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Algebraic cycles on some special hyperkähler varieties

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Abstract. *This note contains some examples of hyperkähler varieties X having a group G of non–symplectic automorphisms, and such that the action of G on certain Chow groups of X is as predicted by Bloch's conjecture. The examples range in dimension from 6 to 132. For each example, the quotient $Y = X/G$ is a Calabi–Yau variety which has interesting Chow–theoretic properties; in particular, the variety Y satisfies (part of) a strong version of the Beauville–Voisin conjecture.*

1 Introduction

Let X be a hyperkähler variety of dimension $n = 2k$ (i.e., a projective irreducible holomorphic symplectic manifold, cf. [3], [4]). Let $G \subset \mathrm{Aut}(X)$ be a finite cyclic group of order k consisting of non–symplectic automorphisms. We will be interested in the action of G on the Chow groups $A^*(X)$. (Here, $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denotes the Chow group of codimension i algebraic cycles modulo rational equivalence with $\mathbb{Q}$–coefficients. We will write $A^i_{hom}(X)$ and $A^i_{AJ}(X) \subset A^i(X)$ to denote the subgroups of homologically trivial (resp. Abel–Jacobi trivial) cycles.)

We will suppose X has a multiplicative Chow–Künneth decomposition, in the sense of [42]. This implies the Chow ring of X is a bigraded ring $A^*_{(*)}(X)$, where each Chow group splits as

$$A^i(X) = A^i_{(0)}(X) \oplus A^i_{(1)}(X) \oplus \cdots \oplus A^i_{(i)}(X) ,$$

and the piece $A^i_{(j)}(X)$ is expected to be the graded $\mathrm{Gr}^j_F A^i(X)$ for the conjectural Bloch–Beilinson filtration F^* on Chow groups. (Conjecturally, all hyperkähler varieties have a multiplicative Chow–Künneth decomposition. This has been checked for Hilbert schemes of $K3$ surfaces [42], [48], and for generalized Kummer varieties [18].)

Since $H^{n,0}(X) = H^{2,0}(X)^{\otimes k}$, the group G acts as the identity on $H^{n,0}(X)$. For $i < n$, we have that $\sum_{g \in G} g^*$ acts as 0 on $H^{i,0}(X)$. The Bloch–Beilinson conjectures [25] thus imply the following conjecture:

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Conjecture 1.1. *Let X be a hyperkähler variety of dimension $n = 2k$, and let $G \subset \mathrm{Aut}(X)$ be a finite cyclic group of order k of non–symplectic automorphisms. Then*

$$\begin{aligned} &A^n_{(n)}(X) \cap A^n(X)^G = A^n_{(n)}(X) \ ; \\ &A^n_{(j)}(X) \cap A^n(X)^G = 0 \quad \textit{for } 0 < j < n \ ; \\ &A^i_{(i)}(X) \cap A^i(X)^G = 0 \quad \textit{for } 0 < i < n \ . \end{aligned}$$

(Here $A^i(X)^G \subset A^i(X)$ denotes the G–invariant part of the Chow group $A^i(X)$.)

The aim of this note is to find examples where conjecture 1.1 is verified. The main result presents an example of dimension $n = 6$ (and so $k = 3$) where most of conjecture 1.1 is true. The example is given by the Hilbert scheme of a certain special $K3$ surface studied by Livné–Schütt–Yui [33]:

Theorem (=theorem 3.1)**.** *Let S_3 be the $K3$ surface as in theorem 2.24, and let X be the Hilbert scheme $X := (S_3)^{[3]}$ of dimension 6. Let $G \subset \mathrm{Aut}(X)$ be the order 3 group of non–symplectic natural automorphisms, corresponding to the group $G_{S_3} \subset \mathrm{Aut}(S_3)$ of definition 2.22. Then*

$$A^i_{(j)}(X) \cap A^i(X)^G = 0 \quad \textit{if } (i,j) \in \ \Big\{(2,2),(4,4),(3,2),(5,2),(6,2),(6,4)\Big\}.$$

The proof of theorem 3.1 is a fairly easy consequence of the fact that the surface S_3 (and hence the Hilbert scheme X) has *finite–dimensional motive* (in the sense of [29]), and is ρ–maximal (in the sense of [6]). Yet, the implications of theorem 3.1 are quite striking. These implications are most conveniently presented in terms of the Chow ring of the quotient $Y = X/G$ (the variety Y is a 6–dimensional "Calabi–Yau variety with quotient singularities"):

Corollary (=corollary 4.3)**.** *Let X and G be as in theorem 3.1, and let $Y := X/G$. For any $r \in \mathbb{N}$, let*

$$E^\ast(Y^r) \subset A^\ast(Y^r)$$

denote the subalgebra generated by (pullbacks of) $A^1(Y), A^2(Y), A^3(Y)$ and the diagonal $\Delta_Y \in A^6(Y \times Y)$ and the small diagonal $\Delta_Y^{sm} \in A^{12}(Y^3)$. Then the cycle class map

$$E^i(Y^r) \ \to \ H^{2i}(Y^r)$$

is injective for $i \geq 6r - 1$.

Corollary (=corollary 4.4)**.** *Let X and G be as in theorem 3.1, and let $Y := X/G$. Let $a \in A^i(Y)$ be a cycle with $i \neq 3$. Assume a is a sum of intersections of 2 cycles of strictly positive codimension, i.e.*

$$a \in \mathrm{Im}\Big(A^m(Y) \otimes A^{i-m}(Y) \ \to \ A^i(Y)\Big) \ , \quad 0 < m < i \ .$$

Then a is rationally trivial if and only if a is homologically trivial.

This behaviour is remarkable, because $A^6(Y)$ is "huge" (it is not supported on any proper subvariety). In a sense, corollary 4.4 is a mixture of the Beauville–Voisin conjecture (concerning the Chow ring of Hilbert schemes of $K3$ surfaces [49, Conjecture 1.3]) on the one hand, and results concerning 0–cycles on certain Calabi–Yau varieties [50], [16], on the other hand (cf. remark 4.6). These corollaries are easily proven; one merely exploits the good properties of multiplicative Chow–Künneth decompositions combined with finite–dimensionality of the motive of X.

We also give a partial generalization of theorem 3.1 to Hilbert schemes of higher dimension. This generalization concerns Hilbert schemes of the other special $K3$ surfaces S_k ($k > 3$) studied by Livné–Schütt–Yui [33]. The surfaces S_k all have finite–dimensional motive, however (apart from $k = 3$) they are *not* ρ–maximal; for this reason, the conclusion is weaker in these cases:

Theorem (=theorem 5.1)**.** *Let S_k be one of the 16 $K3$ surfaces studied in [33]. Let X be the Hilbert scheme $X = (S_k)^{[k]}$ of dimension $n = 2k$. Let $G \subset \mathrm{Aut}(X)$ be the order k group of natural automorphisms induced by the order k automorphisms of S_k. Then*

$$A^i_{(2)}(X) \cap A^i(X)^G = 0 \quad \textit{for } i \in \{2, n\} \ .$$

The $K3$ surfaces S_k of [33] have k ranging from 3 to 66; the dimension n in theorem 5.1 thus ranges from 6 to 132. Theorem 5.1 as proven below is actually more general than the above statement: theorem 5.1 also applies to certain of the $K3$ surfaces studied in [41] (in particular, there exists a one–dimensional family of Hilbert schemes X of dimension 8 for which theorem 5.1 is true).

Again, the quotient $Y := X/G$ is a "Calabi–Yau variety with quotient singularities" (of dimension n up to 132) which has interesting Chow–theoretic behaviour:

Corollary (=corollaries 5.2 and 5.3)**.** *Let X and G be as in theorem 5.1. Let $Y := X/G$.*
(i) Let $a \in A^{n-1}(Y)$ be a 1–cycle which is in the image of the intersection product map

$$A^{i_1}(Y) \otimes A^{i_2}(Y) \otimes \cdots \otimes A^{i_r}(Y) \ \to \ A^{n-1}(Y) \ ,$$

where all i_j are ≤ 2. Then a is rationally trivial if and only if a is homologically trivial.
(ii) Let $a \in A^n(Y)$ be a 0–cycle which is in the image of the intersection product map

$$A^3(Y) \otimes A^{i_1}(Y) \otimes \cdots \otimes A^{i_r}(Y) \ \to \ A^n(Y) \ ,$$

where all i_j are ≤ 2. Then a is rationally trivial if and only if a is homologically trivial.

Results similar in spirit have been obtained for certain other hyperkähler varieties and their Calabi–Yau quotients in [31], [32].

Conventions. In this article, the word *variety* will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A *subvariety* is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A_j(X)$ the Chow group of j–dimensional cycles on X with $\mathbb{Q}$–coefficients; for X smooth of dimension n the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably.

The notations $A^j_{hom}(X)$, $A^j_{AJ}(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f\colon X \to Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of f. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [40], [35]) will be denoted $\mathcal{M}_{\mathrm{rat}}$.

We will write $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$.

Given a group $G \subset \mathrm{Aut}(X)$ of automorphisms of X, we will write $A^j(X)^G$ (and $H^j(X)^G$) for the subgroup of $A^j(X)$ (resp. $H^j(X)$) invariant under G.

2 Preliminaries

2.1 Quotient varieties

Definition 2.1. A *projective quotient variety* is a variety

$$Y = X/G\ ,$$

where X is a smooth projective variety and $G \subset \mathrm{Aut}(X)$ is a finite group.

Proposition 2.2 (Fulton [19])**.** *Let Y be a projective quotient variety of dimension n. Let $A^*(Y)$ denote the operational Chow cohomology ring. The natural map*

$$A^i(Y)\ \to\ A_{n-i}(Y)$$

is an isomorphism for all i.

Proof. This is [19, Example 17.4.10]. □

Remark 2.3. It follows from proposition 2.2 that the formalism of correspondences goes through unchanged for projective quotient varieties (this is also noted in [19, Example 16.1.13]). We can thus consider motives $(Y, p, 0) \in \mathcal{M}_{\mathrm{rat}}$, where Y is a projective quotient variety and $p \in A^n(Y \times Y)$ is a projector. For a projective quotient variety $Y = X/G$, one readily proves (using Manin's identity principle) that there is an isomorphism

$$h(Y) \cong h(X)^G := (X, \Delta^G_X, 0) \quad \text{in } \mathcal{M}_{\mathrm{rat}}\ ,$$

where Δ^G_X denotes the idempotent

$$\Delta^G_X := \frac{1}{|G|}\sum_{g\in G} \Gamma_g \quad \in A^n(X \times X).$$

(NB: Δ^G_X is a projector on the G–invariant part of the Chow groups $A^*(X)^G$.)

2.2 Finite–dimensional motives

We refer to [29], [2], [35], [22], [27] for basics on the notion of finite–dimensional motive. An essential property of varieties with finite–dimensional motive is embodied by the nilpotence theorem:

Theorem 2.4 (Kimura [29])**.** *Let X be a smooth projective variety of dimension n with finite–dimensional motive. Let $\Gamma\in A^n(X\times X)$ be a correspondence which is numerically trivial. Then there is $N\in\mathbb{N}$ such that*

$$\Gamma^{\circ N}=0\qquad \in A^n(X\times X)\ .$$

Actually, the nilpotence property (for all powers of X) could serve as an alternative definition of finite–dimensional motive, as shown by a result of Jannsen [27, Corollary 3.9]. Conjecturally, all smooth projective varieties have finite–dimensional motive [29]. We are still far from knowing this, but at least there are quite a few non–trivial examples:

Remark 2.5. The following varieties have finite–dimensional motive: abelian varieties, varieties dominated by products of curves [29], $K3$ surfaces with Picard number 19 or 20 [36], surfaces not of general type with $p_g=0$ [20, Theorem 2.11], certain surfaces of general type with $p_g=0$ [20], [37], [53], Hilbert schemes of surfaces known to have finite–dimensional motive [14], generalized Kummer varieties [57, Remark 2.9(ii)] (an alternative proof is contained in [18]), threefolds with nef tangent bundle [23] (an alternative proof is given in [45, Example 3.16]), fourfolds with nef tangent bundle [24], certain threefolds of general type [47, Section 8], varieties of dimension ≤ 3 rationally dominated by products of curves [45, Example 3.15], varieties X with $A^i_{AJ}(X)=0$ for all i [44, Theorem 4], products of varieties with finite–dimensional motive [29].

Remark 2.6. It is an embarassing fact that up till now, all examples of finite-dimensional motives happen to lie in the tensor subcategory generated by Chow motives of curves, i.e. they are "motives of abelian type" in the sense of [45]. On the other hand, there exist many motives that lie outside this subcategory, e.g. the motive of a very general quintic hypersurface in $\mathbb{P}^3$ [15, 7.6].

2.3 MCK decomposition

Definition 2.7 (Murre [34])**.** Let X be a projective quotient variety of dimension n. We say that X has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X=\pi_0+\pi_1+\cdots+\pi_{2n}\quad \hbox{in}\ A^n(X\times X)\ ,$$

such that the π_i are mutually orthogonal idempotents and $(\pi_i)_\ast H^\ast(X)=H^i(X)$.

(NB: "CK decomposition" is shorthand for "Chow–Künneth decomposition".)

Remark 2.8. The existence of a CK decomposition for any smooth projective variety is part of Murre's conjectures [34], [25].

Definition 2.9 (Shen–Vial [42])**.** Let X be a projective quotient variety of dimension n. Let $\Delta_X^{sm}\in A^{2n}(X\times X\times X)$ be the class of the small diagonal

$$\Delta_X^{sm}:=\big\{(x,x,x)\mid x\in X\big\}\ \subset\ X\times X\times X\ .$$

An MCK decomposition is a CK decomposition $\{\pi_i\}$ of X that is *multiplicative*, i.e. it satisfies

$$\pi_k\circ\Delta_X^{sm}\circ(\pi_i\times\pi_j)=0\quad\text{in } A^{2n}(X\times X\times X)\quad\text{for all } i+j\neq k\ .$$

(NB: "MCK decomposition" is shorthand for "multiplicative Chow–Künneth decomposition".)

Remark 2.10. The small diagonal (seen as a correspondence from $X\times X$ to X) induces the *multiplication morphism*

$$\Delta_X^{sm}\colon\quad h(X)\otimes h(X)\ \to\ h(X)\quad\text{in } \mathcal{M}_{\mathrm{rat}}\ .$$

Suppose X has a CK decomposition

$$h(X)=\bigoplus_{i=0}^{2n}h^i(X)\quad\text{in } \mathcal{M}_{\mathrm{rat}}\ .$$

By definition, this decomposition is multiplicative if for any i,j the composition

$$h^i(X)\otimes h^j(X)\ \to\ h(X)\otimes h(X)\ \xrightarrow{\Delta_X^{sm}}\ h(X)\quad\text{in } \mathcal{M}_{\mathrm{rat}}$$

factors through $h^{i+j}(X)$. It follows that if X has an MCK decomposition, then setting

$$A^i_{(j)}(X):=(\pi^X_{2i-j})_\ast A^i(X)\ ,$$

one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends $A^i_{(j)}(X)\otimes A^{i'}_{(j')}(X)$ to $A^{i+i'}_{(j+j')}(X)$.

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville's "weak splitting property" [5]. For more ample discussion, and examples of varieties with an MCK decomposition, we refer to [42, Section 8], as well as [48], [43], [18], [31].

Theorem 2.11 (Vial [48])**.** *Let S be an algebraic $K3$ surface, and let $X=S^{[k]}$ be the Hilbert scheme of length k subschemes of S. Then X has a self–dual MCK decomposition.*

Proof. This is [48, Theorem 1]. For later use, we briefly review the construction. First, one takes an MCK decomposition $\{\pi_i^S\}$ for S (this exists, thanks to [42]). Taking products, this induces an MCK decomposition $\{\pi_i^{S^r}\}$ for S^r, $r\in\mathbb{N}$. This product MCK decomposition is invariant under the action of the symmetric group $\mathfrak{S}_r$, and hence it induces an MCK decomposition $\{\pi_i^{S^{(r)}}\}$ for the symmetric products $S^{(r)}$, $r \in \mathbb{N}$. There is the isomorphism of de Cataldo–Migliorini [14]

$$\bigoplus_{\mu\in\mathfrak{B}(k)} ({}^t\hat{\Gamma}_\mu)_*\colon\quad A^i(X) \xrightarrow{\cong} \bigoplus_{\mu\in\mathfrak{B}(k)} A^{i+l(\mu)-k}(S^{(\mu)})\ ,$$

where $\mathfrak{B}(k)$ is the set of partitions of k, $l(\mu)$ is the length of the partition μ, and $S^{(\mu)} = S^{l(\mu)}/\mathfrak{S}_{l(\mu)}$, and ${}^t\hat{\Gamma}_\mu$ is a correspondence in $A^{k+l(\mu)}(S^{[k]}\times S^{(\mu)})$. Using this isomorphism, Vial defines [48, Equation (4)] a natural CK decomposition for X, by setting

$$\pi_i^X := \sum_{\mu\in\mathfrak{B}(k)} \frac{1}{m_\mu}\hat{\Gamma}_\mu \circ \pi^{S^{(\mu)}}_{i-2k+2l(\mu)} \circ {}^t\hat{\Gamma}_\mu\ , \tag{1}$$

where the m_μ are rational numbers coming from the de Cataldo–Migliorini isomorphism. The $\{\pi_i^X\}$ of definition (1) are proven to be an MCK decomposition.

The self–duality of the $\{\pi_i^X\}$ is apparent from definition (1).

□

Remark 2.12. It follows from definition (1) that the de Cataldo–Migliorini isomorphism is compatible with the bigrading of the Chow ring, in the sense that there are induced isomorphisms

$$\bigoplus_{\mu\in\mathfrak{B}(k)} ({}^t\hat{\Gamma}_\mu)_*\colon\quad A^i_{(j)}(X) \xrightarrow{\cong} \bigoplus_{\mu\in\mathfrak{B}(k)} A^{i+l(\mu)-k}_{(j)}(S^{(\mu)})\ .$$

In particular, there are split injections

$$\bigoplus_{\mu\in\mathfrak{B}(k)} ({}^t\Gamma_\mu)_*\colon\quad A^i_{(j)}(X) \xrightarrow{\cong} \bigoplus_{\mu\in\mathfrak{B}(k)} A^{i+l(\mu)-k}_{(j)}(S^{\mu})\ .$$

Lemma 2.13 (Shen–Vial)**.** *Let X be a projective quotient variety of dimension n, and suppose X has a self–dual MCK decomposition. Then*

$$\begin{aligned} \Delta_X &\in A^n_{(0)}(X\times X)\ ,\\ \Delta_X^{sm} &\in A^{2n}_{(0)}(X\times X\times X)\ . \end{aligned}$$

Proof. The first statement follows from [43, Lemma 1.4] when X is smooth. The same argument works for projective quotient varieties; the point is just that

$$\begin{aligned} \Delta_X &= \sum_{i=0}^{2n}\pi_i^X = \sum_{i=0}^{2n}\pi_i^X\circ\pi_i^X\\ &= \sum_{i=0}^{2n}({}^t\pi_i^X\times\pi_i^X)_*\Delta_X\\ &= \sum_{i=0}^{2n}(\pi_{2n-i}^X\times\pi_i^X)_*\Delta_X\\ &= (\pi_{2n}^{X\times X})_*\Delta_X \quad \in A^n_{(0)}(X\times X)\ . \end{aligned}$$

(Here, the second line follows from Lieberman's lemma [45, Lemma 3.3], and the last line is the fact that the product of 2 MCK decompositions is MCK.)

The second statement is proven for smooth X in [42, Proposition 8.4]; the same argument works for projective quotient varieties. □

2.4 Birational invariance

Proposition 2.14 (Rieß[38], Vial [48])**.** *Let X and X' be birational hyperkähler varieties. Assume X has an MCK decomposition. Then also X' has an MCK decomposition, and there are natural isomorphisms*

$$A^i_{(j)}(X) \cong A^i_{(j)}(X') \quad \textit{for all } i, j \ .$$

Proof. As noted by Vial [48, Introduction], this is a consequence of Rieß's result that X and X' have isomorphic Chow motive (as algebras in the category of Chow motives). For more details, cf. [42, Section 6] or [32, Lemma 2.8]. □

2.5 A commutativity lemma

Lemma 2.15. *Let S be an algebraic $K3$ surface, and let $\{\pi^S_i\}$ be the MCK decomposition as above. Let $h \in \mathrm{Aut}(S)$. Then*

$$\Gamma_h \circ \pi^S_i = \pi^S_i \circ \Gamma_h \quad \textit{in } A^2(S \times S) \quad \forall i \ .$$

Proof. It suffices to prove this for $i = 0$. Indeed, by definition of $\{\pi^S_i\}$ we have

$$\begin{aligned} \pi^S_4 &:= {}^t\pi^S_0 \quad \text{in } A^2(S \times S) \ , \\ \pi^S_2 &:= \Delta_S - \pi^S_0 - \pi^S_4 \ . \end{aligned}$$

Supposing the lemma holds for $i = 0$, by taking transpose correspondences we get an equality

$$\Gamma_{h^{-1}} \circ \pi^S_4 = \pi^S_4 \circ \Gamma_{h^{-1}} \quad \text{in } A^2(S \times S) \ .$$

Composing on both sides with Γ_h, we get

$$\pi^S_4 \circ \Gamma_h = \Gamma_h \circ \pi^S_4 \quad \text{in } A^2(S \times S) \ .$$

Next, since obviously the diagonal Δ_S commutes with Γ_h, we also get

$$\Gamma_h \circ \pi^S_2 = \Gamma_h \circ (\Delta_S - \pi^S_0 - \pi^S_4) = (\Delta_S - \pi^S_0 - \pi^S_4) \circ \Gamma_h = \pi^S_2 \circ \Gamma_h \quad \text{in } A^2(S \times S) \ .$$

It remains to prove the lemma for $i = 0$. The projector π^S_0 is defined as

$$\pi^S_0 = \mathfrak{o}_S \times S \quad \in A^2(S \times S) \ ,$$

where $\mathfrak{o}_S \in A^2(S)$ is the "distinguished point" of [7] (any point lying on a rational curve in S equals $\mathfrak{o}_S$ in $A^2(S)$). It is known [7] that

$$\mathrm{Im}\big(A^1(S) \otimes A^1(S) \ \to \ A^2(S)\big) = \mathbb{Q}[\mathfrak{o}_S] \ .$$

It follows that there exist divisors $D_1, D_2 \in A^1(S)$ such that $\mathfrak{o}_S = D_1 \cdot D_2$, and so

$$h^*(\mathfrak{o}_S) = h^*(D_1 \cdot D_2) = h^*(D_1) \cdot h^*(D_2) \quad \in \mathbb{Q}[\mathfrak{o}_S]\ .$$

Since $h^*(\mathfrak{o}_S)$ is the class of a point $h^{-1}(x)$ (where $x \in S$ is any point lying on a rational curve), it has degree 1 and thus

$$h^*(\mathfrak{o}_S) = \mathfrak{o}_S \quad \text{in } A^2(S)\ .$$

Using Lieberman's lemma [48, Lemma 3.3], we find that

$$\begin{aligned} \pi_0^S \circ \Gamma_h &= ({}^t\Gamma_h \times \Delta_S)_*(\pi_0^S) \\ &= ({}^t\Gamma_h \times \Delta_S)_*(\mathfrak{o}_S \times S) \\ &= h^*(\mathfrak{o}_S) \times S \\ &= \mathfrak{o}_S \times S = \pi_0^S \quad \text{in } A^2(S \times S)\ , \end{aligned}$$

whereas obviously

$$\Gamma_h \circ \pi_0^S = (\Delta_S \times \Gamma_h)_*(\mathfrak{o}_S \times S) = \mathfrak{o}_S \times S = \pi_0^S \quad \text{in } A^2(S \times S)\ .$$

This proves the $i = 0$ case of the lemma. $\square$

The following lemmas establish some corollaries of lemma 2.15:

Lemma 2.16. *Let S be an algebraic $K3$ surface, and $G_S \subset \mathrm{Aut}(S)$ a group of finite order k. For any $r \in \mathbb{N}$, let $\{\pi_i^{S^r}\}$ denote the product MCK decomposition of S^r induced by the MCK decomposition of S as above. Let*

$$\Delta^G_{S^r} := \frac{1}{k} \sum_{g \in G_S} \Gamma_g \times \cdots \times \Gamma_g \quad \in A^{2r}(S^r \times S^r)\ .$$

Then

$$\Delta^G_{S^r} \circ \pi_i^{S^r} = \pi_i^{S^r} \circ \Delta^G_{S^r} \quad \in A^{2r}(S^r \times S^r)$$

is an idempotent, for any i.

Proof. It suffices to prove the commutativity statement. (Indeed, since both $\Delta^G_{S^r}$ and $\pi_i^{S^r}$ are idempotent, the idempotence of their composition follows immediately from the stated commutativity relation.) To prove the commutativity statement, we will prove more precisely that for any $h \in \mathrm{Aut}(S)$ we have equality

$$\Gamma_{h^{\times r}} \circ \pi_i^{S^r} = \pi_i^{S^r} \circ \Gamma_{h^{\times r}} \quad \in A^{2r}(S^r \times S^r)\ . \tag{2}$$

This can be seen as follows: we have

$$\begin{aligned}\Gamma_{h^{\times r}}\circ \pi_i^{S^r} &= (\Gamma_h\times\cdots\times\Gamma_h)\circ \Big(\sum_{i_1+\cdots+i_r=i} \pi^S_{i_1}\times\cdots\times\pi^S_{i_r}\Big)\\ &= \sum_{i_1+\cdots+i_r=i} (\Gamma_h\circ\pi^S_{i_1})\times\cdots\times(\Gamma_h\circ\pi^S_{i_r})\\ &= \sum_{i_1+\cdots+i_r=i} (\pi^S_{i_1}\circ\Gamma_h)\times\cdots\times(\pi^S_{i_r}\circ\Gamma_h)\\ &= \sum_{i_1+\cdots+i_r=i} (\pi^S_{i_1}\times\cdots\times\pi^S_{i_r})\circ(\Gamma_h\times\cdots\times\Gamma_h)\\ &= \pi_i^{S^r}\circ\Gamma_{h^{\times r}}\quad \text{in } A^{2r}(S^r\times S^r)\ .\end{aligned}$$

Here, the first and last lines are the definition of the product MCK decomposition for S^r; the second and fourth line are just regrouping, and the third line is lemma 2.15. ☐

Lemma 2.17. *Let S be an algebraic $K3$ surface, and $G_S\subset \mathrm{Aut}(S)$ a group of finite order k. For any $r\in\mathbb{N}$, let $X=S^{[r]}$ and let $G\subset\mathrm{Aut}(X)$ be the group of natural automorphisms induced by G_S. Let $\{\pi_i^X\}$ be the MCK decomposition of theorem 2.11. Let Δ_X^G denote the correspondence*

$$\Delta_X^G := \frac{1}{k}\sum_{g\in G}\Gamma_g\quad \in A^{2r}(X\times X)\ .$$

Then

$$\Delta_X^G\circ\pi_i^X = \pi_i^X\circ\Delta_X^G\quad\in A^{2r}(X\times X)$$

is an idempotent, for any i.

Proof. Again, it suffices to prove the commutativity statement. This can be done as follows: for any $g\in G$, we can write $g=h^{[r]}$ where $h\in\mathrm{Aut}(S)$. Then we have

$$\begin{aligned}\Gamma_g\circ\pi_i^X &= \Gamma_g\circ \sum_{\mu\in\mathfrak{B}(k)}\frac{1}{m_\mu}\Gamma_\mu\circ\pi^{S^\mu}_{i-2k+2l(\mu)}\circ{}^t\Gamma_\mu\\ &= \sum_{\mu\in\mathfrak{B}(k)}\frac{1}{m_\mu}\Gamma_g\circ\Gamma_\mu\circ\pi^{S^\mu}_{i-2k+2l(\mu)}\circ{}^t\Gamma_\mu\\ &= \sum_{\mu\in\mathfrak{B}(k)}\frac{1}{m_\mu}\Gamma_\mu\circ\Gamma_{h^{\times l(\mu)}}\circ\pi^{S^\mu}_{i-2k+2l(\mu)}\circ{}^t\Gamma_\mu\\ &= \sum_{\mu\in\mathfrak{B}(k)}\frac{1}{m_\mu}\Gamma_\mu\circ\pi^{S^\mu}_{i-2k+2l(\mu)}\circ\Gamma_{h^{\times l(\mu)}}\circ{}^t\Gamma_\mu\\ &= \sum_{\mu\in\mathfrak{B}(k)}\frac{1}{m_\mu}\Gamma_\mu\circ\pi^{S^\mu}_{i-2k+2l(\mu)}\circ{}^t\Gamma_\mu\circ\Gamma_g\\ &= \pi_i^X\circ\Gamma_g\quad\text{in } A^{2r}(X\times X)\ .\end{aligned}$$

Here, the first line follows from the definition of π_i^X (definition (1)). The second line is just regrouping, the third line is by construction of natural automorphisms of X, the fourth line is equality (2) above, and the fifth line is again by construction of natural automorphisms. $\square$

Lemma 2.18. *Let S be an algebraic $K3$ surface, and let $X = S^{[r]}$ be the Hilbert scheme of length r subschemes. Let $G \subset \mathrm{Aut}(X)$ a group of finite order k of natural automorphisms. Then the quotient $Y := X/G$ has a self–dual MCK decomposition.*

Proof. Let $p\colon X \to Y$ denote the quotient morphism. One defines

$$\pi_j^Y := \frac{1}{k}\Gamma_p \circ \pi_j^X \circ {}^t\Gamma_p \quad \in A^{2r}(Y \times Y)\ ,$$

where $\{\pi_j^X\}$ is the self–dual MCK decomposition of theorem 2.11. This defines a self–dual CK decomposition $\{\pi_j^Y\}$, since

$$\begin{aligned} \pi_i^Y \circ \pi_j^Y &= \frac{1}{k^2}\Gamma_p \circ \pi_i^X \circ {}^t\Gamma_p \circ \Gamma_p \circ \pi_j^X \circ {}^t\Gamma_p \\ &= \frac{1}{k}\Gamma_p \circ \pi_i^X \circ \Delta_X^G \circ \pi_j^X \circ {}^t\Gamma_p \\ &= \frac{1}{k}\Gamma_p \circ \pi_i^X \circ \pi_j^X \circ \Delta_X^G \circ {}^t\Gamma_p \\ &= \begin{cases} 0 & \text{if } i \neq j\ ; \\ \frac{1}{k}\Gamma_p \circ \pi_i^X \circ {}^t\Gamma_p = \pi_i^Y & \text{if } i = j\ . \end{cases} \end{aligned}$$

(Here, in the third line we have used lemma 2.15.)

It remains to check this CK decomposition is multiplicative. To this end, let i, j, k be integers with $k \neq i + j$. We note that

$$\begin{aligned} \pi_k^Y \circ \Delta_Y^{sm} \circ (\pi_i^Y \times \pi_j^Y) &= \frac{1}{k^3}\ \ \Gamma_p \circ \pi_k^X \circ {}^t\Gamma_p \circ \Delta_{sm}^Y \circ \Gamma_{p\times p} \circ (\pi_i^X \times \pi_j^X) \circ {}^t\Gamma_{p\times p} \\ &= \Gamma_p \circ \pi_k^X \circ \Delta_X^G \circ \Delta_X^{sm} \circ (\Delta_X^G \times \Delta_X^G) \circ (\pi_i^X \times \pi_j^X) \circ {}^t\Gamma_{p\times p} \\ &= \Gamma_p \circ \Delta_X^G \circ \pi_k^X \circ \Delta_X^{sm} \circ (\pi_i^X \times \pi_j^X) \circ (\Delta_X^G \times \Delta_X^G) \circ {}^t\Gamma_{p\times p} \\ &= 0 \quad \text{in } A^{2n}(Y \times Y \times Y)\ . \end{aligned}$$

Here, the first equality is by definition of the π_i^Y, the second equality is lemma 2.19 below, the third equality follows from lemma 2.17, and the fourth equality is the fact that the π_i^X are an MCK decomposition for X.

Lemma 2.19. *There is equality*

$$\begin{aligned} {}^t\Gamma_p \circ \Delta_Y^{sm} \circ \Gamma_{p\times p} &= (\sum_{g\in G} \Gamma_g) \circ \Delta_X^{sm} \circ \big((\sum_{g\in G} \Gamma_g) \times (\sum_{g\in G} \Gamma_g)\big) \\ &= k^3\ \Delta_X^G \circ \Delta_X^{sm} \circ (\Delta_X^G \times \Delta_X^G) \quad \textit{in } A^{2n}(X \times X \times X)\ . \end{aligned}$$

Proof. The second equality is just the definition of Δ_X^G. As to the first equality, we first note that

$$\Delta_Y^{sm} = (p\times p\times p)_\ast(\Delta_X^{sm}) = \Gamma_p\circ\Delta_X^{sm}\circ{}^t\Gamma_{p\times p}\quad \text{in } A^{3n}(Y\times Y\times Y)\ .$$

This implies that

$${}^t\Gamma_p\circ\Delta_Y^{sm}\circ\Gamma_{p\times p} = {}^t\Gamma_p\circ\Gamma_p\circ\Delta_X^{sm}\circ{}^t\Gamma_{p\times p}\circ\Gamma_{p\times p}\ .$$

But ${}^t\Gamma_p\circ\Gamma_p=\sum_{g\in G}\Gamma_g$, and thus

$${}^t\Gamma_p\circ\Delta_Y^{sm}\circ\Gamma_{p\times p} = (\sum_{g\in G}\Gamma_g)\circ\Delta_X^{sm}\circ\big((\sum_{g\in G}\Gamma_g)\times(\sum_{g\in G}\Gamma_g)\big)\quad \text{in } A^{2n}(X\times X\times X)\ ,$$

as claimed. $\square$

$\square$

2.6 An injectivity result

Lemma 2.20 (Vial [48]). *Let S be an algebraic $K3$ surface, and $X=S^{[r]}$ the Hilbert scheme of length r subschemes of S. The cycle class map induces a map*

$$A^i_{(0)}(X)\ \to\ H^{2i}(X)$$

that is injective for $i\ge 2r-1$.

Proof. This is stated without proof in [48, Introduction]. The idea is as follows: let $i\ge 2r-1$. Using remark 2.12, we obtain a commutative diagram

$$\begin{array}{ccc} A^i_{(0)}(X) & \to & A^i_{(0)}(S^r)\\ \downarrow & & \downarrow\\ H^{2i}(X) & \to & H^{2i}(S^r)\ ,\end{array}$$

where horizontal arrows are split injections, and vertical arrows are restrictions of the cycle class map. It thus suffices to prove that restriction of the cycle class map

$$A^i_{(0)}(S^r)\ \to\ H^{2i}(S^r)$$

is injective.

Let $\{\pi_j^{S^r}\}$ denote the product MCK decomposition constructed above. It follows from the definition of $A^i_{(0)}(S^r)$ that

$$(\pi_{2i}^{S^r})_\ast=\mathrm{id}\colon\quad A^i_{(0)}(S^r)\ \to\ A^i(S^r)\ .$$

Let $x\in S$ be a point such that $x=\mathfrak{o}_S$ in $A^2(S)$. Then the projector $\pi_{4r}^{S^r}$ is supported on $S^r\times(x\times\cdots\times x)$, and $\pi_{4r-2}^{S^r}$ is supported on

$$S^r\times(S\times x\times\cdots\times x)\cup S^r\times(x\times S\times x\times\cdots\times x)\cup\cdots\cup S^r\times(x\times\cdots\times x\times S)\quad\subset S^r\times S^r\ .$$

It follows that for $i=2r$ there is a factorization

$$\begin{array}{ccc} A^{2r}_{(0)}(S^r) & \to & H^{4r}(S^r) \\ \downarrow & & \downarrow \\ A^0(x\times\cdots\times x) & \to & H^0(x\times\cdots\times x) \\ \downarrow & & \downarrow \\ A^{2r}_{(0)}(S^r) & \to & H^{4r}(S^r)\ , \end{array}$$

where composition of vertical arrows is $(\pi^{S^r}_{4r})_\ast=\mathrm{id}$. This implies $A^{2r}_{(0)}(S^r)\cong\mathbb{Q}$ and the map to $H^{4r}(S^r)$ is an isomorphism.

Likewise, for $i=2r-1$ there is a factorization

$$\begin{array}{ccc} A^{2r-1}_{(0)}(S^r) & \to & H^{4r-2}(S^r) \\ \downarrow & & \downarrow \\ \bigoplus A^1(S) & \to & \bigoplus H^2(S) \\ \downarrow & & \downarrow \\ A^{2r-1}_{(0)}(S^r) & \to & H^{4r-2}(S^r)\ , \end{array}$$

where composition of vertical maps is $(\pi^{S^r}_{4r-2})_\ast=\mathrm{id}$. Since the middle horizontal arrow is injective, this implies the other horizontal arrows are injective as well. $\square$

Remark 2.21. As explained in [42], conjecturally the restriction of the cycle class map

$$A^i_{(0)}(X)\ \to\ H^{2i}(X)$$

is injective for any variety X having an MCK decomposition. This is related to Murre's "conjecture D" [34], and the expectation that the bigrading $A^\ast_{(\ast)}$ should give a splitting of a Bloch–Beilinson filtration.

As we will see below (lemma 4.5), for Hilbert schemes of special $K3$ surfaces one can prove more than lemma 2.20.

2.7 LSY surfaces

Definition 2.22. An *LSY surface* (short for "Livné–Schütt–Yui surface") is a projective $K3$ surface S, with the following properties:
(i) There is a group $G_S\subset\mathrm{Aut}(S)$ acting trivially on $NS(S)$;
(ii) Let $k:=\mathrm{ord}(G_S)$. There is equality

$$\dim(T_S)=\phi(k)\ ,$$

where $T_S\subset H^2(S)$ denotes the transcendental lattice, and $\phi(k)$ is Euler's totient function.

Remark 2.23. Assumption (i) of definition 2.22 implies that G_S is a finite cyclic group [33], so the definition of the integer k makes sense. Under assumption (i), $\phi(k)$ divides $\dim(T_S)$, so assumption (ii) is equivalent to asking that the Picard number of S is maximal among all $K3$ surfaces satisfying (i) for a given value of $k=ord(G_S)$.

Theorem 2.24 (Livné–Schütt–Yui [33]). *Let S be an LSY surface, and $k :=$ ord(G_S). Then*

$$k \in \Big\{3,5,7,9,11,12,13,17,19,25,27,28,36,42,44,66\Big\}\ .$$

Conversely, for each of these values of k, there exists a unique LSY surface S_k with $k := \mathrm{ord}(G_S)$ up to isomorphism. All these surfaces S_k have finite–dimensional motive.

Proof. This is [33, Theorems 1 and 2], combined with the explicit descriptions given in [33, Sections 3 and 4]. □

Remark 2.25. The study of LSY surfaces was initiated by Vorontsov [56] and Kondo [30]. Livné–Schütt–Yui give explicit equations for all the surfaces S_k [33, Sections 3 and 4]. To give one example, the surface S_{66} can be described as a hypersurface of degree 12

$$x_0^2 + x_1^3 + x_2^{11}x_3 + x_3^{12} = 0$$

in a weighted projective space $\mathbb{P}(6,4,1,1)$. (As explained in [33, Remark 2], the surface S_{66} can also be described as an elliptic surface.).

With the exception of S_3 (which is of maximal Picard rank $\rho(S_3) = 20$), all the S_k are Delsarte surfaces; as such, they are dominated by Fermat surfaces. This immediately implies finite–dimensionality of the S_k.

2.8 Schütt surfaces

Definition 2.26. A *Schütt surface* is a projective $K3$ surface S, with the following properties:
(i) There is a group $G_S \subset \mathrm{Aut}(S)$ acting trivially on $NS(S)$;
(ii) The order $k := \mathrm{ord}(G_S)$ is a 2–power;
(iii) There is equality

$$\dim(T_S) = k\ ,$$

where $T_S \subset H^2(S)$ denotes the transcendental lattice.

Complementing results of [56], [30], [33], Schütt has classified Schütt surfaces:

Theorem 2.27 (Schütt [41]). *Let S be a Schütt surface, and $k = \mathrm{ord}\, G_S$. Then*

$$k \in \big\{2,4,8,16\big\}\ .$$

Conversely:

$k=2$ *there exists a unique Schütt surface S_2 with $k=2$ (up to isomorphism);*

$k=4$ *any Schütt surface with $k=4$ is an element of the one–dimensional family $S^{unimod}_{4,\lambda}$ ($\lambda \in \mathbb{C}$) or the one–dimensional family $S^{non}_{4,\lambda}$ ($\lambda \in \mathbb{C}$);*

$k = 8$ *any Schütt surface with* $k = 8$ *is an element of a one–dimensional family* $S_{8,\lambda}$ *(*$\lambda \in \mathbb{C}$*);*

$k = 16$ *any Schütt surface with* $k = 16$ *is an element of a one–dimensional family* $S_{16,\lambda}$ *(*$\lambda \in \mathbb{C}$*).*

Proof. This is [41, Theorem 1]. □

Remark 2.28. The surfaces in theorem 2.27 are given by explicit equations. For example, the family $S^{unimod}_{4,\lambda}$ is defined by the Weierstrass equation

$$y^2 = x^3 - 3\lambda t^4 x + t^5 + t^7$$

[41, Theorem 1]. For a generic λ, this surface will have $\mathrm{rank}(T_S) = 4$, and so the surface is a Schütt surface.

Contrary to the LSY surfaces, *not* all Schütt surfaces have provably finite–dimensional motive. Some of them do, however:

Proposition 2.29 (Schütt [41])**.** *Let* S *be either a* $S^{unimod}_{4,\lambda}$ *with* λ *generic, or*

$$S \in \Big\{ S_2, S^{non}_{4,0}, S_{8,0}, S_{8,2}, S_{8,\sqrt{3}}, S_{16,0}, S_{16,2}, S_{16,\sqrt{3}} \Big\} \ .$$

Then S *is a Schütt surface with finite–dimensional motive.*

Proof. A generic element of the pencil $S^{unimod}_{4,\lambda}$ is a Schütt surface [41]. It also has a Shioda–Inose structure [41], which implies finite–dimensionality. The surface S_2 has Picard number 20, hence is Kummer. The other surfaces in proposition 2.29 are dominated by Fermat surfaces [41, Lemma 18], hence have finite–dimensional motive. □

2.9 Transcendental part of the motive

Theorem 2.30 (Kahn–Murre–Pedrini [28])**.** *Let* S *be a surface. There exists a decomposition*

$$h_2(S) = t_2(S) \oplus h_2^{alg}(S) \ \in \mathcal{M}_{\mathrm{rat}} \ ,$$

such that

$$H^*(t_2(S), \mathbb{Q}) = H^2_{tr}(S) \ , \quad H^*(h_2^{alg}(S), \mathbb{Q}) = NS(S)_{\mathbb{Q}}$$

(here $H^2_{tr}(S)$ *is defined as the orthogonal complement of the Néron–severi group* $NS(S)_{\mathbb{Q}}$ *in* $H^2(S, \mathbb{Q})$*), and*

$$A^*(t_2(S))_{\mathbb{Q}} = A^2_{AJ}(S)_{\mathbb{Q}} \ .$$

(The motive $t_2(S)$ *is called the* transcendental part of the motive*.)*

Let $h_2^{alg}(S) = (S, \pi_2^{alg}, 0) \in \mathcal{M}_{\mathrm{rat}}$*. The projector* π_2^{alg} *is supported on* $D \times D$*, for* $D \subset S$ *a divisor.*

2.10 Natural automorphisms of Hilbert schemes

Definition 2.31 (Boissière [11])**.** Let S be a surface, and let $X = S^{[k]}$ denote the Hilbert scheme of length k subschemes. An automorphism $\psi \in \mathrm{Aut}(S)$ induces an automorphism $\psi^{[k]}$ of X. This determines a homomorphism

$$\begin{aligned} \mathrm{Aut}(S) &\to \mathrm{Aut}(X) \ , \\ \psi &\mapsto \psi^{[k]} \ , \end{aligned}$$

which is injective [11]. The image of this homomorphism is called the group of *natural automorphisms* of X.

Theorem 2.32 (Boissière–Sarti [13])**.** *Let S be a $K3$ surface, and $X = S^{[k]}$. Let $E \subset X$ denote the exceptional divisor of the Hilbert–Chow morphism. An automorphism $g \in \mathrm{Aut}(X)$ is natural if and only if $g^*(E) = E$ in $NS(X)$.*

Proof. This is [13, Theorem 1]. □

Remark 2.33. To find examples of non–natural automorphisms of a Hilbert scheme X, Boissière and Sarti introduce the notion of *index* of an automorphism of X. For Hilbert schemes of a generic algebraic $K3$ surface, the index of an automorphism is 1 if and only if the automorphism is natural [13, section 4].

2.11 A support lemma

For later use, we establish a lemma:

Lemma 2.34. *Let S be an LSY surface or Schütt surface, and let G_S be the order k group as in definition 2.22, resp. definition 2.26. For any $r \in \mathbb{N}$ let*

$$\Delta^G_{S^r} := \frac{1}{k} \sum_{g \in G_S} \Gamma_g \times \cdots \times \Gamma_g \in A^{2r}(S^r \times S^r) \ .$$

Let $\{\pi^{S^r}_j\}$ denote the product MCK decomposition for S^r as above. There is a homological equivalence

$$\Delta^G_{S^r} \circ \pi^{S^r}_2 = \gamma \quad \text{in } H^{4r}(S^r \times S^r) \ ,$$

where γ is a cycle supported on $C \times D \subset S^r \times S^r$, and $C \subset S^r$ is a curve and $D \subset S^r$ is a divisor.

Proof. Let us first do the $r = 1$ case. Since the group $G_S \subset \mathrm{Aut}(S)$ consists of non–symplectic automorphisms, we have

$$(\Delta^G_S)_* = 0\colon \quad H^{2,0}(S) \to H^{2,0}(S) \ .$$

Let $T \subset H^2(S)$ denote the transcendental lattice. Since T defines an indecomposable Hodge structure (i.e., every Hodge sub–structure of T is either T or 0), we must have

$$(\Delta_S^G)_* = 0\colon \quad T \ \to\ T\ .$$

Since Δ_S^G acts as the identity on $NS(S)$, this implies

$$\Delta_S^G \circ \pi_2^S = \pi_2^{S,alg} \quad \text{in } H^4(S\times S)\ .$$

But $\pi_2^{S,alg}$ is supported on divisor times divisor (theorem 2.30); this proves the case $k=1$.

For arbitrary r, note that (by definition of the product MCK decomposition)

$$\pi_2^{S^r} = \pi_2^S \times \pi_0^S \times \cdots \times \pi_0^S + \cdots + \pi_0^S \times \cdots \times \pi_0^S \times \pi_2^S \quad \in A^{2r}(S^r \times S^r)\ .$$

Thus,

$$\begin{aligned}
&\Delta^G_{S^r} \circ \pi_2^{S^r} = \\
&\quad = \frac{1}{k}\sum_{h\in G_S} (\Gamma_h \times \cdots \times \Gamma_h)\circ (\pi_2^S \times \pi_0^S \times \cdots \times \pi_0^S + \cdots + \pi_0^S \times \cdots \times \pi_0^S \times \pi_2^S)\\
&\quad = \frac{1}{k}\sum_{h\in G_S} (\Gamma_h \circ \pi_2^S)\times (\Gamma_h \circ \pi_0^S)\times \cdots \times (\Gamma_h \circ \pi_0^S)\\
&\qquad\qquad + \cdots + (\Gamma_h \circ \pi_0^S)\times \cdots \times (\Gamma_h \circ \pi_0^S)\times (\Gamma_h \circ \pi_2^S)\\
&\quad = \frac{1}{k}\sum_{h\in G_S} (\Gamma_h \circ \pi_2^S)\times \pi_0^S \times \cdots \times \pi_0^S + \cdots + \pi_0^S \times \cdots \times \pi_0^S \times (\Gamma_h \circ \pi_2^S)\\
&\quad = (\Delta_S^G \circ \pi_2^S)\times \pi_0^S \times \cdots \times \pi_0^S + \cdots + \pi_0^S \times \cdots \times \pi_0^S \times (\Delta_S^G \circ \pi_2^S)\\
&\quad = \pi_2^{S,alg} \times \pi_0^S \times \cdots \times \pi_0^S + \cdots + \pi_0^S \times \cdots \times \pi_0^S \times \pi_2^{S,alg} \quad \text{in } H^{4r}(S^r \times S^r)\ .
\end{aligned}$$

Here, the second line is because $\Gamma_h \circ \pi_0^S = \pi_0^S$ (proof of lemma 2.16), and the last line is the $r=1$ case treated above. The last line is clearly a cycle supported on curve times divisor, and so the lemma is proven. □

3 Main result

Theorem 3.1. *Let S_3 be as in theorem 2.24, and let X be the Hilbert scheme $X = (S_3)^{[3]}$. Let $G \subset \mathrm{Aut}(X)$ be the group of natural automorphisms induced by the order 3 cyclic group $G_{S_3} \subset \mathrm{Aut}(S_3)$ of definition 2.22. Then*

$$(\Delta_X^G)_* = 0\colon\ A^i_{(j)}(X) \to A^i(X)\ \ \textit{for}\,(i,j) \in \Big\{(2,2),(4,4),(3,2),(6,2),(6,4),(5,2)\Big\}.$$

Proof. In the course of this proof, let us write S instead of S_3. The idea is to reduce to the action of automorphisms on $A^i(S^3)$ and $A^i(S^2)$ and $A^i(S)$. This reduction is possible thanks to the commutative diagram

$$\begin{array}{ccccccc} A^i_{(j)}(X) & \hookrightarrow & A^i_{(j)}(S^3) & \oplus & A^{i-1}_{(j)}(S^2) & \oplus & A^{i-2}_{(j)}(S) \\ \downarrow {\scriptstyle (\Delta^G_X)_*} & & \downarrow {\scriptstyle (\Delta^G_{S^3})_*} & & \downarrow {\scriptstyle (\Delta^G_{S^2})_*} & & \downarrow {\scriptstyle (\Delta^G_S)_*} \\ A^i_{(j)}(X) & \hookrightarrow & A^i_{(j)}(S^3) & \oplus & A^{i-1}_{(j)}(S^2) & \oplus & A^{i-2}_{(j)}(S) \end{array} \qquad (3)$$

Here, $\Delta^G_{S^r}$ is as in lemma 2.34. This diagram commutes because of the construction of natural automorphisms. Horizontal arrows are injective because of remark 2.12.

To handle the action of $\Delta^G_{S^r}$ on $A^i_{(j)}(S^r)$ for $r = 1, 2, 3$, we establish two lemmas:

Lemma 3.2. *There are homological equivalences*

$$\begin{aligned} \Delta^G_{S^3} \circ \pi_2^{S^3} &= \gamma_2^{S^3} \quad \text{in } H^{12}(S^3 \times S^3) \ , \\ \Delta^G_{S^2} \circ \pi_2^{S^2} &= \gamma_2^{S^2} \quad \text{in } H^{8}(S^2 \times S^2) \ , \\ \Delta^G_{S} \circ \pi_2^{S} &= \gamma_2^{S} \quad \text{in } H^{4}(S \times S) \ , \end{aligned}$$

where $\gamma_2^{S^3}$ (resp. $\gamma_2^{S^2}$ resp. γ_2^{S}) is a cycle in

$$\begin{aligned} &\mathrm{Im}\Big(A_6(V_{2,3} \times W_{2,3}) \ \to \ A^6(S^3 \times S^3)\Big) \\ \textit{(resp. } &\mathrm{Im}\Big(A_4(V_{2,2} \times W_{2,2}) \ \to \ A^4(S^2 \times S^2)\Big) \ , \\ \textit{(resp. } &\mathrm{Im}\Big(A_2(V_{2,1} \times W_{2,1}) \ \to \ A^2(S \times S)\Big) \ \ ), \end{aligned}$$

and $V_{2,r} \subset S^r$ is a closed subvariety of codimension $2r - 1$, and $W_{2,r} \subset S^r$ is closed of codimension 1.

Proof. This is a special case of lemma 2.34. □

Lemma 3.3. *There are homological equivalences*

$$\begin{aligned} \Delta^G_{S^3} \circ \pi_4^{S^3} &= \gamma_4^{S^3} \quad \text{in } H^{12}(S^3 \times S^3) \ , \\ \Delta^G_{S^2} \circ \pi_4^{S^2} &= \gamma_4^{S^2} \quad \text{in } H^{8}(S^2 \times S^2) \ , \end{aligned}$$

where $\gamma_4^{S^3}$ (resp. $\gamma_4^{S^2}$) is a cycle in

$$\begin{aligned} &\mathrm{Im}\Big(A_6(V_{4,3} \times W_{4,3}) \ \to \ A^6(S^3 \times S^3)\Big) \\ \textit{(resp. } &\mathrm{Im}\Big(A_4(V_{4,2} \times W_{4,2}) \ \to \ A^4(S^2 \times S^2)\Big) \ , \end{aligned}$$

and $V_{4,3}, W_{4,3} \subset S^3$ are closed subvarieties of codimension 4 resp. 2, and $V_{4,2}, W_{4,2} \subset S^2$ are closed subvarieties of codimension 2.

Proof. Here we will use the fact that $S = S_3$ is ρ–maximal (i.e. the Picard number $\rho(S_3)$ is 20). This means that the transcendental lattice $T \subset H^2(S)$ has rank 2 and injects (under the natural map $H^2(S) \to H^2(S,\mathbb{C})$) into $H^{2,0} \oplus H^{0,2}$. It follows that (under the natural map $H^2(S) \to H^2(S,\mathbb{C})$)

$$T \otimes T \ \subset\ H^{4,0}(S^2) \oplus H^{2,2}(S^2) \oplus H^{0,4}(S^2)\ .$$

Let $h \in G_S$ be a generator. Since h is non–symplectic, h^* acts on $H^{2,0}$ as multiplication by a primitive 3rd root of unity ν. It follows that

$$(h \times h)^* = \nu^2 \cdot \mathrm{id}\colon \quad H^{4,0}(S^2)\ \to\ H^{4,0}(S^2)\ ,$$

and hence (since $\nu^2 \neq 1$)

$$(\Delta^G_{S^2})_* = 0\colon \quad H^{4,0}(S^2)\ \to\ H^{4,0}(S^2)\ .$$

For the same reason, we also have

$$(\Delta^G_{S^2})_* = 0\colon \quad H^{0,4}(S^2)\ \to\ H^{0,4}(S^2)\ .$$

It follows that

$$(\Delta^G_{S^2})_*(T \otimes T) = (\Delta^G_{S^2})_*\big((T \otimes T) \cap F^2\big) \quad \subset\ H^4(S^2)$$

(here F^* denotes the Hodge filtration on $H^*(-,\mathbb{C})$). But $H^4(S^2) \cap F^2$ is generated by codimension 2 cycles (indeed, S is a Kummer surface, and so the Hodge conjecture is true for S^r since it is true for self–products of abelian surfaces [1, 7.2.2]). This means that there exist a codimension 2 subvariety $V \subset S^2$ and a cycle γ supported on $V \times V$ such that

$$\Delta^G_{S^2} \circ (\pi_2^{S,tr} \times \pi_2^{S,tr}) - \gamma = 0 \quad \text{in } H^8(S^2 \times S^2)\ .$$

Next, let us write

$$H^2(S) = T \oplus N\ ,$$

where $N := NS(S)$. The action of $\Delta^G_{S^2}$ on $T \otimes N$ and on $N \otimes T$ is 0. Indeed,

$$(h \times h)^* = \nu \cdot \mathrm{id} \times \mathrm{id}\colon \quad T \otimes N\ \to\ T \otimes N\ ,$$

and so

$$(\Delta^G_{S^2})_* = (\Delta^G_S \times \Delta_S)_* = 0\colon \quad T \otimes N\ \to\ T \otimes N\ .$$

This means that

$$\Delta^G_{S^2} \circ (\pi_2^{S,tr} \times \pi_2^{S,alg}) = \Delta^G_{S^2} \circ ((\pi_2^{S,alg} \times \pi_2^{S,tr}) = 0 \quad \text{in } H^8(S^2 \times S^2)\ .$$

The correspondences $\pi^S_0 \times \pi^S_4$ and $\pi^S_4 \times \pi^S_0$ are obviously supported on $V \times V \subset S^2 \times S^2$ for some codimension 2 subvariety $V \subset S^2$. It follows that

$$\Delta^G_{S^2} \circ \pi_4^{S^2} = \Delta^G_{S^2} \circ (\pi_2^{S,tr} \times \pi_2^{S,tr} + \pi_2^{S,alg} \times \pi_2^{S,alg} + \pi^S_0 \times \pi^S_4 + \pi^S_4 \times \pi^S_0) = \gamma' \text{ in } H^8(S^2 \times S^2),$$

where γ' is supported on $V \times V \subset S^2 \times S^2$, for $V \subset S^2$ of codimension 2. This proves the statement for S^2.

The statement for S^3 follows immediately. Indeed, we have

$$\pi_4^{S^3} = \pi_0^S \times \pi_4^{S^2} + \pi_4^{S^2} \times \pi_0^S + \pi_4^{S^2} \times \pi_0^S \quad \text{in } A^6(S^3 \times S^3) ,$$

where π_0^S in the first (resp. second, resp. third) factor lies in the first (resp. second, resp. third) copy of S. But $\Gamma_h \circ \pi_0^S = \pi_0^S$ (proof of lemma 2.16), and so

$$\Delta^G_{S^3} \circ (\pi_0^S \times \pi_4^{S^2}) = \pi_0^S \times (\Delta^G_{S^2} \circ \pi_4^{S^2}) \quad \text{in } A^6(S^3 \times S^3) ,$$

which (by the above) is homologically supported on $V_{4,3} \times W_{4,3} \subset S^3 \times S^3$, where codim. $V_{4,3} = 4$, codim. $W_{4,3} = 2$. □

We are now in position to wrap up the proof of theorem 3.1. Let us first consider 0–cycles, i.e. $i = 6$. The commutative diagram (3) simplifies to

$$\begin{array}{ccc} A^6_{(j)}(X) & \hookrightarrow & A^6_{(j)}(S^3) \\ \downarrow {\scriptstyle (\Delta^G_X)_*} & & \downarrow {\scriptstyle (\Delta^G_{S^3})_*} \\ A^6_{(j)}(X) & \hookrightarrow & A^6_{(j)}(S^3) \end{array} \tag{4}$$

In case $0 < j < 6$ (i.e. $j = 2$ or 4), we need to prove that

$$(\Delta^G_X)_* A^6_{(j)}(X) = 0 ,$$

which (in view of the above diagram) reduces to proving that

$$(\Delta^G_{S^3})_* A^6_{(j)}(S^3) = (\Delta^G_{S^3} \circ \pi^{S^3}_{12-j})_* A^6(S^3) = 0 \quad \text{for } j = 2, 4 . \tag{5}$$

In view of lemma 2.15, we have

$$\Delta^G_{S^3} \circ \pi^{S^3}_{12-j} = \pi^{S^3}_{12-j} \circ \Delta^G_{S^3} = {}^t(\Delta^G_{S^3} \circ \pi^{S^3}_j) \quad \text{for } j = 2, 4 .$$

In view of lemmas 3.2 and 3.3, it follows that

$$\Delta^G_{S^3} \circ \pi^{S^3}_{12-j} - \gamma \quad \in A^6_{hom}(S^3 \times S^3) \quad \text{for } j = 2, 4 , \tag{6}$$

where γ is some cycle with support on $D \times S^3$ with $D \subset S^3$ a divisor. (Indeed, for $j = 2$ one may take $\gamma = {}^t(\gamma_2^{S^3})$, and for $j = 4$ one may take $\gamma = {}^t(\gamma_4^{S^3})$, which is supported on (codim. 2) $\times$ (codim. 4).) Applying the nilpotence theorem (theorem 2.4), it follows that there exists $N \in \mathbb{N}$ such that

$$\Big(\Delta^G_{S^3} \circ \pi^{S^3}_{12-j} - \gamma\Big)^{\circ N} = 0 \quad \text{in } A^6(S^3 \times S^3) .$$

Upon developing, this implies that

$$\big(\Delta^G_{S^3}\circ \pi^{S^3}_{12-j}\big)^{\circ N} = Q_1+Q_2+\cdots +Q_N \quad \hbox{in}\ A^6(S^3\times S^3)\ ,$$

where the Q_i are compositions of correspondences in which γ occurs at least once. The left–hand side is just $\Delta^G_{S^3}\circ \pi^{S^3}_{12-j}$ (since $\Delta^G_{S^3}\circ \pi^{S^3}_{12-j}$ is idempotent, corollary 2.16). The right–hand side is supported on $D\times S^3$ (since γ is), and so does not act on 0–cycles. This proves equality (5).

We now consider the line $i=j$, i.e. the "deepest part" $A^i_{(i)}$ of the Chow groups. Diagram (3) simplifies to

$$\begin{array}{ccc} A^i_{(i)}(X) & \hookrightarrow & A^i_{(i)}(S^3)\\ \downarrow {\scriptstyle (\Delta^G_X)_\ast} & & \downarrow {\scriptstyle (\Delta^G_{S^3})_\ast}\\ A^i_{(i)}(X) & \hookrightarrow & A^i_{(i)}(S^3)\end{array} \qquad (7)$$

In view of lemmas 3.2 and 3.3, it follows that

$$\Delta^G_{S^3}\circ \pi^{S^3}_i -\gamma \quad \in A^6_{hom}(S^3\times S^3)\ ,$$

where γ is some cycle that acts trivially on $A^i(S^3)$. (Indeed, for $i=2$ one may take $\gamma=\gamma_2^{S^3}$, and for $i=4$ one may take $\gamma=\gamma_4^{S^3}$.) Applying the nilpotence theorem, it follows there exists $N\in\mathbb{N}$ such that

$$\Big(\Delta^G_{S^3}\circ \pi^{S^3}_i -\gamma\Big)^{\circ N}=0 \quad \hbox{in}\ A^6(S^3\times S^3)\ .$$

Upon developing, this implies that

$$\big(\Delta^G_{S^3}\circ \pi^{S^3}_{i}\big)^{\circ N} = Q_1+Q_2+\cdots +Q_N \quad \hbox{in}\ A^6(S^3\times S^3)\ , \qquad (8)$$

where the Q_i are correspondences composed with γ. It follows that the right–hand side does not act on $A^6(S^3)$. The left–hand side is $\Delta^G_{S^3}\circ\pi^{S^3}_i$ (corollary 2.16), and so

$$(\Delta^G_{S^3})_\ast=0\colon\quad A^i_{(i)}(S^3)\ \to\ A^i(S^3)\quad \hbox{for}\ i=2,4\ .$$

In view of the commutative diagram (7)), it follows that also

$$(\Delta^G_{X})_\ast=0\colon\quad A^i_{(i)}(X)\ \to\ A^i(X)\quad \hbox{for}\ i=2,4\ .$$

We now consider $i=5$, i.e. 1–cycles A^5. Diagram (3) simplifies to

$$\begin{array}{ccccc} A^5_{(j)}(X) & \hookrightarrow & A^5_{(j)}(S^3) & \oplus & A^4_{(j)}(S^2)\\ \downarrow {\scriptstyle (\Delta^G_X)_\ast} & & \downarrow {\scriptstyle (\Delta^G_{S^3})_\ast} & & \downarrow {\scriptstyle (\Delta^G_{S^2})_\ast}\\ A^5_{(j)}(X) & \hookrightarrow & A^5_{(j)}(S^3) & \oplus & A^4_{(j)}(S^2)\end{array} \qquad (9)$$

For the $j=2$ case, we recall (equation (6)) that

$$\Delta^G_{S^3}\circ \pi_8^{S^3} - \gamma \quad \in A^6_{hom}(S^3\times S^3)\ ,$$

where γ is a cycle supported on (codim. 2) $\times$ (codim. 4). It follows that γ does not act on A^5 (for dimension reasons). As before, applying the nilpotence theorem plus corollary 2.16, we find that

$$(\Delta^G_{S^3}\circ \pi_8^{S^3})_\ast = 0\colon\quad A^5(S^3)\ \to\ A^5(S^3)\ .$$

This is equivalent to

$$(\Delta^G_{S^3})_\ast = 0\colon\quad A^5_{(2)}(S^3)\ \to\ A^5(S^3)\ . \tag{10}$$

Taking the transpose correspondences of lemma 3.2 (and using lemma 2.15), we also find

$$\Delta^G_{S^2}\circ \pi_6^{S^2} - \gamma \quad \in A^4_{hom}(S^2\times S^2)\ ,$$

where γ is a cycle supported on divisor times curve (indeed, one may take $\gamma = {}^t\gamma_2^{S^2}$). Once more applying nilpotence (plus idempotence), we find that

$$(\Delta^G_{S^2}\circ \pi_6^{S^2})_\ast = 0\colon\quad A^4(S^2)\ \to\ A^4(S^2)\ ,$$

which is equivalent to

$$(\Delta^G_{S^2})_\ast = 0\colon\quad A^4_{(2)}(S^2)\ \to\ A^4(S^2)\ . \tag{11}$$

Combining equalities (10) and (11) implies that

$$(\Delta^G_{X})_\ast = 0\colon\quad A^5_{(2)}(X)\ \to\ A^5(X)\ ,$$

in view of commutative diagram (9).

Finally, the statement for $A^3_{(2)}$ follows from the commutative diagram

$$\begin{array}{ccccc} A^3_{(2)}(X) & \hookrightarrow & A^3_{(2)}(S^3) & \oplus & A^2_{(2)}(S^2) \\ \downarrow {\scriptstyle (\Delta^G_X)_\ast} & & \downarrow {\scriptstyle (\Delta^G_{S^3})_\ast} & & \downarrow {\scriptstyle (\Delta^G_{S^2})_\ast} \\ A^3_{(2)}(X) & \hookrightarrow & A^3_{(2)}(S^3) & \oplus & A^2_{(2)}(S^2) \end{array} \tag{12}$$

combined with the corresponding statement for S^3 and for S^2. The statement for S^3 is proven by recalling that (from the $i=4$ case of equality (8) above)

$$\big(\Delta^G_{S^3}\circ \pi_4^{S^3}\big) = Q_1+Q_2+\cdots+Q_N \quad \text{in } A^6(S^3\times S^3)\ ,$$

where the Q_j are (composed with $\gamma_4^{S^3}$ and hence) supported on (codim. 4) $\times$ (codim. 2). For dimension reasons, the Q_j act trivially on $A^3(S^3)$, and so

$$\big(\Delta^G_{S^3}\circ \pi_4^{S^3}\big)_\ast =0\colon\quad A^3(S^3)\ \to\ A^3(S^3)\ .$$

This is equivalent to

$$(\Delta^G_{S^3})_\ast =0\colon\quad A^3_{(2)}(S^3)\ \to\ A^3(S^3)\ . \tag{13}$$

The statement for S^2 is proven by noting that

$$\Delta^G_{S^2}\circ \pi_2^{S^2} -\gamma\quad \in A^4_{hom}(S^2\times S^2)\ ,$$

where $\gamma=\gamma_2^{S^2}$ is supported on divisor times divisor (lemma 3.2). Using nilpotence and idempotence, this implies

$$\Delta^G_{S^2}\circ \pi_2^{S^2} = Q_1+\cdots+Q_N\quad \hbox{in}\ A^4(S^2\times S^2)\ ,$$

where the Q_j (are supported on divisor times divisor and hence) act trivially on $A^2_{(2)}(S^2)\subset A^2_{hom}(S^2)=A^2_{AJ}(S^2)$. It follows that

$$\big(\Delta^G_{S^2}\circ \pi_2^{S^2}\big)_\ast =0\colon\quad A^2_{(2)}(S^2)\ \to\ A^2(S^2)\ ,$$

which is equivalent to

$$(\Delta^G_{S^2})_\ast =0\colon\quad A^2_{(2)}(S^2)\ \to\ A^3(S^2)\ . \tag{14}$$

Taken together, equations (13) and (14) imply that

$$(\Delta^G_X)_\ast =0\colon\quad A^3_{(2)}(X)\ \to\ A^3(X)\ ,$$

in view of diagram (12). □

Remark 3.4. Let X and G be as in theorem 3.1. Presumably, it is also possible to prove

$$A^6_{(6)}(X)\cap A^6(X)^G = A^6_{(6)}(X)\ ,$$

in accordance with conjecture 1.1. Indeed, one can prove that

$$\Gamma:= (\Delta^G_{S^3}-\Delta_{S^3})\circ \pi_6^{S^3}\quad \in A^6(S^3\times S^3)$$

maps to 0 under the restriction

$$H^{12}(S^3\times S^3)\ \to\ H^{12}\big((S^3\times S^3)\setminus (V\times V)\big)\ ,$$

where $V\subset S^3$ is some subvariety of codimension 2. The problem is to find a cycle γ supported on $V\times V$ and such that

$$\Gamma=\gamma\quad \hbox{in}\ H^{12}(S^3\times S^3)\ ;$$

that is, one needs to solve a special case of the "Voisin standard conjecture" [52, Conjecture 1.6]. Perhaps, this can be done using the fact that $\rho(S)=20$? (I have tried a bit, then given up as things got messy...)

4 Some corollaries

Theorem 3.1 can be extended to hyperkähler varieties birational to X:

Corollary 4.1. *Let X and G be as in theorem 3.1. Let X' be a hyperkähler variety birational to X, and let G' denote the group of rational self–maps of X' induced by G. Then*

$$A^i_{(j)}(X') \cap A^i(X')^{G'} = 0 \quad \textit{if} \quad (i,j) \in \Big\{(2,2),(4,4),(3,2),(5,2),(6,2),(6,4)\Big\}.$$

Proof. This follows from theorem 3.1 combined with proposition 2.14. □

Corollary 4.2. *Let X and $G \subset \mathrm{Aut}(X)$ be as in theorem 3.1. Let Y be the quotient variety $Y := X/G$.*
(i) Y has a self–dual MCK decomposition.
(ii)

$$\begin{aligned} A^i(Y) &= \bigoplus_{j\le 0} A^i_{(j)}(Y) \quad \textit{for } i \le 3\ , \\ A^5(Y) &= A^5_{(0)}(Y) \oplus A^5_{(4)}(Y)\ , \\ A^6(Y) &= A^6_{(0)}(Y) \oplus A^6_{(6)}(Y)\ . \end{aligned}$$

Proof. Point (i) follows from lemma 2.18. Point (ii) is just a translation of theorem 3.1, combined with the fact that it is known that

$$A^i_{(j)}(S^{[r]}) = 0 \quad \text{for } i \ge 2r-1 \text{ and } j<0\ .$$

□

Corollary 4.2 has consequences for the multiplicative structure of the Chow ring of the quotient variety Y:

Corollary 4.3. *Let X and $G \subset \mathrm{Aut}(X)$ be as in theorem 3.1. Let Y be the quotient variety $Y := X/G$. For any $r \in \mathbb{N}$, let*

$$E^\ast(Y^r) \subset A^\ast(Y^r)$$

denote the subalgebra generated by (pullbacks of) $A^1(Y), A^2(Y), A^3(Y)$ and the diagonal $\Delta_Y \in A^6(Y\times Y)$ and the small diagonal $\Delta_Y^{sm} \in A^{12}(Y^3)$. Then the cycle class map

$$E^i(Y^r) \ \to\ H^{2i}(Y^r)$$

is injective for $i \ge 6r-1$.

Proof. As we have seen (corollary 4.2(i)), Y has a self–dual MCK decomposition. Since the property of having a self–dual MCK decomposition is stable under products, Y^r has a self–dual MCK decomposition, and so there is a bigraded ring structure $A^*_{(*)}(Y^r)$. We know (lemma 2.13) that the diagonals Δ_Y and Δ_Y^{sm} are "of pure grade 0", i.e.

$$\begin{aligned} \Delta_Y \in &\ \ A^6_{(0)}(Y\times Y)\ ,\\ \Delta_Y^{sm} \in &\ \ A^{12}_{(0)}(Y\times Y\times Y)\ . \end{aligned}$$

We have also seen (corollary 4.2(ii)) that

$$A^i(Y)=\bigoplus_{j\le 0} A^i_{(j)}(Y)\quad \text{for } i\le 3\ .$$

Consider now the projections $p_k\colon Y^r\to Y$ (on the k–th factor), and $p_{kl}\colon Y^r\to Y^2$ (on the k–th and l–th factor), and $p_{klm}\colon Y^r\to Y^3$ (on the k–th and l–th and m–th factor). The projections p_k, p_{kl}, p_{klm} respect the bigrading of the Chow ring. (This follows from [43, Corollary 1.6], or can be readily checked directly.)

It follows there is an inclusion

$$E^*(Y^r)\ \subset\ \bigoplus_{j\le 0} A^*_{(j)}(Y^r)\ ,$$

and so in particular

$$E^i(Y^r)\ \subset\ A^i_{(0)}(Y^r)\quad \text{for } i\ge 6r-1\ .$$

As we have seen (lemma 2.20), the conjectural equality

$$A^i_{(0)}(Y^r)\cap A^i_{hom}(Y^r)\stackrel{??}{=}0 \tag{15}$$

can be proven for $i\ge 6r-1$. This proves the corollary. □

The phenomenon displayed in corollary 4.3 becomes even more pronounced when restricting to the Chow ring of Y (i.e., taking $r=1$):

Corollary 4.4. *Let X and G be as in theorem 3.1, and let $Y:=X/G$. Let $a\in A^i(Y)$ be a cycle with $i\neq 3$. Assume a is a sum of intersections of 2 cycles of strictly positive codimension, i.e.*

$$a\in \operatorname{Im}\Bigl(A^m(Y)\otimes A^{i-m}(Y)\ \to\ A^i(Y)\Bigr)\ ,\quad 0<m<i\ .$$

Then a is rationally trivial if and only if a is homologically trivial.

Proof. Suppose $i=5$ or $i=6$. Since $A^r_{(r)}(Y)=0$ for $0<r<6$ (theorem 3.1), we have

$$\begin{aligned}\mathrm{Im}\Bigl(A^m(Y)\otimes A^{i-m}(Y)\to\ A^i(Y)\Bigr) &=\\ &=\mathrm{Im}\Bigl(\bigl(\bigoplus_{j<m}A^m_{(j)}(Y)\bigr)\otimes\bigl(\bigoplus_{j'<i-m}A^{i-m}(Y)\bigr)\ \to\ A^i(Y)\Bigr)\\ &\subset\bigoplus_{j+j'<i-1}A^i_{(j+j')}(Y)=A^i_{(0)}(Y)\ .\end{aligned}$$

The conclusion now follows from lemma 2.20.

For $i=2$, the corollary follows from a far more general result of Voisin concerning intersections of divisors on Hilbert schemes of $K3$ surfaces [49, Theorem 1.4].

It only remains to treat $i=4$. As both m and $4-m$ are at most 3, we have

$$A^m(Y)=\bigoplus_{j\le 0}A^m_{(j)}(Y)\ ,\quad A^{4-m}(Y)=\bigoplus_{j\le 0}A^{4-m}_{(j)}(Y)$$

(theorem 3.1). It follows that

$$\mathrm{Im}\Bigl(A^m(Y)\otimes A^{4-m}(Y)\ \to\ A^4(Y)\Bigr)\ \subset\ \bigoplus_{j\le 0}A^4_{(j)}(Y)\ ;$$

the conclusion now follows from proposition 4.5. ☐

Proposition 4.5. *Let $X=(S_3)^{[3]}$ and $G\subset\mathrm{Aut}(X)$ be as in theorem 3.1. Let $Y:=X/G$. Then*

$$\begin{aligned}&A^4_{(j)}(Y)=0\quad \textit{for } j<0\ ;\\ &A^4_{(0)}(Y)\cap A^4_{hom}(Y)=0\ .\end{aligned}$$

Proof. First, observe that $A^4_{(j)}(Y)\to A^4_{(j)}(X)$ is split injective for any j (this follows from the construction of the MCK decomposition for Y, lemma 2.18). Consequently, it suffices to prove that we have

$$\begin{aligned}&\bigl(A^4_{(j)}(X)\bigr)^G=0\quad\text{for } j<0\ ;\\ &\bigl(A^4_{(0)}(X)\bigr)^G\cap A^4_{hom}(X)=0\ .\end{aligned}$$

Let us first do the first statement. Using remark 2.12 plus the fact that

$$A^3_{(j)}(S^2)=A^2_{(j)}(S)=0\quad\text{for } j<0\ ,$$

we obtain for $j<0$ a commutative diagram

$$\begin{array}{ccc} A^4_{(j)}(X) & \hookrightarrow & A^4_{(j)}(S^3) \\ \downarrow {\scriptstyle (\Delta^G_X)_\ast} & & \downarrow {\scriptstyle (\Delta^G_{S^3})_\ast} \\ A^4_{(j)}(X) & \hookrightarrow & A^4_{(j)}(S^3)\ , \end{array}$$

where horizontal arrows are split injections. We are thus reduced to proving that

$$(\Delta^G_{S^3})_\ast A^4_{(j)}(S^3)=0\quad \hbox{for}\ j<0\ .$$

Clearly $A^4_{(-4)}(S^3)=(\pi^{S^3}_{12})_\ast A^4(S^3)=0$. It is left to consider $j=-2$, i.e. we need to prove that

$$(\Delta^G_{S^3}\circ\pi^{S^3}_{10})_\ast A^4(S^3)=0\ . \tag{16}$$

But we have seen that

$$\Delta^G_{S^3}\circ\pi^{S^3}_{10}={}^t(\Delta^G_{S^3}\circ\pi^{S^3}_{2})\quad \hbox{in}\ A^6(X\times X)$$

(lemma 2.16), and so it follows from lemma 3.2 that

$$\Delta^G_{S^3}\circ\pi^{S^3}_{10}-\gamma\ \in\ \ A^6_{hom}(X\times X)\ ,$$

where γ is some cycle supported on $D\times C$, and D is a divisor and $C\subset X$ is a curve. Applying the nilpotence theorem (plus the idempotence of lemma 2.16), we find

$$\Delta^G_{S^3}\circ\pi^{S^3}_{10}=Q_1+\cdots+Q_N\quad \hbox{in}\ A^6(X\times X)\ ,$$

where the Q_j are supported on $D\times C$. For dimension reasons, the Q_j act trivially on $A^4(S^3)$ (indeed, the action of Q_j on $A^4(S^3)$ factors over $A^{-1}(\widetilde{C})=0$). It follows that (16) is true, proving the first statement of the proposition.

Next, let us prove the second part of the proposition. Since

$$A^3_{(0)}(S^2)\cap A^3_{hom}(S^2)=A^2_{(0)}(S)\cap A^2_{hom}(S)=0$$

(lemma 2.20), we obtain a commutative diagram

$$\begin{array}{ccc} A^4_{(0)}(X)\cap A^4_{hom}(X) & \hookrightarrow & A^4_{(0)}(S^3)\cap A^4_{hom}(S^3) \\ \downarrow {\scriptstyle (\Delta^G_X)_\ast} & & \downarrow {\scriptstyle (\Delta^G_{S^3})_\ast} \\ A^4_{(0)}(X)\cap A^4_{hom}(X) & \hookrightarrow & A^4_{(0)}(S^3)\cap\ A^4_{hom}(S^3)\ , \end{array}$$

where horizontal arrows are split injections. We are thus reduced to proving that

$$(\Delta^G_{S^3})_\ast\Big(A^4_{(0)}(S^3)\cap A^4_{hom}(S^3)\Big)=0\ ,$$

which is equivalent to proving that

$$(\Delta^G_{S^3} \circ \pi_8^{S^3})_* A^4_{hom}(S^3) = 0 \ . \tag{17}$$

But we have seen that

$$\Delta^G_{S^3} \circ \pi_8^{S^3} = {}^t(\Delta^G_{S^3} \circ \pi_4^{S^3}) \quad \text{in } A^6(X \times X)$$

(lemma 2.16), and so it follows from lemma 3.3 that

$$\Delta^G_{S^3} \circ \pi_8^{S^3} - \gamma \ \in \quad A^6_{hom}(X \times X) \ ,$$

where γ is some cycle supported on $W \times V \subset X \times X$, and $W \subset X$ is codimension 2 and $V \subset X$ is codimension 4. Applying the nilpotence theorem (plus the idempotence of lemma 2.16), we find

$$\Delta^G_{S^3} \circ \pi_8^{S^3} = Q_1 + \cdots + Q_N \quad \text{in } A^6(X \times X) \ ,$$

where the Q_j are supported on $W \times V$. For dimension reasons, the Q_j act trivially on $A^4_{hom}(S^3)$ (indeed, the action of Q_j on $A^4_{hom}(S^3)$ factors over $A^0_{hom}(\widetilde{V}) = 0$). It follows that (17) is true, proving the second statement of the proposition.

$\square$

Remark 4.6. Corollaries 4.3 and 4.4 are similar to the Beauville–Voisin conjecture, on the one hand, and to results of Voisin and L. Fu for Calabi–Yau varieties, on the other hand.

The Beauville–Voisin conjecture [49, Conjecture 1.3] concerns the Chow ring of a hyperkähler variety X. The conjecture is that the subring

$$D^*(X) \ \subset \ A^*(X)$$

generated by divisors and Chern classes injects (via the cycle class map) into cohomology. Partial results towards this conjecture have been obtained in [49], [39], [58].

On the other hand, if Y is a Calabi–Yau variety that is a generic complete intersection, say of dimension n, it has been proven that the image of the intersection product

$$\mathrm{Im}\Big(A^i(Y) \otimes A^{n-i}(Y) \ \to \ A^n(Y) \Big) \ , \quad 0 < i < n \ ,$$

is of dimension 1 and hence injects into cohomology [50], [16].

Results like corollaries 4.3 and 4.4 are presumably *not* true for all Calabi–Yau varieties (since not all Calabi–Yau varieties verify Beauville's weak splitting property [5]); for a general Calabi–Yau variety, one only expects statements about 0–cycles. Conjecturally, statements concerning other codimensions (such as corollaries 4.3 and 4.4) should be true for Calabi–Yau varieties that are finite quotients of hyperkähler varieties.

5 A partial generalization

This section contains a partial generalization of theorem 3.1. We consider Hilbert schemes $X = (S_k)^{[k]}$, where S_k is any of the LSY surfaces. The same result (theorem 5.1) also applies to some of the Schütt surfaces.

Theorem 5.1. *Let S_k be an LSY surface, or a Schütt surface as in proposition 2.29, with $G_{S_k} \subset \mathrm{Aut}(S_k)$ the order k group of non–symplectic automorphisms of definition 2.22, resp. definition 2.26. Let X be the Hilbert scheme $X := (S_k)^{[k]}$ of dimension $n = 2k$. Let $G \subset \mathrm{Aut}(X)$ be the order k group of non–symplectic natural automorphisms, corresponding to $G_{S_k} \subset \mathrm{Aut}(S_k)$. Then*

$$A^i_{(2)}(X) \cap A^i(X)^G = 0 \quad \textit{for } i \in \{2, n\} \ .$$

Proof. Let us write S for the surface S_k. Let $i \in \{2, n\}$. Using remark 2.12, one finds a commutative diagram

$$\begin{array}{ccc} A^i_{(2)}(X) & \to & A^i_{(2)}(S^k) \\ \downarrow (\Delta^G_X)_* & & \downarrow (\Delta^G_{S^k})_* \\ A^i_{(2)}(X) & \to & A^i_{(2)}(S^k) \ , \end{array}$$

where horizontal arrows are split injections. Here $\Delta^G_{S^k}$ is as before defined as the projector

$$\Delta^G_{S^k} := \frac{1}{k} \sum_{g \in G_S} \Gamma_g \times \cdots \times \Gamma_g \in A^{2k}(S^k \times S^k) \ .$$

We are thus reduced to proving that

$$(\Delta^G_{S^k})_* = 0\colon \quad A^i_{(2)}(S^k) \ \to \ A^i_{(2)}(S^k) \quad \text{for } i \in \{2, n\} \ . \tag{18}$$

Let us assume $i = 2$. Lemma 2.34 (with $k = r$) implies that

$$\Delta^G_{S^k} \circ \pi_2^{S^k} - \gamma \ \in \quad A^{2k}_{hom}(S^k \times S^k) \ ,$$

where γ is a cycle supported on (curve)$\times$(divisor). But S^k has finite–dimensional motive, and so there exists $N \in \mathbb{N}$ such that

$$\Big(\Delta^G_{S^k} \circ \pi_2^{S^k} - \gamma \Big)^{\circ N} = 0 \quad \text{in } A^{2k}(S^k \times S^k) \ .$$

Developing, and using that $\Delta^G_{S^k} \circ \pi_2^{S^k}$ is idempotent (lemma 2.16), this implies that

$$\Delta^G_{S^k} \circ \pi_2^{S^k} = Q_1 + \cdots + Q_N \quad \text{in } A^{2k}(S^k \times S^k) \ , \tag{19}$$

where each Q_j is supported on $C\times D$ and hence does not act on $A^2_{hom}(S^k)=A^2_{AJ}(S^k)$. It follows that

$$(\Delta^G_{S^k}\circ\pi_2^{S^k})_\ast=0\colon\quad A^2_{hom}(S^k)\ \to\ A^2(S^k)\ ,$$

and thus

$$(\Delta^G_{S^k})_\ast=0\colon\quad A^2_{(2)}(S^k)\ \to\ A^2(S^k)\ ,$$

proving (18) for $i=2$.

It remains to consider the case $i=n$. Taking the transpose of equality (19) and invoking lemma 2.16, one obtains an equality

$$\Delta^G_{S^k}\circ\pi^{S^k}_{4k-2}=\pi^{S^k}_{4k-2}\circ\Delta^G_{S^k}={}^tQ_1+\cdots+{}^tQ_N\quad\text{in } A^{2k}(S^k\times S^k)\ ,$$

where the ${}^tQ_j$ are supported on $D\times C$. The ${}^tQ_j$ do not act on $A^n(S^k)$ (for dimension reasons), and so

$$(\Delta^G_{S^k}\circ\pi^{S^k}_{4k-2})_\ast=0\colon\quad A^n(S^k)\ \to\ A^n(S^k)\ ,$$

proving (18) for $i=n$. □

Theorem 5.1 has implications for the quotient $Y:=X/G$ (the variety Y is a "Calabi–Yau variety with quotient singularities"):

Corollary 5.2. *Let X and G be as in theorem 5.1, and let $Y:=X/G$ be the quotient. For any $r\in\mathbb{N}$, let*

$$E^\ast(Y^r)\ \subset\ A^\ast(Y^r)$$

be the subalgebra generated by (pullbacks of) $A^1(Y)$ and $A^2(Y)$ and Δ_Y, Δ_Y^{sm}. Then the cycle class map induces maps

$$E^i(Y^r)\ \to\ H^{2i}(Y^r)$$

that are injective for $i\ge nr-1$.

Proof. This is similar to corollary 4.3. First, it follows from lemma 2.18 that Y, and hence Y^r, has a self–dual MCK decomposition. Consequently, the Chow ring $A^\ast(Y^r)$ is a bigraded ring. Theorem 5.1 (plus the fact that $A^1_{hom}(Y)=0$) implies that

$$A^i(Y)=\bigoplus_{j\le 0}A^i_{(j)}(Y)\quad\text{for } i\le 2\ .$$

Lemma 2.13 ensures that

$$\Delta_Y\in A^n_{(0)}(Y)\ ,\quad \Delta_Y^{sm}\in A^{2n}(Y^3)\ .$$

Since pullbacks for projections of type $Y^r \to Y^s, s < r$, preserve the bigrading (this follows from [43, Corollary 1.6], or can be readily checked directly), this implies that

$$E^*(Y^r) \subset \bigoplus_{j\le 0} A^*_{(j)}(Y^r)\ .$$

In particular, this implies

$$E^i(Y^r) \subset A^i_{(0)}(Y^r) \quad \text{for } i \ge nr-1\ .$$

The corollary now follows from the fact that

$$A^i_{(0)}(Y^r) \cap A^i_{hom}(Y^r) \ \to\ A^i_{(0)}(X^r) \cap A^i_{hom}(X^r)$$

is injective (this is true for any i), and

$$A^i_{(0)}(X^r) \cap A^i_{hom}(X^r) = 0 \quad \text{for } i \ge nr-1$$

(lemma 2.20). □

Corollary 5.3. *Let X and G be as in theorem 5.1, and let $Y := X/G$ be the quotient. Let $a \in A^n(Y)$ be a 0–cycle which is in the image of the intersection product map*

$$A^3(Y) \otimes A^{i_1}(Y) \otimes \cdots \otimes A^{i_s}(Y) \ \to\ A^n(Y)\ ,$$

with all $i_m \le 2$ (and $i_1 + \cdots + i_s = n-3$). Then a is rationally trivial if and only if $\deg(a) = 0$.

Proof. The point is that

$$A^3(Y) = \bigoplus_{j\le 0} A^3_{(j)}(Y) \oplus A^3_{(2)}(Y)\ ,$$
$$A^{i_m}(Y) = \bigoplus_{j\le 0} A^{i_m}_{(j)}(Y) \quad \text{for } i_m \le 2$$

(theorem 5.1), and so

$$a \in A^n_{(0)}(Y) \oplus A^n_{(2)}(Y)\ .$$

But we have seen that $A^n_{(2)}(Y) = 0$ (theorem 5.1), and so

$$a \in A^n_{(0)}(Y) \cong \mathbb{Q}\ .$$

□

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