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Recent advances on the theory of Scorza quartics

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Abstract. *We review the proof of the existence of the Scorza quartic for the general element of the moduli space of spin curves.*

1 Introduction

In this work we present the content of two lectures which were held at the Dipartimento di Matematica dell' Università di Catania. They were part of the workshop *Quartiche piane, mappa di Scorza e argomenti correlati*, DMI-UNICT, Catania 19-21/1/2016, marvellously organised by Francesco Russo, and devoted to study geometrical and computational aspects of plane quartics and of the Scorza's construction.

1.1 Historical overview

In the papers [19] and [20], using deep geometrical ideas based on his study of polar polyhedra; see: [18], Gaetano Scorza was able to associate a plane quartic $C = \{F_4 = 0\}$ to a couple (C', θ) where C' is a plane quartic, in general different from C, and θ is a divisor of degree 2 such that 2θ is cut by a line but $\mathcal{O}_{C'}(\theta)$ has no section. Moreover he gave an idea to associate a quartic $\{F_4 = 0\} \subset \mathbb{P}^{g-1}$, nowadays called *Scorza quartic*, to each general couple (Γ, θ) where Γ is a canonical curve of genus g and θ is a divisor of degree $g-1$ such that 2θ is linearly equivalent to the canonical divisor K_Γ and $\mathcal{O}_\Gamma(\theta)$ has no section; a couple (Γ, θ) as above is nowadays called *spin curve* and θ is called an *ineffective theta characteristic*, we learnt this name by Miles Reid. For a modern approach to the theory of theta characteristic see: [13].

In [2] Maurizio Cornalba constructed a compactification of the moduli space S_g^+ of such couples (Γ, θ) and he proved that it is irreducible. The Scorza's construction has been strongly revised by Igor Dolgachev and Vassil Kanev in their paper [3] which is the basis of our first lecture. In particular they strongly clarify Scorza's construction to obtain $\{F_4 = 0\} \subset \mathbb{P}^{g-1}$. Their study led them to find some conditions to prove the existence of the Scorza quartic associated to (Γ, θ); see Subsection 2.3 below. Recently, in his book [4, 5.5 Scorza correspondence,

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pages 212-225] Dolgachev reviews the construction of the Scorza quartics where the conditions to construct them fits with the theory exposed in [21, 22].

Let us briefly recall here the case of plane quartics which is not treated in these notes but it should suggest motivations to read them.

1.2 Plane quartics

We follow the explanation in [3, §6, 7] (see also [17, §3] and [4, Subsection 6.3.4, pages 251-255]).

Assume that V is a 3-dimensional vector space and $\check{V}$ its dual. Let $F \in S^4\check{V}$ be a general ternary quartic form on V. Then the closure of the loci in $\mathbb{P}(V) = \mathbb{P}^2$ at a point of which the first polar of F is a Fermat cubic is again a smooth quartic curve, which is denoted by $S(F)$ and is called the covariant quartic of F: see [3, p. 259]. In symbols:

$$S(F) := \{a \in \mathbb{P}^2 \mid P_aF \text{ is Fermat cubic}\}.$$

By taking the second polars of $S(F)$, we have the following correspondence:

$$T(F) := \{(a, b) \in S(F) \times S(F) \mid \mathrm{rank} P_{a,b}(S(F)) \leq 1\}. \tag{1.1}$$

Actually, this is also equal to $\{(a, b) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \mathrm{rank} P_{a,b}(S(F)) \leq 1\}$ see [3, Corollary 6.6.3 (iv)], [3, Proposition 6.8.1] and [3, Theorem 7.6]. The important point for these notes is that the theory exposed in [3] clarifies that there exists an ineffective theta characteristics θ such that $(a, b) \in T(F)$ if and only if a belongs to the unique effective divisor linearly equivalent to $\theta + b$. By this Scorza was able to construct a map, nowadays called *the Scorza map*: $\mathrm{Sc}\colon [F = 0] \mapsto [S(F), \theta]$, which is defined over the open subset $\mathcal{M}_3^0$ of the coarse moduli space $\mathcal{M}_3$ of genus-3 smooth curves given by those $\{F = 0\}$ such that $S(F)$ is nonsingular.

It is well-known that the forgetful morphism $S_3^+ \to \mathcal{M}_3$, $[C, \theta] \mapsto [C]$ is generically of degree 36, but Scorza was able to see that the map $\mathrm{Sc}\colon \mathcal{M}_3^0 \to S_3^+$ is an injective birational map (cf. [3, Theorem 7.8]).

We also remind the reader that by this construction we can conclude that S_3^+ is rational since $\mathcal{M}_3$ is rational [12] (see also [1]).

By this birational correspondence the curve F corresponding to a couple $(S(F), \theta)$ is called the *Scorza quartic* of $(S(F), \theta)$.

1.3 Primitive Fanos of genus 12

Another motivation to study the geometry of the Scorza quartics comes from Mukai's description of prime Fano threefolds of genus 12. By definition a prime Fano threefold X is a smooth projective variety such that $\dim_{\mathbb{C}} X = 3$, $-K_X$ is ample, the class of $-K_X$ generates $\mathrm{Pic}\, X$, and the number $g(X) := \frac{(-K_X)^3}{2} + 1$, called the genus of X, is equal to 12. These Fano threefolds were quite mysterious objects and the attempt to find a geometrical description of them led Mukai to find their relationship with the concept of varieties of power sums.

First we recall the following object inside the Hilbert scheme $\mathrm{Hilb}^n\mathbb{P}(\check{V})$ given by n-ples of points of $\mathbb{P}(\check{V})$ where V is a $(v+1)$-dimensional vector space and $\check{V}$ is its dual space:

Definition 1.1. Let $F \in S^m\check{V}$ be a homogeneous forms of degree m on V. Set

$$\mathrm{VSP}\,(F,n)^o := \{([H_1],\dots,[H_n]) \mid H_1^m + \cdots + H_n^m = F\} \subset \mathrm{Hilb}^n\mathbb{P}(\check{V}).$$

The closed subset $\mathrm{VSP}\,(F,n) := \overline{\mathrm{VSP}\,(F,n)^o}$ is called the *varieties of power sums* of F.

Mukai discovered the following beautiful description of prime Fano threefolds of genus 12; see: [15, 16].

Theorem 1.2. *Let $\{F_4 = 0\} \subset \mathbb{P}(V)$ be a general plane quartic curve. Then*

(1) $\mathrm{VSP}(F_4, 6) \subset \mathrm{Hilb}^6\mathbb{P}(\check{V})$ *is a general prime Fano threefold of genus* 12*; and conversely,*

(2) *every general prime Fano threefold of genus* 12 *is of this form.*

The above result can be seen as a part of the theory of the ineffective theta characteristics because the Hilbert scheme of lines on X is isomorphic to a smooth curve Γ of genus 3 and Mukai proved that the correspondence on $\Gamma \times \Gamma$ defined by the intersection of lines on X gives an ineffective theta characteristic θ on Γ. More precisely, θ is described via the correspondence

$$I := \{([l],[m]) \in \Gamma \times \Gamma \mid l \cap m \neq \emptyset, l \neq m\}$$

obtained by the geometry of X. Actually in subsection 3.4.1 below we give a more precise description of I in the Fano 3-folds context. Now, by the result of Scorza recalled above, there exists the Scorza quartic $\{F_4 = 0\}$ of the pair (Γ, θ) in the same ambient plane as the canonically embedded Γ. Mukai proved that X is recovered as $\mathrm{VSP}\,(F_4, 6)$. This is the result (2) of Theorem 1.2. The result (1) follows from (2). See also [17]). An evidence for this last result is given by the known fact that the number of the moduli of prime Fano threefolds of genus 12 is equal to $\dim \mathcal{M}_3 = 6$.

1.4 A generalisation of the Mukai's construction

In the papers [21] and [22] Hiromichi Takagi and Francesco Zucconi generalised the Mukai's construction, see [14], to the case of other 3-folds. We explain a relation of their result with Theorem 1.2. We will recall in Subsection 3.1 many properties of the Fano 3-fold of degree 5 and index 2, which we denote by B. In particular its Hilbert schemes of lines and respectively of conics are nicely interplayed to the description of B as a complete intersection of the Grassmannian $\mathbb{G}r(2,5) \subset \mathbb{P}^9$ and of a general $\mathbb{P}^6$ inside $\mathbb{P}^9$.

Let R be a smooth rational curve of degree 5 contained inside B. Let $f\colon A \to B$ be the blow-up of B along R. It is easy to prove that R has three bisecant lines β_1, β_2, β_3 contained inside B and a 1 dimensional family of unisecant lines contained in B. Let $\beta_1', \beta_2', \beta_3'$ be the strict transforms of β_1, β_2, β_3 and $E_R := f^{-1}(R)$ the f-exceptional divisor.

Then it remains defined the following diagram:

$$\begin{tikzcd}
 & & \widetilde{A} \arrow[dl, "\rho"'] \arrow[dr, "\rho'"] \arrow[ddrr, bend left, "\Phi"] & & \\
 & A \arrow[dl, "f"'] \arrow[rr, dashed] & & A' \arrow[dr, "f'"] & \\
B & & & & X,
\end{tikzcd}$$

where

- X is a smooth prime Fano threefold of genus twelve,
- ρ' is the blow-down of the three ρ-exceptional divisors E_i $(i = 1, 2, 3)$ over the strict transforms β_i' in the other direction. In other words, $A \dashrightarrow A'$ is the flops of β_1', β_2' and β_3', and
- the morphism f' contracts the strict transform of the unique hyperplane section S containing R to a general line m on X.

The rational map $X \dashrightarrow B$ is the famous double projection of X from a general line m first discovered by Iskovskih (see [11]).

Now, if we define *line on* A a rational connected curve l contained inside A such that $E_R \cdot l = -K_A \cdot l = 1$, it can be shown that there exists an isomorphism between the Hilbert scheme of lines of X and the Hilbert scheme of lines of A. Almost the same occurs if we suitably define a notion of conic on A. This is recalled in Subsection 3.3. By this correspondence we can translate the geometry over X into the geometry of A and since the Hilbert scheme of lines of X is a genus 3 curve Γ and the incidence correspondence is readable by a theta ineffective divisor, we see a way to associate to a rational curve $R \subset B$ a couple (Γ, θ) such that the geometry of the 3-fold A reads the algebraic relations given by θ. A moment of thought could convince the reader that the restriction on R to be of degree 5 is not so important. In particular this idea leads to associate to a sufficiently general rational curve $R \subset B$ of degree $g+2$ a couple (Γ, θ) where Γ is the Hilbert scheme of lines of A and θ is obtained by line to line intersection on A, where A is the blow-up of B at R. This is the path built in [22], which leads to show the rationality of $\mathcal{S}_4^+$, see [23], the rationality of the moduli space of one-pointed ineffective spin hyperelliptic curves, see [24], and the rationality of the moduli space of ineffective spin hyperelliptic curves, see [25]. Finally our method is applied in [21] to finish the research program started by Scorza and continued by Dolgachev and Kanev, that is to give a complete proof of the existence of the Scorza quartic $\{F_4 = 0\} \subset \mathbb{P}^{g-1}$ associated to a general couple $[(\Gamma, \theta)] \in \mathcal{S}_g^+$.

2 Scorza's construction

2.1 Scorza's correspondence

Let Γ be a smooth curve of genus g and $\omega_\Gamma := \mathcal{O}_\Gamma(K_\Gamma)$ its canonical sheaf. We denote by V the vector space $H^0(\Gamma, \omega_\Gamma)^\vee$ of linear functionals on $H^0(\Gamma, \omega_\Gamma)$ and we set $\mathbb{P}^{g-1} := \mathbb{P}(V)$. An ineffective theta characteristic θ is a divisor of degree $g-1$ such that $h^0(\Gamma, \mathcal{O}_\Gamma(\theta)) = 0$ and 2θ is linearly equivalent to K_Γ. In the sequel we will assume that Γ is a canonical curve inside $\mathbb{P}^{g-1}$.

By Riemann-Roch theorem it follows that $h^0(\Gamma, \mathcal{O}_\Gamma(\theta+a)) = 1$ for every $a \in \Gamma$. The unique effective divisor of $|\theta + a|$ is called *θ-polyhedron attached to a* and it is commonly denoted by $I_\theta(a)$. The points $a_1, \dots, a_g \in \Gamma$ of the support $\mathrm{supp}(\mathrm{I}_\theta(a))$ are called *vertices* of the θ-polyhedron attached to a.

Definition 2.1. The first Scorza correspondence is the following scheme:

$$\mathrm{I}_\theta := \{(a,b) \in \Gamma \times \Gamma \mid a \in \mathrm{supp}(\mathrm{I}_\theta(b))\}$$

Note that I_θ is symmetric. Indeed let $\mathrm{I}_\theta(a) = a_1 + \cdots + a_g \equiv \theta + a$ and $\mathrm{I}_\theta(b) = b_1 + \cdots + b_g \equiv \theta + b$. Assume that there exists $i = 1, \dots, g$ such that $b = a_i$. By Serre duality it holds that $h^0(\Gamma, \mathcal{O}_\Gamma(\theta + b - a)) = 1$.

Consider now the Abel-Jacobi morphism

$$\alpha \colon \Gamma \times \Gamma \to \mathrm{Jac}(\Gamma)$$

which assigns to each couple (x, y) the divisor class $cl(y - x)$. It is known that the locus $W_{g-1} \subset \mathrm{Pic}^{g-1}(\Gamma)$ given by effective divisor is an hypersurface and we set $\Theta := W_{g-1} - \theta \subset \mathrm{Jac}(\Gamma)$.

Let $\sigma \colon \Gamma \times \Gamma \to \Gamma \times \Gamma$ be the exchange map and Δ the diagonal of $\Gamma \times \Gamma$.

Proposition 2.2. *It holds:*

(1) $\mathrm{I}_\theta = \alpha^\star \Theta$.

(2) $\sigma^\star \mathrm{I}_\theta = \mathrm{I}_\theta$.

(3) $\mathrm{I}_\theta \cap \Delta = \emptyset$.

(4) I_θ *is algebraically equivalent to* $\Delta + p_1^\star(\theta)$.

(5) *If* $(x, y) \in \mathrm{I}_\theta$ *then* $h^0(\Gamma, \mathcal{O}_\Gamma(\mathrm{I}_\theta(x) - y)) = 1$.

Proof. (1): $\alpha^*\Theta|\ \{x\}\times\Gamma = \mathrm{I}_\theta(x)$. (2): $\sigma^\star\mathrm{I}_\theta = \alpha^\star(-\mathrm{Id}_{\mathrm{Jac}(\Gamma)}\Theta = \alpha^\star\Theta = \mathrm{I}_\theta$. (3) follows since $h^0(\Gamma, \mathcal{O}_\Gamma(\theta)) = 0$. To show (4) we note that since $\mathrm{I}_\theta(x) - x = \theta$ is independent of $x \in \Gamma$. This means that $\mathrm{I}_\theta - \Delta$ is algebraically equivalent to divisors obtained by pull-back by the two projections. □

We recall that a correspondence $I \subset \Gamma\times\Gamma$ is of valence n, where $n \in \mathbb{N}$ if the divisor class $I(p) + np$ is independent of $p \in \Gamma$. By same arguments used in Proposition 2.2 Dolgachev and Kanev show:

Proposition 2.3. *Let Γ be a smooth curve of genus g and let $I \in \mathrm{Div}(\Gamma\times\Gamma)$ be a symmetric effective correspondence without united points, of valence ν and degree (g, g). Assume that if $x \in \Gamma$ is a general point then $h^0(\Gamma, \mathcal{O}_\Gamma(I_{|\{x\}\times\Gamma})) = 1$. Then there exists a unique ineffective theta characteristic such that $I = \mathrm{I}_\theta$.*

Proof. See: [3, Lemma 7.2.1] and also [4, Proposition 5.5.1]. □

2.2 The discriminant locus

By Proposition 2.2 (5) and by Riemann's theorem on theta divisor, c.f. see: [7, p. 348], we have that the image by α of I_θ is contained inside the open subset $\Theta^{\mathrm{ns}} \subset \Theta$ given by the non singular points of Θ; that is $\alpha(I_\theta) \subset \Theta^{\mathrm{ns}}$. Now recall that $\Theta := W_{g-1} - \theta$ and on Θ^{ns} it is defined the Gauss map $\gamma\colon \epsilon \mapsto D_\epsilon$ where $D_\epsilon \in |K_\Gamma|$ is the unique divisor containing the unique effective divisor of $\epsilon + \theta$ cf. see [7, p. 360]. Consider the composition

$$\pi_\theta\colon \mathrm{I}_\theta \xrightarrow{\alpha} \Theta^{\mathrm{ns}} \xrightarrow{\gamma} |K_\Gamma| = \check{\mathbb{P}}^{g-1} = \mathbb{P}(V^\vee)$$

The following is an important invariant of (Γ, θ):

Definition 2.4. The image $\Gamma(\theta)$ of the above morphism $\pi_\theta\colon \mathrm{I}_\theta \to \check{\mathbb{P}}^{g-1}$ (with reduced structure) is called the *discriminant locus* of the pair (Γ, θ).

Note that if $(a, b) \in \mathrm{I}_\theta$ then $\pi_\theta((a, b)) = [\langle \mathrm{I}_\theta(a) - b\rangle] = [\langle \mathrm{I}_\theta(b) - a\rangle] \in \check{\mathbb{P}}^{g-1}$. Let $\overline{\Theta} := \Theta/\langle -\mathrm{Id}_{\mathrm{Jac}(\Gamma)}\rangle$, $j\colon \Theta \to \overline{\Theta}$ the quotient morphism and $\overline{\gamma}\colon \overline{\Theta} \to \check{\mathbb{P}}^{g-1}$ the induced one. Set $\overline{\Gamma(\theta)} := j(\alpha(I_\theta)) \subset \overline{\Theta}$.

In the sequel we will need the following lemmas.

Lemma 2.5. *The degree of $\pi_\theta\colon \mathrm{I}_\theta \to \Gamma(\theta)$ is $2d(\theta) \le \binom{2g-2}{g-1}$.*

Proof. Since I_θ is symmetric, the morphism $\pi_\theta\colon \mathrm{I}_\theta \to \Gamma(\theta)$ factorizes through $j\circ\alpha$. Since the degree of γ is $\binom{2g-2}{g-1}$ the claim follows. □

Lemma 2.6. *The number $2d(\theta)$ is the number of θ-polyhedra of Γ having a common face.*

Proof. If $[H] \in \Gamma(\theta)$ is a general element then

$$\pi_\theta^{-1}([H]) = \{(x, y) \in \mathrm{I}_\theta \mid H = \langle \mathrm{I}_\theta(x) - y\rangle\}\,.$$

□

2.3 Conditional existence of Scorza's quartics

Dolgachev and Kanev showed that three conditions, overlooked by Scorza, are needed to construct the Scorza quartic: (see [3, (9.1) (A1)–(A3)]). They are modified in [4, Definition 5.5.15]:

(A1) the degree of the map $\mathrm{I}_\theta \to \Gamma(\theta)$ is two, namely, $\langle \mathrm{I}_\theta(x') - y'\rangle = \langle \mathrm{I}_\theta(x) - y\rangle$ implies $(x', y') = (x, y)$ or (y, x),

(A2) $\Gamma(\theta)$ is not contained in a quadric, and

(A3) I_θ is smooth and connected.

From now on we assume that these conditions hold for (Γ, θ).

By Definition 2.4, we have the following diagram:

$$\begin{tikzcd} & \mathrm{I}_\theta \subset \Gamma \times \Gamma \arrow[dl, "\pi_\theta"'] \arrow[dr, "p"] & \\ \Gamma(\theta) \subset \check{\mathbb{P}}^{g-1} & & \Gamma \subset \mathbb{P}^{g-1}. \end{tikzcd} \tag{2.1}$$

Let $|K_\Gamma| \ni [H] \in \check{\mathbb{P}}^{g-1}$ or in other words H is an hyperplane of $\mathbb{P}^{g-1}$.

Definition 2.7. We call the Scorza-trasform of $[H]$ the following divisor on $\Gamma(\theta)$

$$\overline{D}_H := \pi_{\theta *} p^*(H \cap \mathcal{H}).$$

Proposition 2.8. *Assume that* (Γ, θ) *is a spin curve which satisfies* (A1), (A2) *and* (A3). *Then the following hold:*

(1) $\deg \Gamma(\theta) = g(g-1)$;

(2) $\rho_a(\mathrm{I}_\theta/\langle\tau\rangle) = \frac{3}{2}g(g-1) + 1$;

(3) *The Scorza transform* $\overline{D}_H$ *is cut by a (unique) quadric in* $\check{\mathbb{P}}^{g-1}$.

Proof. Let $\mathrm{S}^2\Gamma$ be the symmetric product and $\sigma\colon \Gamma \times \Gamma \to \mathrm{S}^2\Gamma$ the natural morphism. Since $\mathrm{I}_\theta \cap \Delta = \emptyset$ then $\sigma_{|\mathrm{I}_\theta}\colon \mathrm{I}_\theta \to \mathrm{I}_\theta/\langle\tau\rangle$ in an unramified double cover. Since $d(\theta) = 1$ then the induced morphism $h\colon \mathrm{I}_\theta/\langle\tau\rangle \to \Gamma(\theta)$ is a birational morphism. Consider an hyperplane $\check{H} \subset \check{\mathbb{P}}^{g-1}$. Then there exists $a \in \mathbb{P}^{g-1}$ such that $\check{H} = \check{H}_a$ is the set of the hyperplanes containing a. Fix a point $a \in \Gamma \subset \mathbb{P}^{g-1}$. Then $[H] \in \check{H}_a \cap \Gamma(\theta)$ means that there exists $(b, c) \in \mathrm{I}_\theta$ such that $H = \langle \mathrm{I}_\theta(b) - c\rangle$ and $a \in H$. Set $\mathrm{I}_\theta(b) = b_1 + \cdots + b_g$ and $\mathrm{I}_\theta(c) = c_1 + \cdots + c_g$; see also the remark below this proof. Assume $a \neq b$ and $a \neq c$. We can write $b = c_1$ and $c = b_1$. This implies that there exists $j = 2, \ldots, g-1$ such that $a = b_j$ or $a = c_j$ since $a \in \langle b_2, \ldots, b_g, c_2, \ldots, c_g\rangle = H \cap \Gamma$. Hence $a \in \mathrm{I}_\theta(b)$ or $a \in \mathrm{I}_\theta(c)$. In both cases this means that there exists $b \in \mathrm{I}_\theta(a)$ such that H is a face of $\mathrm{I}_\theta(b)$ and $a \in H$.

Hence set $\mathrm{I}_\theta(a) = b_1 + \cdots + b_g$. Then $[H] \in \check{H}_a \cap \Gamma(\theta)$ iff there exists $i = 1, \ldots, g$ such that $H = \langle \mathrm{I}_\theta(b_i) - c\rangle = \pi_\theta(b_i, c)$ and $c \neq a$. Consider the two projections $p_i \colon \Gamma \times \Gamma \to \Gamma$, $i = 1, 2$. Then

$$\begin{aligned} \pi_\theta^\star(\check{H}_a \cap \Gamma(\theta)) &= p_1^\star(\mathrm{I}_\theta(a)) - p_2^\star(a) + p_2^\star(\mathrm{I}_\theta(a)) - p_1^\star(a) \\ &= p_1^\star(\mathrm{I}_\theta(a) - a) + p_2^\star(\mathrm{I}_\theta(a) - a) = p_1^\star(\theta) + p_2^\star(\theta). \end{aligned} \tag{2.2}$$

Finally notice that $\mathrm{I}_\theta \sim \Delta + p_1^\star(\theta) + p_2^\star(\theta) = \alpha^\star(\Theta)$ and since $\mathrm{I}_\theta \cdot \Delta = 0$ it follows that:

$$\pi_\theta^\star(\mathcal{O}_{\Gamma(\theta)}(1) = \mathcal{O}_{I_\theta}(p_1^\star(\theta) + p_2^\star(\theta)) \sim \mathcal{O}_{I_\theta}(\alpha^\star(\Theta)).$$

In particular $\deg \Gamma(\theta) = \frac{1}{2} \deg \mathcal{O}_{I_\theta}(p_1^\star(\theta) + p_2^\star(\theta)) = g(g-1)$. To show (1) and (2) we have used (A1) and (A3). Now we use (A2) to show (3). Claim (3) follows if we show that there exists an hyperplane H such that $\overline{D}_H$ is cut by a quadric in $\check{\mathbb{P}}^{g-1}$. Now choose two points $a, b \in \Gamma$ such that $a \in \mathrm{I}_\theta(b)$. Let H be a common face to $\mathrm{I}_\theta(a)$ and $\mathrm{I}_\theta(b)$. We set $\mathrm{I}_\theta(b) = a + b_1 + \cdots + b_{g-1}$ and $\mathrm{I}_\theta(a) = b + a_1 + \cdots + a_{g-1}$. Then $H = \langle b_1, \ldots, b_{g-1}\rangle = \langle a_1, \ldots, a_{g-1}\rangle$. Now by the same argument used to obtain above the equation (2.2) we can write that $[Z]$ is in $\overline{D}_H$ if Z is a face of $\mathrm{I}_\theta(b_j)$, $j = 1, \ldots, g-1$ or Z is a face of $\mathrm{I}_\theta(a_j)$, $j = 1, \ldots, g-1$. Since the claim (1) is true then the quadric $\Phi = \check{H}_a \check{H}_b$ does the job. Note that by the condition (A2) it follows that Φ is unique.

$\square$

Remark 1. It is not clear if there are two such non-degenerate polyhedra $\mathrm{I}_\theta(a)$ and $\mathrm{I}_\theta(b)$ whose existence is claimed in the above proof from the conditions (A1)–(A3) only. For *general spin curves* this is true since we will show that it is true for a general (Γ, θ) where Γ is trigonal. This can be seen as an application of our geometrical construction.

2.3.1 The second Scorza's correspondence

To define the Scorza quartic of (Γ, θ) we need to consider the correspondence:

$$\mathcal{D} := \{(q_1, q_2) \mid q_1 \in \overline{D}_{H_{q_2}}\} \subset \Gamma(\theta) \times \Gamma(\theta), \tag{2.3}$$

where H_q is the hyperplane of $\mathbb{P}^{g-1}$ corresponding to $q \in \check{\mathbb{P}}^{g-1}$. It is easy to see that $\mathcal{D}$ is symmetric. The correspondence $\mathcal{D}$ is called *the second Scorza's correspondence*. Proposition 2.8 suggests that $\mathcal{D}$ is the restriction of a symmetric $(2,2)$ divisor $\{\mathcal{D}' = 0\}$ of $\check{\mathbb{P}}^{g-1} \times \check{\mathbb{P}}^{g-1}$; this too will be proved in Theorem 3.25. For the moment assume that $\mathcal{D}'$ exists. By [3] we may take the equation $\mathcal{D}'$ so that it is the bi-homogeneization of an equation $\check{F}_4$ such that $\{\check{F}_4 = 0\} \subset \check{\mathbb{P}}^{g-1}$ is a quartic hypersurface. Actually this quartic hypersurface is obtained by restricting

$\mathcal{D}'$ to the diagonal of $\check{\mathbb{P}}^{g-1} \times \check{\mathbb{P}}^{g-1}$. Moreover by definition of the fiber of $\{\mathcal{D}' = 0\}$ over a point $q \in \check{\mathbb{P}}^{g-1}$ we obtain the homomorphism:

$$\lambda\colon H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2)) \to H^0(\check{\mathbb{P}}^{g-1}, \mathcal{O}_{\check{\mathbb{P}}^{g-1}}(2)) \tag{2.4}$$

such that for all $q \in \Gamma(\theta)$

$$\lambda\colon [H_q^2] \mapsto [\check{H}_a \check{H}_b]$$

where the hyperplane $\{H_q = 0\} = \widetilde{H}_q \subset \mathbb{P}^{g-1}$ is the one corresponding to the point $\pi_\theta(a,b) = q \subset \Gamma(\theta) \subset \check{\mathbb{P}}^{g-1}$ and $(a,b) \in I_\theta$.

2.3.2 The definition of the Scorza quartic

Proposition 2.9. *The homomorphism*

$$\begin{aligned} \lambda\colon S^2\check{V} &\to S^2 V \\ Q &\mapsto \lambda(Q). \end{aligned}$$

is an isomorphism.

Proof. We have that for a point $q \in \Gamma(\theta)$ it holds that $\lambda([H_q^2]) = \check{H}_a \check{H}_b$ where $(a,b) \in \mathrm{I}_\theta$. The claim is equivalent to show that the quadrics $\check{H}_a \check{H}_b$ where $(a,b) \in \mathrm{I}_\theta$ generates the space of the quadrics of $\check{\mathbb{P}}^{g-1}$. By contradiction assume that this is not the case. This means that by the pairing $(Q_1, Q_2) \mapsto \langle Q_1, \lambda(Q_2)\rangle = P_{\lambda(Q_2)}(Q_1)$ there exists a quadric Q of $\mathbb{P}^{g-1}$ such that $\langle Q, \lambda(H_q^2\rangle = \langle Q, \check{H}_a \check{H}_b\rangle = 0$ for every $(a,b) \in \mathrm{I}_\theta$. Now fix $a \in \Gamma$. Consider $\mathrm{I}_\theta(a) = a_1 + \cdots + a_g$. Then if H is the polar hyperplane of a with respect to Q then $a_i \in H$ for every $i = 1, \dots, g$. This means that the span of $\mathrm{I}_\theta(a)$ is a hyperplane, hence $|K_\Gamma - \theta - a| \neq \emptyset$: a contradiction. □

Remark 2. The inverse $\lambda^{-1}\colon S^2V \to S^2\check{V}$ defines an element $\check{\mathcal{D}}_2 \in S^2\check{V} \otimes S^2\check{V}$. We consider the polarization map $\mathrm{pl}_2\colon S^2\check{V} \to \mathrm{Sym}_2 V$. Set $\widetilde{U} := \mathrm{pl}_2 \otimes \mathrm{pl}_2(\check{\mathcal{D}}_2) \in \mathrm{Sym}_2 V \otimes \mathrm{Sym}_2 V \subset \check{V}^{\otimes 4}$. In the next Proposition 2.10 we will show that $\widetilde{U}$ is contained in $\mathrm{Sym}_4 V$.

We need to clarify a subtle point. The isomorphism $\lambda\colon S^2\check{V} \to S^2V$ sends the square of a point $\Gamma(\theta) \ni [H_q]$ to the quadric $\check{H}_a\check{H}_b$ where $(a,b) \in \mathrm{I}_\theta$ and $\pi_\theta((a,b)) = q = [H_q]$. In the proof of proposition 2.10 we will see that if $x \in \Gamma$ is a general point and $\mathrm{I}_\theta(x) = x_1 + \cdots + x_g$ then letting $q_1 = \pi_\theta(x, x_1)$, $q_2 = \pi_\theta(x, x_2), \dots$, $q_g = \pi_\theta(x, x_g)$ it holds that $\check{\mathbb{P}}^{g-1}$ coincides with the span $\langle q_1, q_2, \dots, q_g\rangle$. By our geometric description of the Scorza quartic associated to (Γ, θ) where Γ is a trigonal curve we will see that the above fact is geometrically evident in the case of trigonal curves.

Proposition 2.10. *The tensor* $\widetilde{U} := \mathrm{pl}_2 \otimes \mathrm{pl}_2(\check{\mathcal{D}}_2) \in \mathrm{Sym}_2 V \otimes \mathrm{Sym}_2 V \subset \check{V}^{\otimes 4}$ *is contained in* $\mathrm{Sym}_4 V$. *In particular* $\widetilde{U}$ *is the image of a quartic form* $\in S^4\check{V}$ *by* pl_4.

Proof. Our argument is almost identical with the one by Dolgachev and Kanev; see: [3, Theorem 9.3.1].

Let (Γ,θ) be a canonical curve and $x\in\Gamma$ a general point in it. Set $\Gamma(\theta)\ni q_i:=\pi_\theta(x,x_i)$, $i=1,\dots,g$ where $\mathrm{I}_\theta(x)=x_1+\cdots+x_g$. We point out the reader that only for this proof we denote by $H_q\in\check{V}$ an equation associated to the hyperplane inside $\mathbb{P}^{g-1}$ which correspond to the point $q\in\check{\mathbb{P}}^{g-1}$ and we denote by $\check{H}_x\in V$ an equation associated to the hyperplane inside $\check{\mathbb{P}}^{g-1}$ which correspond to the point $x\in\mathbb{P}^{g-1}$; that is, the notation $\check{H}_x(q)\neq 0$ means that the point q is not in the hyperplane with equation $\check{H}_x$ and $\check{H}_x(q)=0$ means that the point q belongs to the hyperplane with equation $\check{H}_x$. The same holds for the dual notation. By of Proposition 2.9 it holds that

$$\lambda(H^2_{q_i})=c_i\cdot\check{H}_x\check{H}_{x_i},\qquad c_i\in\mathbb{C}^\star,\quad i=1,\dots,g\,.$$

Note that $\check{H}_{x_i}(q_i)\neq 0$, $i=1,\dots,g$ since $x_i\notin\langle \mathrm{I}_\theta(x)-x_i\rangle$. Moreover by the same argument used in the proof of Proposition 2.8 (1) and (3) it also holds that $\check{H}_{x_i}(q_j)=0$ where $i\neq j$, $i,j=1,\dots,g$. In other words, it holds $\langle\check{H}_{x_i},H_{q_i}\rangle\neq 0$ and $\langle\check{H}_{x_i},H_{q_j}\rangle=0$ for $i\neq j$, where $\langle,\rangle$ is now the natural dual pairing. Since it is easy to show that $\mathbb{P}^{g-1}=\langle x_1,\dots,x_g\rangle$ we easily see that the linear forms $\check{H}_{x_i}$, $i=1,\dots,g$ give a basis and then $\{H_{q_i}\}_{i=1}^g$ gives a basis for the vector space of linear forms of V, that is $\langle q_1,q_2,\dots,q_g\rangle$ coincides with $\check{\mathbb{P}}^{g-1}$. More precisely, not only $\check{H}_{x_1},\dots,\check{H}_{x_g}$ and $H_{q_1},\dots,H_{q_g}$ span $\check{V}$ and V, respectively but $\{H_{q_i}\}_{i=1}^g$ and $\{\check{H}_{x_i}\}_{i=1}^g$ can be taken dual to each other. Choose coordinates of V and $\check{V}$ such that H_{q_i} and $\check{H}_{x_i}$ are coordinate hyperplanes $\{x_i=0\}$ and $\{u_i=0\}$ respectively. Set $L=\sum_{i=1}^g a_iu_i$ for the point $\check{H}_x$. For any $y=(y_1,\dots,y_g)\in V$, we have $\lambda(\sum y_ix_i^2)=(\sum a_iu_i)(\sum y_iu_i)$ since $\lambda(x_i^2)=u_iL$. We consider now $\lambda^{-1}\colon S^2V\to S^2\check{V}$. We have seen that it defines an element $\check{\mathcal{D}}_2\in S^2\check{V}\otimes S^2\check{V}$ and by the polarization map $\mathrm{pl}_2\colon S^2\check{V}\to\mathrm{Sym}_2V$ we construct $\widetilde{U}:=\mathrm{pl}_2\otimes\mathrm{pl}_2(\check{\mathcal{D}}_2)\in\mathrm{Sym}_2V\otimes\mathrm{Sym}_2V\subset\check{V}^{\otimes 4}$. By considering $\widetilde{U}\in\check{V}^{\otimes 4}$, we can write: $\widetilde{U}(L,y,x,x)=\sum y_ix_i^2=P_y(\frac{1}{3}\sum x_i^3)$, where $x=(x_1,x_2,\dots,x_g)$ and P_y is the polar with respect to y. Thus we have $\widetilde{U}(L,y,x,z)=\sum y_ix_iz_i$ for $z=(z_1,z_2,\dots,z_g)$, hence $\widetilde{U}(L,y,x,z)$ is symmetric for y, x and z. Since $\widetilde{U}\in\mathrm{Sym}_2\check{V}\otimes\mathrm{Sym}_2\check{V}$ and $\widetilde{\mathcal{D}}_2$ is symmetric, we have shown that $\widetilde{U}\in\mathrm{Sym}_4\check{V}$. $\square$

By Proposition 2.10 there exists a quartic $\{F_4=0\}$ in $\mathbb{P}^{g-1}$ associated to $\widetilde{U}$, namely, $F_4:=\widetilde{U}(x,x,x,x)$.

By the construction, we obtain that the double polarity with respect to F_4 gives back the inverse $\lambda^{-1}\colon S^2V\to S^2\check{V}$. In other words F_4 satisfies a rather important property to study ineffective theta characteristics:

Proposition 2.11. *Let $(\Gamma,\theta)\in S_g^+$ satisfying the assumptions* (A1), (A2) *and* (A3)*. Let $\{F_4=0\}\subset\mathbb{P}^{g-1}$ be the quartic constructed in Proposition 2.10. Let*

$$\mathrm{ap}^{(2)}_{F_4}\colon S^2V\to S^2\check{V},$$

$$\mathrm{ap}^{(2)}_{F_4}\colon \Phi \mapsto P_\Phi(F_4),$$

be the second polarisation homomorphism associated to F_4. Then for any $(x, y) \in \mathrm{I}_\theta$ it holds that

$$P_{\check{H}_x \check{H}_y}(F_4) = H_q^2 \tag{2.5}$$

where $q = \pi_\theta(x, y)$. Moreover F_4 is a non degenerate quartic in the sense of [3, Definition 2.8].

Proof. By the theory of polarity we can interpret what we have done as follows: $\lambda^{-1} = \mathrm{ap}^2_{F_4}$. Since λ^{-1} is an isomorphism, F_4 is non-degenerate. □

The quartics obtained from elements $(\Gamma, \theta) \in S_g^+$ deserve a name because it is trivial to show that if $g > 3$ then they are very special in the space of all the quartics and because in the case where $g = 3$ a rich geometry is attached to their construction.

Definition 2.12. The Scorza quartic of the pair (Γ, θ) is the quartic $\{F_4 = 0\}$ in $\mathbb{P}^{g-1} = \mathbb{P}(V)$ such that $\mathrm{ap}^2_{F_4}\colon S^2V \to S^2\check{V}$ satisfies the Equation (2.5).

Note that by construction we have a quartic $\{\check{F}_4 = 0\} \subset \check{\mathbb{P}}^{g-1}$ which is induced by the restriction to the diagonal of a symmetric $(2, 2)$ form of $\check{\mathbb{P}}^{g-1} \times \check{\mathbb{P}}^{g-1}$. Clearly once we have the Scorza quartic $\{F_4 = 0\} \subset \mathbb{P}^{g-1} = \mathbb{P}(V)$ by the inverse of $\mathrm{ap}^2_{F_4}\colon S^2V \to S^2\check{V}$ we reconstruct the isomorphism $\lambda\colon S^2\check{V} \to S^2V$ of Proposition 2.9 and from it we obtain the $(2, 2)$ form $\mathcal{D}'$ whose restriction to the diagonal is $\{\check{F}_4 = 0\} \subset \check{\mathbb{P}}^{g-1}$. Even if, in other contexts, this can cause a misunderstanding we choose to call $\{\check{F}_4 = 0\} \subset \check{\mathbb{P}}^{g-1}$ the dual of the Scorza quartic.

We can summarises the above discussion in the following theorem.

Theorem 2.13. *Let (Γ, θ) be a pair satisfying the assumptions* (A1)*,* (A2) *and* (A3)*. If the correspondence $\mathcal{D}$ is the restriction of a symmetric $(2, 2)$ divisor $\{\mathcal{D}' = 0\}$ of $\check{\mathbb{P}}^{g-1} \times \check{\mathbb{P}}^{g-1}$, the Scorza quartic of (Γ, θ) exists.*

2.4 On Dolgachev and Kanev construction of Scorza quartics

In their proof (see [3, pag 296-298]) about the conditional existence of the Scorza quartic, Dolgachev and Kanev require I_θ to be only reduced. Actually we will see below that I_θ is smooth for a general pair (Γ, θ) since, by our geometrical reconstruction of trigonal spin curves, I_θ is smooth for a general pair $(\mathcal{H}, \theta)$ where $\mathcal{H}$ is trigonal. Moreover we have presented a slight different definition of Scorza quartic.

Indeed for their definition of Scorza quartic they need the following proposition whose proof is on [3, page 296].

Proposition 2.14. *Let S be the set of hyperplanes H in $\mathbb{P}^{g-1}$ such that $H \cap \Gamma$ is reduced. The divisors $\overline{D}_H$ $(H \in S)$ span a linear system L such that*

$$L = \mathrm{Im}\left(|\mathcal{O}_{\check{\mathbb{P}}^{g-1}}(2)| \to |\mathcal{O}_{\Gamma(\theta)}(2)|\right).$$

Proof. First Step. By Proposition 2.8 if H is a common face of two non-degenerate polyhedra $\mathrm{I}_\theta(a)$ and $\mathrm{I}_\theta(b)$ then $\overline{D}_H = (\check{H}_a \cup \check{H}_b) \cap \Gamma(\theta)$.

Second Step. Let U be the Zariski open subset of $\check{\mathbb{P}}^{g-1}$ consisting of hyperplanes such that $p^*(H \cap \Gamma)$ contains neither singular points of I_θ nor ramification points of the map p. Clearly U is not empty.

Let $Z \subset \mathrm{Div}^{2g(g-1)}\big(\Gamma(\theta)\big)$ be the variety of divisors of degree $2g(g-1)$ on $\Gamma(\theta)$ with support outside of $\mathrm{Sing}\,\Gamma(\theta)$. If $H \in U$, then $[\overline{D}_H] \in Z$ since π is étale outside $\mathrm{Sing}\,\mathrm{I}_\theta$. The subvariety $W \subset Z$ of the class of divisors $\overline{D}_H$ with $H \in U$ is unirational since it is dominated by $\check{\mathbb{P}}^{g-1}$. We have denoted by L the linear system spanned by W. The linear system L has no base point since divisors $\overline{D}_H$ have no point in common.

Third Step. Set $\widehat{L} := \mathrm{Im}\,(|\mathcal{O}_{\check{\mathbb{P}}^{g-1}}(2)| \to |\mathcal{O}_{\Gamma(\theta)}(2)|)$. Now we show $L = \widehat{L}$.

By the first step, the linear system $L \cap |\mathcal{O}_{\Gamma(\theta)}(2)|$ contains the divisors $\overline{D}_H$ ($[H] \in \Gamma(\theta)$) since they are the restrictions of $\check{H}_a\check{H}_b$ for $(a,b) \in I_\theta$.

By the proof of Proposition 2.9 the quadrics $\check{H}_a\check{H}_b$ spans the space of quadrics in $\check{\mathbb{P}}^{g-1}$. Thus by the assumption (A2), we have

$$\dim L \cap |\mathcal{O}_{\Gamma(\theta)}(2)| \geq \dim |\mathcal{O}_{\check{\mathbb{P}}^{g-1}}(2)| = \dim \widehat{L}\,.$$

We show the inequality in the other direction. Consider the following map

$$\begin{aligned} f\colon \check{\mathbb{P}}^{g-1} &\to L. \\ [H] &\mapsto \overline{D}_H \end{aligned} \tag{2.6}$$

Fix a point $[H_0] \in \Gamma(\theta)$ and let $L_0 < L$ be the hyperplane of L which consists of the members containing a point $[H_0] \in \Gamma(\theta)$. We can write $H_0 = \langle I_\theta(a) - b\rangle$ as a hyperplane of $\mathbb{P}^{g-1}$. We show that $f^{-1}L_0 = \check{H}_a\check{H}_b$. Indeed, if $[H] \in \check{H}_a\check{H}_b$, equivalently, $a \in H$ or $b \in H$, then clearly $[H_0] \in \overline{D}_H$. Conversely, if $[H] \notin \check{H}_a\check{H}_b$, then $[H_0] \notin \overline{D}_H$ by Proposition 2.8 $\deg \Gamma(\theta) = g(g-1)$. Thus the inverse image of any hyperplane of L by f is a quadric of $\check{\mathbb{P}}^{g-1}$. The induced map $\check{L} \to |\check{\mathbb{P}}^{g-1}(2)|$ is clearly injective. This implies that $\dim L \leq \dim |\mathcal{O}_{\check{\mathbb{P}}^{g-1}}(2)|$. Consequently, we have $L = \widehat{L}$. $\square$

Remark 3. In the argument of the above Proposition 2.14 is not clear why we can restrict onto U since we need that the map $\check{\mathbb{P}}^{g-1} \to L$, $[H] \mapsto \overline{D}_H$ is a morphism rather than a rational map. For this, we have to assume that I_θ is *smooth* rather than reduced. If so, we can pull back $\Gamma \cap H$ to I_θ as a divisor for *any* H. Since $\mathrm{I}_\theta \to \Gamma(\theta)$ is étale (here we need to assume that $\mathrm{I}_\theta/\langle\tau\rangle \simeq \Gamma(\theta)$; a condition which holds true for a general spin since it is so for general $(\mathcal{H}_1, \theta)$), $\Gamma(\theta)$ is also smooth, thus we can push forward $p^*(\mathcal{H} \cap H)$ to $\Gamma(\theta)$ as a divisor. Since $\pi_*p^*(\mathcal{H} \cap H)$ is a quadric section for a special H as above, so is for any H without using the unirationality argument. Theorem 2.13 is shown in a slightly different manner in [4, Theorem 5.5.17]. In any case which are the most general conditions to have Scorza quartics is still an open problem; this is why we have presented here some comments on the Dolgachev and Kanev construction.

3 Scorza's quartics and rational curves on the del Pezzo 3-fold.

To show the existence of the Scorza quartic for a general pair (Γ, θ) we show first its existence in the particular case of a pair (Γ, θ) where Γ is a general trigonal curve. Then the existence for the general couple (Γ, θ) follows by a general argument on the moduli space of genus g curves; see Theorem 3.23 below.

In [21, 22] it is showed that there is a rich geometry associated to (Γ, θ) if Γ is trigonal, and it is exactly because of this geometry that we can explicitly describe the discriminant locus $\Gamma(\theta)$ and then show the theorem. We stress that the foundational paper is [22], but here we can review only some results of [21].

3.1 Lines on the quintic del Pezzo's threefold

Let B be the smooth quintic del Pezzo threefold, that is B is a smooth projective threefold such that $-K_B = 2H$, where H is the ample generator of $\operatorname{Pic} B$ and $H^3 = 5$. It is well known that the linear system $|H|$ embeds B into $\mathbb{P}^6$ and this image of B can be seen as $B = G(2,5) \cap \mathbb{P}^6$, where $\mathbb{P}^6 \subset \mathbb{P}^9$ is transversal to the embedded Grassmannian of the 2-dimensional vector subspaces of a 5-dimensional vector space; see: [6] and [10, Thm 4.2 (iii), the proof p.511-p.514].

Let $\pi\colon \mathbb{P} \to \mathcal{H}^B_1$ be the universal family of lines on B and let $\varphi\colon \mathbb{P} \to B$ be the natural projection. By [5, Lemma 2.3 and Theorem I], $\mathcal{H}^B_1$ is isomorphic to $\mathbb{P}^2$ and φ is a finite morphism of degree three. In particular the number of lines passing through a point is three counted with multiplicities.

Denote by $M(C)$ the locus $\subset \mathbb{P}^2$ of lines intersecting an irreducible curve C on B, namely, $M(C) := \pi(\varphi^{-1}(C))$ with reduced structure. Since φ is flat, $\varphi^{-1}(C)$ is purely one-dimensional. If $\deg C \geq 2$, then $\varphi^{-1}(C)$ does not contain a fiber of π, thus $M(C)$ is a curve.

A line l on B is called a *special line* if $\mathcal{N}_{l/B} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Note that $\mathcal{N}_{l/B} = \mathcal{O}_l \oplus \mathcal{O}_l$ if l is not a special line on B.

Proposition 3.1. *It holds*:

(1) *Special lines are parameterised by a conic Ω on $\mathcal{H}^B_1$,*

(2) *if l is a special line, then $M(l)$ is the tangent line to Ω at $[l]$. If l is not a special line, then $\varphi^{-1}(l)$ is the disjoint union of the fiber of π corresponding to l and the smooth rational curve dominating a line on $\mathbb{P}^2$. In particular, $M(l)$ is the disjoint union of a line and the point $[l]$.* By abuse of notation, *we denote by $M(l)$ the one-dimensional part of $M(l)$ for any line l. Vice-versa, any line in $\mathcal{H}^B_1$ is of the form $M(l)$ for some line l, and*

(3) *the locus swept by lines intersecting l is a hyperplane section T_l of B whose singular locus is l. For every point b of $T_l \setminus l$, there exists exactly one line which belongs to $M(l)$ and passes through b. Moreover, if l is not special, then the normalisation of T_l is $\mathbb{F}_1$ and the inverse image of the singular locus is the*

negative section of $\mathbb{F}_1$, or, if l is special, then the normalisation of T_l is $\mathbb{F}_3$ and the inverse image of the singular locus is the union of the negative section and a fiber.

Proof. See [5, §2] and [9, §1]. □

3.2 Smooth rational curves on the del Pezzo's threefold

In [22, 2.2 and 2.3], we constructed a smooth rational curve C_d on B of degree d by smoothing the union of a smooth rational curve C_{d-1} of degree $d-1$ and a general uni-secant line of C_{d-1} which lives on B.

Let $\mathcal{H}_d^B$ be the Hilbert scheme of general smooth rational curves of degree d on B obtained inductively as smoothing of unions of general smooth rational curves of degree $d-1$ on B and their general uni-secant lines according the smoothing process described in [8]. In fact by [23, Proposition 3.7], $\mathcal{H}_d^B$ is irreducible.

A general C_d belonging to $\mathcal{H}_d^B$ has the following several nice properties:

Proposition 3.2. (1) $\mathcal{N}_{C_d/B} \simeq \mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)$. *In particular* $h^1(\mathcal{N}_{C_d/B}) = 0$ *and* $h^0(\mathcal{N}_{C_d/B}) = 2d$,

(2) *there exist no k-secant lines of C_d on B with $k \geq 3$,*

(3) *there exist at most finitely many bi-secant lines of C_d on B, any of them intersects C_d simply, and they are mutually disjoint,*

(4) *neither a bi-secant line nor a line through the intersection point between a bi-secant line and C_d is a special line,*

(5) *$M(C_d)$ intersects Ω simply,*

(6) *$M(C_d)$ is an irreducible curve of degree d with only simple nodes (recall that we abuse the notation by denoting the one-dimensional part of $\pi(\varphi^{-1}(C_1))$ by $M(C_1)$), and*

(7) *by letting l be a general line intersecting C_d or any bi-secant line of C_d, $M(C_d) \cup M(l)$ has only simple nodes as its singularities.*

Proof. See [22, Propositions 2.2.2, 2.3.1, 2.3.3 and 2.4.4]. □

We have similar results for the conics which are multisecants to C_d where $[C_d] \in \mathcal{H}_d^B$. Its proof is similar to the proof of Proposition 3.2.

Proposition 3.3. *A general C_d as in Proposition 3.2 satisfies the following conditions:*

(1) *there exist no k-secant conics of C_d with $k \geq 5$,*

(2) *there exist at most finitely many quadri-secant conics of* C_d *on* B, *and no quadri-secant conic is tangent to* C_d, *and*

(3) $q_{|C_d}$ *has no point of multiplicity greater than two for any multi-secant conic* q.

Proof. See [22, Propositions 2.3.4.]. □

Finally we want to stress that we have a clear picture of degenerates conics: see [22, Proposition 4.2.6].

3.3 Lines and conics on certain blow-ups of the del Pezzo's threefold

Let C be a smooth rational curve of degree d on B, where d is an arbitrary integer greater than or equal to 6, as in Proposition 3.2. Let $f\colon A \to B$ be the blow-up along C and E_C the f-exceptional divisor. We define:

Definition 3.4. A connected curve $l \subset A$ is called a *line* on A if $-K_A \cdot l = 1$ and $E_C \cdot l = 1$.

We point out that since $-K_A = f^*(-K_B) - E_C$ and $E_C \cdot l = 1$ then $f(l)$ is a line on B *intersecting* C. The classification of lines on A is simple:

Proposition 3.5. *A line* l *on* A *is one of the following curves on* A :

(i) *the strict transform of a uni-secant line of* C *on* B, *or*

(ii) *the union* $l_{ij} = \beta'_i \cup \zeta_{ij}$ $(i = 1, \dots, s,\ j = 1, 2)$, *where* β'_i *is a bi-secant line* β_i *of* C *and* ζ_{ij} *is the fiber of* E_C *over a point in* $C \cap \beta_i$.

In particular l *is reduced and* $p_a(l) = 0$.

Proposition 3.6. *The Hilbert scheme of lines on* A *is a smooth trigonal curve* $\mathcal{H}_1$ *of genus* $d - 2$.

Proof. See [22, Corollary 4.18]. □

Definition 3.7. We say that a connected and reduced curve $q \subset A$ is a *conic* on A if $-K_A \cdot q = 2$ and $E_C \cdot q = 2$.

In [22, Corollary 4.2.10] it is showed that the Hilbert scheme of conics on A is an irreducible surface and the normalisation morphism is injective, namely, the normalisation $\mathcal{H}_2$ parameterises conics on A in one to one way.

Moreover we have the full description of $\mathcal{H}_2$ as follows ([22, Theorem 4.2.11]). For this, let $D_l \subset \mathcal{H}_2$ be the locus parameterising conics on A which intersect a fixed line l on A.

Theorem 3.8. *The normalization $\mathcal{H}_2$ of the Hilbert scheme of conics of A is smooth and is a so-called* White surface *obtained by blowing up $S^2C \simeq \mathbb{P}^2$ at $s := \binom{d-2}{2}$ points. The locus D_l is a divisor linearly equivalent to $(d-3)h - \sum_{i=1}^{s} e_i$ on $\mathcal{H}_2$, where h is the pull-back of a line, e_i are the exceptional curves of $\mathcal{H}_2 \to \mathbb{P}^2$, and $|D_l|$ embeds $\mathcal{H}_2$ into $\check{\mathbb{P}}^{d-3}$. The scheme $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$ is projectively Cohen-Macaulay, equivalently, $h^i(\check{\mathbb{P}}^{d-3}, \mathcal{I}_{\mathcal{H}_2}(j)) = 0$ for $i = 1, 2$ and $j \in \mathbb{Z}$, where $\mathcal{I}_{\mathcal{H}_2}$ is the ideal sheaf of $\mathcal{H}_2$ in $\check{\mathbb{P}}^{d-3}$. Moreover, $\mathcal{H}_2$ is given by intersection of cubics.*

Here we use the notation $\check{\mathbb{P}}^{d-3}$ since the ambient projective space of $\mathcal{H}_2$ and that of the canonical embedding of $\mathcal{H}_1$ can be considered as reciprocally dual (see the line-conic duality described in the following subsection 3.5); hence we write the ambient of $\mathcal{H}_1$ by $\mathbb{P}^{d-3}$ and that of $\mathcal{H}_2$ by $\check{\mathbb{P}}^{d-3}$.

3.4 Scorza correspondence for trigonal curves

We are going to attach a pair $(\mathcal{H}_1, \theta)$ to the blow-up $f \colon A \to B$ along the rational curve C. In particular $\mathcal{H}_1$ is a trigonal curve but we are able to give a scheme theoretic definition of the correspondence I_θ which is easily geometrical reflected into the geometry of lines and conics of the 3-fold A. This will make possible to show that for $(\mathcal{H}_1, \theta)$ the discriminant locus $\Gamma(\theta)$ is smooth and the θ-correspondence I_θ is smooth too.

There is a natural morphism $\mathcal{H}_1 \to \mathcal{H}_1^B \simeq \mathbb{P}^2$ mapping the class of a line l on A to that of the image $\bar{l}$ of l on B. The image of $\mathcal{H}_1$ on $\mathcal{H}_1^B$ is nothing but $M := M(C)$ defined in 3.1, and $\mathcal{H}_1 \to M$ is the normalisation. By Proposition 3.2 (6), M has only nodes as its singularities. By Proposition 3.5, singularities of M correspond to bi-secant lines of C. Since $p_a(M) = \frac{(d-1)(d-2)}{2}$ and we have seen in Proposition 3.6 that $g(\mathcal{H}_1) = d-2$, the number of nodes of M, is $s := \frac{(d-2)(d-3)}{2}$. Hence $\frac{(d-2)(d-3)}{2}$ is also equal to the number of bi-secant lines of C.

We have shown:

Proposition 3.9. *The number s of nodes of M is equal to the number of the bisecants of C, that is $s = \frac{(d-2)(d-3)}{2}$.*

Lemma 3.10. *For a general $[C] \in \mathcal{H}_B^d$ it holds that $h^0(\mathcal{H}_1, (\pi_{|\mathcal{H}_1})^* \mathcal{O}_M(1)) = 3$.*

Proof. Let $h \colon S \to \mathcal{H}_1^B \simeq \mathbb{P}^2$ be the blow-up of $\mathcal{H}_1^B$ at the $s = \binom{d-2}{2}$ nodes of M. Then $\mathcal{H}_1 \sim d\lambda - 2\sum_{i=1}^{s} \varepsilon_i$, where λ is the pull-back of a general line and ε_i are exceptional curves. By the exact sequence

$$0 \to \mathcal{O}_S(\lambda - \mathcal{H}_1) \to \mathcal{O}_S(\lambda) \to (\pi_{|\mathcal{H}_1})^* \mathcal{O}_M(1) \to 0$$

together with $h^0(\mathcal{O}_S(\lambda)) = 3$ and $h^0(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = h^1(\mathcal{O}_S(\lambda)) = 0$, we see that to have $h^0(\mathcal{H}_1, (\pi_{|\mathcal{H}_1})^* \mathcal{O}_M(1)) = 3$ is equivalent to have $h^1(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$. By the Riemann-Roch theorem, we have $\chi(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$. Thus by $h^0(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$, $h^1(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$ is equivalent to $h^2(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$. By the Serre duality,

$h^2(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = h^0(\mathcal{O}_S((d-4)\lambda - \sum_{i=1}^{s} \varepsilon_i)$. Thus we have only to prove that there exists no plane curve of degree $d-4$ through s nodes of M. We prove this fact by using the inductive construction of $C = C_d$. By subsection 3.2 we know that C_{d+1} is obtained as the smoothing of the union of C_d and a general uni-secant line $\bar{l}$ of C_d. From now on in the proof, we put the suffix d to the object depending on d. For example, $s_d := \binom{d-2}{2}$. If $d = 1$, the assertion is obvious. Assuming $h^0(\mathcal{O}_{S_d}((d-4)\lambda_d - \sum_{i=1}^{s_d} \varepsilon_{i,d}) = 0$, we prove $h^0(\mathcal{O}_{S_{d+1}}((d-3)\lambda_{d+1} - \sum_{i=1}^{s_{d+1}} \varepsilon_{i,d+1}) = 0$. By a standard degeneration argument, we have only to prove that there exists no plane curve of degree $d-3$ through s_{d+1} nodes of $M_d \cup M(\bar{l})$, where s_d of s_{d+1} nodes are those of M_d and the remaining $s_{d+1} - s_d = d-2$ nodes are $M_d \cap M(\bar{l})$ except the two points corresponding to the two other lines $\bar{l}'$, $\bar{l}''$ through $C_d \cap \bar{l}$. Assume that there exists a plane curve G of degree $d-3$ through s_{d+1} nodes of $M_d \cup M(\bar{l})$. Then $G \cap M(\bar{l})$ contains at least $d-2$ points. Since $\deg G = d-3$, this implies $M(\bar{l}) \subset G$. Thus there exists a plane curve of degree $d-4$ through s_d nodes of M_d, a contradiction. □

3.4.1 Schematic definition of I_θ for trigonal curves

We denote by δ the g^1_3 on $\mathcal{H}_1$ which defines the trigonal morphism $\varphi_{|\mathcal{H}_1} \colon \mathcal{H}_1 \to C$.

Let l, l' and l'' be three lines on A such that $[l] + [l'] + [l''] \sim \delta$. Then $\bar{l}$, $\bar{l}'$ and $\bar{l}''$ are lines through one point of C. Set $\lambda_{\mathcal{H}_1}$ to be an effective divisor associated to the line bundle $(\pi_{|\mathcal{H}_1})^* \mathcal{O}_M(1)$. We define the following divisor:

$$\theta := \lambda_{\mathcal{H}_1} - \delta.$$

Note that for the moment we only know that $\deg \theta = d-3$. By definition of θ it holds that $\theta + [l] = \pi^*_{|\mathcal{H}_1} \mathcal{O}_M(1) - [l'] - [l'']$. By Lemma 3.10 we have $h^0(\mathcal{H}_1, \mathcal{O}_{\mathcal{H}_1}(\theta + [l])) = 1$.

Consider now the two natural projections $p_i \colon \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}_1$ $(i = 1, 2)$ and denote by Δ the diagonal of $\mathcal{H}_1 \times \mathcal{H}_1$. Set $\mathcal{L} := \mathcal{O}_{\mathcal{H}_1 \times \mathcal{H}_1}(p_2{}^*\theta + \Delta)$. By $h^0(\mathcal{H}_1, \mathcal{O}_{\mathcal{H}_1}(\theta + [l])) = 1$ for any $[l] \in \mathcal{H}_1$, we see that $p_{1*}\mathcal{L}$ is an invertible sheaf. Define an ideal sheaf $\mathcal{I}$ by $p_1{}^* p_{1*}\mathcal{L} = \mathcal{L} \otimes \mathcal{I}$. $\mathcal{I}$ is an invertible sheaf and let I be the divisor defined by $\mathcal{I}$. We will denote by $I([l])$ the fiber of $I \to \mathcal{H}_1$ over $[l]$. By definition, $I([l])$ consists of the points in the support of $|\theta + [l]|$. Since $\pi^*_{|\mathcal{H}_1} \mathcal{O}_M(1) - [l'] - [l'']$, they correspond to lines on B intersecting both C and $\bar{l}$ except $\overline{l'}$ and $\overline{l''}$. The number of them is at most $d-3$. By Proposition 3.2 (7), the number is actually $d-2$. Thus the fiber of $I \to \mathcal{H}_1$ over a general $[l]$ is reduced. Hence I is reduced.

In our setting we show the following generalization of Mukai's result [16, §4]:

Proposition 3.11. *The class of θ is an ineffective theta characteristic and $I = \mathrm{I}_\theta$.*

Proof. By Proposition 2.3 and the definition of I, it suffices to prove the following:

(a) $h^0(\mathcal{H}_1, \mathcal{O}_{\mathcal{H}_1}(\theta + [l])) = 1$ for any $[l] \in \mathcal{H}_1$,

(b) I is reduced,

(c) I is disjoint from the diagonal, equivalently, $([l],[m]) \in I$ implies $l \neq m$,

(d) I is symmetric, and

(e) I is a $(g(\mathcal{H}_1), g(\mathcal{H}_1))$-correspondence.

I have proved (a) and (b) already in the above discussion. (c) is equivalent to show that the support of $I([l])$ does not contain $[l]$. By definition $\theta + [l] = \pi^*_{|\mathcal{H}_1}\mathcal{O}_M(1) - [l'] - [l'']$. Two cases occur: $[\bar{l}] \in \Omega$ or $[\bar{l}] \notin \Omega$ where Ω is the conic of special lines introduced in Proposition 3.1. If $\bar{l}$ is special, then by Proposition 3.2 (4) it is uni-secant to C and by Proposition 3.2 (5) M is not tangent at $[\bar{l}]$ to the conic Ω. Hence we are done. If $\bar{l}$ is not special, then $M(\bar{l})$ does not contain $[\bar{l}]$, thus we are done.

We prove (d). Let m be a line on A such that $[m]$ is contained in the support of $I([l])$. It suffices to prove that for a general l, $[l]$ is contained in the support of $I([m])$. For a general l, we may assume that $m \neq l'$ or l''. Then it is easy to verify this fact.

Finally we prove (e). Since I is symmetric and $\deg(\theta + [l]) = d - 2 = g(\mathcal{H}_1)$, the divisor is a $(g(\mathcal{H}_1), g(\mathcal{H}_1))$-correspondence. By Proposition 2.3 there exists an ineffective theta characteristic θ' such that $I = I_{\theta'}$. On the general point $[l] \in \mathcal{H}_1$ it holds that $\mathrm{I}_\theta + [l] = \theta' + [l]$. Then $\theta = \theta'$ and θ is an ineffective theta characteristic. $\square$

From now on the general couple of trigonal curve and ineffective theta characteristic to which Proposition 3.11 applies is denoted by $(\mathcal{H}_1, \theta)$, while (Γ, θ) denote a general spin curve where Γ is not necessarily a trigonal curve.

3.5 Line-conic duality

We consider the embedding of $\mathcal{H}_1$ by the canonical linear system and the embedding of $\mathcal{H}_2$ by the linear system $|D_l|$, see Theorem 3.8. We show the ambient projective spaces of respectively $\mathcal{H}_1$ and $\mathcal{H}_2$ are reciprocally dual.

Convention 3.12. We denote by $\bar{l}$ the image in B of a line l on A.

Here we give a more precise definition of the divisor D_l which appear above in Theorem 3.8. Inside $\mathcal{H}_2 \times \mathcal{H}_1$, we can define the incidence loci:

$$\widehat{\mathcal{D}}_1 := \{([q],[l]) \in \mathcal{H}_2 \times \mathcal{H}_1 \mid q \cap l \neq \emptyset\}.$$

Definition 3.13. The divisorial part of $\widehat{\mathcal{D}}_1$ is denoted by $\mathcal{D}_1$, and the divisor $\mathcal{D}_1 \subset \mathcal{H}_2 \times \mathcal{H}_1$ is called the *line-conic correspondence*.

Since $\mathcal{H}_1$ is a smooth curve the morphism $\mathcal{D}_1 \to \mathcal{H}_1$ induced by the canonical projection is flat. Let D_l be the fiber of $\mathcal{D}_1 \to \mathcal{H}_1$ over $[l] \in \mathcal{H}_1$. Clearly we can write $D_l \hookrightarrow \mathcal{H}_2$.

The following result contains the nontrivial result that for a general $[l] \in \mathcal{H}_1$, D_l parameterises conics which properly intersect l.

Proposition 3.14. *Let $[l] \in \mathcal{H}_1$ be general. Then D_l does not contain any point corresponding to the line pairs $l \cup m$ with $[m] \in \mathcal{H}_1$, and hence D_l parameterises all conics which properly interesect l.*

Proof. See [22, Corollary 4.2.17]. □

Lemma 3.15. *The projection $\mathcal{D}_1 \to \mathcal{H}_2$ is finite and flat.*

Proof. Since $\mathcal{D}_1$ is a Cartier divisor in a smooth threefold $\mathcal{H}_1 \times \mathcal{H}_2$, $\mathcal{D}_1$ is Cohen-Macaulay. Since $M = M(C)$ is irreducible, no conic on A intersects infinitely many lines on A. Therfore $\mathcal{D}_1 \to \mathcal{H}_2$ is finite, hence $\mathcal{D}_1 \to \mathcal{H}_2$ is flat since $\mathcal{H}_2$ is smooth. □

Denote by $\widetilde{H}_q$ the fiber of the projection $\mathcal{D}_1 \to \mathcal{H}_2$ over $[q]$. For a general q, lines intersecting q are general. Thus, by Proposition 3.14, $\widetilde{H}_q$ parameterises all the lines intersecting a general q.

We remind the reader that general conics of A are parameterised by a general point $q \in \mathcal{H}_2$.

Lemma 3.16. *Let $q \in \mathcal{H}_2$ be a general point. Then $\widetilde{H}_q \in |\pi^*\mathcal{O}_M(2) - 2\delta|$, namely,*

$$\widetilde{H}_q \sim 2\theta \sim K_{\mathcal{H}_1}.$$

Proof. Since q is general, the image $\overline{q}$ of q is a bi-secant conic of C. Let $\bar{l}_i$ and $\overline{m}_j$ $(i = 1, 2, 3, j = 1, 2, 3)$ be the lines on B through each point of $C \cap \overline{q}$ respectively. Denote by l_i and m_j the lines on A corresponding to $\bar{l}_i$ and $\overline{m}_j$. Since q is general, lines l_i and m_j are also general. By definition of δ, we have $[l_1]+[l_2]+[l_3] \sim [m_1]+[m_2]+[m_3] \sim \delta$. The lines on A intersecting q come from lines on B intersecting C and $\overline{q}$ except $\bar{l}_i$ and $\overline{m}_j$ $(i = 1, 2, 3, j = 1, 2, 3)$. Therefore $\widetilde{H}_q \in |\pi^*\mathcal{O}_M(2) - 2\delta|$. □

By the flatness of $\mathcal{D}_1 \to \mathcal{H}_2$ showed in Lemma 3.15 it holds $\widetilde{H}_q \sim K_{\mathcal{H}_1}$ for any q.

By Theorem 3.8 D_l is a hyperplane section of $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$. Thus, using the universal property of the Hilbert scheme $\mathcal{H}_1$, the family

$$\begin{tikzcd} \mathcal{D}_1 \arrow[r] \arrow[d] & \mathcal{H}_2 \times \mathcal{H}_1 \arrow[dl] \\ \mathcal{H}_1 & \end{tikzcd}$$

induces the morphism

$$\begin{aligned}\mathcal{H}_1 &\to \mathbb{P}^{d-3}\\ [l] &\mapsto [D_l]\end{aligned}$$

where we stress that $\mathbb{P}^{d-3}$ is the dual projective space of $\check{\mathbb{P}}^{d-3}$. Since $D_l \neq D_{l'}$ for general $l \neq l'$, $\mathcal{H}_1 \to \mathbb{P}^{d-3}$ is birational. We denote by $\{H_q = 0\}$ the hyperplane in $\mathbb{P}^{d-3}$ corresponding to the point $[q] \in \check{\mathbb{P}}^{d-3}$. Note that by the definition of H_q it holds that for $[l] \in \mathcal{H}_1$ and $[q] \in \mathcal{H}_2$, $[D_l] \in \{H_q = 0\}$ if and only if $D_l([q]) = 0$. Thus $\widetilde{H}_q = \{H_q = 0\}$ for a general q. Consequently, by Lemma 3.16 and by the linear equivalence $\widetilde{H}_q \sim K_{\mathcal{H}_1}$ the morphism $\mathcal{H}_1 \to \mathbb{P}^{d-3}$ coincides with the canonical embedding $\Phi_{|K_{\mathcal{H}_1}|}\colon \mathcal{H}_1 \to \mathbb{P}^{d-3}$.

3.6 Discriminant locus of trigonal spin curves.

We consider $\mathcal{H}_1 \subset \mathbb{P}^{d-3}$ and $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$. For the pair $(\mathcal{H}_1, \theta)$, we can interpret $\Gamma(\theta)$ via the geometry of lines and conics on A:

Proposition 3.17. *For the pair $(\mathcal{H}_1, \theta)$, the discriminant locus $\Gamma(\theta)$ is contained in $\mathcal{H}_2$, and the generic point of the curve $\Gamma(\theta)$ parameterises line pairs on A.*

Proof. Take a general point $([l_1], [l_2]) \in I$, equivalently, take two general intersecting lines l_1 and l_2. The union $l_1 \cup l_2$ is a conic and the lines corresponding to the points of $I([l_1]) - [l_2]$ are lines intersecting l_1 except l_2. Thus by discussions in 3.5, the point in $\check{\mathbb{P}}^{d-3}$ corresponding to the hyperplane $\langle I([l_1]) - [l_2]\rangle$ is nothing but $[l_1 \cup l_2] \in \mathcal{H}_2$. This implies the assertion. □

Proposition 3.18. *We use the notation of Theorem 3.8. For the discriminant $\Gamma(\theta)$ of the pair $(\mathcal{H}_1, \theta)$ it holds that:*

$$\Gamma(\theta) \in \left|3(d-2)h - 4\sum_{i=1}^{s} e_i\right| .$$

In particular $\Gamma(\theta)$ is not contained in a cubic section of $\mathcal{H}_2$.

Proof. We consider a point b contained in the smooth rational curve $C \subset B$ such that $[C] \in \mathcal{H}^B_d$. Set

$$L_b := \overline{\{q \in \mathcal{H}_2 \mid \exists b' \neq b, f(q) \cap C = \{b, b'\}\}}\,.$$

By Theorem 3.8 we know that $\mathcal{H}_2$ is obtained by the blow-up $\eta\colon \mathcal{H}_2 \to S^2C = \mathbb{P}^2$ at the $s = \binom{d-2}{2}$ points which corresponds to the couple $e_i \cap C$ where $i = 1, \dots, s$ are the bisecants of C counted in Proposition 3.9. We first show that the image $\eta(L_b) \subset S^2C$ is a line. Choose $b' \in C$ such that no line on B exists through b and b'. By [22, Corollary 3.2.1] there exists a unique conic on B_5 through b and b'. This implies that $\eta(L_b)$ is a line.

We can write inside $\mathrm{Pic}(\mathcal{H}_2)$:

$$\Gamma(\theta) \sim ah - \sum m_i e_i,$$

where $a \in \mathbb{Z}$ and $m_i \in \mathbb{Z}$.

For a general $b \in C$, L_b intersects $\Gamma(\theta)$ simply. Thus a is the number of line pairs whose images on B pass through b. By noting there exists three lines l_1, l_2 and l_3 through b, it suffices to count the number of reducible conics on B having one of l_i's as a component except $l_1 \cup l_2$, $l_2 \cup l_3$ and $l_3 \cup l_1$. Thus $a = 3(d-2)$.

Now we count the number of line pairs belonging to e_i. Each of such line pairs is of the form $l_{ij;k} \cup l_{ij}$, where $l_{ij;k}$ $(k = 1, 2)$ is the strict transform of the line through one of the two points in $\beta_i \cap C$ distinct from β_i, and l_{ij} is defined in Proposition 3.5. Thus the number of such pairs is four, whence $m_i \geq 4$.

Finally we count the number of line pairs intersecting a general line l. By Proposition 3.14, D_l does not contain any line pair having l as a component. Since the number of lines on A intersecting a fixed line on A is $d-2$, we see that $D_l \cdot \Gamma(\theta) \geq (d-2)(d-3)$. Then

$$(d-2)(d-3) \leq \Gamma(\theta) \cdot D_l = (d-3)a - \sum_{i=1}^{s} m_i.$$

where $s = \frac{(d-2)(d-3)}{2}$. Since we have shown $m_i \geq 4$, this implies $m_i = 4$. □

We obtain that:

Corollary 3.19. *For $(\mathcal{H}_1, \theta)$, it holds that $\deg \Gamma(\theta) = g(g-1)$ and $p_a(\Gamma(\theta)) = \frac{3}{2}g(g-1)+1$. Moreover, $K_{\Gamma(\theta)} = \mathcal{O}_{\Gamma(\theta)}(3)$.*

Proof. The invariants of $\Gamma(\theta)$ are easily calculated by Proposition 3.18. □

Corollary 3.20. *The restriction map $H^0(\mathcal{O}_{\check{\mathbb{P}}^{d-3}}(2)) \to H^0(\mathcal{O}_{\Gamma(\theta)}(2))$ is an isomorphism.*

Proof. By Theorem 3.8, $H^0(\mathcal{O}_{\check{\mathbb{P}}^{d-3}}(2)) \to H^0(\mathcal{O}_{\mathcal{H}_2}(2))$ is an isomorphism. To see $H^0(\mathcal{O}_{\mathcal{H}_2}(2)) \to H^0(\mathcal{O}_{\Gamma(\theta)}(2))$ is an isomorphism, we have only to show that $H^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(2) \otimes \mathcal{O}_{\mathcal{H}_2}(-\Gamma(\theta))) = \{0\}$. By the Serre duality, the last cohomology group is isomorphic to $H^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(-2) \otimes \mathcal{O}_{\mathcal{H}_2}(K_{\mathcal{H}_2} + \Gamma(\theta))$, and moreover, by $K_{\mathcal{H}_2} + \Gamma(\theta) = \mathcal{O}_{\mathcal{H}_2}(3)$, it is isomorphic to $H^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(1))$, which vanishes by Theorem 3.8. □

3.7 Conditions (A1), (A2), (A3) are satisfied for a general spin curve

In this section we will use the geometries of the trigonal curve $\mathcal{H}_1$ (see Proposition 3.6) and of the White surface $\mathcal{H}_2$ (see Theorem 3.8), respectively to give an affirmative answer to the conjecture of Dolgachev and Kanev [3, Introduction p. 218] (see Theorem 3.23).

First we show that for our trigonal curve $\mathcal{H}_1$ and the ineffective theta characteristic θ defined by intersecting lines on A the above conditions hold.

Proposition 3.21. *$(\mathcal{H}_1, \theta)$ satisfies* (A1), (A2) *and* (A3).

Proof. (A1) The condition $d(\theta) = 1$ means that for general lines l and l' on A such that $([l], [l']) \in I$ the face $\langle I([l]) - [l'] \rangle$ belongs only to $I([l])$ and to $I([l'])$.

By contradiction assume that there exists a line m on A such that $m \neq l$, $m \neq l'$ and $\langle I([l]) - [l'] \rangle$ is a face of $I([m])$. Then some $d - 3$ points of $I([m])$ lie on the hyperplane $\langle I([l]) - [l'] \rangle$, equivalently, m intersects $d - 3$ lines on A corresponding to $d-3$ points of $I([l]) \cup I([l'])$ except l and l'. Since $d \geq 6$, it holds that, for l or l', say, l, there exist two lines intersecting both l and m.

Consider the projection $B \dashrightarrow Q$ from the line $f(l) = \bar{l}$. By [6] the target of the projection is the smooth quadric threefold Q and the projection is decomposed as follows:

$$\begin{tikzcd} & B_{\bar{l}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B & & Q, \end{tikzcd}$$

where π_1 is the blow-up along $\bar{l}$. Moreover, the image $E'_{\bar{l}}$ of the π_1-exceptional divisor $E_{\bar{l}}$ on Q is a hyperplane section.

Now notice that, by generality of l, $\bar{l} \neq \overline{m} := f(m)$ is equivalent to have $l \neq m$. Assume by contradiction that $\bar{l} \cap \overline{m} \neq \emptyset$. Then they span a plane P, which contains two lines intersecting both $\bar{l}$ and $\overline{m}$. This implies that $P \subset B$, but it is well known that B is the intersection of quadrics passing through it: a contradiction. Thus $\bar{l} \cap \overline{m} = \emptyset$, whence the strict transform $\overline{m}'$ of $\overline{m}$ on Q is a line. Since there exist two lines intersecting both $\bar{l}$ and $\overline{m}$, $\overline{m}'$ intersects the image $E'_{\bar{l}}$ of $E_{\bar{l}}$ at two points. Since $E'_{\bar{l}}$ is a hyperplane section on Q, this implies that $\overline{m}' \subset E'_{\bar{l}}$, a contradiction.
(A2) This condition is satisfied by Theorem 3.8 and Proposition 3.18.
(A3) By [3, Lemma 7.1.3], $([m_1], [m_2]) \in I$ is a singular point of I if and only if $|I([m_1]) - 2[m_2]| \neq \emptyset$ and $|I([m_2]) - 2[m_1]| \neq \emptyset$.

Let m be a line on A, and l_1 and l_2 two lines on A such that $\delta \sim [m]+[l_1]+[l_2]$. By definition of θ, $I([m]) \sim \theta + [m] \sim (\pi_{|\mathcal{H}_1})^* \mathcal{O}_M(1) - [l_1] - [l_2]$. Therefore $|I([m]) - 2[n]| \neq \emptyset$ if and only if one of the following holds:

(1) $[\overline{n}]$ is a smooth point of M. In this case, $\overline{n}$ is a uni-secant line of C. If $\overline{n} \neq \bar{l}_1$ nor $\bar{l}_2$, then $M(\overline{m})$ is tangent to M at $[\overline{n}]$. If $\overline{n} = \bar{l}_1$ or $\bar{l}_2$, then $M(\overline{m})$ is tangent to M at $[\overline{n}]$ with multiplicities three, or

(2) $[\overline{n}]$ is a singular point of M, which is a node. In this case, $\overline{n}$ is a bi-secant line of C. Correspondingly, there is another line n' on A, see proposition 3.5 (ii). The two branches of M at $[\overline{n}]$ correspond to n and n' respectively since

$\mathcal{H}_1 \to M$ is the normalisation. If $\overline{n} \neq \bar{l}_1$ nor $\bar{l}_2$, then $M(\overline{m})$ is tangent at $[\overline{n}]$ to the branch of M corresponding to n. If $\overline{n} = \bar{l}_1$ or $\bar{l}_2$, then $M(\overline{m})$ is tangent at $[\overline{n}]$ to the branch of M corresponding to n with multiplicity three.

Recall that, for a line $\bar{l}$ on B, we denote by $T_{\bar{l}}$ the hyperplane section swept out by lines intersecting $\bar{l}$ (Proposition 3.1 (3)). We can restate the above conditions as follows:

(1) If $\overline{n} \neq \bar{l}_1$ nor $\bar{l}_2$, then C is tangent to $T_{\overline{m}}$ at $C \cap \overline{n}$. Assume that $\overline{n} = \bar{l}_1$ or $\bar{l}_2$. If $\overline{n}$ is not a special line, then C is tangent at $C \cap \overline{n}$ with multiplicity three to the branch of $T_{\overline{m}}$ corresponding to $\overline{n}$. If $\overline{n}$ is a special line, then C intersects $T_{\overline{m}}$ at $C \cap \overline{n}$ with multiplicity three.

(2) Note that, by Proposition 3.5, n corresponds to one of a point p_n of $C \cap \overline{n}$. By Proposition 3.2 (4), $\overline{n}$ is not a special line. If $\overline{n} \neq \bar{l}_1$ nor $\bar{l}_2$, then C is tangent to $T_{\overline{m}}$ at p_n. If $\overline{n} = \bar{l}_1$ or $\bar{l}_2$, then C is tangent at p_n with multiplicity three to the branch of $T_{\overline{m}}$ corresponding to $\overline{n}$.

Bearing this in mind, we prove that I is smooth for a general C by simple dimension count. We only prove I is smooth at $([\overline{m}_1], [\overline{m}_2])$ with both $\overline{m}_1$ and $\overline{m}_2$ non-special. The remaining cases can be treated similarly. Let $\overline{m}_1$ and $\overline{m}_2$ be two intersecting non-special lines on B. We estimate the codimension in $\mathcal{H}_d^B$ of the locus $\mathcal{H}'$ of C such that C intersects both $\overline{m}_1$ and $\overline{m}_2$ and is tangent to both $T_{\overline{m}_1}$ and $T_{\overline{m}_2}$. By Proposition 3.2 (1), passing through one point is a codimension two condition. Moreover, being tangent to a surface along its smooth locus is a codimension one condition. The choice of two points one on $\overline{m}_1$ and the other on $\overline{m}_2$ has two parameters. Thus $\operatorname{codim} \mathcal{H}' = 4$. Since the choice of $\overline{m}_1$ and $\overline{m}_2$ has three parameters, we have the claim for a general C. □

For any spin curve $(\mathcal{H}, \theta)$ with ineffective θ, let

$$\Gamma'(\theta) := I_\theta/(\tau),$$

where τ is the involution on I_θ induced by that of $\mathcal{H} \times \mathcal{H}$ permuting the factors. Note that $I_\theta \to \Gamma(\theta)$ factor through $\Gamma'(\theta)$.

Corollary 3.22. *For a general pair $(\mathcal{H}_1, \theta)$ obtained by a general $[C] \in \mathcal{H}_d^B$ it holds that $\Gamma'(\theta) \simeq \Gamma(\theta)$. In particular, $\Gamma(\theta)$ is a smooth curve.*

Proof. By Proposition 3.21, (A1) and (A3) hold for $(\mathcal{H}_1, \theta)$. Thus we have $p_a(\Gamma'(\theta)) = \frac{3}{2}g(g-1)+1$ by [3, Corollary 7.1.7]. Thus $p_a(\Gamma'(\theta)) = p_a(\Gamma(\theta))$ by Corollary 3.19. By (A1) again, the natural morphism $\Gamma'(\theta) \to \Gamma(\theta)$ is birational. Therefore it holds $\Gamma'(\theta) \simeq \Gamma(\theta)$.

Since I is smooth, and I is disjoint from the diagonal, the map $I \to \Gamma'(\theta)$ is étale. Thus $\Gamma(\theta) \simeq \Gamma'(\theta)$ is a smooth curve. □

By a moduli theoretic argument we can obtain that a general pair (Γ, θ) satisfies the conditions (A1)–(A3) since they hold for a general pair $(\mathcal{H}_1, \theta)$ obtained by $\mathcal{H}_d^B$.

Theorem 3.23. *A general spin curve satisfies the conditions* (A1)–(A3).

Proof. It is known that the moduli space $\mathcal{S}_g^+$ of even spin curves of genus g is irreducible (see: [2]). Let U be a suitable finite cover of an open neighborhood of a general $[(\mathcal{H}_1, \theta)] \in \mathcal{S}_g^+$ such that there exists the family $\mathcal{C} \to U$ of pairs of canonical curves and ineffective theta characteristics. Denote by (Γ_u, θ_u) the fiber of $\mathcal{C} \to U$ over $u \in U$. By Proposition 3.21, $(\mathcal{H}_1, \theta)$ satisfies (A1)–(A3). Since the conditions (A1) and (A3) are open conditions, these are true on U. Thus we have only to prove that the condition (A2) is still true on U. Let $\mathcal{J} \to U$ be the family of Jacobians and $\Theta \to U$ the corresponding family of theta divisors. By [3, p.279-282], the family $\mathcal{I}$ of the Scorza correspondences embeds into Θ, and by the family of Gauss maps $\Theta \to \check{\mathbb{P}}^{g-1} \times U$, we can construct the family $\mathcal{G} \to U$ whose fiber $\mathcal{G}_u \subset \check{\mathbb{P}}^{g-1}$ is the discriminant $\Gamma_u(\theta_u)$. Set $\Gamma(\theta_u) := \Gamma_u(\theta_u)$, and $\Gamma'(\theta_u) := \Gamma'_u(\theta_u)$. By Corollary 3.22, it holds $\Gamma'(\theta) \simeq \Gamma(\theta)$ for $(\mathcal{H}_1, \theta)$. Thus up to shrink U we have also $\Gamma'(\theta_u) \simeq \Gamma(\theta_u)$ for $u \in U$. By [3, Corollary 7.1.7] we see that $p_a(\Gamma(\theta_u))$ and $\deg \Gamma(\theta_u)$ are constant for $u \in U$. Thus $\mathcal{G} \to U$ is a flat family since the Hilbert polynomials of fibers are constant. Since no quadrics contain $\Gamma(\theta)$ for $(\mathcal{H}_1, \theta)$, neither does $\Gamma(\theta_u)$ for $u \in U$ by the upper semi-continuity theorem. □

We have the following corollary of the proof of Theorem 3.23:

Corollary 3.24. *Let (Γ, θ) be a general pair of a canonical curve Γ and an ineffective theta characteristic θ.*

(1) *$\Gamma(\theta)$ is smooth.*

(2) *$\Gamma'(\theta) \simeq \Gamma(\theta)$.*

(3) *$K_{\Gamma(\theta)} = \mathcal{O}_{\Gamma(\theta)}(3)$.*

(4) *The restriction morphism $H^0(\mathcal{O}_{\check{\mathbb{P}}^{d-3}}(2)) \to H^0(\mathcal{O}_{\Gamma(\theta)}(2))$ is an isomorphism.*

Proof. (1) follows from (A3) for (Γ, θ). For the other, by the deformation theoretic argument in the proof of Theorem 3.23, we have only to show the assertion for a general $(\mathcal{H}_1, \theta)$ constructed from the incidence correspondence of lines on A. This is true by Corollaries 3.19, 3.20, and 3.22. □

3.8 Existence of the Scorza quartic

Now we see that the conditions to apply Theorem 2.13 holds for a general pair (Γ, θ)

Theorem 3.25. *The Scorza quartic exists for a general spin curve.*

Proof. Let (Γ, θ) be a general spin curve. By Theorem 3.23 it satisfies the conditions (A1)–(A3). Now consider the associated discriminant locus $\Gamma(\theta) \subset \check{\mathbb{P}}^{g-1}$ and the second Scorza correspondence

$$\mathcal{D} := \{(q_1, q_2) \mid q_1 \in \overline{D}_{H_{q_2}}\} \subset \Gamma(\theta) \times \Gamma(\theta) \,. \tag{3.1}$$

As in the proof of Theorem 3.23 and using the notation of its proof, by flatness we obtain that the map $H^0(\mathcal{O}_{\check{\mathbb{P}}^{d-3}}(2)) \to H^0(\mathcal{O}_{\Gamma(\theta)}(2))$ is an isomorphism if Γ is general. Then the space $H^0(\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}, \mathcal{O}_{\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}}(2,2))$ is isomorphic to $H^0(\Gamma(\theta) \times \Gamma(\theta), \mathcal{O}_{\Gamma(\theta) \times \Gamma(\theta)}(2,2))$. Then by Theorem 2.13 the claim follows. $\square$

3.8.1 Conic-conic duality

A problem posed by Igor Dolgachev is about a more explicit construction of Scorza's quartics. Indeed even if we have shown that the Scorza quartic exists for a general spin curve (Γ, θ) its construction is not explicit. For a general trigonal pair $(\mathcal{H}_1, \theta)$ obtained by $\mathcal{H}_d^B$ we have at least a very geometrical description of the dual (in our sense) of the Scorza quartic. More precisely consider the following correspondence:

$$\mathcal{D}_2 := \{([q_1], [q_2]) \in \mathcal{H}_2 \times \mathcal{H}_2 \mid q_1 \cap q_2 \neq \emptyset\} \tag{3.2}$$

and denote by D_q the fiber of $\mathcal{D}_2 \to \mathcal{H}_2$ over a point $[q]$. Then $D_q \sim 2D_l$ and it holds that $\mathcal{D}_2 \sim p_1^* D_q + p_2^* D_q$. In particular since $\mathcal{H}_2$ is not contained in a quadric, it holds:

$$H^0(\mathcal{H}_2 \times \mathcal{H}_2, \mathcal{D}_2) \simeq H^0(\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}, \mathcal{O}(2,2)) \,. \tag{3.3}$$

Thus $\mathcal{D}_2$ is the restriction of a unique $(2,2)$-divisor $\mathcal{D}_2'$ on $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$. Since $\mathcal{D}_2'$ is symmetric, we may assume its equation $\widetilde{\mathcal{D}}_2$ is also symmetric. The restriction of $\widetilde{\mathcal{D}}_2$ to the diagonal is a quartic hypersurface $\{\check{F}_4' = 0\}$ in $\check{\mathbb{P}}^{d-3}$. We showed that $\check{F}_4'$ is non-degenerate. Then the desired quartic is the unique quartic hypersurface $\{F_4' = 0\}$ in $\mathbb{P}^{d-3}$ dual to $\check{F}_4'$.

3.8.2 On the geometric construction of the Scorza quartics in the trigonal case.

The construction of $\check{F}_4'$ is quite similar to that of the Scorza quartic. Indeed they coincides.

Theorem 3.26. *The special quartic F_4' obtained by the conic-conic correspondence defined via the Equation (3.2) is the Scorza quartic for $(\mathcal{H}_1, \theta)$.*

Proof. We recall that by Theorem 3.8, $H^0(\mathcal{O}_{\check{\mathbb{P}}^{d-3}}(2)) \to H^0(\mathcal{O}_{\mathcal{H}_2}(2))$ is an isomorphism and that by Corollary 3.24 the restriction morphism $H^0(\mathcal{O}_{\check{\mathbb{P}}^{d-3}}(2)) \to H^0(\mathcal{O}_{\Gamma(\theta)}(2))$ is an isomorphism. Moreover we recall that the dual $\check{F}_4$ of the Scorza quartic is obtained by restricting $\check{\mathcal{D}}$ to the diagonal, where $\{\check{\mathcal{D}} = 0\}$ is the unique divisor on $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$ which restricts to the correspondence

$$\mathcal{D} := \{([q_1], [q_2]) \mid [q_1] \in \overline{D}_{H_{q_2}}\} \subset \Gamma(\theta) \times \Gamma(\theta).$$

On the other hand, the special quartic $\check{F}'_4$ is obtained by restricting $\widetilde{\mathcal{D}_2}$ to the diagonal where $\{\widetilde{\mathcal{D}_2} = 0\}$ is the divisor of $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$, which restricts to the conic-conic correspondence (3.2)

$$\mathcal{D}_2 := \{([q_1], [q_2]) \mid q_1 \cap q_2 \neq \emptyset\} \subset \mathcal{H}_2 \times \mathcal{H}_2 \,.$$

Therefore the assertion is equivalent to show $\overline{D}_{H_q} = \{\widetilde{D}_q = 0\} \cap \Gamma(\theta)$ for a general q. The set $\{\widetilde{D}_q = 0\} \cap \Gamma(\theta)$ consists of points corresponding to the line pairs on A intersecting q. By definition of $\overline{D}_{H_q}$, it is rather straightforward to see the set $\overline{D}_{H_q}$ also consists of points corresponding to the line pairs intersecting q. □

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