (2.6) and of the Hölder inequality is

$$\begin{aligned}\int_\Omega |\nabla G_k(u_n)| &= \int_\Omega \frac{|\nabla G_k(u_n)|}{(b(x)+|z_n|)}\,(b(x)+|z_n|)\\ &\le \left[\int_{\{k\le|u_n|\}} \frac{|f|^2}{\alpha}\right]^{\frac{1}{2}} \big(\|b\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\big).\end{aligned}$$

Similar estimates hold true for z_n. In particular, with $k=0$, we have

$$\begin{aligned}\int_\Omega |\nabla u_n| &\le \frac{\|f\|_{L^2(\Omega)}\,\big(\|b\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\big)}{\alpha^{\frac{1}{2}}},\\ \int_\Omega |\nabla z_n| &\le \frac{\|F\|_{L^2(\Omega)}\,\big(\|b\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\big)}{\alpha^{\frac{1}{2}}}.\end{aligned} \tag{2.7}$$

□

Now we improve the convergence (2.5).

LEMMA 2.4. *The sequences $\{u_n\}$ and $\{z_n\}$ are compact in $L^2(\Omega)$.*

PROOF. The estimates (2.7) imply, thanks to the Rellich embedding Theorem, the L^1 compactenss and then the a.e. convergences

$$u_n(x) \to u(x), \qquad z_n(x) \to z(x). \tag{2.8}$$

Now we use the Vitali Theorem: since we have the a.e. convergences (2.8), the compactness is achieved if we prove the equiintegrability.

Let E be a measurable subset of Ω. Since $u_n = T_k(u_n) + G_k(u_n)$, we have (we use (2.4))

$$\begin{aligned}\int_E |u_n|^2 &\le 2\int_E |T_k(u_n)|^2 + 2\int_E |G_k(u_n)|^2\\ &\le 2\,k^2\,|E| + 2\int_\Omega |G_k(u_n)|^2\\ &\le 2\,k^2\,|E| + 2\int_{\{k\le|u_n|\}} |f|^2,\end{aligned}$$

where $|E|$ denotes the measure of E. Now we recall that a consequence of Lemma 2.3 is that the sequence $\{u_n\}$ is bounded in $L^1(\Omega)$, so that if we fix $\epsilon > 0$, there exists k_ϵ such that (uniformly with respect to n)

$$\int_{\{k\le|u_n|\}} |f|^2 \le \epsilon, \quad k \ge k_\epsilon.$$

Then

$$\int_E |u_n|^2 \le 2\,k^2\,|E| + 2\epsilon$$

implies

$$\lim_{|E|\to 0}\int_E |u_n|^2 \le 2\epsilon, \text{ uniformly with respect to } n.$$

Similar inequality holds true for z_n. □

LEMMA 2.5. *The sequences $\{u_n\}$ and $\{z_n\}$ are weakly compact in $W_0^{1,1}(\Omega)$.*

PROOF. Here we follow [4, 5]. Let again E be a measurable subset of Ω, and let i be in $\{1, \ldots, N\}$. Then

$$\begin{aligned}\int_E |\partial_i u_n| \le \int_E |\nabla u_n| &= \int_E \frac{|\nabla u_n|}{b(x)+|z_n|}\,(b(x)+|z_n|)\\ &\le \left[\int_\Omega \frac{|\nabla u_n|^2}{(b(x)+|z_n|)^2}\right]^{\frac12}\left[\int_E (b(x)+|z_n|)^2\right]^{\frac12}\\ &\le \left[\frac{1}{\alpha}\int_\Omega |f|^2\right]^{\frac12}\left\{\left[\int_E b(x)\right]^{\frac12} + \left[\int_E |z_n|^2\right]^{\frac12}\right\},\end{aligned}$$

where we have used the inequality (2.6) in the last passage. Since the sequence $\{u_n\}$ is compact in $L^2(\Omega)$, we have that the sequence $\{\partial_i u_n\}$ is equiintegrable. Thus, by Dunford-Pettis theorem, and up to subsequences, there exists Y_i in $L^1(\Omega)$ such that $\partial_i u_n$ weakly converges to Y_i in $L^1(\Omega)$. Since $\partial_i u_n$ is the distributional derivative of u_n, we have, for every n in $\mathbb{N}$,

$$\int_\Omega \partial_i u_n \, \phi = -\int_\Omega u_n \, \partial_i \phi \,, \quad \forall \, \phi \in C_0^\infty(\Omega) \,.$$

We now pass to the limit in the above identities, using that $\partial_i u_n$ weakly converges to Y_i in $L^1(\Omega)$, and that u_n strongly converges to u in $L^2(\Omega)$; we obtain

$$\int_\Omega Y_i \, \phi = -\int_\Omega u \, \partial_i \phi \,, \quad \forall \, \phi \in C_0^\infty(\Omega) \,,$$

which implies that $Y_i = \partial_i u$, and this result is true for every i. Since Y_i belongs to $L^1(\Omega)$ for every i, u belongs to $W_0^{1,1}(\Omega)$. A similar result holds true for z_n. □

Thus, thanks to Lemma 2.4 and Lemma 2.5, we can improve the convergence (2.5):

$$\begin{cases} u_n \text{ converges weakly in } W_0^{1,1}(\Omega) \text{ and strongly in } L^2(\Omega) \text{ to } u, \\ z_n \text{ converges weakly in } W_0^{1,1}(\Omega) \text{ and strongly in } L^2(\Omega) \text{ to } z. \end{cases} \tag{2.9}$$

2.2 – Proof of Theorem 1.1

First of all, we use the equiintegrability proved in Lemma 2.5: fix $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for every measurable subset E with $|E| \le \delta(\varepsilon)$, we have

$$\int_E |\nabla u_n| \le \varepsilon.$$

Taking into account the absolute continuty of the Lebesgue integral, we have, for some $\tilde{\delta}(\varepsilon) > 0$,

$$\int_E |\nabla u_n| \le \varepsilon, \quad \int_E |\nabla u| \le \varepsilon,$$

for every measurable subset E with $|E| \le \tilde{\delta}(\varepsilon)$.

On the other hand, since $|\Omega|$ is finite and the sequence

$$D_n = \frac{a(x)}{(b(x) + |z_n|)^2}$$

converges almost everywhere (recall (2.9)), the Egorov theorem says that for every $q > 0$, there exists a measurable subset F of Ω such that $|F| < q$, and D_n converges to D uniformly on $\Omega \setminus F$. We choose $q = \tilde{\delta}$ so that we have, for every $\varphi \in \mathrm{Lip}(\Omega)$,

$$
\begin{aligned}
&\left| \int_\Omega [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| \\
[-1pt] &\leq \left| \int_{\Omega \setminus F} [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| + \left| \int_F [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| \\
&\leq \left| \int_{\Omega \setminus F} [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| + \frac{\beta}{\lambda^2} \| \, |\nabla \varphi| \, \|_{L^\infty(\Omega)} \left[\int_F |\nabla u_n| + \int_F |\nabla u| \right] \\
&\leq \left| \int_{\Omega \setminus F} [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| + 2\varepsilon \frac{\beta}{\lambda^2} \| \, |\nabla \varphi| \, \|_{L^\infty(\Omega)} ,
\end{aligned}
$$

which proves that

$$\int_\Omega \frac{a(x) \, \nabla u_n \nabla \varphi}{(b(x) + |z_n|)^2} \ \to \ \int_\Omega \frac{a(x) \, \nabla u \nabla \varphi}{(b(x) + |z|)^2}. \tag{2.10}$$

Thus, thanks to the above limit, (2.1) and Lemma 2.4, it is possible to pass to the limit in the weak formulation of (2.2), for every φ, $\psi \in \mathrm{Lip}(\Omega)$,

$$\begin{cases} \displaystyle\int_\Omega \frac{a(x) \nabla u_n \nabla \varphi}{(b(x) + |z_n|)^2} + \int_\Omega u_n \, \varphi = \int_\Omega f_n(x) \, \varphi, \\ \displaystyle\int_\Omega \frac{A(x) \nabla z_n \nabla \psi}{(B(x) + |u_n|)^2} + \int_\Omega z_n \, \psi = \int_\Omega F_n(x); \end{cases} \tag{2.11}$$

and we prove that u and z are solutions of our system, in the following distributional sense

$$\begin{cases} \displaystyle\int_\Omega \frac{a(x) \nabla u \nabla \varphi}{(b(x) + |z|)^2} + \int_\Omega u \, \varphi = \int_\Omega f(x) \, \varphi, \quad \forall \, \varphi \in \mathrm{Lip}(\Omega); \\ \displaystyle\int_\Omega \frac{A(x) \nabla z \nabla \psi}{(B(x) + |u|)^2} + \int_\Omega z \, \psi = \int_\Omega F(x) \, \psi, \quad \forall \, \psi \in \mathrm{Lip}(\Omega). \end{cases} \tag{2.12}$$

□

Now we show that, in the above definition of solution, it is possible to use less regular test functions: it possible to use functions only belonging to $W_0^{1,2}(\Omega)$.

Proposition 2.6. *The above functions u and z are solutions of our system, in the following sense*

$$\begin{cases} \displaystyle\int_\Omega \frac{a(x) \nabla u \nabla v}{(b(x) + |z|)^2} + \int_\Omega u \, v = \int_\Omega f(x) \, v, \quad \forall \, v \in W_0^{1,2}(\Omega); \\ \displaystyle\int_\Omega \frac{A(x) \nabla z \nabla w}{(B(x) + |u|)^2} + \int_\Omega z \, w = \int_\Omega F(x) \, w, \quad \forall \, w \in W_0^{1,2}(\Omega). \end{cases} \tag{2.13}$$

Proof. In order to avoid technicalities, here we also assume that

$$a(x) \text{ and } A(x) \text{ are scalar functions.} \tag{2.14}$$

We start with the following inequalities (we use (2.6) with $k = 0$)

$$\int_\Omega \left| \frac{a(x)\nabla u_n}{(b(x)+|z_n|)^2} \right|^2 \le \frac{\alpha^2}{\lambda^2} \int_\Omega \frac{|\nabla u_n|^2}{(b(x)+|z_n|)^2} \le \frac{\alpha^2}{\lambda^2} \int_\Omega |f|^2.$$

Thus, up to subsequences, we can say that, for some $\Psi \in (L^2(\Omega))^N$,

$$\int_\Omega \frac{a(x)\nabla u_n}{(b(x)+|z_n|)^2}\,\Phi \;\to\; \int_\Omega \Psi\,\Phi, \tag{2.15}$$

for every $\Phi \in (L^2(\Omega))^N$. Now we compare the limit (2.10) with the limit (2.15), taking $\Phi = \nabla\varphi$, and we deduce that

$$\int_\Omega \left[\frac{a(x)\,\nabla u}{(b(x)+|z|)^2} - \Psi \right] \Phi = 0.$$

Thus we proved that

$$\frac{a(x)\nabla u_n}{(b(x)+|z_n|)^2} \text{ weakly converges in } (L^2(\Omega))^N \text{ to } \frac{a(x)\nabla u}{(b(x)+|z|)^2},$$

which allows us to pass to the limit in (2.11) only assuming $\varphi,\ \psi \in W_0^{1,2}(\Omega)$.

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