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Chan's Cubic Analogue of Ramanujan's "most beautiful identity" for $p(5n+4)$

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Abstract: *New proof for Chan's cubic analogue of Ramanujan's "most beautiful identity" about the generating function for $p(5n+4)$ is presented and four equations satisfied by Ramanujan's cubic continued fraction are reviewed by means of Weierstrass' three-term relation for classical theta functions.*

1 – Introduction and Motivation

For the two indeterminates q and z with $|q| < 1$, the q-shifted factorial of infinite order and the modified Jacobi theta function are defined respectively by

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n) \quad \text{and} \quad \langle z;q\rangle_\infty = (z;q)_\infty (q/z;q)_\infty.$$

Their product forms are abbreviated respectively as

$$[\alpha,\beta,\cdots,\gamma;q]_\infty = (\alpha;q)_\infty(\beta;q)_\infty\cdots(\gamma;q)_\infty,$$
$$\langle\alpha,\beta,\cdots,\gamma;q\rangle_\infty = \langle\alpha;q\rangle_\infty\langle\beta;q\rangle_\infty\cdots\langle\gamma;q\rangle_\infty.$$

Let $p(n)$ be the number of unrestricted partitions of n with the generating function

$$\sum_{n=0}^{\infty} p(n)q^n \;=\; \frac{1}{(q;q)_\infty}.$$

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Ramanujan [25] discovered the following "most beautiful identity"

$$\sum_{n=0}^{\infty} p(5n+4)q^n \;=\; 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.$$

There exist many proofs in mathematical literature, for which one can find, for example, in [3, 4, 8, 16, 20, 21, 22, 23, 24].

Similarly define the function $a(n)$ by

$$\sum_{n=0}^{\infty} a(n)q^n \;=\; \frac{1}{(q;q)_\infty(q^2;q^2)_\infty}.$$

By means of the product of two identities involving Ramanujan's cubic continued fraction, Chan [10, 11] finds the following cubic analogue

$$\sum_{n=0}^{\infty} a(3n+2)q^n \;=\; 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}.$$

Another proof via trisection series has recently been given by Cao [7].

The main purpose of this paper is to present a new proof for this remarkable generating function. This will be done by showing that Chan's key lemma (see Lemma 2.1) is implied by one of the extended quintuple product identity due to Hirschhorn [19]. Then we shall review four equations satisfied by Ramanujan's cubic continued fraction due to Ramanujan himself and Chan [12] via Weierstrass' three–term relation for classical theta functions.

2 – Identities of triple and quintuple products

In this section, we present a direct proof for the following equation, which has been the key lemma for Chan [11, Theorem 2] to prove her cubic analogue of Ramanujan's "most beautiful identity".

Lemma 2.1 (Chan [10, Theorem 1] and [11, Theorem 2]).

$$\frac{1}{x(q^3)} - q - 2q^2x(q^3) = \frac{(q;q)_\infty(q^2;q^2)_\infty}{(q^9;q^9)_\infty(q^{18};q^{18})_\infty} \quad \textit{where} \quad x(q) = \frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3}.$$

For this sake, recall first Jacobi's triple product identity

$$[q,x,q/x;q]_\infty = \sum_{k=-\infty}^{\infty}(-1)^k q^{\binom{k}{2}}x^k$$

and the quintuple product identity

$$[q,x,q/x;q]_\infty\left[qx^2,q/x^2;q^2\right]_\infty=\sum_{k=-\infty}^{\infty}q^{3\binom{k}{2}+k}(1-q^kx)x^{3k}.$$

Hirschhorn [19] extended the latter with one of its special cases having been highlighted by Chu and Yan [15, Example 7] as follows:

$$\begin{aligned}[q,x,q/x;q]_\infty\left[q^2,x^2,q^2/x^2;q^2\right]_\infty=\left[q^6,q^3,q^3;q^6\right]_\infty&\sum_{k=-\infty}^{\infty}q^{3\binom{k}{2}}x^{3k}\\ -\left[q^6,q,q^5;q^6\right]_\infty&\sum_{k=-\infty}^{\infty}q^{3\binom{k}{2}+k}\left(1+q^kx\right)x^{3k+1}.\end{aligned}$$

Under the replacements $q\to q^3$ and $x\to q$, this identity can be reformulated as

$$\begin{aligned}(q;q)_\infty(q^2;q^2)_\infty=&\left[q^{18},q^9,q^9;q^{18}\right]_\infty\sum_{k=-\infty}^{\infty}q^{9\binom{k}{2}+3k}\\ &-q\left[q^{18},q^3,q^{15};q^{18}\right]_\infty\sum_{k=-\infty}^{\infty}\left(1+q^{3k+1}\right)q^{9\binom{k}{2}+6k}\end{aligned}$$

which can be expressed in terms of triple products

$$\begin{aligned}(q;q)_\infty(q^2;q^2)_\infty=&\left[q^{18},q^9,q^9;q^{18}\right]_\infty\left[q^9,-q^3,-q^6;q^9\right]_\infty\\ &-q\left[q^{18},q^3,q^{15};q^{18}\right]_\infty\left[q^9,-q^3,-q^6;q^9\right]_\infty\\ &-2q^2\left[q^{18},q^3,q^{15};q^{18}\right]_\infty\left[q^9,-q^9,-q^9;q^9\right]_\infty.\end{aligned}$$

Dividing across this equation by $(q^9;q^9)_\infty(q^{18};q^{18})_\infty$ and observing that

$$\begin{aligned}\frac{\left[q^{18},q^3,q^{15};q^{18}\right]_\infty}{(q^{18};q^{18})_\infty}\frac{\left[q^9,-q^3,-q^6;q^9\right]_\infty}{(q^9;q^9)_\infty}&=\left[q^3,q^{15};q^{18}\right]_\infty\left[-q^3,-q^6;q^9\right]_\infty\\ &=\frac{(q^3;q^6)_\infty}{(q^9;q^{18})_\infty}\frac{(-q^3;q^3)_\infty}{(-q^9;q^9)_\infty}\;=\;1\end{aligned}$$

we get the following equalities

$$\begin{aligned}\frac{(q;q)_\infty(q^2;q^2)_\infty}{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}&=\frac{\left[q^{18},q^9,q^9;q^{18}\right]_\infty}{\left[q^{18},q^3,q^{15};q^{18}\right]_\infty}-q-2q^2\frac{\left[q^9,-q^9,-q^9;q^9\right]_\infty}{\left[q^9,-q^3,-q^6;q^9\right]_\infty}\\ &=\frac{(q^9;q^{18})^3_\infty}{(q^3,q^6)_\infty}-q-2q^2\frac{(-q^9;q^9)^3_\infty}{(-q^3;q^3)_\infty}\\ &=\frac{(-q^3;q^3)_\infty}{(-q^9;q^9)^3_\infty}-q-2q^2\frac{(-q^9;q^9)^3_\infty}{(-q^3;q^3)_\infty}\end{aligned}$$

which is equivalent to the equation stated in Lemma 2.1.

3 – Computational proof of Chan's cubic analogue

By means of Lemma 2.1, we are going to produce a computational proof for Chan's following cubic analogue of Ramanujan's "Most Beautiful Identity".

Theorem 3.1 (Chan [10, Theorem 2] and [11, Theorem 1]).

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}.$$

Proof. According to the definition of $a(n)$, it is obvious that

$$\frac{q}{(q;q)_\infty(q^2;q^2)_\infty} = q\sum_{n=0}^{\infty} a(n)q^n.$$

Denote by $\omega = e^{\frac{2\pi i}{3}}$ the cubic root of unity. Replacing q by $q\omega^k$ in the last equation and then summing the resulting equation with respect to k over $0 \le k \le 2$, we have

$$\begin{aligned}\sum_{k=0}^{2} \frac{q\omega^k}{(q\omega^k;q\omega^k)_\infty(q^2\omega^{2k};q^2\omega^{2k})_\infty} &= \sum_{k=0}^{2} q\omega^k \sum_{n=0}^{\infty} a(n)(q\omega^k)^n \\ &= \sum_{n=0}^{\infty} a(n)q^{n+1}\sum_{k=0}^{2}\omega^{k(n+1)}.\end{aligned}$$

Observing the trivial equality

$$\sum_{k=0}^{2}\omega^{k(n+1)} = \begin{cases} 3, & n+1 \equiv_3 0; \\ 0, & n+1 \not\equiv_3 0; \end{cases}$$

and then making the replacement $n \to 3m+2$, we get the following equation

$$\sum_{k=0}^{2} \frac{q\omega^k}{(q\omega^k;q\omega^k)_\infty(q^2\omega^{2k};q^2\omega^{2k})_\infty} = 3\sum_{m=0}^{\infty} a(3m+2)q^{3m+3}.$$

For $p := q^{1/3}$, the last equation can equivalently be restated as

$$\sum_{k=0}^{2} \frac{p\omega^k}{(p\omega^k;p\omega^k)_\infty(p^2\omega^{2k};p^2\omega^{2k})_\infty} = 3\sum_{m=0}^{\infty} a(3m+2)q^{m+1}. \tag{3.1}$$

In Lemma 2.1, replacing q by $p\omega^k$ yields the equation

$$\frac{x(q)}{1-p\omega^k x(q) - 2p^2\omega^{2k}x^2(q)} = \frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(p\omega^k;p\omega^k)_\infty(p^2\omega^{2k};p^2\omega^{2k})_\infty}. \tag{3.2}$$

We can therefore express (3.1) alternatively as

$$\sum_{n=0}^{\infty} a(3n+2)q^n = \frac{1}{3q(q^3;q^3)_\infty (q^6;q^6)_\infty} \sum_{k=0}^{2} \frac{p\omega^k x(q)}{1-p\omega^k x(q) - 2p^2\omega^{2k}x^2(q)}. \tag{3.3}$$

By means of the partial fraction decomposition

$$\frac{y}{1-y-2y^2} = \frac{1}{3}\left\{\frac{1}{1-2y} - \frac{1}{1+y}\right\}$$

the sum of three terms displayed in (3.3) can be reformulated as

$$\sum_{k=0}^{2} \frac{p\omega^k x(q)}{1-p\omega^k x(q) - 2p^2\omega^{2k}x^2(q)} = \frac{1}{3}\sum_{k=0}^{2}\left\{\frac{1}{1-2p\omega^k x(q)} - \frac{1}{1+p\omega^k x(q)}\right\}.$$

Denote by σ_ℓ the ℓth elementary symmetric function of $\{\omega^k\}_{k=0}^{2}$. They can be evaluated explicitly as

$$\sigma_0 = \sigma_3 = 1 \quad \text{and} \quad \sigma_1 = \sigma_2 = 0.$$

Computing the two finite sums

$$\sum_{k=0}^{2} \frac{1}{1-2p\omega^k x(q)} = \frac{3-4px(q)\sigma_1 + 4p^2x^2(q)\sigma_2}{\prod_{k=0}^{2}\{1-2p\omega^k x(q)\}} = 3\prod_{k=0}^{2}\frac{1}{1-2p\omega^k x(q)},$$

$$\sum_{k=0}^{2} \frac{1}{1+p\omega^k x(q)} = \frac{3+2px(q)\sigma_1 + p^2x^2(q)\sigma_2}{\prod_{k=0}^{2}\{1+p\omega^k x(q)\}} = 3\prod_{k=0}^{2}\frac{1}{1+p\omega^k x(q)};$$

and then expanding further the two products

$$\prod_{k=0}^{2}\{1+p\omega^k x(q)\} = 1+px(q)\sigma_1 + p^2x^2(q)\sigma_2 + p^3x^3(q)\sigma_3 = 1+qx^3(q),$$

$$\prod_{k=0}^{2}\{1-2p\omega^k x(q)\} = 1-2px(q)\sigma_1 + 4p^2x^2(q)\sigma_2 - 8p^3x^3(q)\sigma_3 = 1-8qx^3(q);$$

we can factorize the following difference

$$\begin{aligned}
&\sum_{k=0}^{2}\frac{1}{1-2p\omega^kx(q)}-\sum_{k=0}^{2}\frac{1}{1+p\omega^kx(q)}\\
&=3\prod_{k=0}^{2}\frac{1}{1-2p\omega^kx(q)}-3\prod_{k=0}^{2}\frac{1}{1+p\omega^kx(q)}\\
&=\frac{27qx^3(q)}{\prod_{k=0}^{2}\{1-p\omega^kx(q)-2p^2\omega^{2k}x^2(q)\}}.
\end{aligned}$$

Hence this leads us to the following expression

$$\sum_{n=0}^{\infty}a(3n+2)q^n=\frac{3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}\prod_{k=0}^{2}\frac{x(q)}{1-p\omega^kx(q)-2p^2\omega^{2k}x^2(q)}.$$

Then Theorem 3.1 will be confirmed if we can simplify the last product to

$$\prod_{k=0}^{2}\frac{x(q)}{1-p\omega^kx(q)-2p^2\omega^{2k}x^2(q)}=\frac{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}{(q;q)_\infty^4(q^2;q^2)_\infty^4}.$$

According to (3.2), this is equivalent to

$$\prod_{k=0}^{2}\frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(p\omega^k;p\omega^k)_\infty(p^2\omega^{2k};p^2\omega^{2k})_\infty}=\frac{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}{(q;q)_\infty^4(q^2;q^2)_\infty^4}$$

which follows from the two expressions

$$\begin{aligned}
\prod_{k=0}^{2}(p\omega^k;p\omega^k)_\infty&=\frac{(p^3;p^3)_\infty^4}{(p^9;p^9)_\infty}=\frac{(q;q)_\infty^4}{(q^3;q^3)_\infty},\\
\prod_{k=0}^{2}(p^2\omega^{2k};p^2\omega^{2k})_\infty&=\frac{(p^6;p^6)_\infty^4}{(p^{18};p^{18})_\infty}=\frac{(q^2;q^2)_\infty^4}{(q^6;q^6)_\infty}.
\end{aligned}$$

They are justified by the following two almost trivial relations

$$\prod_{k=0}^{2}\{1-(p\omega^k)^n\}=\begin{cases}(1-p^n)^3, & n\equiv_3 0;\\ 1-p^{3n}, & n\not\equiv_3 0;\end{cases}$$

$$\prod_{k=0}^{2}\{1-(p^2\omega^{2k})^n\}=\begin{cases}(1-p^{2n})^3, & n\equiv_3 0;\\ 1-p^{6n}, & n\not\equiv_3 0.\end{cases}$$

This completes the proof of the generating function displayed in Theorem 3.1. □

4 – Weierstrass' three–term relation and implications

In this section, we shall review four identities satisfied by Ramanujan's cubic continued fraction. This is accomplished by utilizing exclusively the fundamental relation of three–terms for classical theta functions, originally due to Weierstrass (*cf.* Whittaker–Watson [30, page 451]). In order to facilitate the subsequent applications, we reproduce it equivalently as follows. For the five complex parameters a, b, c, d, e satisfying $a^2 = bcde$, there holds the theta function identity:

$$\langle a/b, a/c, a/d, a/e; q\rangle_\infty \;-\; \langle b, c, d, e; q\rangle_\infty \;=\; b\,\langle a, a/bc, a/bd, a/be; q\rangle_\infty. \tag{4.1}$$

Chu [13, 14] has systematically explored its applications to Ramanujan's congruences on the partition function and theta function identities. Three identities are reproduced for exemplification. Further examples can be found in Chu [14].

- Jacobi: see Chan [9] and Whittaker–Watson [30, Page 470].

$$(-q;q^2)_\infty^8 - (q;q^2)_\infty^8 \;=\; 16q(-q^2;q^2)_\infty^8. \tag{4.2}$$

- Farkas and Kra [17]: see Chu [14, Example 12] and Warnaar [28, Equation 1].

$$(-q;q^2)_\infty(-q^7;q^{14})_\infty - (q;q^2)_\infty(q^7;q^{14})_\infty = 2q(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty.$$

- Chu [14, Example 13].

$$(-q;q^2)_\infty(q^9;q^{18})_\infty - (q;q^2)_\infty(-q^9;q^{18})_\infty = 2q\frac{(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}(-q^{18};q^{18})_\infty.$$

Denote by $v(q)$ Ramanujan's cubic continued fraction

$$v(q) = \frac{q^{1/3}}{1} \;+\; \frac{q+q^2}{1} \;+\; \frac{q^2+q^4}{1} \;+\; \frac{q^3+q^6}{1} \;+\; \cdots.$$

Ramanujan [26, page 366] discovered the following infinite product expression

$$v(q) = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3} \tag{4.3}$$

and two additional relations

$$1 + \frac{1}{v} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}, \tag{4.4}$$

$$1 - 2v = \frac{\phi(-q^{1/3})}{\phi(-q^3)}; \tag{4.5}$$

where ϕ and ψ are respectively given by

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$
$$\psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

For these identities and other properties of Ramanujan's cubic continued fraction $v(q)$, wonderful introductions have been given by Andrews-Berndt [2, Section 3.3] and Berndt [6, Chapter 20]. Further related works can be found in the papers by Andrews [1], Baruah [5], Chan [12], Gordon [18], Selberg [27], and Watson [29] as well as the references cited by Chan [10].

The product of (4.4) and (4.5) results in the key Lemma 2.1 for Chan [11] to prove Theorem 3.1 for which she gives also separate proofs in [10]. Interestingly enough, we are going to show that both (4.4) and (4.5) follow easily from Weierstrass' Three–Term Relation (4.1).

Proof. First of all, it is not hard to check that (4.4) and (4.5) can equivalently be reformulated as the following respective equations

$$\frac{\psi(q^{1/3})}{\psi(q^3)} - \frac{q^{1/3}}{v} = q^{1/3} \implies \frac{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty(q^3;q^6)_\infty}{(q^{\frac{1}{3}};q^{\frac{2}{3}})_\infty(q^6;q^6)_\infty} - \frac{[q^3,q^3;q^6]_\infty}{[q,q^5;q^6]_\infty} = q^{\frac{1}{3}}, \tag{4.6}$$
$$1 - \frac{\phi(-q^{1/3})}{\phi(-q^3)} = 2v \implies 1 - \frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(-q^3;q^3)_\infty}{(-q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^3;q^3)_\infty} = 2q^{\frac{1}{3}}\frac{[q,q^5;q^6]_\infty}{[q^3,q^3;q^6]_\infty}. \tag{4.7}$$

Putting the left hand side over the common denominator $(q^{\frac{1}{3}};q^{\frac{2}{3}})_\infty\big[q,q^5,q^6;q^6\big]_\infty$ and then rewriting the numerator in the base q^6, we can express (4.6) as

$$\begin{aligned}
&(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty(q;q^2)_\infty - (q^{\frac{1}{3}};q^{\frac{2}{3}})_\infty(q^3;q^3)_\infty(q^3;q^6)_\infty\\
&= \Big[q^{\frac{2}{3}},q^{\frac{4}{3}},q^2,q^{\frac{8}{3}},q^{\frac{10}{3}},q^4,q^{\frac{14}{3}},q^{\frac{16}{3}},q^6,q,q^3,q^5;q^6\Big]_\infty\\
&\quad - \Big[q^{\frac{1}{3}},q,q^{\frac{5}{3}},q^{\frac{7}{3}},q^3,q^{\frac{11}{3}},q^{\frac{13}{3}},q^5,q^{\frac{17}{3}},q^3,q^3,q^6;q^6\Big]_\infty\\
&= \big[q,q^3,q^5,q^6;q^6\big]_\infty\Big\{\langle q^2,q^{\frac{14}{3}},q^{\frac{8}{3}},q^{\frac{2}{3}};q^6\rangle_\infty - \langle q^3,q^{\frac{1}{3}},q^{\frac{7}{3}},q^{\frac{13}{3}};q^6\rangle_\infty\Big\}.
\end{aligned}$$

Under $a \to q^5,\ b \to q^{\frac{1}{3}},\ c \to q^{\frac{7}{3}},\ d \to q^{\frac{13}{3}},\ e \to q^3$, (4.1) can be used to factorize the difference inside the braces into

$$\langle q^{\frac{14}{3}},q^{\frac{8}{3}},q^{\frac{2}{3}},q^2;q^6\rangle_\infty - \langle q^{\frac{1}{3}},q^{\frac{7}{3}},q^{\frac{13}{3}},q^3;q^6\rangle_\infty = q^{\frac{1}{3}}\langle q^5,q^{\frac{1}{3}},q^{\frac{5}{3}},q^{\frac{7}{3}};q^6\rangle_\infty = q^{\frac{1}{3}}\frac{(q^{\frac{1}{3}};q^{\frac{2}{3}})_\infty}{(q^3;q^6)_\infty}.$$

Consequently, the left hand side of (4.6) can be simplified as follows:

$$\begin{aligned}\frac{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty(q^3;q^6)_\infty}{(q^{\frac{1}{3}};q^{\frac{2}{3}})_\infty(q^6;q^6)_\infty}-\frac{[q^3,q^3;q^6]_\infty}{[q,q^5;q^6]_\infty}&=\frac{(q^{\frac{2}{3}};q^{\frac{2}{3}})_\infty(q;q^2)_\infty-(q^{\frac{1}{3}};q^{\frac{2}{3}})_\infty(q^3;q^3)_\infty(q^3;q^6)_\infty}{(q^{\frac{1}{3}};q^{\frac{2}{3}})_\infty\,[q,q^5,q^6;q^6]_\infty}\\&=q^{\frac{1}{3}}\frac{\big[q,q^3,q^5,q^6;q^6\big]_\infty}{(q^{\frac{1}{3}};q^{\frac{2}{3}})_\infty\,[q,q^5,q^6;q^6]_\infty}\frac{(q^{\frac{1}{3}};q^{\frac{2}{3}})_\infty}{(q^3;q^6)_\infty}=q^{\frac{1}{3}}.\end{aligned}$$

This confirms the identity (4.6) and (4.4) consequently.

Similarly for the left hand side of (4.7), multiplying it by $(-q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^3;q^3)_\infty$ and then rewriting the resulting difference in the base q^3 yield the following expression

$$\begin{aligned}&(-q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^3;q^3)_\infty-(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(-q^3;q^3)_\infty\\&=\Big[-q^{\frac{1}{3}},-q^{\frac{2}{3}},-q,-q^{\frac{4}{3}},-q^{\frac{5}{3}},-q^2,-q^{\frac{7}{3}},-q^{\frac{8}{3}},-q^3;q^3\Big]_\infty(q^3;q^3)_\infty\\&\quad-\Big[q^{\frac{1}{3}},q^{\frac{2}{3}},q,q^{\frac{4}{3}},q^{\frac{5}{3}},q^2,q^{\frac{7}{3}},q^{\frac{8}{3}},q^3;q^3\Big]_\infty(-q^3;q^3)_\infty\\&=(q^6;q^6)_\infty\Big\{\langle -q^{\frac{8}{3}},-q^{\frac{5}{3}},-q^{\frac{2}{3}},-q;q^3\rangle_\infty-\langle q^{\frac{1}{3}},q^{\frac{4}{3}},q^{\frac{7}{3}},q^2;q^3\rangle_\infty\Big\}.\end{aligned}$$

Applying (4.1) specified with $a\to -q^3,\ b\to q^{\frac{1}{3}},\ c\to q^{\frac{4}{3}},\ d\to q^{\frac{7}{3}},\ e\to q^2$ leads to the following factorization

$$\begin{aligned}\langle -q^{\frac{8}{3}},-q^{\frac{5}{3}},-q^{\frac{2}{3}},-q;q^3\rangle_\infty-\langle q^{\frac{1}{3}},q^{\frac{4}{3}},q^{\frac{7}{3}},q^2;q^3\rangle_\infty&=q^{\frac{1}{3}}\langle -q^3,-q^{\frac{1}{3}},-q^{\frac{2}{3}},-q^{\frac{4}{3}};q^3\rangle_\infty\\&=2q^{\frac{1}{3}}\frac{(-q^3;q^3)^2_\infty}{(-q;q)_\infty}(-q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty.\end{aligned}$$

Therefore, the left hand side of (4.7) can be reduced to the following expression

$$\begin{aligned}1-\frac{(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(-q^3;q^3)_\infty}{(-q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^3;q^3)_\infty}&=\frac{(-q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^3;q^3)_\infty-(q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(-q^3;q^3)_\infty}{(-q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^3;q^3)_\infty}\\&=2q^{\frac{1}{3}}\frac{(q^6;q^6)_\infty}{(-q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty(q^3;q^3)_\infty}\frac{(-q^3;q^3)^2_\infty}{(-q;q)_\infty}(-q^{\frac{1}{3}};q^{\frac{1}{3}})_\infty\\&=2q^{\frac{1}{3}}\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty}\frac{(-q^3;q^3)^2_\infty}{(-q;q)_\infty}=2q^{\frac{1}{3}}\frac{(q;q^6)_\infty(q^5;q^6)_\infty}{(q^3;q^6)_\infty(q^3;q^6)_\infty}\end{aligned}$$

which proves the identity (4.7) and equivalently (4.5). □

Two further three–term relations that can be wiped out by (4.1) read as

$$v(q)+v(-q)=-2v^2(q^2), \tag{4.8}$$

$$v^2(q)-v(q^2)=-2v(q)\,v^2(q^2). \tag{4.9}$$

They are found by Chan [12, Equations 2.1 and 2.2] in course of determining explicit evaluations for the cubic continued fraction.

Proof. According to (4.1), it is almost trivial to check

$$\begin{aligned}&\langle q,q^5,-q^3,-q^3;q^{12}\rangle_\infty-\langle -q,-q^5,q^3,q^3;q^{12}\rangle_\infty\\&=-q\langle -1,q^2,q^2,-q^6;q^{12}\rangle_\infty=-2q\left[q^2,q^{10},-q^6,-q^{12};q^{12}\right]^2_\infty\end{aligned}$$

which is equivalent to the relation

$$\langle q,-q^3;q^6\rangle_\infty-\langle -q,q^3;q^6\rangle_\infty=-2q^{-\frac13}\,v^2(q^2)\langle q^6;q^{12}\rangle_\infty. \tag{4.10}$$

Then (4.8) and (4.9) follow immediately from the two expressions

$$\begin{aligned}v(q)+v(-q)&=q^{\frac13}\left\{\left[\begin{matrix}q,q^5\\q^3,q^3\end{matrix}\,\middle|\,q^6\right]_\infty-\left[\begin{matrix}-q,-q^5\\-q^3,-q^3\end{matrix}\,\middle|\,q^6\right]_\infty\right\}\\&=\frac{q^{1/3}}{\langle q^6;q^{12}\rangle_\infty}\Big\{\langle q,-q^3;q^6\rangle_\infty-\langle -q,q^3;q^6\rangle_\infty\Big\},\\v^2(q)-v(q^2)&=q^{\frac23}\left\{\left[\begin{matrix}q,q^5\\q^3,q^3\end{matrix}\,\middle|\,q^6\right]^2_\infty-\left[\begin{matrix}q^2,q^{10}\\q^6,q^6\end{matrix}\,\middle|\,q^{12}\right]_\infty\right\}\\&=\frac{q^{1/3}v(q)}{\langle q^6;q^{12}\rangle_\infty}\Big\{\langle q,-q^3;q^6\rangle_\infty-\langle -q,q^3;q^6\rangle_\infty\Big\}.\end{aligned}$$

In addition, there is another theta function identity

$$\frac{(-q;q^2)^2_\infty}{(-q^2;q^2)^2_\infty}-4q^{1/2}\frac{(-q^2;q^2)^2_\infty}{(-q;q^2)^2_\infty}=\frac{(q^{1/2};q^{1/2})^4_\infty}{(q^2;q^2)^4_\infty} \tag{4.11}$$

due to Chan [9, Theorem 1.2], where she uses it to give a new proof of Jacobi's identity displayed in (4.2). This is again implied by the three-term relation (4.1).

In fact, rewriting the difference on the left hand side of (4.11) as the fraction with the common denominator

$$\frac{(-q;q^2)^2_\infty}{(-q^2;q^2)^2_\infty}-4q^{1/2}\frac{(-q^2;q^2)^2_\infty}{(-q;q^2)^2_\infty}=\frac{(-q;q^2)^4_\infty-4q^{1/2}(-q^2;q^2)^4_\infty}{(-q;q)^2_\infty}. \tag{4.12}$$

According to the factorizations

$$\begin{aligned}(-q;q^2)^4_\infty&=(q^{\frac12}\sqrt{-1};q)^4_\infty(-q^{\frac12}\sqrt{-1};q)^4_\infty=\langle q^{\frac12}i;q\rangle^3_\infty\langle -q^{\frac12}i;q\rangle_\infty,\\4(-q^2;q^2)^4_\infty&=(-1;q^2)^2_\infty(-q^2;q^2)^2_\infty=\langle i;q\rangle^2_\infty\langle -i;q\rangle_\infty\langle qi;q\rangle_\infty;\end{aligned}$$

we can therefore factorize the difference displayed in (4.12)

$$\begin{aligned}(-q;q^2)^4_\infty - 4q^{1/2}(-q^2;q^2)^4_\infty =&\langle q^{\frac{1}{2}}i;q\rangle^3_\infty\langle -q^{\frac{1}{2}}i;q\rangle_\infty - q^{1/2}\langle i;q\rangle^2_\infty\langle -i;q\rangle_\infty\langle qi;q\rangle_\infty\\ =&\langle q^{\frac{1}{2}};q\rangle^3_\infty\langle -q^{\frac{1}{2}};q\rangle_\infty = (q;q^2)^2_\infty(q^{\frac{1}{2}};q)^4_\infty\end{aligned}$$

where the last step is justified by (4.1) after having carried out the parameter replacements $a \to qi, b \to q^{\frac{1}{2}}, c \to q^{\frac{1}{2}}, d \to q^{\frac{1}{2}}, e \to -q^{\frac{1}{2}}$. Finally, the fraction displayed in (4.12) can be simplified as follows:

$$\frac{(-q;q^2)^4_\infty - 4q^{1/2}(-q^2;q^2)^4_\infty}{(-q;q)^2_\infty} = \frac{(q;q^2)^2_\infty(q^{\frac{1}{2}};q)^4_\infty}{(-q;q)^2_\infty} = \frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})^4_\infty}{(q^2;q^2)^4_\infty}.$$

□

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