Rend. Mat. Appl. (7).
Volume 39, (2018), 29 – 78

Rendiconti di Matematica
e delle sue Applicazioni

Long time evolution of fluids with concentrated vorticity and convergence to the point-vortex model

Daniele Cetrone* and **Gabriele Serafini**

Abstract. *In this paper we study the evolution of vorticity in Navier-Stokes planar equations (with a small viscosity) and in Euler's axisymmetric tridimensional equations (with a large distance from the axis), when the initial vorticity is sharply concentrated around N points. We show that, in both cases, this evolution is close to the point-vortex dynamics for long times.*

1 Introduction

The point-vortex model is the dynamical system defined by the following differential equations:

$$\dot{z}_i(t) = \sum_{\substack{j=1 \\ j\neq i}}^{N} a_j K(z_i(t) - z_j(t)), \qquad z_i(0) = z_i \tag{1.1}$$

for $i = 1, \ldots, N$, with $a_i \in \mathbb{R}$, $z_i(t) \in \mathbb{R}^2$ and

$$K(x) = \nabla^{\perp} G(x), \qquad \nabla^{\perp} = (\partial_2, -\partial_1), \tag{1.2}$$

where $G(x) = -1/2\pi \log |x|$ is the fundamental solution of the Laplace operator in $\mathbb{R}^2$.

This model was first introduced by Helmholtz in order to study the motion of vortices in an incompressible, nonviscous, planar fluid, governed by the Euler's equations. Denote by $u(x,t) \in \mathbb{R}^2$ the velocity field of the fluid. The presence of a vortex corresponds to a high value of the vorticity $\omega := \partial_{x_1} u_2 - \partial_{x_2} u_1$; in fact the vorticity is an index of the rotation movement of the fluid. Euler's equations for vorticity in $\mathbb{R}^2$ read:

$$\begin{cases} \partial_t \omega(x,t) + (u \cdot \nabla)\omega(x,t) = 0 \\ \nabla \cdot u = 0 \\ |u(x,t)| \to 0 \ \text{ for } \ |x| \to \infty \end{cases} \tag{1.3}$$

2010 Mathematics Subject Classification: 76B47, 76D17, 37N10.
Keywords: Incompressible Euler flow with axisymmetry, incompressible viscous fluid, point vortex model, long time behaviour.

*Corresponding author.

Moreover, it is possible to reconstruct the velocity field u from ω by means of the formula

$$u(x,t) = \int K(x-y)\,\omega(y,t)\,dy\,. \tag{1.4}$$

It is quite interesting to study the evolution of the vorticity according to (1.3), when the initial datum is close (in the sense of weak convergence) to:

$$\omega_0 = \sum_{i=1}^{N} a_i \delta_{z_i} \tag{1.5}$$

If we try to apply (1.4) directly to this vorticity, we get a velocity field that diverges in each point z_i, $i = 1, \dots, N$. However our physical intuition suggests that, if $N = 1$, the only vortex should not move, i.e. we should have $u(z_1) = 0$. We conclude that the velocity field u in this particular case should not take into account the self-interaction, i.e. the presence of a vortex in the point z_i should not influence $u(z_i)$. Therefore, for a vorticity as in (1.5), we think to define:

$$u(z_i) = \int K(z_i - y) \sum_{\substack{j=1 \\ j\neq i}}^{N} a_j \delta_{z_j}(dy) = \sum_{\substack{j=1 \\ j\neq i}}^{N} a_j K(z_i - z_j).$$

Using this convention, the evolution of vorticity is

$$\omega_t = \sum_{i=1}^{N} a_i \delta_{z_i(t)} \tag{1.6}$$

where the points $z_i(t)$ evolve according to (1.1).

The previous paragraph gives an idea of the motivation that took to the introduction of (1.1) and obviously it is not a proof of its validity. However a precise statement is proved for example in [14] (here one can find also a more detailed introduction to the model and its principal properties): if we have a sequence of regular functions ω_0^n converging (in weak sense for measures) to $\sum_{i=1}^N a_i \delta_{z_i}$ for $n \to \infty$, and if we call ω_t^n the evolution of vorticity for Euler's equation in $\mathbb{R}^2$ with initial data ω_0^n, then, for each fixed t, ω_t^n converges to $\sum_{i=1}^N a_i \delta_{z_i(t)}$.

Moreover an analogous result was proved for the evolution of vorticity in Navier-Stokes planar equations (in [11]) and in Euler's equations in $\mathbb{R}^3$ with axial symmetry (in [10]), with suitable assumptions. This means that the point-vortex model approximates the motion of vortices also in these cases and not only for an Eulerian planar fluid.

We now want to state more quantitative results: taking an initial datum that approximates (1.5) and with support contained in N disjoint disks of radius ε, for ε small, we wonder how long the support of the vorticity is still contained in disjoint disks with small radius, whose centres move according to the point-vortex model. In fact, in the applications, the parameter ε is small, but not really zero: in this case it is interesting to quantify the time scale in which the approximation with the point-vortex dynamics is valid. It was recently proved (in [3]), under suitable assumptions, that the approximation holds for times of order $|\log \varepsilon|$ for Euler's equation in $\mathbb{R}^2$. The goal of this paper is to show analogous results for Navier-Stokes planar equations with vanishing viscosity and for Euler's equations in $\mathbb{R}^3$ with axial symmetry, extending in this way the results contained in [11] and in [10].

In the next section we first introduce weak formulations for the two equations, then we state the main results, whose proofs are object of Section 3 and 4 respectively.

2 Notation and statements

2.1 Euler's axisymmetric equations

Euler's equations for an incompressible and nonviscous fluid in $\mathbb{R}^3$ read:

$$\begin{cases} \partial_t u(x,t) + (u\cdot\nabla)u(x,t) = -\nabla p(x,t) \\ \nabla\cdot u = 0 \\ |u(x,t)| \to 0 \ \text{ for } \ |x|\to\infty \end{cases} \tag{2.1}$$

where $u(x,t)\in\mathbb{R}^3$ is the velocity field and $p(x,t)\in\mathbb{R}$ is the pressure. We introduce the vorticity $\omega := \operatorname{rot} u$, which is related to the rotation movement of the fluid. Equations (2.1) are equivalent to

$$\begin{cases} \partial_t \omega(x,t) + (u\cdot\nabla)\omega(x,t) = (\omega\cdot\nabla)u(x,t) \\ u(x,t) = -\frac{1}{4\pi}\int \frac{x-y}{|x-y|^3}\wedge\omega(y,t)\,dy \end{cases}. \tag{2.2}$$

Given an initial datum $u_0 \in C^s \cap L^p$, $s>1$, $p\in(1,\infty)$ (C^s denotes the Hölder space), this problem has a unique solution, which is only local in time (see for example [6, Thm. 2.9]).

The vector u is said to be axisymmetric without swirl if in cylindrical coordinates (r,z,θ) it has the form:

$$u = u_r(r,z)\mathbf{e}_r + u_z(r,z)\mathbf{e}_z$$

i.e. it is independent from θ and $u\cdot\mathbf{e}_\theta = 0$. It is known that, if the initial datum of Euler's equation is axisymmetric without swirl, then the solution is axisymmetric without swirl at each time (see [16, Prop. 2.2]) and that $\omega = \omega_\theta \mathbf{e}_\theta$ at each time (this follows from a straightforward computation). Using these facts, we get the following equation for ω in this case:

$$\partial_t\left(\frac{\omega_\theta}{r}\right) + u\cdot\nabla\left(\frac{\omega_\theta}{r}\right) = 0$$

which is a transport equation. Therefore we can reformulate Euler's equations in a weak sense, when the initial data is axisymmetric without swirl, as follows:

$$\begin{cases} u_t(x) = \int H(x-y) \wedge \omega_t(y)\, dy \\ \phi_t(x) = x + \int_0^t u_s(\phi_s(x))ds \\ \alpha_t(x) = \alpha_0(\phi_{-t}(x)) \end{cases} \tag{2.3}$$

where $H(x-y)$ is the integral kernel $-\frac{x-y}{4\pi|x-y|^3}$, $\alpha := \frac{\omega}{\delta}$ and $\delta(x)$ denotes the distance of x from the symmetry axis[1]. This problem admits a unique solution which is also global in time, provided $u_0 \in L^2$ and $\omega_0, \alpha_0 \in L^q \cap L^\infty$ for some $q < 3$ (see for example [16, Thm. 3.3]). Observe that ϕ_t is defined via an integral equation, because the continuity with respect to the time of u is not guaranteed, and therefore we cannot write the differential equation. However in the appendix we show that, under some extra assumptions, u is also continuous with respect to the time, and then ϕ_t is differentiable.

In this paper we treat the problem (2.3) when the initial vorticity has support contained in N small disks in the half-plane rz. These disks are rings in $\mathbb{R}^3$, therefore we talk about "smoke rings". We will also refer to this problem as (EA), i.e. Euler's Axisymmetric equations (without swirl).

2.2 Navier-Stokes planar equations

It is well known that the dynamics of an incompressible two-dimensional viscous fluid is described by the Navier-Stokes equations

$$\begin{cases} \partial_t u(x,t) + (u \cdot \nabla)\, u(x,t) = \nu \Delta\, u(x,t) - \nabla p(x,t) \\ \nabla \cdot u = 0 \\ |u(x,t)| \to 0 \;\text{ for }\; |x| \to \infty \end{cases} \tag{2.4}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $\nu > 0$ is the viscosity, u is the velocity field of the fluid and p is the pressure. By introducing the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$, the Navier-Stokes equations can be reformulated in terms of the vorticity, similarly to (1.3), i.e.,

$$\begin{cases} \partial_t \omega(x,t) + \big(u \cdot \nabla\big)\omega(x,t) = \nu \Delta \omega(x,t) \\ u(x,t) = \int dy\, K(x-y)\, \omega(y,t) \end{cases} \tag{2.5}$$

with K already defined in (1.2). In order to consider non-smooth initial data, we introduce a weak formulation of the Navier-Stokes equations, which can be obtained from (2.5) by integrating by parts; let $t \to \omega_t(dx)$ be a measure-valued function, then the weak formulation reads:

$$\begin{cases} \frac{d}{dt}\omega_t(f) = \omega_t(u \cdot \nabla f) + \nu \omega_t(\Delta f) + \omega_t(\partial_t f) \\ u(x,t) = \int K(x-y)\omega_t(dy) \\ \omega_{t=0}(dx) = \omega^0(dx) \end{cases} \tag{2.6}$$

[1] See [13] for this result; the equivalence of the problem (2.3) to Euler's equation for regular solutions can be proved as in the 2D case

where $f = f(x,t)$ is a $C^{2,1}(\mathbb{R}^2 \times [0,+\infty))$ function such that $|f|, |\nabla f|, |\Delta f|$ and $|\partial_t f|$ are bounded, and $\omega_t(f) := \int \omega_t(dx)\, f(x,t)$. If the initial datum is absolutely continuous with respect to the Lebesgue measure with density $\omega_0(x) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, it is shown in [12] that (2.6) admits a unique global solution which is absolutely continuous with respect to the Lebesgue measure at any time, with density $\omega(x,t)$ such that

$$\|\omega(\cdot,t)\|_{L^1} \le \|\omega_0\|_{L^1}, \qquad \|\omega(\cdot,t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}.$$

Moreover, for each bounded and measurable function $f = f(x,t)$, the solution $\omega(x,t)$ satisfies the equality

$$\int dx\, \omega(x,t)\, f(x,t) = \int dx\, \omega_0(x)\, \mathbb{E}(f(X_t^x, t))$$

i.e. it is the density function of the stochastic process $\{X_t^x\}$, which is the unique solution to the stochastic differential equation

$$\begin{cases} dX_t^x = u(X_t^x, t)\, dt + \sqrt{2\nu} dW_t \\ X_0^x = x \end{cases} \tag{2.7}$$

where $\{W_t\}$ is the canonical realization of a two dimensional Brownian motion.[2]

Observe that the process $\{X_t^x\}$ is the analogue for the viscous case of the characteristic flow $\phi_t(x)$ defined in (2.3).

2.3 Statement of main results

We now introduce some tools in order to state the main results. Concerning the axisymmetric case, we have to define a suitable change of variables to get the convergence to this dynamics: the smoke rings should increase their radius while their support becomes smaller. In fact, the interaction of the N vortices with the axis should be negligible in the limit if we want a convergence to point-vortex dynamics (which is valid for a fluid in $\mathbb{R}^2$, with no boundary). Let (r, z, θ) be the cylindrical coordinates in $\mathbb{R}^3$ and define:

$$x = z; \quad y = r - r_0\,. \tag{2.8}$$

The parameter r_0 will diverge for $\varepsilon \to 0$. These are the coordinates we will use in the sequel for the study of (EA), while for (NS) we use the canonical coordinates of $\mathbb{R}^2$.

Consider for both (NS) and (EA) an initial vorticity of the form:

$$\omega_\varepsilon(x,0) = \sum_{i=1}^N \omega_{i,\varepsilon}(x,0), \qquad \operatorname{supp} \omega_{i,\varepsilon} \subseteq \Sigma(z_i|\varepsilon). \tag{2.9}$$

[2]Let $(\mathcal{C}, \mathcal{B}(\mathcal{C}), P)$ be the Wiener space in two dimensions, where $\mathcal{C}$ denotes the set of continuous functions $[0,+\infty) \ni t \mapsto w(t) \in \mathbb{R}^2$, $\mathcal{B}(\mathcal{C})$ denotes the Borel sets in $\mathcal{C}$, and P the product of two one dimensional Wiener measures. Then, the coordinate mapping process $W_t(w) := w(t)$, $w \in \mathcal{C}$, is the desired object.

The points $z_i \in \mathbb{R}^2$ are chosen in such way, that the N disks are disjoint; the functions $\omega_{i,\varepsilon} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ have definite sign and their integral, which we call a_i, does not depend on ε. In both cases we state that the evolution of the vorticity is close to the solution to the point vortex model.

Observe that the decomposition in N vortices, which holds at time $t = 0$, can be extended in a natural way for all time, i.e.,

$$\omega_\varepsilon(x,t) = \sum_{i=1}^{N} \omega_{i,\varepsilon}(x,t),$$

for both (EA) and (NS): more precisely, for (EA) we define

$$\omega_{i,\varepsilon}(x,t) := \frac{r_0 + x_2}{r_0 + (\phi_{-t}(x))_2} \omega_{i,\varepsilon}(\phi_{-t}(x), 0); \tag{2.10}$$

as far as concerned (NS), we define the measure $\omega_t^{i,\varepsilon}(dx)$ by setting

$$\omega_t^{i,\varepsilon}(f) := \int dx\, \omega_{i,\varepsilon}(x,0)\, \mathbb{E}(f(X_t^x, t)), \quad \forall f \in C_b^{2,1}, \tag{2.11}$$

where $\phi_t(x)$ and X_t^x are defined in (2.3) and (2.7). Following [12], it can be shown that the measure $\omega_t^{i,\varepsilon}(dx)$ is absolutely continuous with respect to the Lebesgue measure, with density $\omega_{i,\varepsilon}(x,t)$ such that

$$||\omega_{i,\varepsilon}(\cdot,t)||_\infty \leq ||\omega_{i,\varepsilon}(\cdot,0)||_\infty, \qquad ||\omega_{i,\varepsilon}(\cdot,t)||_1 \leq ||\omega_{i,\varepsilon}(\cdot,0)||_1$$

for all $t \geq 0$. Moreover, for both (EA) and (NS), $\omega_{i,\varepsilon}(x,t)$ preserves the initial sign and the total mass a_i, as immediately follows from the definitions.

Furthermore, for each index i, we can decompose the velocity field u as follows:

$$u(x,t) = u^i(x,t) + F_\varepsilon^i(x,t),$$

where

$$u^i(x,t) = \int dy\, J(x,y)\, \omega_{i,\varepsilon}(y,t)$$

is the velocity field generated by the vortex $\omega_{i,\varepsilon}$, and

$$F_\varepsilon^i(x,t) = \sum_{j \neq i} \int dy\, J(x,y)\, \omega_{j,\varepsilon}(y,t)$$

is the one generated by the remaining $N-1$ vortices; here $J(x,y)$ denotes the integral kernel $G(x,y)$ (which is H written in the new coordinates (2.8), as we will see in the next section) for (EA) and $K(x-y)$ for (NS).

Let's call $\{z_i(t)\}_{i=1,\dots,N}$ the solution to point vortex dynamics (1.1) with intensities a_i and initial data z_i. Define:

$$T_{\varepsilon,\beta} := \sup\{t > 0 :\ \operatorname{supp} \omega_{i,\varepsilon}(s) \subseteq \Sigma(z_i(s)|\varepsilon^\beta)\ \ \forall i = 1,\dots,N\,,\ \forall s \in [0,t]\}$$

The result for (EA) is the following.

Theorem 2.1. *Consider the solution of problem* (2.3) *with initial vorticity as in* (2.9) *and* $r_0 = \varepsilon^{-\alpha}$, *for some* $\alpha > 0$, *and assume that:*

$$R_m := \min_{i \neq j} \inf_{t \in [0,+\infty)} |z_i(t) - z_j(t)| > 0$$

$$|\omega_{i,\varepsilon}(x,0)| \leq M\varepsilon^{-\gamma} \quad \textit{for some } M, \gamma > 0$$

Then if $\beta < \frac{1}{2}\min(1,\alpha)$, *there are* $\zeta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that*

$$T_{\varepsilon,\beta} > \zeta_0 |\log \varepsilon| \quad \forall \varepsilon \in (0, \varepsilon_0)$$

We cannot hope that an analogous result holds also in the (NS) case, because of the presence of the diffusion term $\nu\Delta\omega$ that immediately makes the support of the vorticity the whole plane, even if we start with a compactly supported datum. However, since the motion of the i-th vortex $z_i(t)$ is influenced only by the interactions with the other $N-1$ vortices and since the initial vorticity $\omega_{i,\varepsilon}(x,0)$ is sharply concentrated around the point z_i, we can expect that, in the regime of vanishing viscosity, the "centre" of the i-th term $\omega_{i,\varepsilon}(x,t)$ of the vorticity behaves in a similar way to $z_i(t)$; in other words, we can expect that the i-th centre is influenced only by the velocity field produced by the other $N-1$ terms, and not by itself. Hence, a natural candidate as "centre" of $\omega_{i,\varepsilon}(x,t)$ is the solution $t \mapsto B^i_\varepsilon(t)$ to the ordinary differential equation

$$\frac{d}{dt} B^i_\varepsilon(t) = F^i_\varepsilon(B^i_\varepsilon(t), t), \qquad B^i_\varepsilon(0) = z_i. \tag{2.12}$$

We observe that the above ordinary differential equation has a unique global solution since $F^i_\varepsilon(x,t)$ is a uniformly bounded and quasi-Lipschitz vector field[3] and it is continuous w.r.t. (x,t)[4].
We shall see that the motion of $B^i_\varepsilon(t)$ is very close to $z_i(t)$ and that the main part of the i-th term $\omega_{i,\varepsilon}(x,t)$ of the vorticity remains concentrated around $B^i_\varepsilon(t)$, for long times. To this purpose we define, for any $R > 0$, the function $t \mapsto m^i_\varepsilon(R,t)$ by setting

$$m^i_\varepsilon(R,t) := \int_{|x - B^i_\varepsilon(t)| > R} dx\, |\omega_{i,\varepsilon}(x,t)|, \tag{2.13}$$

i.e., the mass at time t of the i-th term of the vorticity outside the disk of centre $B^i_\varepsilon(t)$ and radius R, and the variables

$$T_{\varepsilon,\overline{\alpha},\beta} := \sup\{t > 0 : m^i_\varepsilon(\varepsilon^\beta, s) < \varepsilon^{\overline{\alpha}} \ \forall\, s \in [0,t],\ \forall\, i = 1, \dots, N\}$$

[3] A vector field $v(x) \in \mathbb{R}^d$ is said to be quasi-Lipschitz if there exists a constant $C > 0$ such that for any $x, x' \in \mathbb{R}^d$, $|v(x) - v(x')| \leq C\varphi(|x - x'|)$, where $\varphi = \varphi(r)$ is defined as $\varphi(r) = r(1 - \log r)$ if $0 < r < 1$ and $\varphi(r) = 1$ if $r \geq 1$.

[4] The quasi-Lipschitz bound in the case $d = 2$ can be found in [13]; the case $d = 3$ is discussed in the Appendix together with the continuity property w.r.t. (x,t). The proof of the existence of a unique global flow under these hypothesis can also be found in [13].

and

$$\overline{T}_{\varepsilon,\overline{\delta}} := \sup\{t > 0 : |z_i(s) - B^i_\varepsilon(s)| < \varepsilon^{\overline{\delta}} \ \forall\ s \in [0,t],\ \forall\ i = 1,\dots,N\}.$$

The result for (NS) is the following.

Theorem 2.2. *Consider the solution of problem* (2.6) *with initial vorticity as in* (2.9) *and* $\nu \le \nu_0\,\varepsilon^\alpha$, *for some* $\nu_0 > 0$ *and* $0 < \alpha \le 2$, *and assume that:*

$$R_m := \min_{i\neq j} \inf_{t\in[0,+\infty)} |z_i(t) - z_j(t)| > 0,$$

$$|\omega_{i,\varepsilon}(x,0)| \le M\varepsilon^{-\gamma} \quad \textit{for some } M,\gamma > 0.$$

Then if $\bar\alpha > 4\alpha + \gamma$, $\overline{\delta} \in (0, \alpha/3)$ *and* $\beta \in (0, \alpha/14)$, *there exist* $\zeta > 0$ *and* $\varepsilon_0 > 0$ *such that*

$$T_{\varepsilon,\overline{\alpha},\beta} > \zeta|\log\varepsilon| \quad \forall\,\varepsilon \in (0,\varepsilon_0)$$

and

$$\overline{T}_{\varepsilon,\overline{\delta}} > \zeta|\log\varepsilon| \quad \forall\,\varepsilon \in (0,\varepsilon_0).$$

In particular, for any $\beta' \in (0, \beta \wedge \overline{\delta})$ *and* $\varepsilon \in (0,\varepsilon_0)$,

$$\sup_{t\in[0,\zeta|\log\varepsilon|]} \int_{|x-z_i(t)|>\varepsilon^{\beta'}} dx\,|\omega^i_\varepsilon(x,t)| < \varepsilon^{\overline{\alpha}}.$$

Remark 2.3. Obviously the upper bound for α and the lower bound for α' are pleonastic. Indeed for any $\varepsilon \in (0,1)$, $\varepsilon^\alpha < \varepsilon^{\alpha'}$ if $\alpha > \alpha'$, and hence Theorem 2.2 holds for any choice of α and α'.

Note that the two theorems are extensions of the results stated, respectively, in [10] and [11], which treat the limit $\varepsilon \to 0$ when $t \in [0,T]$, for some fixed T.
The proofs of the theorems are obtained by adapting the strategies given in [3], [11] and [10], and they are based on a bootstrap argument. Indeed, if ε is small, then $T_{\varepsilon,\beta}$, $T_{\varepsilon,\overline{\alpha},\beta}$ and $\overline{T}_{\varepsilon,\overline{\delta}}$ are positive by continuity. We then analyse the evolution up to times smaller than $\hat{T}_1 := T_{\varepsilon,\beta} \wedge \zeta|\log\varepsilon|$ or $\hat{T}_2 := T_{\varepsilon,\overline{\alpha},\beta} \wedge \overline{T}_{\varepsilon,\overline{\delta}} \wedge \zeta|\log\varepsilon|$, getting better estimates which imply $\hat{T}_i = \zeta|\log\varepsilon|$, $i = 1,2$, for ζ and ε small enough.

3 Smoke rings

In this section we discuss the axisymmetric case: first we estimate the convolution kernel in the new cooridnates (2.8) and show that, under suitable assumptions, this is near to K; then we give the proof of Theorem 2.1.

3.1 Estimate of the axisymmetric convolution kernel

We know that, in the $\mathbb{R}^3$ coordinates[5]:

$$\mathbf{u}(\mathbf{x},t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(\mathbf{x}-\mathbf{y})\wedge\omega(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|^3}\,d\mathbf{y}\,.$$

Writing this with respect to the new coordinate system (2.8), we get

$$u(x,t) = \int_{\mathbb{R}^2} G(x,y)\,\omega(y,t)\,dy$$

where the convolution kernel $G(x,y)$ is defined by:

$$G_1(x,y) = \frac{1}{2\pi}\int_0^\pi d\theta\,\frac{(r_0+y_2)\big[(r_0+y_2)-(r_0+x_2)\cos\theta\big]}{\{|x-y|^2+2(r_0+x_2)(r_0+y_2)(1-\cos\theta)\}^{3/2}} \tag{3.1}$$

$$G_2(x,y) = \frac{1}{2\pi}\int_0^\pi d\theta\,\frac{(r_0+y_2)(x_1-y_1)\cos\theta}{\{|x-y|^2+2(r_0+x_2)(r_0+y_2)(1-\cos\theta)\}^{3/2}}\,. \tag{3.2}$$

We now want to give an estimate for this convolution kernel, in particular we want to show that, for small enough ε, the vector field u is near to the vector field $\widetilde{u}$ corresponding to the planar case, namely

$$\widetilde{u}(x,t) = \int_{\mathbb{R}^2} K(x-y)\,\omega(y,t)\,dy\,.$$

We need the following lemma.

A warning on the notation. In this paper we denote by C, C_i, with $i\in\mathbb{N}$, L, $\tilde{L}$ any constant which is *independent* on the parameters R, α, ε, ν, t, α, $\overline{\alpha}$, δ, $\overline{\delta}$, β. These constants may differ from line to line.

Lemma 3.1. *Let, for $a>0$:*

$$I_1(a) := \int_0^\pi d\theta\,\frac{\cos\theta}{[a^2+2(1-\cos\theta)]^{3/2}},$$

$$I_2(a) := \int_0^\pi d\theta\,\frac{1-\cos\theta}{[a^2+2(1-\cos\theta)]^{3/2}}.$$

Denoting by $\chi_{(0,1)}$ the indicator function of the interval $(0,1)$, the following equalities hold:

$$I_1(a) = a^{-2} + R_1(a), \qquad I_2(a) = -\frac{1}{2}\log a\cdot\chi_{(0,1)}(a) + R_2(a),$$

where $a\cdot R_1(a)$ is bounded and $|R_2(a)|\le C\min(1,\frac{1}{a})$.

[5] Here we denote by $\mathbf{x}$ the canonical coordinates of $\mathbb{R}^3$, while x stands for the coordinates introduced in (2.8).

Proof. We start from I_2. We recall that $1 - \cos\theta = 2[\sin(\theta/2)]^2$ and we write the integral as:

$$I_2 = \int_0^\pi d\theta \frac{2[\sin(\theta/2)]^2\,\cos(\theta/2)}{\{a^2 + 4[\sin(\theta/2)]^2\,\}^{3/2}} + \int_0^\pi d\theta \frac{2[\sin(\theta/2)]^2(1 - \cos(\theta/2))}{\{a^2 + 4[\sin(\theta/2)]^2\,\}^{3/2}}. \tag{3.3}$$

The first addend can be evaluated with the substitution $z = 2\sin(\theta/2)$:

$$\int_0^2 dz \frac{z^2}{2[a^2 + z^2]^{3/2}} = \frac{1}{2}\left[\log(\sqrt{a^2 + z^2} + z) - \frac{z}{\sqrt{a^2 + z^2}}\right]_{z=0}^{z=2}$$
$$= -(a^2 + 4)^{-1/2} + \frac{1}{2}\log(2 + \sqrt{a^2 + 4}) - \frac{1}{2}\log a.$$

We observe that for $a \to 0$ this is equal to $-\frac{1}{2}\log a$ plus a bounded rest, while for $a \to \infty$ this goes to zero like $\frac{1}{a}$.
Concerning the second addend in (3.3), we observe that

$$\int_0^\pi d\theta \frac{2[\sin(\theta/2)]^2(1 - \cos(\theta/2))}{\{a^2 + 4[\sin(\theta/2)]^2\,\}^{3/2}} \le \frac{1}{4}\int_0^\pi d\theta\,\frac{1 - \cos(\theta/2)}{\sin(\theta/2)}$$

which is a bounded integral; on the other hand

$$\int_0^\pi d\theta \frac{2[\sin(\theta/2)]^2(1 - \cos(\theta/2))}{\{a^2 + 4[\sin(\theta/2)]^2\,\}^{3/2}} \le \frac{2}{a^3}\int_0^\pi d\theta \sin^2(\theta/2)(1 - \cos(\theta/2)) \le Ca^{-3}.$$

So $R_2(a)$ is bounded by a constant for small a, by $\frac{C}{a}$ for large a: we got the desired estimate for I_2.
To evaluate I_1, when $a < 1$, we decompose the integral in the following way:

$$I_1 = \int_0^\pi d\theta \frac{\cos(\theta/2)}{\{a^2 + 2[\sin(\theta/2)]^2\,\}^{3/2}} + \int_0^\pi d\theta \frac{\cos\theta - \cos(\theta/2)}{\{a^2 + 2[\sin(\theta/2)]^2\,\}^{3/2}}. \tag{3.4}$$

The first addend can be computed again with the substitution $z = 2\sin(\theta/2)$:

$$\int_0^2 dz \frac{1}{[a^2 + z^2]^{3/2}} = \left[\frac{z}{a^2\sqrt{a^2 + z^2}}\right]_{z=0}^{z=2} = \frac{2}{a^2\sqrt{a^2 + 4}}$$

which, for $a \to 0$, is equal to a^{-2} plus a bounded rest. The second addend in (3.4) can be bounded by observing that

$$0 \le \cos(\theta/2) - \cos\theta \le 1 - \cos\theta \qquad \text{per } 0 \le \theta \le \pi$$

obtaining in this way I_2.
Instead, if $a > 1$, we observe that:

$$|I_1(a)| \le a^{-3}\int_0^\pi d\theta\,|\cos\theta|$$

and so $|R_1(a)| \le |I_1(a)| + a^{-2} \le Ca^{-2}$. In both cases we have that $a \cdot R_1(a)$ is bounded (it goes to zero like $a\log a$ when $a \to 0$, and like a^{-1} when $a \to \infty$) $\square$

Proposition 3.2. *Let x, y such that:*

$$|x_2| \le \frac{r_0}{2} \qquad |y_2| \le \frac{r_0}{2}$$

and let $r_0 = \varepsilon^{-\alpha}$. Then, for small enough ε:

$$|G(x,y) - K(x-y)| \le C \, \left(\varepsilon^\alpha + \varepsilon^\alpha |\log \varepsilon| + \varepsilon^\alpha \Big| \log |x-y| \Big| \cdot \chi_{(0,1)}(|x-y|)\right).$$

Proof. Let $a := |x-y| \, (r_0 + x_2)^{-1/2} (r_0 + y_2)^{-1/2}$.

$$\begin{aligned}
2\pi G_1(x,y) &= \int_0^\pi d\theta \frac{(r_0+y_2)(y_2 - x_2 \, \cos\theta + r_0(1-\cos\theta))}{(r_0+x_2)^{3/2} \, (r_0+y_2)^{3/2} \, \{a^2 + 2(1-\cos\theta)\}^{3/2}} \\
&= \frac{y_2 \cdot \big(I_1(a) + I_2(a)\big) - x_2 \cdot I_1(a) + r_0 \cdot I_2(a)}{(r_0+x_2)^{3/2} (r_0+y_2)^{1/2}} \\
&= \frac{y_2 - x_2}{(r_0+x_2)^{3/2}(r_0+y_2)^{1/2}} \cdot I_1(a) + \frac{(r_0+y_2)^{1/2}}{(r_0+x_2)^{3/2}} \cdot I_2(a) \\
&= \sqrt{\frac{r_0+y_2}{r_0+x_2}} \cdot \frac{y_2-x_2}{|x-y|^2} + \frac{y_2-x_2}{(r_0+x_2)^{3/2}(r_0+y_2)^{1/2}} \cdot R_1(a) \\
&\qquad + \sqrt{\frac{r_0+y_2}{(r_0+x_2)^3}} \cdot I_2(a).
\end{aligned}$$

We point out that $\frac{y_2-x_2}{|x-y|^2}$ is the first component of $2\pi K(x-y)$, so we subtract this quantity and estimate the remaining terms.

Defined $A = \sqrt{\frac{r_0+y_2}{r_0+x_2}}$, we observe that:

$$|A-1| = \frac{|A^2-1|}{|A+1|} = \left(\frac{|y_2-x_2|}{r_0+x_2}\right)(1+A)^{-1} \le \frac{2|x-y|}{r_0}$$

where the last inequality is due to the assumption $x \ge -\frac{r_0}{2}$. Moreover

$$\begin{aligned}
\sqrt{\frac{r_0+y_2}{(r_0+x_2)^3}} &= \frac{1}{(r_0+x_2)^{1/2} \, (r_0+y_2)^{1/2}} \cdot \frac{r_0+y_2}{r_0+x_2} \\
&\le \frac{1}{(r_0+x_2)^{1/2} \, (r_0+y_2)^{1/2}} \cdot \frac{(r_0+x_2) + |x-y|}{r_0+x_2} \\
&\le \frac{1}{(r_0+x_2)^{1/2} \, (r_0+y_2)^{1/2}} + \frac{a}{r_0+x_2} \le \frac{2}{r_0}(1+a).
\end{aligned}$$

Putting this into the above equation, and calling $D := G - K$, we get:

$$\begin{aligned}
2\pi |D_1(x,y)| &\le \frac{2|x-y| \, |y_2-x_2|}{r_0 \, |x-y|^2} + \frac{a \cdot R_1(a)}{r_0+x_2} + \frac{2}{r_0}(1+a) \cdot I_2(a) \\
&\le \frac{2}{r_0} + \frac{2C}{r_0} + \frac{2}{r_0}\left(C - \frac{1}{2} \log a \cdot \chi_{(0,1)}(a)\right).
\end{aligned}$$

Now if $a \in (0,1)$

$$\begin{aligned} 0 \leq -\log a =& \log(r_0+x_2)^{1/2} + \log(r_0+y_2)^{1/2} - \log|x-y| \\ \leq& \log r_0 + \log(3/2) + \big|\log|x-y|\big| \cdot \chi_{(0,1)}(|x-y|) \end{aligned}$$

using that, by the hypothesis, $\frac{1}{2}r_0 \leq r_0 + x_2 \leq \frac{3}{2}r_0$. Recalling that $r_0 = \varepsilon^{-\alpha}$ we obtain:

$$2\pi|D_1(x,y)| \leq C\varepsilon^{\alpha}\Big[1+\alpha|\log\varepsilon| + \big|\log|x-y|\big| \cdot \chi_{(0,1)}(|x-y|)\Big].$$

Let's now write the second component:

$$\begin{aligned} 2\pi G_2(x,y) =& \int_0^{\pi} \frac{(r_0+y_2)(x_1-y_1)\,\cos\theta}{(r_0+x_2)^{3/2}\,(r_0+y_2)^{3/2}\,\{a^2+2(1-\cos\theta)\}^{3/2}} \\ =& \frac{x_1-y_1}{(r_0+x_2)^{3/2}\,(r_0+y_2)^{1/2}} \cdot I_1(a) \\ =& \sqrt{\frac{r_0+y_2}{r_0+x_2}} \cdot \frac{x_1-y_1}{|x-y|^2} + \frac{x_1-y_1}{(r_0+x_2)^{3/2}\,(r_0+y_2)^{1/2}} R_1(a). \end{aligned}$$

Arguing in the same way:

$$2\pi|D_2(x,y)| \leq \frac{2|x-y|\,|x_1-y_1|}{r_0\,|x-y|^2} + \frac{a\cdot R_1(a)}{r_0+x_2} \leq \frac{2}{r_0}(1+C).$$

Since $|D| \leq |D_1| + |D_2|$, we get

$$|D(x,y)| \leq C\varepsilon^{\alpha}\left(1+|\log\varepsilon| + \big|\log|x-y|\big| \cdot \chi_{(0,1)}(|x-y|)\right),$$

which is the thesis. □

3.2 Proof of main result for (EA)

Now we state some preliminary lemmas in order to prove the result for (EA).

Lemma 3.3. *Let $\omega_{i,\varepsilon}(x,0)$ as defined in Section 2. Then*

$$\int_{\mathbb{R}^2} dx\,\omega_{i,\varepsilon}(x,t) = \int_{\mathbb{R}^2} dx\,\omega_{i,\varepsilon}(x,0) = a_i$$

and for each time $\omega_{i,\varepsilon}(x,t)$ has the same sign of $\omega_{i,\varepsilon}(x,0)$. Moreover, given $\zeta > 0$, for $t \leq T_{\varepsilon,\beta} \wedge \zeta|\log\varepsilon|$ and for small enough ε,

$$|\,\omega_{i,\varepsilon}(x,t)| \leq 3M\,\varepsilon^{-\gamma}.$$

Proof. The conservation of the integral is a direct consequence of Corollary 5.6 in Appendix, with $f = 1$. The conservation of sign is evident by the definition of $\omega_{i,\varepsilon}(x,t)$. To give a bound on the L^∞ norm, we observe that, if $x \in \Lambda_{i,\varepsilon}(0) := \operatorname{supp}\omega_{i,\varepsilon}(0)$, then $|x - z^i| \le \varepsilon$ and

$$|r_0 + x_2| = |\varepsilon^{-\alpha} + x_2| \ge \varepsilon^{-\alpha} - |z^i| - \varepsilon \ge \frac{1}{2}\varepsilon^{-\alpha}.$$

In the same way, since $\phi_t(x) \in \Lambda_{i,\varepsilon}(t)$, $|\phi_t(x) - z^i(t)| \le \varepsilon^\alpha$ and then

$$|r_0 + \phi_2^t(x)| \le \varepsilon^{-\alpha} + |z^i(t)| + \varepsilon^\beta.$$

Moreover

$$|z^i(t)| \le |z^i| + \int_0^t ds \sum_{j\neq i} \frac{|a_j|}{|z^i(s) - z^j(s)|} \le |z^i| + \frac{t}{R_m}\sum_{j\neq i}|a_j|,$$

and then, since $t \le \zeta|\log\varepsilon|$, for small enough ε,

$$|r_0 + \phi_2^t(x)| \le \frac{3}{2}\varepsilon^{-\alpha}.$$

Therefore the claim follows from the equality

$$\omega_{i,\varepsilon}(\phi_t(x),t) = \frac{r_0 + \phi_2^t(x)}{r_0 + x_2}\omega_{i,\varepsilon}(x,0).$$

□

We can now prove that the difference $u - \widetilde{u}$ is small

Proposition 3.4. *Let ω_ε as in (2.9), and let $t \le T_{\varepsilon,\beta} \wedge \zeta|\log\varepsilon|$, $x \in \Lambda_\varepsilon(t)$. Then, if ε is small enough,*

$$|u(x,t) - \widetilde{u}(x,t)| \le C\,\varepsilon^\alpha|\log\varepsilon|.$$

Proof. We start observing that, if $x \in \Lambda_\varepsilon(t)$, then $|x_2| \le r_0/2$, as seen in the proof of Lemma 3.3. In view of this lemma and of Proposition 3.2, we only need to bound the following:

$$\int_{|x-y|<1} dy\,\big|\log|x-y|\big|\,\omega_\varepsilon(y,t).$$

To this purpose, we use a rearrangement: we bound this integral with the one obtained by concentrating as much as possible the vorticity around the singularity of $\log|x-y|$, namely $y = x$ (since we integrate only in the domain $|x-y| < 1$, the function is bounded for y outside a disk centred in x). Since the integral of ω_ε is constant in time, and its L^∞ norm is less or equal to $3M\varepsilon^{-\gamma}$, we get the rearrangement replacing ω_ε with the function equal to the constant $3M\varepsilon^{-\gamma}$ in the disk of centre x and radius r, and equal to zero outside this disk. The radius r is chosen such that the total mass of vorticity is 1, so $\pi r^2 \cdot 3M\varepsilon^{-\gamma} = 1$. We have then:

$$\begin{aligned}\int_{|x-y|<1} dy \left|\log|x-y|\right| \omega_\varepsilon(y,t) \leq\ & 3M\varepsilon^{-\gamma} \int_{\{|x-y|\leq r\}} dy \left|\log|x-y|\right| \\ =& - 6\pi M\varepsilon^{-\gamma} \int_0^r \rho \log(\rho) d\rho \\ =& - 6\pi M\varepsilon^{-\gamma} \left[\frac{\rho^2}{2}\log\rho - \frac{\rho^2}{4}\right]_{\rho=0}^{\rho=r} \\ =& - 3\pi M\varepsilon^{-\gamma} r^2 \log r + \frac{3}{2}\pi M\varepsilon^{-\gamma} r^2.\end{aligned}$$

Since $r = \sqrt{\frac{\varepsilon^\gamma}{3\pi M}}$, we get finally:

$$\int_{|x-y|<1} dy \left|\log|x-y|\right| \omega_\varepsilon(y,t) \leq C + C\,|\log\varepsilon|,$$

which concludes the proof □

Lemma 3.5. *Recalling the definition of F^i_ε given in Section 2, we can write, for $t \leq T_{\varepsilon,\beta} \wedge \zeta|\log\varepsilon|$*

$$F^i_\varepsilon = F^i_{\varepsilon,1} + F^i_{\varepsilon,2}$$

where $F^i_{\varepsilon,1}$ is Lipschitz and bounded uniformly in ε, and $F^i_{\varepsilon,2}$ is small, i.e.

$$\|F^i_{\varepsilon,2}\|_{L^\infty} \leq C\varepsilon^\alpha |\log\varepsilon|.$$

Proof. Define

$$\begin{aligned}F^i_{\varepsilon,1}(x,t) =& \sum_{j\neq i} \int dy\, K(x-y)\, \omega_{j,\varepsilon}(y,t), \\ F^i_{\varepsilon,2}(x,t) =& \sum_{j\neq i} \int dy\, [G(x,y) - K(x-y)]\, \omega_{j,\varepsilon}(y,t).\end{aligned}$$

Then $F^i_{\varepsilon,1}$ is Lipschitz and bounded uniformly in ε because K is Lipschitz and bounded outside the disk $\Sigma(0|R_m/2)$: in fact, if $x \in \Lambda_{i,\varepsilon}(t)$ and $y \in \Lambda_{j,\varepsilon}(t)$, for $i \neq j$, we have

$$|x-y| \geq |z_i(t) - z_j(t)| - |x - z_i(t)| - |y - z_j(t)| \geq R_m - 2\varepsilon^\beta \geq \frac{R_m}{2}$$

for ε small enough. $F^i_{\varepsilon,2}$ is small because of Proposition 3.4. □

Let's define

$$B^i_\varepsilon(t) := a_i^{-1} \int_{\mathbb{R}^2} dx\, x\, \omega_{i,\varepsilon}(x,t) \qquad \text{center of vorticity}$$
$$I^i_\varepsilon(t) := \int_{\mathbb{R}^2} dx\, |x - B^i_\varepsilon(t)|^2 |\omega_{i,\varepsilon}(x,t)| \qquad \text{moment of inertia}$$

In the following lemmas, for simplicity, we omit the index i from the notation and we assume, without lost of generality, $a_i = 1$. This is equivalent to consider a "reduced system" with only one vortex and an external field acting on it, which has the properties stated in Lemma 3.5. It is easy to verify that in this case, the following equation holds (instead of (5.3), see the Appendix):

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t(f) = \omega_t\big(((u + F_\varepsilon)\cdot\nabla)f + \partial_t f\big). \tag{3.5}$$

The results we will prove hold obviously for each i.

Lemma 3.6. *There exist $\zeta > 0$ and $\delta > 4\beta$ such that, for $t \le T_{\varepsilon,\beta} \wedge \zeta|\log\varepsilon|$ and for small enough ε,*

$$I_\varepsilon(t) \le C_1 \varepsilon^\delta \tag{3.6}$$

Proof. We estimate the derivative of $I_\varepsilon(t)$, using (3.5):

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(t) = \int dx\, \omega_\varepsilon(x,t)\, \Big[(u + F_\varepsilon)\cdot 2(x - B_\varepsilon(t)) - \dot{B}_\varepsilon(t)\cdot 2(x - B_\varepsilon(t))\Big].$$

Moreover

$$\frac{\mathrm{d}}{\mathrm{d}t} B_\varepsilon(t) = \int dx\, \omega_\varepsilon(x,t)\, (u(x,t) + F_\varepsilon(x,t)),$$

then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} I_\varepsilon(t) =& 2\int dx\, \omega_\varepsilon(x,t)\left[u(x,t) - \int dy\, \omega_\varepsilon(y,t)\, u(y,t)\right]\cdot(x - B_\varepsilon(t))\\ &+ 2\int dx\, \omega_\varepsilon(x,t)\left[F_\varepsilon(x,t) - \int dy\, \omega_\varepsilon(y,t)\, F_\varepsilon(y,t)\right]\cdot(x - B_\varepsilon(t)).\end{aligned}$$

Consider, first of all, the term containing F_ε: we observe that, by the definition of $B_\varepsilon(t)$,

$$\int dx\, \omega_\varepsilon(x,t)(x - B_\varepsilon(t))\cdot\int dy\, \omega_\varepsilon(y,t) F_\varepsilon(y,t) = 0$$

$$\int dx\, \omega_\varepsilon(x,t)(x - B_\varepsilon(t))\cdot F_{\varepsilon,1}(B_\varepsilon(t),t) = 0.$$

Then we have:

$$
\begin{aligned}
&2\left|\int dx\,\omega_\varepsilon(x,t)\left[F_\varepsilon(x,t)-\int_{\mathbb{R}^2} dy\,\omega_\varepsilon(y,t)\,F_\varepsilon(y,t)\right]\cdot(x-B_\varepsilon(t))\right| \\
&\quad= 2\left|\int dx\,\omega_\varepsilon(x,t)\,[F_{\varepsilon,1}(x,t)-F_{\varepsilon,1}(B_\varepsilon(t),t)]\cdot(x-B_\varepsilon(t))\right| \\
&\qquad+2\left|\int dx\,\omega_\varepsilon(x,t)\,F_{\varepsilon,2}(x,t)\cdot(x-B_\varepsilon(t))\right| \\
&\quad\le 2\int dx\,\omega_\varepsilon(x,t)\,L|x-B_\varepsilon(t)|^2+C\varepsilon^\alpha|\log\varepsilon|\int dx\,|x-B_\varepsilon(t)|\,\omega_\varepsilon(x,t) \\
&\quad\le 2L\,I_\varepsilon(t)+2\,C\varepsilon^\alpha|\log\varepsilon|[I_\varepsilon(t)]^{1/2}
\end{aligned}
$$

where, in the last line, we used Cauchy-Schwarz inequality. Concerning the term which contains u, we have analogously:

$$
\int dx\,\omega_\varepsilon(x,t)(x-B_\varepsilon(t))\cdot\int dy\,\omega_\varepsilon(y,t)u(y,t)=0.
$$

Moreover, thanks to the antisymmetry of K:

$$
\int dx\,\omega_\varepsilon(x,t)\,\widetilde{u}(x,t)=\int dx\int dy\,\omega_\varepsilon(x,t)\,\omega_\varepsilon(y,t)\,K(x-y)=0
$$

and recalling that, by definition, $(x-y)\cdot K(x-y)=0$, we get

$$
\begin{aligned}
\int dx\,\omega_\varepsilon(x,t)\,x\cdot\widetilde{u}(x,t)&=\int dx\int dy\,\omega_\varepsilon(x,t)\,\omega_\varepsilon(y,t)\,x\cdot K(x-y) \\
&=\int dx\int dy\,\omega_\varepsilon(x,t)\,\omega_\varepsilon(y,t)\,y\cdot K(x-y)
\end{aligned}
$$

so this integral is also 0 for antisymmetry of K. Using Proposition 3.4 we get then:

$$
\begin{aligned}
&2\left|\int dx\,\omega_\varepsilon(x,t)\left[u(x,t)-\int dy\,\omega_\varepsilon(y,t)\,u(y,t)\right]\cdot(x-B_\varepsilon(t))\right| \\
&\quad\le\ 2\int dx\,\omega_\varepsilon(x,t)\left|u(x,t)-\widetilde{u}(x,t)\right|\cdot|x-B_\varepsilon(t)| \\
&\quad\le\ 2\,C\,\varepsilon^\alpha|\log\varepsilon|\int dx\,\omega_\varepsilon(x,t)\cdot|x-B_\varepsilon(t)|\le\ 2C\,\varepsilon^\alpha|\log\varepsilon|\,[I_\varepsilon(t)]^{1/2}
\end{aligned}
$$

where we used again Cauchy-Schwarz inequality in the last line.

Therefore we obtain:

$$
|\dot{I}_\varepsilon(t)|\le 2L\,I_\varepsilon(t)+C\,\varepsilon^\alpha|\log\varepsilon|\,[I_\varepsilon(t)]^{1/2}\,.
$$

We apply Gronwall's inequality to $H_\varepsilon(t):=[I_\varepsilon(t)]^{1/2}$, and use that $I_\varepsilon(0)\le 4\varepsilon^2$. We get:

$$
H_\varepsilon(t)\le\left(2\varepsilon+\frac{C}{2L}\varepsilon^\alpha|\log\varepsilon|\right)e^{Lt}\le C_1\varepsilon^{\alpha'}e^{Lt} \tag{3.7}
$$

for each $\alpha' < \min(\alpha, 1)$. Taking the square, and recalling that t is less or equal to $\zeta|\log\varepsilon|$, for ζ to be fixed:

$$I_\varepsilon(t) \le C_1^2\,\varepsilon^{2\alpha'-2L\,\zeta}$$

then, in order to get the thesis, it's enough to choose ζ such that $\delta := 2\alpha' - 2L\,\zeta > 4\beta$; this is possible because $\beta < \frac{1}{2}\min(\alpha, 1)$, choosing a suitable α' (i.e. $\alpha' > 2\beta$). $\square$

Remark 3.7. By the definition of $T_{\varepsilon,\beta}$, it follows immediately:

$$I_\varepsilon(t) \le 4\varepsilon^{2\beta} \qquad \forall t \in [0, T_{\varepsilon,\beta}].$$

The estimate just shown, which holds for logarithmic times, is much better, since $\delta > 4\beta$.

Lemma 3.8. *Define*

$$R_t := \max\{|x - B_\varepsilon(t)| \,:\, x \in \Lambda_\varepsilon(t)\}$$

and let $x_0 \in \Lambda_\varepsilon(0)$ *be such that, at the time* $t \le T_{\varepsilon,\beta} \wedge \zeta|\log\varepsilon|$*,*

$$|\phi_t(x_0) - B_\varepsilon(t)| = R_t.$$

Then at this time t *the following inequality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}|\phi_t(x_0) - B_\varepsilon(t)| \le 2LR_t + \frac{5I_\varepsilon(t)}{\pi R_t^3} + \sqrt{\frac{3\,M\varepsilon^{-\gamma}m_\varepsilon(R_t/2, t)}{\pi}} + C\varepsilon^\alpha|\log\varepsilon| \tag{3.8}$$

where the function m_ε *is defined by:*

$$m_\varepsilon(R, t) := \int_{|y-B_\varepsilon(t)|>R} dy\,\omega_\varepsilon(y, t) \qquad \textit{for } R \in (0, +\infty).$$

Remark 3.9. The definition of m_ε is analogous to the one given in the first section for (NS). However note that here B_ε is the center of vorticity, while in (NS) B_ε is defined as the flow generated by F_ε.

Proof. Call $x = \phi_t(x_0)$. We have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\phi_t(x_0) - B_\varepsilon(t)| &= [u(x, t) + F_\varepsilon(x, t) - \dot{B}_\varepsilon(t)] \cdot \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \\
&= \left[\int dy\,(F_\varepsilon(x, t) - F_\varepsilon(y, t))\,\omega_\varepsilon(y, t)\right] \cdot \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \\
&\quad + \left[\int dy\,(u(x, t) - u(y, t))\,\omega_\varepsilon(y, t)\right] \cdot \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|}.
\end{aligned}$$

The first addend is easily bounded, taking absolute values, since $F_{\varepsilon,1}$ is Lipschitz:

$$\begin{aligned}\int dy\,|F(x,t)-F(y,t)|\,\omega_\varepsilon(y,t) \le& L\int dy\,|x-y|\,\omega_\varepsilon(y,t)+2C\varepsilon^\alpha|\log\varepsilon|\\ \le& 2L\,R_t+2C\varepsilon^\alpha|\log\varepsilon|.\end{aligned}$$

Concerning the second addend, we split it into three terms, recalling that $\int dy\,\widetilde{u}(y)\,\omega_\varepsilon(y)=0$:

$$|u(x,t)-\widetilde{u}(x,t)|\le\, C\,\varepsilon^\alpha|\log\varepsilon|,$$

$$\left|\int dy\,(u(y,t)-\widetilde{u}(y,t))\,\omega_\varepsilon(y,t)\right|\le\, C\,\varepsilon^\alpha|\log\varepsilon|.$$

It remains only the term

$$\widetilde{u}(x,t)\cdot\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}=\frac{x-B_\varepsilon(t)}{|x-B_\varepsilon(t)|}\cdot\int dy\,K(x-y)\omega_\varepsilon(y,t)$$

We divide the integration domain in two regions: $A_1:=\Sigma\big(B_\varepsilon(t)\big|R_t/2\big)$ and $A_2:=\mathbb{R}^2\setminus A_1$. We call H_1 and H_2 the resultant addends; following the proof of [3, Lemma 2.5] we have:

$$|H_1|\le\frac{5}{\pi R_t^3}I_\varepsilon(t)$$

Then we study H_2. First observe that:

$$|H_2|\le\frac{1}{2\pi}\int_{|y-B_\varepsilon(t)|>R_t/2}dy\,\frac{1}{|x-y|}\,\omega_\varepsilon(y,t)$$

so we can estimate H_2 with a rearrangement as in the proof of Proposition 3.4; we bound the integral taking a vorticity concentrated, as much as possible, around the singularity of $\frac{1}{|x-y|}$. Therefore, the rearrangement is achieved defining ω_ε equal to $3\,M\varepsilon^{-\gamma}$ for $|x-y|<r$, and equal to 0 for $|x-y|\ge r$, where r is chosen such that $3\,M\varepsilon^{-\gamma}\cdot\pi r^2=m_\varepsilon(R_t/2,t)$ (which is the "total mass" of ω in the integration domain A_2). Hence:

$$|H_2|\le\frac{3\,M\varepsilon^{-\gamma}}{2\pi}\int_{|z|<r}\frac{dz}{z}=3\,M\varepsilon^{-\gamma}\int_0^r\frac{\rho}{\rho}\,d\rho=3\,M\varepsilon^{-\gamma}\,r=\sqrt{\frac{3\,Mm_t(R_t/2)}{\pi\,\varepsilon^\gamma}}.$$

Putting together all the terms, we get the thesis. □

We introduced the function $m_\varepsilon(\cdot,t):\mathbb{R}^+\to\mathbb{R}^+$; now we investigate its behaviour near to 0.

Lemma 3.10. *For each* $0<\beta<\frac{1}{2}\min(1,\alpha)$, *and for each* $\ell>0$, *there exists* $\zeta>0$ *such that:*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T_{\varepsilon,\beta}\wedge\zeta|\log\varepsilon|]}\varepsilon^{-\ell}m_\varepsilon(\varepsilon^\beta,t)=0.$$

Proof. Given $R > 0$, let $W_R : \mathbb{R}^2 \to [0,1]$ be a radial function such that

$$W_R(x) = \begin{cases} 1 & \text{if } |x| \le R \\ 0 & \text{if } |x| \ge 2R \end{cases},$$

that $0 \le W_{R_1}(x) \le W_{R_2}(x)$ for any $0 < R_1 \le R_2$, and that, for some constant C_1, the following conditions hold:

$$|\nabla W_R(x)| \le \frac{C_1}{R}, \qquad |\Delta W_R(x)| < \frac{C_1}{R^2}, \qquad |\nabla W_R(x) - \nabla W_R(x')| < \frac{C_1}{R^2}|x - x'|.$$

We define a mollified version of m_ε:

$$\mu_\varepsilon(R,t) := 1 - \int dx\, W_R(x - B_\varepsilon(t))\, \omega_\varepsilon(x,t)$$

It follows immediately by definition that:

$$\mu_\varepsilon(R,t) \le m_\varepsilon(R,t) \le \mu_\varepsilon(R/2,t), \tag{3.9}$$

then it's enough to prove the claim with μ_ε instead of m_ε. The convenience is that the function μ_ε is differentiable (with respect to t); therefore we compute its derivative:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu_\varepsilon(R,t) &= -\int dx\, \nabla W_R(x - B_\varepsilon(t)) \cdot [u(x,t) + F(x,t) - \dot{B}_\varepsilon(t)]\, \omega_\varepsilon(x,t) \\ &= -H_3 - H_4 - H_5 \end{aligned}$$

with

$$\begin{aligned} H_3 &= \int dx\, \nabla W_R(x - B_\varepsilon(t)) \cdot \widetilde{u}(x,t)\, \omega_\varepsilon(x,t) \\ H_4 &= \int dx\, \nabla W_R(x - B_\varepsilon(t)) \cdot \left[F_{\varepsilon,1}(x,t) - \int dy\, F_{\varepsilon,1}(y,t)\, \omega_\varepsilon(y,t)\right] \omega_\varepsilon(x,t) \\ H_5 &= \int dx\, \nabla W_R(x - B_\varepsilon(t)) \\ &\qquad \cdot \left[u(x,t) - \widetilde{u}(x,t) - \int dy\, [u(y,t) - \widetilde{u}(y,t)]\, \omega_\varepsilon(y,t)\right] \omega_\varepsilon(x,t) \\ &\quad + \int dx\, \nabla W_R(x - B_\varepsilon(t)) \cdot \left[F_{\varepsilon,2}(x,t) - \int dy\, F_{\varepsilon,2}(y,t)\, \omega_\varepsilon(y,t)\right] \omega_\varepsilon(x,t) \end{aligned}$$

since $\dot{B}_\varepsilon(t) = \int dy\, \omega_\varepsilon(y,t)[F_\varepsilon(y,t) + u(y,t) - \widetilde{u}(y,t)]$. We immediately observe that, thanks to Proposition 3.4, to Lemma 3.5 and to the fact that $\nabla W_R(z)$ is zero if $|z| \le R$,

$$|H_5| \le \frac{C_1}{R} \cdot C\varepsilon^\alpha |\log \varepsilon| \cdot m_\varepsilon(R,t). \tag{3.10}$$

Following the proof of [3, Lemma 2.6] we find:

$$|H_3| \leq \frac{11C_1}{\pi R^4} I_\varepsilon(t)\, m_\varepsilon(R,t); \tag{3.11}$$

$$|H_4| \leq 3L\, C_1 \cdot m_\varepsilon(R,t) + 2C_1\, \|F_{\varepsilon,1}\|_{L^\infty} \cdot \frac{I_\varepsilon(t)}{R^3} m_\varepsilon(R,t). \tag{3.12}$$

Now, given $\beta < \frac{1}{2}\min(1,\alpha)$, we fix β_* such that $\beta < \beta_* < \frac{1}{2}\min(1,\alpha)$ and we choose $\zeta > 0$ small enough, such that (3.6) holds with $\delta > 4\beta_*$ (it is possible choosing, in the proof, $\alpha' > 2\beta_*$). It follows then from estimates (3.10), (3.11), (3.12):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_\varepsilon(R,t) \leq C_2 \left(\frac{\varepsilon^\delta}{R^4} + \frac{\varepsilon^\delta}{R^3} + \frac{\varepsilon^\alpha |\log\varepsilon|}{R} + 1 \right) m_\varepsilon(R,t)$$

for each $t \in [0, T_{\varepsilon,\beta} \wedge \zeta|\log\varepsilon|]$. Define $A(R)$ the expression in the parenthesis in last inequality; since $\delta > 4\beta_*$, there exists A_* such that $C_2 A(R) \leq A_*$ for each $R \geq \varepsilon^{\beta_*}$. Therefore recalling (3.9):

$$\mu_\varepsilon(R,t) \leq \mu_\varepsilon(R,0) + A_* \int_0^t ds\, \mu_\varepsilon(R/2,s) \qquad \forall R \geq \varepsilon^{\beta_*}.$$

The proof can be now achieved iterating this estimates n times and choosing a suitable n, as in [3, Lemma 2.6]. □

We are now ready to prove Theorem 2.1

Proof of Theorem 2.1. In view of (3.6) and (3.8), there exist $\zeta > 0$ and $\delta > 4\beta$ such that, if $x_0 \in \Lambda_{i,\varepsilon}(0)$ and $|\phi_t(x_0) - B^i_\varepsilon(t)| = R^i_t$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\phi_t(x_0) - B^i_\varepsilon(t)| \leq 2L\, R^i_t + C\varepsilon^\delta (R^i_t)^{-3} + C\sqrt{\varepsilon^{-\gamma} m^i_\varepsilon(R^i_t/2,t)} + C\varepsilon^\alpha |\log\varepsilon|$$

for each $t \leq T_{\varepsilon,\beta} \wedge \zeta|\log\varepsilon|$ and for each index i. Therefore, in this time interval:

$$\Lambda_{i,\varepsilon}(t) \subseteq \Sigma(B^i_\varepsilon(t)|R(t)) \tag{3.13}$$

where $R(t)$ solves:

$$\dot{R}(t) = 2LR(t) + C\varepsilon^\delta R(t)^{-3} + C\sqrt{\varepsilon^{-\gamma} m_t(R(t)/2)} + C\varepsilon^\alpha |\log\varepsilon| \tag{3.14}$$

with initial data $R(0) = \varepsilon$. In fact, this is obviously true at time 0; moreover, if it happens, at time t, that $|\phi_t(x_0) - B^i_\varepsilon| = R(t)$, for some $x_0 \in \Lambda_{i,\varepsilon}(0)$, then at this time $R(t) = R^i_t$ and hence, thanks to (3.8), the radial velocity of $|\phi_t(x_0) - B^i_\varepsilon|$ is less or equal to $\dot{R}(t)$.

We fix β' such that $\beta < \beta' < \frac{1}{2}\min(1,\alpha)$ and we show that, for a suitable $\zeta_1 > 0$ and for ε small enough:

$$R(t) < \varepsilon^{\beta'} \quad \forall\, t \leq T_{\varepsilon,\beta} \wedge \zeta_1 |\log\varepsilon|. \tag{3.15}$$

Then we show that, for a suitable $\zeta_2 > 0$ and ε small enough:

$$|B^i_\varepsilon(t) - z_i(t)| \le C\varepsilon^{\delta_1} \quad \forall\, t \le T_{\varepsilon,\beta} \wedge \zeta_2 |\log \varepsilon| \tag{3.16}$$

for some $\delta_1 > \beta'$ and for each index i. With these two bounds, together with (3.13), we can conclude that, for ε small enough and $\zeta_0 := \min(\zeta_1, \zeta_2)$:

$$\sup_{x\in\Lambda_{i,\varepsilon}(t)} |x - z_i(t)| = \sup_{x_0\in\Lambda_{i,\varepsilon}(0)} |\phi_t(x_0) - z_i(t)| < \varepsilon^\beta \quad \forall t \le T_{\varepsilon,\beta} \wedge \zeta_0 |\log \varepsilon|.$$

By continuity of ϕ_t and $z_i(t)$ and by definition of $T_{\varepsilon,\beta}$, we deduce:

$$T_{\varepsilon,\beta} \wedge \zeta |\log \varepsilon| < T_{\varepsilon,\beta}$$

and hence $T_{\varepsilon,\beta} > \zeta_0 |\log \varepsilon|$, which is the thesis.

We now prove (3.15) by contradiction. Assume that (3.15) is false for every $\zeta > 0$. Then, given a $\xi > 0$ to be chosen later, there exists $t_1 \in [0, T_{\varepsilon,\beta} \wedge \xi |\log \varepsilon|]$ such that $R(t_1) = \varepsilon^{\beta'}$; fixed β_* such that $\beta' < \beta_* < \frac{1}{2}\min(1,\alpha)$, we define:

$$t_0 := \inf\{t \in [0, t_1] :\ R(s) > \varepsilon^{\beta_*}\ \ \forall s \in [t, t_1]\}\,.$$

Then, for $t \in [t_0, t_1]$, $R(t) \ge \varepsilon^{\beta_*}$; therefore $m_\varepsilon(R(t)/2, t) \le m_\varepsilon(\varepsilon^{\beta_*}/2, t)$. We choose ξ such that $\delta > 4\beta_*$ and that Lemma 3.10 holds with $\ell = \gamma + 2\beta_*$; then the following estimates hold:

$$\sqrt{\varepsilon^{-\gamma} m_t(R(t)/2)} \le \varepsilon^{\beta_*}\,; \quad \varepsilon^\delta R(t)^{-3} \le \varepsilon^{\delta - 3\beta*} \le \varepsilon^{\beta_*}\,; \quad \varepsilon^\alpha |\log \varepsilon| \le \varepsilon^{\beta_*}.$$

Putting these into (3.14), we get, for a suitable C_3:

$$\dot R(t) \le 2LR(t) + C_3\, \varepsilon^{\beta_*} \qquad \forall t \in [t_0, t_1].$$

Integrating we obtain:

$$R(t_1) \le e^{2L(t_1 - t_0)}[R(t_0) + (t_1 - t_0) C_3\, \varepsilon^{\beta_*}] \le \varepsilon^{-2L\xi}[\varepsilon^{\beta*} + C_3 \xi |\log \varepsilon| \varepsilon^{\beta_*}].$$

Then, if we fix ξ small enough, such that $\beta_* - 2L\xi > \beta'$, we find $R(t_1) < \varepsilon^{\beta'}$, which contradicts the assumption $R(t_1) = \varepsilon^{\beta'}$. We point out that such a choice of $\xi > 0$ is possible, because it depends only on β, β' and β_*, which are fixed a priori providing $\beta < \beta' < \beta_* < \frac{1}{2}\min(1,\alpha)$.

We shall now prove (3.16); we compute the time derivatives:

$$\begin{aligned}\dot B^i_\varepsilon(t) - \dot z^i(t) = a_i^{-1} \int dx\, \big(u^i(x,t) + F^i_\varepsilon(x,t)\big)\, \omega_{i,\varepsilon}(x,t) \\ - \sum_{j\neq i} a_j K(z^i(t) - z^j(t)).\end{aligned}$$

By adding and subtracting suitable terms, and recalling the definition of F^i_ε, we get:

$$\begin{aligned}\dot B_{i,\varepsilon}(t) - \dot z^i(t) = a_i^{-1}\int dx\, u^i(x,t)\,\omega_{i,\varepsilon}(x,t) + a_i^{-1}\int dx\, F^i_{\varepsilon,2}(x,t)\,\omega_{i,\varepsilon}(x,t)\\ + a_i^{-1}\int dx[F^i_{\varepsilon,1}(x,t) - F^i_{\varepsilon,1}(B^i_\varepsilon(t),t)]\,\omega_{i,\varepsilon}(x,t)\\ + \sum_{j\neq i}\int dy\,[K(B^i_\varepsilon(t)-y) - K(B^i_\varepsilon(t)-B^j_\varepsilon(t))]\,\omega_{j,\varepsilon}(y,t)\\ + \sum_{j\neq i} a_j[K(B^i_\varepsilon(t)-B^j_\varepsilon(t)) - K(B^i_\varepsilon(t)-z^j(t))]\\ + \sum_{j\neq i} a_j[K(B^i_\varepsilon(t)-z^j(t)) - K(z^i(t)-z^j(t))].\end{aligned}$$

Now we take absolute values, we use the triangle inequality, the Lipschitz property of K outside the disk $\Sigma\big(0\big|R_{\min}/2\big)$ (we call L_1 the Lipschitz constant of K in this region), and Lemma 3.5; then we obtain:

$$\begin{aligned}|\dot B^i_\varepsilon(t) - \dot z^i(t)| \le |a_i|^{-1}C\varepsilon^\alpha|\log\varepsilon| + |a_i|^{-1}L\, I^i_\varepsilon(t)^{1/2}\,|a_i|^{1/2}\\ + \sum_{j\neq i} L_1 I_{j,\varepsilon}(t)^{1/2}\,|a_j|^{1/2} + \sum_{j\neq i}|a_j|\,L_1|B^j_\varepsilon(t) - z^j(t)|\\ + \sum_{j\neq i}|a_j|\,L_1|B^i_\varepsilon(t) - z^i(t)|.\end{aligned}$$

Define $\Delta(t) := \max_{i=1,\dots,N}|B_{i,\varepsilon}(t) - z^i(t)|$: then

$$\begin{aligned}\dot\Delta(t) &\le \max_{i=1,\dots,N}|\dot B^i_\varepsilon(t) - \dot z^i(t)|\\ &\le C\varepsilon^\alpha|\log\varepsilon| + C\sum_{j=1}^N\sqrt{I^j_\varepsilon(t)} + 2L_1\sum_{j=1}^N|a_j|\,\Delta(t).\end{aligned}$$

We find easily that, by definition of $F^i_{\varepsilon,1}$, $L \ge L_1\sum_j|a_j|$. Integrating the previous inequality we find:

$$\Delta(t) \le \Delta(0)e^{2Lt} + C\int_0^t\sum_{j=1}^N\sqrt{I^j_\varepsilon(s)}e^{L(t-s)}\,ds + C\varepsilon^\alpha|\log\varepsilon|\cdot(e^{2Lt}-1).$$

We use the bound (3.7) and choose $\alpha' < \min(\alpha,1)$. If $t \le T_{\varepsilon,\beta}\wedge\zeta|\log\varepsilon|$ we get:

$$\Delta(t) \le C\,\varepsilon^{\alpha'-2L\zeta} = C\varepsilon^{\delta_1} \qquad \forall t\in[0, T_{\varepsilon,\beta}\wedge\zeta|\log\varepsilon|]. \tag{3.17}$$

We choose ζ_2 such that $\delta_1 := \alpha' - 2L\zeta_2 > \beta'$, so the proof is achieved. □

3.3 Better estimate for $T_{\varepsilon,\beta}$

An example, for which a better estimate for $T_{\varepsilon,\beta}$ can be proved, is presented in [3] for the planar case. It consists on three vortices with intensities a_i and initial points $z_i(0)$ chosen in such a way, that at each time t the following relation holds:

$$|z_i(t) - z_j(t)| = |z_i(0) - z_j(0)| \cdot \sqrt{1 + gt} \quad \text{for some } g > 0 \tag{3.18}$$

i.e. the three vortices moves away one from another, and the triangle formed by them remains similar to itself at each time. Under this extra assumption, in the planar case, there exist ζ_0, $\varepsilon_0 > 0$ such that $T_{\varepsilon,\beta} > \varepsilon^{-\zeta_0}$ for each $\varepsilon \in (0, \varepsilon_0)$.

This result is still valid for (EA), and the proof is completely analogous to the one given in [3]: we give here only a sketch. The crucial remark is that in this case, the Lipschitz constant of $F^i_{\varepsilon,1}$ decreases with t, i.e.:

$$|F^i_{\varepsilon,1}(x,t) - F^i_{\varepsilon,1}(z,t)| \leq \frac{L}{1+t}\,|x - z| \tag{3.19}$$

for each $x, z \in \Lambda_{i,\varepsilon}(t)$ and for each $t \leq T_{\varepsilon,\beta}$. This can be easily proved observing that the Lipschitz constant of K outside the disk $\Sigma(0|r)$ goes like $1/r^2$ and taking account of (3.18).

We now work in the time interval $(0, T_{\varepsilon,\beta} \wedge \varepsilon^{-\zeta})$, for some $\zeta > 0$. Thanks to (3.19) we can prove that Lemmas 3.6, 3.8 (replacing, in (3.8), L with $\frac{L}{1+t}$) and 3.10 holds true in this new time interval, provided ζ small enough. We can now proceed as in the proof of Theorem 2.1, proving that (3.15) and (3.16) hold in the time interval $(0, T_{\varepsilon,\beta} \wedge \varepsilon^{-\zeta})$.

4 Vortices in viscous fluids

In this section we discuss the viscous case; in subsection 4.1 we study the motion of the i-th term of the vorticity $\omega_{i,\varepsilon}(x,t)$ in (2.11). In particular, we prove some a priori estimates which allow to conclude the proof of Theorem 2.2 in subsection 4.2 via a bootstrap argument.

4.1 The reduced system

Recalling the definition of R_m in the statement of Theorem 2.2 and the function $W_R(x)$ in the proof of Lemma 3.10 and defining $R_* := R_m/10$, we decompose the vector field $F^i_\varepsilon(x,t)$ into a sum of two terms, namely

$$F^i_\varepsilon(x,t) = F^i_{\varepsilon,1}(x,t) + F^i_{\varepsilon,2}(x,t)$$

where

$$F^i_{\varepsilon,1}(x,t) := \sum_{j \neq i} \int dy\, K(x-y)\,(1 - W_{R_*}(x-y))\,\omega_{j,\varepsilon}(y,t) \tag{4.1}$$

is a smooth, divergence free, uniformly bounded, time dependent vector field which satisfies the Lipschitz condition

$$|F^i_{\varepsilon,1}(x,t) - F^i_{\varepsilon,1}(x',t)| \le L|x-x'|, \quad \forall\, x,x' \in \mathbb{R}^2 \tag{4.2}$$

for some constant L depending only on R_*, and

$$F^i_{\varepsilon,2}(x,t) := \sum_{j\neq i} \int dy\, K(x-y)\, W_{R_*}(x-y)\, \omega_{j,\varepsilon}(y,t). \tag{4.3}$$

Recalling the definition of $B^i_\varepsilon(t)$ in (2.12), for any $\alpha > 0$ we define

$$T^* = T^*(\varepsilon,\alpha) := \min_{i=1,\dots,N} \sup\left\{ t>0 : \sup_{|x-B^i_\varepsilon(s)|\le 3R_*} |F^i_{\varepsilon,2}(x,s)| \le \varepsilon^\alpha \ \forall\, s\in[0,t] \right\};$$

Since $F^i_{\varepsilon,2}(x,0)=0$ for any $|x-B^i_\varepsilon(0)| \le 3R_*$, then, by continuity[6] $T^*>0$, and hence $F^i_{\varepsilon,2}(x,t)$ satisfies, for $t\in[0,T^*]$ and $|x-B^i_\varepsilon(t)|\le 3R_*$, the estimate

$$|F^i_{\varepsilon,2}(x,t)| \le \varepsilon^\alpha. \tag{4.4}$$

Therefore, for $t \in [0,T^*]$ the i-th term of vorticity moves according to the evolution equation

$$\begin{cases} \dfrac{d}{dt}\omega^{i,\varepsilon}_t(f) = \omega^{i,\varepsilon}_t((u^i_\varepsilon + F^i_\varepsilon)\cdot\nabla f) + \nu\omega^{i,\varepsilon}_t(\Delta f) + \omega^{i,\varepsilon}_t(\partial_t f) \\ \omega^{i,\varepsilon}_0(dx) = \omega_{i,\varepsilon}(x,0)\, dx. \end{cases} \tag{4.5}$$

The latter can be thought as the evolution equation of a *single* compactly-supported blob of initial vorticity $\omega_{i,\varepsilon}(x,0)$ under the action of an external field $F^i_\varepsilon(x,t)$ satisfying the conditions (4.2) and (4.4) till time T^*. For the sake of brevity, in this section, we suppress the index i, so that (4.5) reads

$$\begin{cases} \dfrac{d}{dt}\omega^{\varepsilon}_t(f) = \omega^{\varepsilon}_t((u_\varepsilon + F_\varepsilon)\cdot\nabla f) + \nu\omega^{\varepsilon}_t(\Delta f) + \omega^{\varepsilon}_t(\partial_t f) \\ \omega^{\varepsilon}_0 = \omega^0_\varepsilon(x)\, dx \end{cases} \tag{4.6}$$

for all $C^{2,1}$ bounded functions $f = f(x,t)$ with bounded first and second derivatives.

Since each term $\omega_{i,\varepsilon}(x,t)$ preserves at any time the initial sign and the total mass (as immediately follows from the definition), we can suppose $\omega_\varepsilon(x,t) \ge 0 \ \ \forall\, t\ge 0$, $\int dx\, \omega_\varepsilon(x,t) = 1$ and that $\operatorname{supp}\omega_\varepsilon(x,0) \subset \Sigma(z_*|\varepsilon)$.

The field $u_\varepsilon(x,t)$ is the velocity field generated by $\omega_{i,\varepsilon}(x,t)$, which here reads

$$u_\varepsilon(x,t) = \int dy\, K(x-y)\, \omega_\varepsilon(y,t).$$

[6] The continuity property of $F^i_{\varepsilon,2}$ w.r.t. (x,t) is shown in the Appendix.

The field $F_\varepsilon(x,t)$ is the sum of the two terms $F_{\varepsilon,1}(x,t) + F_{\varepsilon,2}(x,t)$ in (4.1) and (4.3), and they respectively satisfy the conditions

$$|F_{\varepsilon,1}(x,t) - F_{\varepsilon,1}(x',t)| \le L|x-x'|, \quad \forall\, x,x' \in \mathbb{R}^2$$

and, for $t \in [0,T^*]$,

$$|F_{\varepsilon,2}(x,t)| \le \varepsilon^\alpha, \qquad \text{if } \ |x - B_\varepsilon(t)| \le 3R_*$$

where $B_\varepsilon(t)$ is the solution to the ordinary differential equation generated by F_ε with initial condition $B_\varepsilon(0) = z_*$ (here z_* plays the role of z_i).
So, in our convention of suppressing the index i, (2.13) here reads

$$m_\varepsilon(R,t) = \int_{|x-B_\varepsilon(t)|>R} dx\, \omega_\varepsilon(x,t).$$

Finally, we introduce a truncated moment of inertia with respect to $B_\varepsilon(t)$, by setting

$$I_\varepsilon(R,t) := \int dx\, \omega_\varepsilon(x,t)\, |x - B_\varepsilon(t)|^2\, W_R(x - B_\varepsilon(t)).$$

We start by studying the growth in time of the moment of inertia. Note that we can perform the time derivative of the moment of inertia by following the weak formulation (4.6), thanks to the identity

$$I_\varepsilon(R,t) = \omega_t^\varepsilon(f_R),$$

with $f_R = f_R(x,t) = |x - B_\varepsilon(t)|^2\, W_R(x - B_\varepsilon(t))$. Without introducing the function W_R, the "total" moment of inertia

$$\int dx\, \omega_\varepsilon(x,t)\, |x - B_\varepsilon(t)|^2,$$

used in the previous section, could be a priori divergent because $\omega_\varepsilon(x,t)$ has not a compact support like in the inviscid case, and hence the application of the weak formulation is not guaranteed.

Lemma 4.1. *There exists a constant C_1 such that, for every $R \le \min\{R_*/2, 1\}$, $0 \le t \le T^*$, and $\varepsilon > 0$,*

$$\frac{d}{dt} I_\varepsilon(R,t) \le C_1 \big[I_\varepsilon(R,t) + \frac{1}{R} m_\varepsilon(R,t) + \varepsilon^\alpha \big]. \tag{4.7}$$

Proof. As claimed before, we use (4.6) with $f(x,t) = |x - B_\varepsilon(t)|^2 \, W_R(x - B_\varepsilon(t))$:

$$\begin{aligned}
\frac{d}{dt} I_\varepsilon(R,t) &= \frac{d}{dt}\omega_t^\varepsilon\Big(|x - B_\varepsilon(t)|^2 \, W_R(x - B_\varepsilon(t))\Big) \\
&= \omega_t^\varepsilon\Big(\big(u_\varepsilon(x,t) + F_\varepsilon(x,t)\big) \cdot \nabla\big(|x - B_\varepsilon(t)|^2 \, W_R(x - B_\varepsilon(t))\big)\Big) \\
&+ \nu\omega_t^\varepsilon\Big(\Delta\big(|x - B_\varepsilon(t)|^2 \, W_R(x - B_\varepsilon(t))\big)\Big) \\
&+ \omega_t^\varepsilon\Big(\partial_t\big(|x - B_\varepsilon(t)|^2 \, W_R(x - B_\varepsilon(t))\big)\Big) \\
&= A + D + E,
\end{aligned}$$

where

$$A = \omega_t^\varepsilon\Big(2\big(x - B_\varepsilon(t)\big) \cdot \big[u_\varepsilon(x,t) + F_\varepsilon(x,t) - \frac{d}{dt}B_\varepsilon(t)\big] \, W_R(x - B_\varepsilon(t))\Big)$$

$$D = \omega_t^\varepsilon\Big(|x - B_\varepsilon(t)|^2 \, \big[u_\varepsilon(x,t) + F_\varepsilon(x,t) - \frac{d}{dt}B_\varepsilon(t)\big] \cdot \nabla W_R(x - B_\varepsilon(t))\Big)$$

$$\begin{aligned}
E = \nu\omega_t^\varepsilon\Big(&|x - B_\varepsilon(t)|^2 \, \Delta W_R(x - B_\varepsilon(t)) + 4(x - B_\varepsilon(t)) \cdot \nabla W_R(x - B_\varepsilon(t)) \\
&+ 4W_R(x - B_\varepsilon(t))\Big).
\end{aligned}$$

We estimate separately the terms A, D and E, starting with A. We decompose,

$$\begin{aligned}
A &= 2\int dx \, \omega_\varepsilon(x,t) \, W_R(x - B_\varepsilon(t)) \, \big(x - B_\varepsilon(t)\big) \\
&\quad \cdot \int dy \, \omega_\varepsilon(y,t) \, \big\{K(x-y) + F_\varepsilon(x,t) - F_\varepsilon(B_\varepsilon(t),t)\big\} \\
&= A_1 + A_2 + A_3
\end{aligned}$$

where

$$\begin{aligned}
A_1 = &\int dx \int dy \, \omega_\varepsilon(x,t) \, \omega_\varepsilon(y,t) \, K(x-y) \\
&\cdot \big\{W_R(x - B_\varepsilon(t))(x - B_\varepsilon(t)) - W_R(y - B_\varepsilon(t))(y - B_\varepsilon(t))\big\}
\end{aligned}$$

$$A_2 = 2\int dx \, \omega_\varepsilon(x,t) \, W_R(x - B_\varepsilon(t)) \, (x - B_\varepsilon(t)) \cdot \Big[F_{\varepsilon,1}(x,t) - F_{\varepsilon,1}(B_\varepsilon(t),t)\Big]$$

$$A_3 = 2\int dx \, \omega_\varepsilon(x,t) \, W_R(x - B_\varepsilon(t)) \, (x - B_\varepsilon(t)) \cdot \Big[F_{\varepsilon,2}(x,t) - F_{\varepsilon,2}(B_\varepsilon(t),t)\Big].$$

Above, we used the antisymmetry of $K(x-y)$, and the decomposition of the external field F_ε.

Let's start from A_1; we first observe that the integrand vanishes when both $|x-B_\varepsilon(t)|$ and $|y-B_\varepsilon(t)|$ are less than R (because $W_R \equiv 1$ and $K(x-y)\cdot(x-y)=0$) or larger than $2R$ (because $W_R \equiv 0$). Moreover, the integrand is bounded where x is near y; this is not trivial because of the singularity of $K(x-y)$, but this singularity is compensated by a zero of the same order. More precisely, we notice that

$$\begin{aligned}
&\big|W_R(x-B_\varepsilon(t))\,(x-B_\varepsilon(t)) - W_R(y-B_\varepsilon(t))\,(y-B_\varepsilon(t))\big| \\
&= \frac{1}{2}\big|(W_R(x-B_\varepsilon(t)) + W_R(y-B_\varepsilon(t)))\,\big[(x-B_\varepsilon(t))-(y-B_\varepsilon(t))\big] \\
&\quad + (W_R(x-B_\varepsilon(t)) - W_R(y-B_\varepsilon(t)))\,\big[(x-B_\varepsilon(t))+(y-B_\varepsilon(t))\big]\big| \\
&\le |x-y| + \frac{C}{2R}|x-y|\big[|x-B_\varepsilon(t)|+|y-B_\varepsilon(t)|\big]
\end{aligned} \tag{4.8}$$

and decompose

$$\begin{aligned}
A_1 = \Bigg(&\int_{|x-B_\varepsilon(t)|<R} dx \int_{2R>|y-B_\varepsilon(t)|>R} dy + \int_{|x-B_\varepsilon(t)|<R} dx \int_{|y-B_\varepsilon(t)|>2R} dy \\
&+ \int_{2R>|x-B_\varepsilon(t)|>R} dx \int_{|y-B_\varepsilon(t)|<R} dy + \int_{2R>|x-B_\varepsilon(t)|>R} dx \int_{2R>|y-B_\varepsilon(t)|>R} dy \\
&+ \int_{2R>|x-B_\varepsilon(t)|>R} dx \int_{|y-B_\varepsilon(t)|>2R} dy + \int_{|x-B_\varepsilon(t)|>2R} dx \int_{|y-B_\varepsilon(t)|<2R} dy\Bigg) \\
&\Bigg(\omega_\varepsilon(x,t)\,\omega_\varepsilon(y,t)\,K(x-y)\cdot\big\{W_R(x-B_\varepsilon(t))\,(x-B_\varepsilon(t)) \\
&\quad - W_R(y-B_\varepsilon(t))\,(y-B_\varepsilon(t))\big\}\Bigg).
\end{aligned}$$

In the first, third and fourth term we apply (4.8) and obtain that each one, in absolute value, is bounded by $(1+2C)m_\varepsilon(R,t)$.
In the second term $W_R(y-B_\varepsilon(t))=0$, so it is equal to

$$\begin{aligned}
&\int_{|x-B_\varepsilon(t)|<R} dx \int_{|y-B_\varepsilon(t)|>2R} dy\,\omega_\varepsilon(x,t)\,\omega_\varepsilon(y,t)\,K(x-y)\cdot \\
&\cdot \big\{W_R(x-B_\varepsilon(t))\,(x-B_\varepsilon(t)) - W_R(y-B_\varepsilon(t))\,(x-B_\varepsilon(t))\big\}
\end{aligned}$$

and then, taking the absolute value and using the Lipschitz condition, we get that it's bounded by

$$\begin{aligned}
&\int_{|x-B_\varepsilon(t)|<R} dx \int_{|y-B_\varepsilon(t)|>2R} dy\,\omega_\varepsilon(x,t)\,\omega_\varepsilon(y,t)\,\frac{C}{2\pi R}|x-B_\varepsilon(t)| \\
&\quad \le \frac{C}{2\pi} m_\varepsilon(2R,t) \le \frac{C}{2\pi} m_\varepsilon(R,t).
\end{aligned}$$

For the fifth term, we apply the same arguments used for the second one. In the sixth term $W_R(x - B_\varepsilon(t)) = 0$ and so we can repeat also here the arguments used for the second one. So we found that there exist a constant C such that $|A_1| \leq C m_\varepsilon(R,t)$.
Concerning the term A_2, we use the Lipschitz condition getting

$$|A_2| \leq 2L \int dx\, \omega_\varepsilon(x,t)\, |x - B_\varepsilon(t)|^2\, W_R(x - B_\varepsilon(t)) = 2L I_\varepsilon(R,t)$$

while $|A_3|$ is bounded by $4RC\varepsilon^\alpha \leq 4C\varepsilon^\alpha$. So

$$|A| \leq C m_\varepsilon(R,t) + 2L I_\varepsilon(R,t) + 4C\varepsilon^\alpha. \tag{4.9}$$

We now study the term D that we decompose as $D = D_1 + D_2 + D_3$, where

$$D_1 = \int dx \int dy\, \omega_\varepsilon(x,t)\, \omega_\varepsilon(y,t)\, |x - B_\varepsilon(t)|^2\, K(x-y) \cdot \nabla W_R(x - B_\varepsilon(t))$$

$$D_2 = \int dx\, \omega_\varepsilon(x,t)\, |x - B_\varepsilon(t)|^2 \left[F_{\varepsilon,1}(x,t) - F_{\varepsilon,1}(B_\varepsilon(t),t)\right] \cdot \nabla W_R(x - B_\varepsilon(t))$$

$$D_3 = \int dx\, \omega_\varepsilon(x,t)\, |x - B_\varepsilon(t)|^2 \left[F_{\varepsilon,2}(x,t) - F_{\varepsilon,2}(B_\varepsilon(t),t)\right] \cdot \nabla W_R(x - B_\varepsilon(t)).$$

Using the Lipschitz condition and that $|\nabla W_R| \leq \dfrac{C}{R}$ we have,

$$\begin{aligned}|D_2| = &\int_{2R > |x - B_\varepsilon(t)| > R} dx\, \omega_\varepsilon(x,t)\, |x - B_\varepsilon(t)|^2 \left\{\left[F_{\varepsilon,1}(x,t) - F_{\varepsilon,1}(B_\varepsilon(t),t)\right] \cdot \right.\\ &\left. \cdot \nabla W_R(x - B_\varepsilon(t))\right\} \leq \frac{LC}{R} \int_{2R > |x - B_\varepsilon(t)| > R} dx\, \omega_\varepsilon(x,t)\, |x - B_\varepsilon(t)|^3 \\ &\leq 8LCR^2 m_\varepsilon(R,t).\end{aligned}$$

The term $|D_3|$ can be easily bounded by $8CR\varepsilon^\alpha m_\varepsilon(R,t) \leq 8C\varepsilon^\alpha$.
To bound the term $|D_1|$ we use the same trick used for A_1: using the antisymmetry of $K(x-y)$ we write

$$\begin{aligned}D_1 = &\frac{1}{2} \int dx \int dy\, \omega_\varepsilon(x,t)\, \omega_\varepsilon(y,t)\, K(x-y) \\ &\cdot \left\{|x - B_\varepsilon(t)|^2\, \nabla W_R(x - B_\varepsilon(t)) - |y - B_\varepsilon(t)|^2\, \nabla W_R(y - B_\varepsilon(t))\right\}.\end{aligned}$$

We first observe that the integrand vanishes when both $|x - B_\varepsilon(t)|$ and $|y - B_\varepsilon(t)|$ are less than R or greater than $2R$. Moreover, the integrand is bounded where x

is near y; more precisely,

$$
\begin{aligned}
&\big[|x-B_\varepsilon(t)|^2\,\nabla W_R(x-B_\varepsilon(t)) - |y-B_\varepsilon(t)|^2\,\nabla W_R(y-B_\varepsilon(t))\big] \\
&= \frac{1}{2}\big\{\big[\nabla W_R(x-B_\varepsilon(t)) + \nabla W_R(y-B_\varepsilon(t))\big]\big[|x-B_\varepsilon(t)|^2 - |y-B_\varepsilon(t)|^2\big] \\
&\quad + \big[\nabla W_R(x-B_\varepsilon(t)) - \nabla W_R(y-B_\varepsilon(t))\big]\big[|x-B_\varepsilon(t)|^2 + |y-B_\varepsilon(t)|^2\big]\big\} \\
&\le \frac{C}{R^2}|x-y|\big[|x-B_\varepsilon(t)| + |y-B_\varepsilon(t)| + |x-B_\varepsilon(t)|^2 + |y-B_\varepsilon(t)|^2\big].
\end{aligned}
$$

We thus obtain that $|D_1| \le \dfrac{C}{R} m_\varepsilon(R,t)$ and so

$$
|D| \le \left(\frac{C}{R} + 8LC\right) m_\varepsilon(R,t) + 8C^2\varepsilon^\alpha. \tag{4.10}
$$

Finally, recalling that $\nu \le \nu_0\varepsilon^\alpha$, the term $|E|$ is easily bounded,

$$
|E| \le (4 + 12C)\nu_0\varepsilon^\alpha. \tag{4.11}
$$

The lemma follows by (4.9), (4.10) and (4.11). □

We now need a bound for $m_\varepsilon(R,t)$. To do this, we use the mollified version of $m_\varepsilon(R,t)$ introduced in Lemma (3.10), namely

$$
\mu_\varepsilon(R,t) := 1 - \int dx\, \omega_\varepsilon(x,t)\, W_R(x - B_\varepsilon(t));
$$

as we shall see, a control on $t \mapsto \mu_\varepsilon(R,t)$ gives us a control on $t \mapsto m_\varepsilon(R,t)$. In the next lemma we perform the derivative in time of μ_ε to obtain a bound for m_ε.

Lemma 4.2. *There exists a constant C_2 such that, for every $R \le \min\{R_*/2, 1\}$, $0 \le t \le T^*$, and $0 < \varepsilon < \dfrac{R}{2}$,*

$$
m_\varepsilon(R,t) \le C_2\left(\frac{1}{R^4} + \frac{\varepsilon^\alpha}{R^4}\right)\int_0^t ds\, I_\varepsilon(R,s). \tag{4.12}
$$

Proof. We investigate the growth in time of $\mu_\varepsilon(R,t)$: using the weak formulation we have

$$
\begin{aligned}
\frac{d}{dt}\mu_\varepsilon(R,t) = -\frac{d}{dt}\omega_t^\varepsilon\Big(W_R(x-B_\varepsilon(t))\Big) &= -\omega_t^\varepsilon\Big(\big(u_\varepsilon(x,t)+F_\varepsilon(x,t)\big) \\
&\cdot \nabla W_R(x-B_\varepsilon(t))\Big) - \nu\omega_t^\varepsilon\Big(\Delta W_R(x-B_\varepsilon(t))\Big) \\
&+ \omega_t^\varepsilon\Big(\nabla W_R(x-B_\varepsilon(t))\cdot F_\varepsilon(B_\varepsilon(t),t)\Big) \\
&= \omega_t^\varepsilon\Big(\big[F_\varepsilon(B_\varepsilon(t),t)-u_\varepsilon(x,t)-F_\varepsilon(x,t)\big]\cdot\nabla W_R(x-B_\varepsilon(t))\Big) \\
&- \nu\omega_t^\varepsilon\Big(\Delta W_R(x-B_\varepsilon(t))\Big) = H_1+H_2+H_3+H_4
\end{aligned}
$$

where

$$
\begin{aligned}
H_1 &= \int dx\,\omega_\varepsilon(x,t)\,\big[F_{\varepsilon,1}(B_\varepsilon(t),t)-F_{\varepsilon,1}(x,t)\big]\cdot\nabla W_R(x-B_\varepsilon(t)) \\
H_2 &= \int dx\,\omega_\varepsilon(x,t)\,\big[F_{\varepsilon,2}(B_\varepsilon(t),t)-F_{\varepsilon,2}(x,t)\big]\cdot\nabla W_R(x-B_\varepsilon(t)) \\
H_3 &= -\int dx\,\omega_\varepsilon(x,t)\,u_\varepsilon(x,t)\cdot\nabla W_R(x-B_\varepsilon(t)) \\
H_4 &= -\nu\int dx\,\omega_\varepsilon(x,t)\,\Delta W_R(x-B_\varepsilon(t)).
\end{aligned}
$$

Using the Lipschitz condition and the bound on $|\nabla W_R|$ we have

$$
\begin{aligned}
|H_1| = \Bigg|\int_{2R>|x-B_\varepsilon(t)|>R} &dx\,\omega_\varepsilon(x,t)\,\big[F_{\varepsilon,1}(B_\varepsilon(t),t)-F_{\varepsilon,1}(x,t)\big] \\
&\cdot\nabla W_R(x-B_\varepsilon(t))\Bigg| \\
&\le \frac{LC}{R}\int_{2R>|x-B_\varepsilon(t)|>R} dx\,\omega_\varepsilon(x,t)\,|x-B_\varepsilon(t)| \\
&\le 2LC\big(m_\varepsilon(R,t)-m_\varepsilon(2R,t)\big).
\end{aligned}
\tag{4.13}
$$

Similarly

$$
|H_2| \le \frac{2C}{R}\varepsilon^\alpha\big(m_\varepsilon(R,t)-m_\varepsilon(2R,t)\big) \tag{4.14}
$$

and

$$
|H_4| \le \frac{C\nu_0}{R^2}\varepsilon^\alpha\big(m_\varepsilon(R,t)-m_\varepsilon(2R,t)\big). \tag{4.15}
$$

We now study the term H_3. Using the antisymmetry of $K(x-y)$ we obtain

$$
\begin{aligned}
H_3 = \frac{1}{2} \int dx \int dy\, \omega_\varepsilon(x,t)\, \omega_\varepsilon(y,t) \left[\nabla W_R(x - B_\varepsilon(t)) - \nabla W_R(y - B_\varepsilon(t)) \right] \cdot \\
\cdot K(x-y) = \frac{1}{2} \Big(\int_{|x-B_\varepsilon(t)|<R} dx \int_{2R>|y-B_\varepsilon(t)|>R} dy \\
+ \int_{2R>|x-B_\varepsilon(t)|>R} dx \int dy + \int_{|x-B_\varepsilon(t)|>2R} dx \int_{2R>|y-B_\varepsilon(t)|>R} dy \Big) \\
\Big(\omega_\varepsilon(x,t)\, \omega_\varepsilon(y,t) \left[\nabla W_R(x - B_\varepsilon(t)) - \nabla W_R(y - B_\varepsilon(t)) \right] \cdot K(x-y) \Big);
\end{aligned}
$$

the Lipschitz condition and the bound on $|\Delta W_R|$ imply

$$
|H_3| \le \frac{3C}{4\pi R^2} \big(m_\varepsilon(R,t) - m_\varepsilon(2R,t) \big). \tag{4.16}
$$

Putting together the bounds (4.13), (4.14), (4.15), (4.16) we get

$$
\frac{d}{dt} \mu_\varepsilon(R,t) \le C \left(1 + \frac{1}{R^2} + \frac{\varepsilon^\alpha}{R^2} \right) \Big(m_\varepsilon(R,t) - m_\varepsilon(2R,t) \Big). \tag{4.17}
$$

Since $\varepsilon < R/2$, $\mu_\varepsilon(R,0) = 0$, and so integrating the above inequality, we obtain

$$
\mu_\varepsilon(R,t) \le C \left(1 + \frac{1}{R^2} + \frac{\varepsilon^\alpha}{R^2} \right) \int_0^t ds \, \big(m_\varepsilon(R,s) - m_\varepsilon(2R,s) \big). \tag{4.18}
$$

We now observe two simple facts:

$$
\begin{aligned}
I_\varepsilon(R,t) &= \int dx\, \omega_\varepsilon(x,t)\, |x - B_\varepsilon(t)|^2\, W_R(x - B_\varepsilon(t)) \\
&\ge \int_{R>|x-B_\varepsilon(t)|>\frac{R}{2}} dx\, \omega_\varepsilon(x,t)\, |x - B_\varepsilon(t)|^2 \\
&\ge \frac{R^2}{4} \Big(m_\varepsilon(R/2,t) - m_\varepsilon(R,t) \Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_\varepsilon(R/2,t) &= 1 - \int dx\, \omega_\varepsilon(x,t)\, W_{\frac{R}{2}}(x - B_\varepsilon(t)) \\
&= 1 - \int_{|x-B_\varepsilon(t)| \le R} dx\, \omega_\varepsilon(x,t)\, W_{\frac{R}{2}}(x - B_\varepsilon(t)) \\
&\ge 1 - \int_{|x-B_\varepsilon(t)| \le R} dx\, \omega_\varepsilon(x,t) = m_\varepsilon(R,t).
\end{aligned} \tag{4.19}
$$

So, evaluating (4.18) in $\dfrac{R}{2}$ and using the above two facts, we obtain that there exists a constant C_2 such that

$$m_\varepsilon(R,t) \le C_2\left(\frac{1}{R^4}+\frac{\varepsilon^\alpha}{R^4}\right)\int_0^t ds\, I_\varepsilon(R,s)$$

for all $\varepsilon < \dfrac{R}{2}$ and $0 \le t \le T^*$ and this completes the proof. $\square$

In the sequel, we will also need to control the time derivative of $\mu_\varepsilon(R,t)$ by means of the function $m_\varepsilon(R,t)$. As a matter of fact, the estimate (4.17) is not useful for our purposes because of the term $\frac{1}{R^2}$. To get the right bound, we have to improve the estimate of the term H_3 in Lemma 4.2. This is the content of the next lemma.

Lemma 4.3. *There exists a constant C_3 such that, for every $\varepsilon > 0$, $2\varepsilon^{\frac{\alpha}{10}} < R \le \min\{R_*/2, 1\}$, and $0 \le t \le T^*$,*

$$\frac{d}{dt}\mu_\varepsilon(R,t) \le C_3\left(1+\frac{\varepsilon^\alpha}{R^2}+\frac{\varepsilon^{-\frac{\alpha}{10}}I_\varepsilon(R,t)}{R^3}+\frac{I_\varepsilon(R,t)}{R^4}+\frac{m_\varepsilon(R,t)}{R^2}\right)m_\varepsilon(R,t). \tag{4.20}$$

Proof. As already mentioned, we have to improve the estimate of the term H_3. Using the antisymmetry of $K(x-y)$ we have,

$$H_3 = \frac{1}{2}\int dx \int dy\, Q_\varepsilon^R(x,y,t), \tag{4.21}$$

where

$$Q_\varepsilon^R(x,y,t) := \omega_\varepsilon(x,t)\,\omega_\varepsilon(y,t)\left[\nabla W_R(x-B_\varepsilon(t)) - \nabla W_R(y-B_\varepsilon(t))\right]\cdot K(x-y).$$

We now split the integration domain in several sets defined as follows. Let $h \in \mathbb{N}$ and set

$$T_h = \Big\{(x,y) : x \notin \Sigma_R(B_\varepsilon(t)),\, y \in \Sigma_{a_h}(B_\varepsilon(t)) \setminus \Sigma_{a_{h-1}}(B_\varepsilon(t))\Big\} \;\text{ if }\; h<n$$

$$T_n = \Big\{(x,y) : x \notin \Sigma_R(B_\varepsilon(t)),\, y \notin \Sigma_{a_{n-1}}(B_\varepsilon(t))\Big\} \;\text{ if }\; h=n$$

$$S_h = \Big\{(x,y) : y \notin \Sigma_R(B_\varepsilon(t)),\, x \in \Sigma_{a_h}(B_\varepsilon(t)) \setminus \Sigma_{a_{h-1}}(B_\varepsilon(t))\Big\} \;\text{ if }\; h<n$$

$$S_n = \Big\{(x,y) : y \notin \Sigma_R(B_\varepsilon(t)),\, x \notin \Sigma_{a_{n-1}}(B_\varepsilon(t))\Big\} \;\text{ if }\; h=n$$

where a_h is defined as

$$a_0 = 0, \quad a_1 = \varepsilon^{\frac{\alpha}{10}}, \quad a_{h+1} = 2a_h$$

and $n = n(\varepsilon, R) \in \mathbb{N}$ is chosen in such a way that $a_{n+1} \le R$ and $a_{n+2} > R$ (recall the assumptions on R). We notice that the integrand in (4.21) vanishes in the complement of $\bigcup_{h=1}^{n} \big(T_h \cup S_h\big)$, so we only have to consider the integration on T_h and S_h $\forall\, h = 1, \dots, n$.
By using that $K(x - B_\varepsilon(t)) \cdot \nabla W_R(x - B_\varepsilon(t)) = 0$ and that $\nabla W_R(y - B_\varepsilon(t)) = 0$ if $y \in \Sigma_{a_h}(B_\varepsilon(t)) \setminus \Sigma_{a_{h-1}}(B_\varepsilon(t))$ and $h < n$, we have, for any $h < n$,

$$\int_{T_h} dx\, dy\, Q_\varepsilon^R(x, y, t) = \int_{|x - B_\varepsilon(t)| > R} dx \int_{\Sigma_{a_h}(B_\varepsilon(t)) \setminus \Sigma_{a_{h-1}}(B_\varepsilon(t))} dy\, \omega_\varepsilon(x, t)\, \omega_\varepsilon(y, t) \big[K(x - y) - K(x - B_\varepsilon(t))\big] \cdot \nabla W_R(x - B_\varepsilon(t)). \tag{4.22}$$

We now observe that, by the explicit form of $K(x)$,

$$\big|K(x' - y') - K(x')\big| \le \frac{3}{2\pi} \frac{\gamma}{|x'|(|x'| - \gamma)} \quad \text{if } |y'| < \gamma < |x'| \tag{4.23}$$

(the proof of this fact is postponed to the end of this lemma), so

$$\begin{aligned} \big|K(x - y) - K(x - B_\varepsilon(t))\big| &= \big|K(x - B_\varepsilon(t) + B_\varepsilon(t) - y) - K(x - B_\varepsilon(t))\big| \\ &\le \frac{a_h}{R(R - a_h)} \end{aligned} \tag{4.24}$$

where we used (4.23) with $x' = (x - B_\varepsilon(t))$, $y' = (y - B_\varepsilon(t))$ and $\gamma = a_h$ and that $|x'| > R$. From (4.22), (4.24), and using that $|\nabla W_R| \le \frac{C}{R}$, we obtain

$$\begin{aligned} \left|\int_{T_h} dx\, dy\, Q_\varepsilon^R(x, y, t)\right| &\le \frac{C a_h}{R^2(R - a_h)} \big[m_\varepsilon(a_{h-1}, t) - m_\varepsilon(a_h, t)\big] m_\varepsilon(R, t) \\ &\le \frac{C a_h}{R^2(R - a_h)} \frac{1}{a_{h-1}^2} I_\varepsilon(R, t) m_\varepsilon(R, t). \end{aligned}$$

where we also used in the last inequality that for $h < n$

$$\begin{aligned} m_\varepsilon(a_{h-1}, t) - m_\varepsilon(a_h, t) &= \int_{a_h > |x - B_\varepsilon(t)| > a_{h-1}} dx\, \omega_\varepsilon(x, t)\, W_R(x - B_\varepsilon(t)) \\ &\le \frac{1}{a_{h-1}^2} I_\varepsilon(R, t). \end{aligned}$$

Summing on $h = 1, \dots, n-1$ we get

$$\begin{aligned} \left|\int_{\bigcup_{h=1}^{n-1} T_h} dx\, dy\, Q_\varepsilon^R(x, y, t)\right| &\le \left\{\sum_{h=1}^{n-1} \frac{C a_h}{R^2(R - a_h) a_{h-1}^2}\right\} I_\varepsilon(R, t) m_\varepsilon(R, t) \\ &\le \frac{16 \varepsilon^{-\frac{\alpha}{10}}}{R^3} I_\varepsilon(R, t) m_\varepsilon(R, t) \end{aligned}$$

because

$$\sum_{h=1}^{n-1} \frac{Ca_h}{R^2(R-a_h)a_{h-1}^2} = \frac{4}{R^2} \sum_{h=1}^{n-1} \frac{a_h}{(R-a_h)a_h^2} = \frac{4\varepsilon^{-\frac{\alpha}{10}}}{R^2} \sum_{h=1}^{n-1} \frac{1}{2^{h-1}} \frac{1}{R-a_h}$$
$$\leq \frac{4\varepsilon^{-\frac{\alpha}{10}}}{R^2(R-a_n)} \sum_{h=1}^{n-1} \frac{1}{2^{h-1}} \leq \frac{16\varepsilon^{-\frac{\alpha}{10}}}{R^3},$$

where we used that $R - a_n = R - a_{n+1}/2 \geq R - R/2 = R/2$.
Concerning the integration on T_n, we decompose

$$\int_{T_n} dx\, dy\, Q_\varepsilon^R(x,y,t) = J_1 + J_2$$

where

$$J_1 = \int_{|x-B_\varepsilon(t)|>R} dx \int_{R\geq|y-B_\varepsilon(t)|>a_{n-1}} dy\, Q_\varepsilon^R(x,y,t)$$

and

$$J_2 = \int_{|x-B_\varepsilon(t)|>R} dx \int_{|y-B_\varepsilon(t)|>R} dy\, Q_\varepsilon^R(x,y,t).$$

Using the Lipschitz condition, we have

$$|J_1| \leq \int_{|x-B_\varepsilon(t)|>R} dx \int_{R\geq|y-B_\varepsilon(t)|>a_{n-1}} dy\, \omega_\varepsilon(x,t)\, \omega_\varepsilon(y,t)\, \frac{C}{2\pi R^2}$$
$$\leq \frac{C}{2\pi R^2 a_{n-1}^2} I_\varepsilon(R,t) m_\varepsilon(R,t) \leq \frac{32C}{\pi R^4} I_\varepsilon(R,t) m_\varepsilon(R,t)$$

where in the last inequality we used that $a_{n-1} = a_n/2 = a_{n+1}/4 = a_{n+2}/8 > R/8$.
Again, by the Lipschitz condition,

$$|J_2| \leq \frac{C}{2\pi R^2} m_\varepsilon(R,t) m_\varepsilon(R,t).$$

The contribution to H_3 coming from the integration on the sets S_h, $h \leq n$, can be treated in the same way and we omit the details. In conclusion,

$$|H_3| \leq \frac{16\varepsilon^{-\frac{\alpha}{10}}}{R^3} I_\varepsilon(R,t) m_\varepsilon(R,t) + \frac{32C}{\pi R^4} I_\varepsilon(R,t) m_\varepsilon(R,t) + \frac{C}{2\pi R^2} m_\varepsilon(R,t) m_\varepsilon(R,t). \tag{4.25}$$

The estimate (4.20) now follows from (4.13), (4.14), (4.15) (in which we remove the term $m_\varepsilon(2R,t)$) and (4.25), for a suitable costant C_3.

We are left with the proof of (4.23). To simplify the notation we use x and y instead of x' and y', and we observe that

$$\big|K(x-y)-K(x)\big| = \frac{1}{2\pi}\left|\frac{(x-y)^\perp}{|x-y|^2}-\frac{x^\perp}{|x|^2}\right| = \frac{1}{2\pi}\left|\frac{|x|^2(x-y)^\perp-|x-y|^2x^\perp}{|x|^2|x-y|^2}\right|. \tag{4.26}$$

We have,

$$\begin{aligned}
&|x|^2(x-y)^\perp-|x-y|^2x^\perp = -|x|^2y^\perp-|y|^2x^\perp+2(x\cdot y)x^\perp\\
&= -|x|^2y^\perp + x^\perp(y\cdot(2x-y))\\
&= -|x|^2y^\perp + x^\perp(y\cdot x)+x^\perp(y\cdot(x-y))\\
&= (x-y)^\perp(y\cdot x)+y^\perp(y\cdot x-|x|^2)+x^\perp(y\cdot(x-y))\\
&= (x-y)^\perp(y\cdot x)+y^\perp((y-x)\cdot x)+x^\perp(y\cdot(x-y)).
\end{aligned} \tag{4.27}$$

Inserting (4.27) in (4.26) and using the Cauchy-Schwarz inequality we get

$$\big|K(x-y)-K(x)\big| \le \frac{3}{2\pi}\frac{|x||y||x-y|}{|x|^2|x-y|^2} \le \frac{3}{2\pi}\frac{\gamma}{|x|(|x|-\gamma)}$$

since $|y|<\gamma<|x|$ and $|x-y|>|x|-|y|>|x|-\gamma$, which completes the proof. □

4.2 Proof of main result for (NS)

In this section we come back to the notation of Section 3.1, which means that we reintroduce the index i to distinguish the N terms of the vorticity, and we use the estimates found in the previous section, which are valid for each $i=1,\dots,N$, to prove Theorem 2.2. In other words, the lemmas of Subsection 3.1 have to be read replacing $I_\varepsilon(R,t)$ by

$$I_\varepsilon^i(R,t) := \int dx\,|\omega_{i,\varepsilon}(x,t)|\,|x-B_\varepsilon^i(t)|^2\,W_R(x-B_\varepsilon^i(t)),$$

$m_\varepsilon(R,t)$ by

$$m_\varepsilon^i(R,t) = \int_{|x-B_\varepsilon^i(t)|>R} dx\,|\omega_{i,\varepsilon}(x,t)|,$$

and $\mu_\varepsilon(R,t)$ by

$$\mu_\varepsilon^i(R,t) := |a_i| - \int dx\,|\omega_{i,\varepsilon}(x,t)|\,W_R(x-B_\varepsilon^i(t))$$

for any $i=1,\dots,N$.

Proof of Theorem 2.2. We fix once for all $\alpha \le 2$, $\overline{\alpha} > 4\alpha + \gamma$, $\beta \in (0, \alpha/14)$, $\beta_* \in (\beta, \alpha/14)$ and $\overline{\delta} \in (0, \alpha/3)$ and we work with $R \in [\varepsilon^{\beta_*}, \varepsilon^{\beta}]$.

We preliminary observe that, since the function $t \mapsto m^i_\varepsilon(R, t)$ is continuous (see Appendix) and $m_\varepsilon(R, 0) = 0$ for $\varepsilon < R$, then $T_{\varepsilon,\overline{\alpha},\beta} > 0$. Obviously, since $B^i_\varepsilon(0) = z_i$, $\overline{T}_{\varepsilon,\overline{\delta}} > 0$ by continuity.

We split the proof into three steps: in the first one, we show that the estimates found in Subsection 3.1 are valid till time $T_{\varepsilon,\overline{\alpha},\beta} \wedge \overline{T}_{\varepsilon,\overline{\delta}}$ by proving that $T^* \ge T_{\varepsilon,\overline{\alpha},\beta} \wedge \overline{T}_{\varepsilon,\overline{\delta}}$ for ε sufficiently small. In the second step, we study the distances between the centre $B^i_\varepsilon(t)$ and the solution $z_i(t)$ to the point vortex-model, showing that $\overline{T}_{\varepsilon,\overline{\delta}} > T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_1 |\log \varepsilon|$ for some $\xi_1 > 0$ and ε small enough. Finally, in the third step, we study the vorticity mass outside the disk of centre $B^i_\varepsilon(t)$ and radius ε^β proving that $T_{\varepsilon,\overline{\alpha},\beta} > \zeta |\log \varepsilon|$ for some $\zeta > 0$ and ε small enough.

Step 1. We take $\overline{\varepsilon} \in (0, 1)$ sufficiently small such that $\varepsilon^{\overline{\delta}} < 2R_*$ and $\varepsilon^\beta < R_*$ for $\varepsilon < \overline{\varepsilon}$, and we prove that there exists $0 < \varepsilon_1 < \overline{\varepsilon}$ such that $T^* \ge T_{\varepsilon,\overline{\alpha},\beta} \wedge \overline{T}_{\varepsilon,\overline{\delta}}$ for any $0 < \varepsilon < \varepsilon_1$.

Take $t \le T_{\varepsilon,\overline{\alpha},\beta} \wedge \overline{T}_{\varepsilon,\overline{\delta}}$ and $|x - B^i_\varepsilon(t)| \le 3R_*$, then

$$
\begin{aligned}
|F^i_{\varepsilon,2}(x,t)| &\le \sum_{j \ne i} \int_{|y-x| \le 2R_*} dy\, |K(x-y)|\, |\omega_{j,\varepsilon}(y,t)| \\
&\le \sum_{j \ne i} \int_{|y-B^i_\varepsilon(t)| \le 5R_*} dy\, |K(x-y)|\, |\omega_{j,\varepsilon}(y,t)| \\
&\le \sum_{j \ne i} \int_{|y-B^j_\varepsilon(t)| > \varepsilon^\beta} dy\, |K(x-y)|\, |\omega_{j,\varepsilon}(y,t)|
\end{aligned}
$$

where in the last inequality we used that for any $j \ne i$ and $t \le \overline{T}_{\varepsilon,\overline{\delta}}$

$$
\begin{aligned}
|y - B^j_\varepsilon(t)| &\ge |y - z_j(t)| - |z_j(t) - B^j_\varepsilon(t)| \\
&\ge |z_j(t) - z_i(t)| - |z_i(t) - y| - |z_j(t) - B^j_\varepsilon(t)| \\
&\ge |z_j(t) - z_i(t)| - |z_j(t) - B^j_\varepsilon(t)| - |z_i(t) - B^i_\varepsilon(t)| \\
&\quad - |B^i_\varepsilon(t) - y| > R_m - 2R_* - 2R_* - 5R_* = R_* \\
&> \varepsilon^\beta.
\end{aligned}
$$

For any $j \ne i$, we have,

$$
\begin{aligned}
\int_{|y-B^j_\varepsilon(t)| > \varepsilon^\beta} dy\, |K(x-y)|\, |\omega_{j,\varepsilon}(y,t)| &\le \frac{1}{2\pi} \int_{|y-B^j_\varepsilon(t)| > \varepsilon^\beta} dy\, \frac{|\omega_{j,\varepsilon}(y,t)|}{|x-y|} \\
&\le \sqrt{\frac{M\varepsilon^{-\gamma} m^j_\varepsilon(\varepsilon^\beta, t)}{\pi}}
\end{aligned}
$$

where in the last inequality we applied the same argument used in the bound of the term H_2 in Lemma 3.8, consisting in the rearrangement of the vorticity as close as possible to the singularity of $1/|x-y|$.

Since for $t \le T_{\varepsilon,\overline{\alpha},\beta} \wedge \overline{T}_{\varepsilon,\overline{\delta}}$

$$m_\varepsilon^j(\varepsilon^\beta, t) \le \varepsilon^{\overline{\alpha}} \quad \forall j$$

and since $\overline{\alpha} > 4\alpha + \gamma$, we obtain

$$|F^i_{\varepsilon,2}(x,t)| \le (N-1)\sqrt{\frac{M}{\pi}}\varepsilon^{2\alpha}.$$

Choosing now ε_1 sufficiently small to make $(N-1)\sqrt{M/\pi}\varepsilon^\alpha < 1/2$ for any $0 < \varepsilon < \varepsilon_1$, by definition of T^* we get

$$T^* \ge T_{\varepsilon,\overline{\alpha},\beta} \wedge \overline{T}_{\varepsilon,\overline{\delta}} \quad \text{for any} \quad 0 < \varepsilon < \varepsilon_1. \tag{4.28}$$

At this point we observe that there exists $\varepsilon_2 < \varepsilon_1$ such that the hypothesis of the three lemmas in Section 3.3 are satisfied for each $0 < \varepsilon < \varepsilon_2$ and $R \in [\varepsilon^{\beta_*}, \varepsilon^\beta]$.
Step 2. Let $\xi_1 > 0$ (to be chosen later), and work with $0 \le t \le \overline{T}_{\varepsilon,\overline{\delta}} \wedge T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_1|\log\varepsilon|$, $0 < \varepsilon < \varepsilon_2$, and $R \in [\varepsilon^{\beta_*}, \varepsilon^\beta]$; in view of (4.28), for t, ε and R in these intervals we can use all the estimates of Subsection 3.1. Define also, for $i = 1, \dots, N$,

$$\Delta_i(t) = |B^i_\varepsilon(t) - z_i(t)|.$$

We have,

$$\begin{aligned}
\frac{d}{dt}B^i_\varepsilon(t) - \frac{d}{dt}z_i(t) &= F^i_\varepsilon(B^i_\varepsilon(t), t) - \sum_{j\ne i} a_j K(z_i(t) - z_j(t)) \\
&= \sum_{j\ne i} \int dy\, K(B^i_\varepsilon(t) - y)\, \omega_{j,\varepsilon}(y,t) - \sum_{j\ne i} a_j K(z_i(t) - z_j(t)) \\
&= Y_1 + Y_2 + Y_3 + Y_4
\end{aligned}$$

where

$$\begin{aligned}
Y_1 = &\sum_{j\ne i} \int dy\, K(B^i_\varepsilon(t) - y)\,(1 - W_{R_*}(B^i_\varepsilon(t) - y))\,\omega_{j,\varepsilon}(y,t) \\
&- \sum_{j\ne i} \int dy\, K(z_i(t) - y)\,(1 - W_{R_*}(z_i(t) - y))\,\omega_{j,\varepsilon}(y,t),
\end{aligned}$$

$$Y_2 = \sum_{j\ne i} \int dy\, K(B^i_\varepsilon(t) - y)\, W_{R_*}(B^i_\varepsilon(t) - y)\,\omega_{j,\varepsilon}(y,t),$$

$$\begin{aligned}
Y_3 = &\sum_{j\ne i} \int dy\, K(z_i(t) - y)\,(1 - W_{R_*}(z_i(t) - y))\,\omega_{j,\varepsilon}(y,t) \\
&- \sum_{j\ne i} \int dy\, K(z_i(t) - z_j(t))\,(1 - W_{R_*}(z_i(t) - y))\,\omega_{j,\varepsilon}(y,t),
\end{aligned}$$

$$Y_4 = -\sum_{j\neq i} \int dy\, K(z_i(t) - z_j(t))\, W_{R_*}(z_i(t) - y)\, \omega_{j,\varepsilon}(y,t).$$

We have

$$\begin{aligned} |Y_1| &= \left| F^i_{\varepsilon,1}(B^i_\varepsilon(t), t) - F^i_{\varepsilon,1}(z_i(t), t) \right| \leq L |B^i_\varepsilon(t) - z_i(t)| \\ &\leq L \max_{j=1,\dots,N} \Delta_j(t). \end{aligned}$$

The term $|Y_2| = \left| F^i_{\varepsilon,2}(B^i_\varepsilon(t), t) \right|$ is bounded by ε^α since $t \leq T^*$.
The term Y_3 can be written as

$$Y_3 = Y_3^{'} + Y_3^{''}$$

where

$$\begin{aligned} Y_3^{'} = \sum_{j\neq i} \int_{|y - B^j_\varepsilon(t)| > \varepsilon^\beta} & dy \left[K(z_i(t) - y) - K(z_i(t) - z_j(t)) \right] \\ & \times (1 - W_{R_*}(z_i(t) - y))\, \omega_{j,\varepsilon}(y,t) \end{aligned}$$

and

$$\begin{aligned} Y_3^{''} = \sum_{j\neq i} \int_{|y - B^j_\varepsilon(t)| \leq \varepsilon^\beta} & dy \left[K(z_i(t) - y) - K(z_i(t) - z_j(t)) \right] \\ & \times (1 - W_{R_*}(z_i(t) - y))\, \omega_{j,\varepsilon}(y,t); \end{aligned}$$

the j-th integrand in $Y_3^{'}$ is bounded by $11/2\pi R_m |\omega_{j,\varepsilon}(y,t)|$ so that,

$$|Y_3^{'}| \leq \tilde{C} \sum_{j\neq i} m^j_\varepsilon(\varepsilon^\beta, t) \leq N \tilde{C} \varepsilon^{\overline{\alpha}}$$

since $t \leq T_{\varepsilon,\overline{\alpha},\beta}$. To bound the term $Y_3^{''}$ we use the Lipschitz condition: indeed, if $\tilde{x}$ is a point on the segment $\overline{y z_j(t)}$ where $|y - B^j_\varepsilon(t)| \leq \varepsilon^\beta$, then $|\nabla K^h(z_i(t) - \tilde{x})| \leq C/|z_i(t) - \tilde{x}|^2 \leq C/36 R_*^2$ ($K^h, h = 1, 2$ denotes the first and second component of the vector field K), because

$$\begin{aligned} |z_i(t) - \tilde{x}| &\geq |z_i(t) - z_j(t)| - |z_j(t) - \tilde{x}| \geq |z_i(t) - z_j(t)| - |z_j(t) - y| \\ &\geq |z_i(t) - z_j(t)| - |z_j(t) - B^j_\varepsilon(t)| - |B^j_\varepsilon(t) - y| > 6R_*, \end{aligned}$$

where we used that for $t \leq \overline{T}_{\varepsilon,\overline{\delta}} \wedge T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_1 |\log \varepsilon|$, $|z_j(t) - B^j_\varepsilon(t)| < 2R_*$ for any $j = 1, \dots, N$.

Therefore, for some constant $\tilde L$ and $a := \max\{|a_1|, \dots, |a_N|\}$,

$$
\begin{aligned}
|Y_3''| &\le \tilde L \sum_{j\neq i} \int_{|y-B^j_\varepsilon(t)|\le \varepsilon^\beta} dy\, |y - z_j(t)|\, |\omega_{j,\varepsilon}(y,t)| \\
&\le \tilde L \sum_{j\neq i} \int_{|y-B^j_\varepsilon(t)|\le \varepsilon^\beta} dy\, |y - B^j_\varepsilon(t)|\, |\omega_{j,\varepsilon}(y,t)| + \tilde L \sum_{j\neq i} |a_j||B^j_\varepsilon(t) - z_j(t)| \\
&\le \tilde L \sum_{j\neq i} \int_{|y-B^j_\varepsilon(t)|\le \varepsilon^\beta} dy\, |y - B^j_\varepsilon(t)|\, |\omega_{j,\varepsilon}(y,t)| + a\tilde L (N-1) \max_{j=1,\dots,N} \Delta_j(t).
\end{aligned}
$$

Consider the first term in the r.h.s. of the above inequality: since $W_{\varepsilon^\beta}(y - B^j_\varepsilon(t)) \equiv 1$ for $|y - B^j_\varepsilon(t)| \le \varepsilon^\beta$, it is equal to

$$
\sum_{j\neq i} \int_{|y-B^j_\varepsilon(t)|\le \varepsilon^\beta} dy\, |y - B^j_\varepsilon(t)|\, |\omega_{j,\varepsilon}(y,t)|\, W_{\varepsilon^\beta}(y - B^j_\varepsilon(t))
$$

and by applying the Cauchy-Schwarz inequality we obtain that each term of the sum is bounded by

$$
\begin{aligned}
&\left(\int_{|y-B^j_\varepsilon(t)|\le \varepsilon^\beta} dy\, |y - B^j_\varepsilon(t)|^2\, |\omega_{j,\varepsilon}(y,t)|\, W_{\varepsilon^\beta}(y - B^j_\varepsilon(t)) \right)^{\frac12} \\
&\qquad \times \left(\int_{|y-B^j_\varepsilon(t)|\le \varepsilon^\beta} dy\, |\omega_{j,\varepsilon}(y,t)| \right)^{\frac12}
\end{aligned}
$$

so that

$$
|Y_3''| \le \tilde L \sum_{j\neq i} \sqrt{|a_j|} \sqrt{I^j_\varepsilon(\varepsilon^\beta, t)} + a\tilde L (N-1) \max_{j=1,\dots,N} \Delta_j(t).
$$

Finally, since $|K(z_i(t) - z_j(t))| \le 1/2\pi R_m$, we have,

$$
|Y_4| \le C_* \sum_{j\neq i} \int_{|y-z_i(t)|\le 2R_*} dy\, |\omega_{j,\varepsilon}(y,t)|.
$$

Now, if $|y - z_i(t)| \le 2R_*$ then

$$
\begin{aligned}
|y - B^j_\varepsilon(t)| &> |y - z_j(t)| - |z_j(t) - B^j_\varepsilon(t)| > |z_j(t) - z_i(t)| - |z_i(t) - y| \\
&\quad - |z_j(t) - B^j_\varepsilon(t)| > 10R_* - 2R_* - 2R_* = 6R_* > \varepsilon^\beta,
\end{aligned}
$$

where we used that for $t \le \overline{T}_{\varepsilon,\overline{\delta}} \wedge T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_1 |\log \varepsilon|$, $|z_j(t) - B^j_\varepsilon(t)| < 2R_*$ for any $j = 1, \dots, N$. Hence,

$$
|Y_4| \le C_* \sum_{j\neq i} m^j_\varepsilon(\varepsilon^\beta, t).
$$

Putting together the estimates on the terms Y_1, Y_2, Y_3, Y_4 we obtain that for $0 < \varepsilon < \varepsilon_2$ and $t \le \overline{T}_{\varepsilon,\overline{\delta}} \wedge T_{\varepsilon,\overline{\alpha},\varepsilon^\beta} \wedge \xi_1 |\log\varepsilon|$

$$\left|\frac{d}{dt}B^i_\varepsilon(t) - \frac{d}{dt}z_i(t)\right| \le C_4\Big[\sum_{j\neq i} m^j_\varepsilon(\varepsilon^\beta, t) + \sum_{j\neq i}\sqrt{I^j_\varepsilon(\varepsilon^\beta, t)} + \max_{j=1,\dots,N}\Delta_j(t)\Big]. \tag{4.29}$$

Now, from (4.7) with $R = \varepsilon^\beta$, since $t \le T_{\varepsilon,\overline{\alpha},\beta}$ and $\overline{\alpha} - \beta > \alpha$ we obtain

$$\frac{d}{dt}I^j_\varepsilon(\varepsilon^\beta, t) \le C_1\big[I^j_\varepsilon(\varepsilon^\beta, t) + 2\varepsilon^\alpha\big].$$

Applying Gronwall's inequality and using that $I^j_\varepsilon(\varepsilon^\beta, 0) \le a\varepsilon^2$ we get

$$I^j_\varepsilon(\varepsilon^\beta, t) \le (a\varepsilon^2 + \varepsilon^\alpha)e^{Ct} \le (1+a)\varepsilon^\alpha e^{Ct} \tag{4.30}$$

and inserting (4.30) in (4.29) we have

$$\left|\frac{d}{dt}B^i_\varepsilon(t) - \frac{d}{dt}z_i(t)\right| \le C_5\Big[\varepsilon^{\overline{\alpha}} + \varepsilon^{\frac{\alpha}{2}}e^{\frac{Ct}{2}} + \max_{j=1,\dots,N}\Delta_j(t)\Big].$$

Integrating the above inequality for $t \le \overline{T}_{\varepsilon,\overline{\delta}} \wedge T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_1|\log\varepsilon|$ and taking the max over j in the l.h.s. we get

$$\max_{j=1,\dots,N}\Delta_j(t) \le \varepsilon^{\overline{\alpha}}t + \frac{2}{C}\varepsilon^{\frac{\alpha}{2}}e^{\frac{Ct}{2}} + \int_0^t ds \max_{j=1,\dots,N}\Delta_j(s),$$

where we used that $\Delta_j(0) = 0$ for any $j = 1, \dots, N$. Applying again Gronwall's inequality we get, for $t \le \overline{T}_{\varepsilon,\overline{\delta}} \wedge T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_1|\log\varepsilon|$,

$$\begin{aligned}\max_{j=1,\dots,N}\Delta_j(t) &\le C_6(\varepsilon^{\overline{\alpha}}t + \varepsilon^{\frac{\alpha}{2}}e^{\frac{Ct}{2}})e^t \\ &\le C_6\big(\xi_1\varepsilon^{\overline{\alpha}-\xi_1}|\log\varepsilon| + 2\varepsilon^{\frac{\alpha-\xi_1(C+2)}{2}}\big).\end{aligned}$$

By choosing ξ_1 sufficiently small, there exists $0 < \varepsilon_3 < \varepsilon_2$ such that for any $0 < \varepsilon < \varepsilon_3$ this quantity can be made smaller than $\varepsilon^{\overline{\delta}}/2$; hence, according to the definition of $\overline{T}_{\varepsilon,\overline{\delta}}$ we conclude that

$$\overline{T}_{\varepsilon,\overline{\delta}} > T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_1|\log\varepsilon| \quad \text{for any} \quad 0 < \varepsilon < \varepsilon_3. \tag{4.31}$$

Thus, in view of (4.31), we have only to show that there exist $0 < \varepsilon_0 < \varepsilon_3$ and $0 < \zeta < \xi_1$ such that

$$T_{\varepsilon,\overline{\alpha},\beta} > \zeta|\log\varepsilon| \quad \text{for any} \quad 0 < \varepsilon < \varepsilon_0. \tag{4.32}$$

and the proof will be complete. This is the content of the next step.

Step 3. We first observe that (4.28) and (4.31) imply that for any $0 < \varepsilon < \varepsilon_3$

$$T^* \geq T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_1 |\log \varepsilon|; \tag{4.33}$$

so if we take $0 < \xi_2 < \xi_1$ and work with $t \leq T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_2 |\log \varepsilon|$ and $0 < \varepsilon < \varepsilon_3$, then, thanks to (4.33), all the estimates found in the lemmas of Subsection 3.1 hold in this time interval for any choice of $R \in [\varepsilon^{\beta_*}, \varepsilon^{\beta}]$.
Since $\varepsilon^{\overline{\alpha}} \leq \varepsilon^{\alpha} < \varepsilon^{\alpha-\beta_*}$, from (4.7) for $R = \varepsilon^{\beta}$ we get

$$\frac{d}{dt} I^i_\varepsilon(\varepsilon^\beta, t) \leq C I^i_\varepsilon(\varepsilon^\beta, t) + C\varepsilon^{\alpha-\beta_*}$$

where we have defined here $C := 2C_1$. By Gronwall's inequality and $I^i_\varepsilon(\varepsilon^\beta, 0) \leq a\varepsilon^2$, we get, for $t \leq T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_2 |\log \varepsilon|$,

$$\begin{aligned} I^i_\varepsilon(\varepsilon^\beta, t) &\leq (I^i_\varepsilon(\varepsilon^\beta, 0) + \varepsilon^{\alpha-\beta_*}) e^{Ct} \leq 2\varepsilon^{\alpha-\beta_*} e^{Ct} \\ &\leq 2\varepsilon^{\alpha-\beta_*} e^{-C\xi_2 \log \varepsilon} = 2\varepsilon^{\alpha-\beta_*-C\xi_2} \end{aligned}$$

and we choose ξ_2 sufficiently small such that $\delta := \alpha - \beta_* - C\xi_2 > 0$. Thus,

$$I^i_\varepsilon(\varepsilon^\beta, t) \leq 2\varepsilon^{\delta} \quad \forall\, t \leq T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_2 |\log \varepsilon|. \tag{4.34}$$

Using now that $I^i_\varepsilon(R, s) \leq I^i_\varepsilon(\varepsilon^\beta, s)$ (because $R \leq \varepsilon^\beta$ and hence $W_R(x - B^i_\varepsilon(t)) \leq W_{\varepsilon^\beta}(x - B^i_\varepsilon(t))$), and (4.34), from (4.12) we get

$$\begin{aligned} m_\varepsilon(R, t) &\leq C_2 \left(\frac{1}{R^4} + \frac{\varepsilon^\alpha}{R^4} \right) \int_0^{T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_2 |\log \varepsilon|} ds\, I_\varepsilon(R, s) \\ &\leq 4C_2 \frac{\varepsilon^\delta}{R^4} \xi_2 |\log \varepsilon| \leq \frac{\varepsilon^{\frac{\delta}{2}}}{R^4} \end{aligned} \tag{4.35}$$

where we used that for ε sufficiently small $4C_2 \xi_2 \varepsilon^{\frac{\delta}{2}} |\log \varepsilon| < 1$.
We now insert (4.34) and (4.35) in (4.20) and we obtain for $t \leq T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_2 |\log \varepsilon|$ and $R \in [\varepsilon^{\beta_*}, \varepsilon^\beta]$,

$$\begin{aligned} \frac{d}{dt} \mu^i_\varepsilon(R, t) &\leq \tilde{C} \left(1 + \frac{\varepsilon^\alpha}{R^2} + \frac{\varepsilon^{-\frac{\alpha}{10}+\delta}}{R^3} + \frac{\varepsilon^\delta}{R^4} + \frac{\varepsilon^{\frac{\delta}{2}}}{R^6} \right) m^i_\varepsilon(R, t) \\ &\leq \tilde{C} \left(1 + \varepsilon^{\alpha-2\beta_*} + \varepsilon^{-\frac{\alpha}{10}+\delta-3\beta_*} + \varepsilon^{\delta-4\beta_*} + \varepsilon^{\frac{\delta}{2}-6\beta_*} \right) m^i_\varepsilon(R, t). \end{aligned}$$

Thanks to the choice of β, β_* and ξ_2 there exists a constant $A_* > 0$ such that

$$A_* \geq \tilde{C} \left(1 + \varepsilon^{\alpha-2\beta_*} + \varepsilon^{-\frac{\alpha}{10}+\delta-3\beta_*} + \varepsilon^{\delta-4\beta_*} + \varepsilon^{\frac{\delta}{2}-6\beta_*} \right)$$

for any $\varepsilon < \varepsilon_3$ and $R \in [\varepsilon^{\beta_*}, \varepsilon^{\beta}]$, so we finally obtain

$$\frac{d}{dt}\mu_\varepsilon^i(R,t) \le A_* m_\varepsilon^i(R,t) \tag{4.36}$$

for any $\varepsilon < \varepsilon_3$, $R \in [\varepsilon^{\beta_*}, \varepsilon^{\beta}]$ and $t \le T_{\varepsilon,\overline{\alpha},\beta} \wedge \xi_2 |\log\varepsilon|$. Integrating (4.36),

$$\mu_\varepsilon^i(R,t) \le A_* \int_0^t ds\, \mu_\varepsilon^i(R/2,s),$$

where we used (4.19) and that $\mu_\varepsilon^i(R,0) = 0$. We can now proceed as in the proof of [3, Lemma 2.6] to prove that for each $\beta \in (0, \alpha/14)$ and $\ell > 0$ there exists $\tilde{\xi} > 0$ such that

$$\lim_{\varepsilon\to 0} \sup_{t\in[0, T_{\varepsilon,\overline{\alpha},\beta}\wedge\tilde{\xi}|\log\varepsilon|]} \varepsilon^{-\ell} m_\varepsilon^i(\varepsilon^\beta, t) = 0;$$

taking $\ell = 2\overline{\alpha}$, we conclude that there exist $0 < \zeta < \xi_2$ and $0 < \varepsilon_0 < \varepsilon_3$ such that

$$m_\varepsilon^i(\varepsilon^\beta, t) < \varepsilon^{2\overline{\alpha}} < \varepsilon^{\overline{\alpha}}$$

for any $t \le T_{\varepsilon,\overline{\alpha},\beta} \wedge \zeta|\log\varepsilon|$, $0 < \varepsilon < \varepsilon_0$, and $i = 1,\dots,N$. By the definition of $T_{\varepsilon,\overline{\alpha},\beta}$ this clearly implies $T_{\varepsilon,\overline{\alpha},\beta} > \zeta|\log\varepsilon|$ for $0 < \varepsilon < \varepsilon_0$ and this completes the proof. □

We conclude giving the probabilistic interpretation of Theorem 2.2.

Corollary 4.4. *Suppose that $\omega_{\varepsilon,i}^0(x) \ge 0$, $\int dx\, \omega_{\varepsilon,i}^0(x) = 1$ for any $i = 1,\dots,N$ and denote by $\{X_t^i\}$ the solution of the stochastic differential equation*

$$dX_t^i = u_\varepsilon(X_t^i, t)dt + \sigma dW_t$$

with $X_0^{x,i}$ distributed according to $\omega_{\varepsilon,i}^0(x)\, dx$, where $u_\varepsilon(x,t) = \int dy\, K(x-y)\,\omega_\varepsilon(y,t)$. Then, for any $\beta' \in (0, \beta\wedge\overline{\delta})$ there exists $\varepsilon_* > 0$ such that, for any $i = 1,\dots,N$ and $0 < \varepsilon < \varepsilon_*$,

$$\mathbb{P}_{\omega_{\varepsilon,i}^0}\left\{ \left|X_t^i - z_i(t)\right| > \varepsilon^{\beta'} \right\} < \varepsilon^{\overline{\alpha}} \qquad \forall\, t \in [0, \zeta|\log\varepsilon|] \tag{4.37}$$

where $\mathbb{P}_{\omega_{\varepsilon,i}^0}$ denotes the law of the process $\{X_t^i\}$.

Proof. We know that

$$\mathbb{P}_{\omega_{i,\varepsilon}^0}\left\{ \left|X_t^i - z_i(t)\right| > \varepsilon^{\beta'} \right\} = \int_{|x-z_i(t)|>\varepsilon^{\beta'}} dx\, \omega_{i,\varepsilon}(x,t).$$

On the other hand, there exists $\varepsilon_* > 0$ such that $\varepsilon^{\beta'} > \varepsilon^{\beta} + \varepsilon^{\overline{\delta}}$ for any $0 < \varepsilon < \varepsilon_*$; thus, for any $0 < \varepsilon < \varepsilon_*$ and $t \in [0, \zeta|\log\varepsilon|]$, if $|x - z_i(t)| > \varepsilon^{\beta'}$ then $|x - B_\varepsilon^i(t)| > |x - z_i(t)| - |z_i(t) - B_\varepsilon^i(t)| > \varepsilon^\beta$, so that

$$\int_{|x-z_i(t)|>\varepsilon^{\beta'}} dx\, \omega_{i,\varepsilon}(x,t) \le \int_{|x-B_\varepsilon^i(t)|>\varepsilon^{\beta}} dx\, \omega_{i,\varepsilon}(x,t) < \varepsilon^{\overline{\alpha}},$$

hence (4.37). □

5 Appendix

As said in Subsection 2.1, we now show that, if the support of the initial vorticity has positive distance from the symmetry axis, a regular solution to problem (2.3) exists. In this appendix we come back to the original notation and write simply x meaning the canonical coordinates of $\mathbb{R}^3$.

We first state the following important lemmas.

Lemma 5.1. *Let $u \in L^2$ be such that $\omega \in L^q \cap L^\infty$ for some $q < 3$, then u is quasi-Lipschitz, i.e.:*

$$|u(x) - u(x')| \le C\,\varphi(|x - x'|)$$

and the constant C depends only on the L^q and L^∞ norms of ω.

Lemma 5.2. *Let ϕ, ψ homeomorphisms in $\mathbb{R}^3$ which preserve the Lebesgue measure, namely such that, for each function f*

$$\int f(\phi(x))dx = \int f(x)dx = \int f(\psi(x))dx$$

and let $\omega \in L^q \cap L^\infty$. Then

$$\int \big[H(x - \phi(y)) - H(x - \psi(y))\big] \wedge \omega(y)dy \le C\,\varphi\left(\sup_{z\in\mathbb{R}^3} |\phi(z) - \psi(z)|\right).$$

Lemma 5.3. *Let $b \in L^\infty\big([0, T^*]; C(\mathbb{R}^3)\big)$ be a quasi-Lipschitz vector field, uniformly in t. Then, for each $x_0 \in \mathbb{R}^3$, there exists a unique $x(t)$ which solves:*

$$x(t) = x_0 + \int_0^t b(x(s), s)ds\,.$$

The proof of Lemma 5.1 is completely analogous to the one for the planar case contained in [13, App. 2.3]; concerning Lemma 5.2 we only have to define $r := \sup_z |\phi(z) - \psi(z)|$ and argue again as in [13, App. 2.3]; finally Lemma 5.3 is a simple adjustment of [13, Lemma 3.2, p.67].

The proof of the following theorem is an adjustment of the one for the analogous planar result [13, Thm. 3.1, p.72].

Theorem 5.4. *Let $u_0 \in L^2$ be an axisymmetric, divergence-free vector field such that $\omega_0 \in L^q \cap L^\infty$ for some $q < 3$, and assume that $r_{\min} := \inf_{x\in \operatorname{supp}\omega_0} \delta(x) > 0$ (then $\alpha_0 \in L^q \cap L^\infty$).[7] Then the unique solution of (2.3) satisfies $u \in C([0,T] \times \mathbb{R}^3)$, and then $\phi_t(x)$ is C^1 with respect to t.*

Proof. We define $\omega_t^0(x) := \omega_0(x)$ for every time, and for $n \ge 1$:

$$\begin{cases} u_t^n(x) = \int H(x-y) \wedge \omega_t^{n-1}(y)\,dy \\ \phi_t^n(x) = x + \int_0^t u_s^n(\phi_s^n(x))ds \\ \omega_t^n(x) = \frac{\delta(x)}{\delta(\phi_{-t}^n(x))}\,\omega_0(\phi_{-t}^n(x)) \end{cases}. \tag{5.1}$$

[7] We recall that $\delta(x)$ denotes the distance of x from the axis and that $\alpha := \omega/\delta$.

We should prove, first of all, that these objects are well defined (in fact, ϕ_t^n is defined by an integral equation).

Step 1: Well-posedness of (5.1) *and a priori estimates.*
We observe that u_t^1 is a quasi-Lipschitz vector field, uniformly in t thanks to Lemma 5.1, so ϕ_t^1 is well defined in view of Lemma 5.3. Let's assume that $(u_t^n, \phi_t^n, \omega_t^n)$ are well defined. For each $j \le n$ let:

$$R^j(t) := 1 + \frac{1}{r_{\min}} \int_0^t \|u_s^j\|_{L^\infty} ds.$$

Moreover we denote $D_t^j(y) := \frac{\delta(\phi_t^j(y))}{\delta(y)}$. We have:

$$\sup_{y \in \operatorname{supp} \omega_0} D_t^j(y) \le \frac{\delta(y) + \int_0^t \|u_s^j\|_{L^\infty} ds}{\delta(y)} \le R^j(t). \tag{5.2}$$

Arguing as in [16, Prop. 1.1 and 2.3], we get the following estimates for each $j \le n$:

$$\|u_t^{j+1}\|_{L^\infty} \le C \|\omega_t^j\|_{L^q}^{3/q} \|\omega_t^j\|_{L^\infty}^{1-3/q},$$

$$\|\omega_t^j\|_{L^q} \le (\|\alpha_0\|_{L^q}^q + \|\omega_0\|_{L^q}^q)^{1/q} R^j(t),$$
$$\|\omega_t^j\|_{L^\infty} \le \max(\|\alpha_0\|_{L^\infty}, \|\omega_0\|_{L^\infty}) R^j(t).$$

Then:

$$R^{j+1}(t) \le 1 + C_0 \int_0^t R^j(s) ds$$

where C_0 depends on $r_{\min}$ and on the L^q and L^∞ norms of ω_0 and α_0, but not on the index j. Since $R^1(t) = 1 + \frac{t}{r_{\min}} \|u_0\|_{L^\infty}$, by iteration we have:

$$R^{j+1}(t) \le \sum_{k=0}^{j} C_0^k \frac{t^k}{k!} + \|u_0\|_{L^\infty} \frac{C_0^j \, t^{j+1}}{r_{\min}(j+1)!} \le \max\left(1, \frac{\|u_0\|_{L^\infty}}{C_0 \, r_{\min}}\right) e^{C_0 t}$$

independently from $j \le n$. Then $\|\omega_t^j\|_{L^q} \le C_1 e^{C_0 t}$ and $\|\omega_t^j\|_{L^\infty} \le C_2 e^{C_0 t}$ independently from j. So, thanks to Lemma 5.1, u_t^{n+1} is a quasi-Lipschitz vector field (uniformly in $t \in [0, T]$) and then the flow ϕ_t^{n+1} is well defined.

By induction $(u_t^n, \phi_t^n, \omega_t^n)$ are well defined for each n, and the following estimates hold:

$$R^n(t) \le C_3 e^{C_0 t}, \qquad \|\omega_t^n\|_{L^q} \le C_1 e^{C_0 t}, \qquad \|\omega_t^n\|_{L^\infty} \le C_2 e^{C_0 t}.$$

Step 2: Continuity in time of u_n.
Observe that

$$u_t^n(x) = \int H(x-y) \wedge \omega_t^{n-1}(y) dy = \int H(x - \phi_t^{n-1}(y)) \wedge \left[D_t^{n-1}(y)\, \omega_0(y) \right] dy.$$

Let $t_2, t_1 \in [0,T]$, $t_2 > t_1$; then:

$$\begin{aligned}
&|u^n_{t_2}(x) - u^n_{t_1}(x)| = \\
&= \left| \int \left[H(x - \phi^{n-1}_{t_2}(y)) D^{n-1}_{t_2}(y) - H(x - \phi^{n-1}_{t_1}(y)) D^{n-1}_{t_1}(y) \right] \wedge \omega_0(y)\, dy \right| \\
&\leq \left| \int D^{n-1}_{t_2}(y) \left[H(x - \phi^{n-1}_{t_2}(y)) - H(x - \phi^{n-1}_{t_1}(y)) \right] \wedge \omega_0(y)\, dy \right| \\
&\quad + \left| \int \left[D^{n-1}_{t_2}(y) - D^{n-1}_{t_1}(y) \right] H(x - \phi^{n-1}_{t_1}(y)) \wedge \omega_0(y)\, dy \right|.
\end{aligned}$$

Using the triangle inequality, we have $|\delta(x) - \delta(y)| \leq |x-y|$. In view of this fact, of Lemmas 5.1 and 5.2 (which holds because the homeomorphisms ϕ^n_t preserve the Lebesgue measure: in fact the vector fields u^n_t, which generate them, are divergence-free) and of the estimate (5.2), we get:

$$\begin{aligned}
|u^n_{t_2}(x) - u^n_{t_1}(x)| &\leq R^{n-1}(t_2) \cdot C\, \varphi \left(\sup_{z \in \mathbb{R}^3} |\phi^{n-1}_{t_2}(z) - \phi^{n-1}_{t_1}(z)| \right) \\
&\quad + \frac{1}{r_{\min}} \sup_{z \in \mathbb{R}^3} |\phi^{n-1}_{t_2}(z) - \phi^{n-1}_{t_1}(z)| \cdot C\, \|\omega_0\|^{3/q}_{L^q} \|\omega_0\|^{1-3/q}_{L^\infty}.
\end{aligned}$$

Since $\sup_z |\phi^{n-1}_{t_2}(z) - \phi^{n-1}_{t_1}(z)| \leq \int_{t_1}^{t_2} \|u^{n-1}_s\|_{L^\infty}$, which goes to 0 if $t_2 \to t_1$, we proved the continuity of u^n with respect to t.

Step 3: Uniform convergence of $\phi^n_t(x)$ for $n \to \infty$
We now show that ϕ^n is a Cauchy sequence in $C([0,T] \times \mathbb{R}^3)$.

$$\begin{aligned}
\phi^n_t(x) &= x + \int_0^t ds\, u^n_s(\phi^n_s(x)) \\
&= x + \int_0^t ds \int_{\mathbb{R}^3} dy\, H(\phi^n_s(x) - \phi^{n-1}_s(y)) \wedge \left[D^{n-1}_s(y)\, \omega_0(y) \right]
\end{aligned}$$

$$\begin{aligned}
&|\phi^{n+1}_t(x) - \phi^n_t(x)| \\
&\leq \int_0^t ds \left| \int_{\mathbb{R}^3} dy\, D^n_s(y) \left[H(\phi^{n+1}_s(x) - \phi^n_s(y)) - H(\phi^n_s(x) - \phi^n_s(y)) \right] \wedge \omega_0(y) \right| \\
&\quad + \int_0^t ds \left| \int_{\mathbb{R}^3} dy\, D^n_s(y) \left[H(\phi^n_s(x) - \phi^n_s(y)) - H(\phi^n_s(x) - \phi^{n-1}_s(y)) \right] \wedge \omega_0(y) \right| \\
&\quad + \int_0^t ds \left| \int_{\mathbb{R}^3} dy \left[D^n_s(y) - D^{n-1}_s(y)\, H(\phi^n_s(x) - \phi^{n-1}_s(y)) \wedge \omega_0(y) \right| \right.
\end{aligned}$$

Using again Lemmas 5.1 and 5.2, we have:

$$\begin{aligned}|\phi_t^{n+1}(x)-\phi_t^n(x)| \le & \int_0^t ds\, R^n(s)\cdot C\varphi\left(|\phi_s^{n+1}(x)-\phi_s^n(x)|\right)\\ & + \int_0^t ds\, R^n(s)\cdot C\varphi\left(\sup_z |\phi_s^n(z)-\phi_s^{n-1}(z)|\right)\\ & + \int_0^t ds\, \frac{C}{r_{\min}} \sup_z |\phi_s^n(z)-\phi_s^{n-1}(z)|.\end{aligned}$$

Then, defining $\delta_T^n := \sup_{t\in[0,T]} \sup_z |\phi_t^{n+1}(z)-\phi_t^n(z)|$, we get

$$\delta_T^n \le C\int_0^T \varphi(\delta_s^n)+\varphi(\delta_s^{n-1})ds.$$

We call $\rho_T^n := \sup_{m\ge n} \delta_T^m$; since $\rho_T^n \le \rho_T^{n-1}$, we obtain:

$$\rho_T^n \le 2C\int_0^T \varphi(\rho_s^{n-1})ds.$$

which is the same estimate obtained in the proof of [13, Lemma 3.2, p.67]. Arguing in the same way we deduce that ϕ^n is a Cauchy sequence in $C([0,T]\times\mathbb{R}^3)$, and then it has limit ϕ. Moreover, ϕ_t preserves the Lebesgue measure t, because the homeomorphisms ϕ_t^n do, for each n.

Step 4: The limit is a solution to the equations.
We define, according to (2.3):

$$\omega_t(x) := \frac{\delta(x)}{\delta(\phi_{-t}(x))}\,\omega_0(\phi_{-t}(x)), \qquad u_t(x) := \int H(x,y)\wedge\omega_t(y)dy$$

With a change of variable in the integral that defines u, we get:

$$u_t(x) = \int H(x-\phi_t(y)\wedge[D_t(y)\,\omega_0(y)]\,dy.$$

Then:

$$\begin{aligned}&|u_t(x)-u_t^n(x)|\\ &= \left|\int \left[H(x-\phi_t(y))D_t(y)-H(x-\phi_t^{n-1}(y))D_t^{n-1}(y)\right]\wedge\omega_0(y)\,dy\right|\\ &\le \left|\int D_t^{n-1}(y)\left[H(x-\phi_t(y))-H(x-\phi_t^{n-1}(y))\right]\wedge\omega_0(y)\,dy\right|\\ &\quad + \left|\int \left[D_t(y)-D_t^{n-1}(y)\right]H(x-\phi_t(y))\wedge\omega_0(y)\,dy\right|\\ &\le\; R^{n-1}(t)\cdot C\varphi\left(\sup_z |\phi_t(z)-\phi_t^{n-1}(z)|\right)+\frac{C}{r_{\min}}\cdot\sup_z|\phi_t(z)-\phi_t^{n-1}(z)|.\end{aligned}$$

Since ϕ^n converges to ϕ uniformly in $[0,T] \times \mathbb{R}^3$, and since $R^{n-1}(t)$ is uniformly bounded in n, for t in $[0,T]$, we find that u^n converges to u uniformly in $[0,T] \times \mathbb{R}^3$ and then $u \in C([0,T] \times \mathbb{R}^3)$.

Finally, taking the limit in the integral equation

$$\phi_t^n(x) = x + \int_0^t u_s^n(\phi_s^n(x), s)ds$$

thanks to the uniform convergence of u^n to u and since it is uniformly bounded (in view of this facts, we are able to apply the dominated convergence theorem), we get:

$$\phi_t(x) = x + \int_0^t u_s(\phi_s(s), s)ds,$$

and since u is continuous, $\phi_t(x)$ is C^1 with respect to the time and satisfies the differential equation $\dot{\phi}_t(x) = u(\phi_t(x), t)$. □

Remark 5.5. We point out that, on the z axis, the vector field u has only the component u_z (this is due to the axisymmetry of u and to its continuity with respect to x). Therefore the flow ϕ_t maps the z axis in itself, and so does its inverse (this is due to the two-parameters group property of ϕ); calling S the set $\mathbb{R}^3$ without the z axis, ϕ_t maps S in itself. Then the support of ω cannot reach the axis in finite time; however we don't have a control on its distance from the axis for positive times.

We now state an important consequence of Theorem 5.4.

Corollary 5.6. *Under the same assumptions of the previous theorem, let ω_t be the solution to the equations, and let $f \in C^1([0,T] \times S)$. Then:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \alpha(x,t) f(x,t)\, dx = \int_{\mathbb{R}^3} \alpha(x,t) \left[u \cdot \nabla f + \partial_t f\right](x,t)\, dx$$

Moreover if, in cylindrical coordinates, f does not depend on θ, define

$$\omega_t(f) := \int_{-\infty}^{+\infty} dz \int_0^{+\infty} dr\, \omega(r,z,t) f(r,z,t)$$

Then:

$$\frac{\mathrm{d}}{\mathrm{d}t} \omega_t(f) = \omega_t\big(u_r \partial_r f + u_z \partial_z f + \partial_t f\big) \tag{5.3}$$

Proof. With a change of variable,

$$\int_{\mathbb{R}^3} \alpha(x,t) f(x,t)\, dx = \int_{\mathbb{R}^3} \alpha_0(x) f(\phi_t(x), t) dx$$

Then the expression is differentiable, thanks to the regularity of f. Furthermore

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\alpha_0(x)f(\phi_t(x),t)dx = \int_{\mathbb{R}^3}\alpha_0(x)\left[u\cdot\nabla f+\partial_t f\right](\phi_t(x),t)dx$$
$$= \int_{\mathbb{R}^3}\alpha(x,t)\left[u\cdot\nabla f+\partial_t f\right](x,t)dx$$

Now, if f does not depend on θ, writing the integral in cylindrical coordinates we have:

$$\int_{\mathbb{R}^3}\alpha(x,t)f(x,t)\,dx = 2\pi\int_{-\infty}^{+\infty}dz\int_0^{+\infty}dr\,\omega(r,z,t)f(r,z,t) = 2\pi\omega_t(f)$$

Then, recalling the expression of gradient in cylindrical coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t(f) = \omega_t\big(u\cdot\nabla f+\partial_t f\big) = \omega_t\big(u_r\partial_r f+u_z\partial_z f+\partial_t f\big)$$

□

We move to the NS equations and we prove the continuity property of the functions $t\mapsto m^i_\varepsilon(R,t)$ and $(x,t)\mapsto F^i_{\varepsilon,h}(x,t), h=1,2$.

Lemma 5.7. *Let $m^i_\varepsilon(R,t)$ be defined in (2.13). For any $R>0$ and $\varepsilon>0$ the function $t\mapsto m^i_\varepsilon(R,t)$ is continuous.*

Proof. Let $t\geq 0$, $t_n\to t$ as $n\to+\infty$, and denote by $\chi(A)$ the indicator function of the set A. We recall that $\omega_{i,\varepsilon}(x,t)$ preserves the initial sign, and, without loss of generality we can suppose $\omega_{i,\varepsilon}(x,t)\geq 0$. Then,

$$|m^i_\varepsilon(R,t_n)-m^i_\varepsilon(R,t)| = \bigg|\int dx\,\omega_{i,\varepsilon}(x,t_n)\,\chi(|x-B^i_\varepsilon(t_n)|>R)$$
$$-\int dx\,\omega_{i,\varepsilon}(x,t)\,\chi(|x-B^i_\varepsilon(t)|>R)\bigg|$$
$$\leq\bigg|\int dx\,\omega_{i,\varepsilon}(x,t_n)\left[\chi(|x-B^i_\varepsilon(t_n)|>R)-\chi(|x-B^i_\varepsilon(t)|>R)\right]\bigg|$$
$$+\bigg|\int dx\left[\omega_{i,\varepsilon}(x,t_n)-\omega_{i,\varepsilon}(x,t)\right]\chi(|x-B^i_\varepsilon(t)|>R)\bigg|.$$

The first term in the r.h.s. of the above inequality is bounded by

$$||\omega_{i,\varepsilon}(\cdot,0)||_\infty|\Sigma(B^i_\varepsilon(t_n)|R)\triangle\Sigma(B^i_\varepsilon(t)|R)|$$

where $|\Sigma(B^i_\varepsilon(t_n)|R)\triangle\Sigma(B^i_\varepsilon(t)|R)|$ denotes the Lebesgue measure of the symmetric difference between the two disks, which tends to 0 as $n\to+\infty$ by continuity of $B^i_\varepsilon(t)$.

The second term goes to 0 as $n\to+\infty$ because $\mathbb{R}^2\setminus\Sigma(B^i_\varepsilon(t)|R)$ is a $\omega_{i,\varepsilon}(x,t)\,dx-$ continuity set and $t\mapsto\omega_{i,\varepsilon}(x,t)\,dx$ is continuous w.r.t. the topology induced by the weak convergence. □

Lemma 5.8. *For any $\varepsilon > 0$, the vector fields $(x,t) \mapsto F^i_{\varepsilon,2}(x,t)$ and $(x,t) \mapsto F^i_{\varepsilon,1}(x,t)$ are uniformly bounded and continuous.*

Proof. We only prove the statement concerning $F^i_{\varepsilon,2}(x,t)$ which is more delicate to handle, the other can be treated analogously. The uniform boundedness is a consequence of the statement 1) of [12, Lemma 3.1]. To show the continuity w.r.t. (x,t), let us consider a sequence $(x_n,t_n) \to (x,t)$ as $n \to +\infty$. Then,

$$\begin{aligned} |F^i_{\varepsilon,2}(x_n,t_n) - F^i_{\varepsilon,2}(x,t)| \le |F^i_{\varepsilon,2}(x_n,t_n) - F^i_{\varepsilon,2}(x,t_n)| \\ + |F^i_{\varepsilon,2}(x,t_n) - F^i_{\varepsilon,2}(x,t)|. \end{aligned} \tag{5.4}$$

Arguing as in the proof of the statement 2) in [12, Lemma 3.1], we obtain that the first term in the r.h.s. is bounded by

$$|F^i_{\varepsilon,2}(x_n,t_n) - F^i_{\varepsilon,2}(x,t_n)| \le C\varphi(|x_n - x|)$$

where C is a constant independent on n, and hence it goes to 0 as $n \to +\infty$. We now consider the second term; fix $\eta > 0$ and using that

$$\int_{|x-y|<\eta} dy\, \big|K(x-y)\big|\big|[\omega_{j,\varepsilon}(y,t_n) - \omega_{j,\varepsilon}(y,t)]\big| \le 2||\omega_{j,\varepsilon}(\cdot,0)||_\infty \eta,$$

we get,

$$\begin{aligned} |F^i_{\varepsilon,2}(x,t_n) - F^i_{\varepsilon,2}(x,t)| &= \sum_{j\neq i} \int dy\, K(x-y)\, W_{R_*}(x-y)[\omega_{j,\varepsilon}(y,t_n) - \omega_{j,\varepsilon}(y,t)] \\ &\le 2||\omega_{j,\varepsilon}(\cdot,0)||_\infty \eta + \sum_{j\neq i} \int_{|x-y|>\eta} dy\, K(x-y)\, W_{R_*}(x-y) \\ &\quad \times [\omega_{j,\varepsilon}(y,t_n) - \omega_{j,\varepsilon}(y,t)]; \end{aligned}$$

hence, if $n \to +\infty$ the second term in the r.h.s. of the above inequality goes to 0 by weak convergence, and by arbitrariness of η, the thesis follows. □

References

[1] P. Billingsley, *Convergence of probability measures*, John Wiley and Sons, Inc., New York, 1999.

[2] V.I.Bogachev, M. Röckner, S. V. Shaposhnikov, *On uniqueness problems related to the Fokker-Planck-Kolmogorov equations for measures*, Journal of Mathematical Sciences **179**, 1, 2011, pp. 7-44.

[3] P. Buttà, C. Marchioro, *Long time evolution of concentrated Euler flows with planar simmetry*, SIAM J. Math. Anal. **50**, 2018, pp. 735-760.

[4] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[5] A. Friedman, *Stochastic differential equations and applications Volume 1*, Academic Press, New York, 1975.
[6] P. Gamblin, X. Saint-Raymond, *On Three-Dimensional Vortex Patches*, Bull. Soc. math. France **123**, 1995, pp. 375-424.
[7] I. Karatsas, S. E. Shreve, *Brownian motion and stochastic calculus*, Springer, New York, 1998.
[8] A. Maass et al (eds.), *Dynamics and Randomness* II, 17, 79, Kluer Academic Publisher, 2004.
[9] O. A. Manita, S. V. Shaposhnikov, *On the well-posedness of the Cauchy problem for Fokker-Planck-Kolmogorov equations with potential terms on arbitrary domains*, arXiv: 1307.3662 [math.AP].
[10] C. Marchioro, *Large smoke rings with concentrated vorticity*, Journal of Mathemtical Physics **40**, 1999, pp. 869-883.
[11] C. Marchioro, *On the Inviscid Limit for a Fluid with a Concentrated Vorticity*, Commun. Math. Phys. **196**, 1998, pp. 53-65.
[12] C. Marchioro, M. Pulvirenti, *Hydrodynamics in two dimensions and vortex theory*, Commun. Math. Phys. **84**, 1982, pp. 483-503.
[13] C. Marchioro, M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Springer-Verlag New York, 1994.
[14] C. Marchioro, M. Pulvirenti, *Vortex methods in two-dimensional fluid dynamics*, Lecture Notes in Physics, 203, Springer-Verlag, Berlin, 1984.
[15] D. Revuz, M. Yor, *Continuous martingales and Brownian motion*, Springer-Verlag, Berlin, 1990.
[16] X. Saint-Raymond, *Remarks on axisymmetric solutions of the incompressible Euler system*, Commun. in Partial Differential Equations **19**, 1994, pp. 321-334.
[17] S. V. Shaposhnikov, *On uniqueness of a probability solution to the Cauchy problem for the Fokker-Planck-Kolmogorov equation*, Theory Probab. Appl. **56**, 1, 2012, pp. 96-115.

Received: 23 July 2018.
Accepted: 9 September 2018.

Daniele Cetrone

Dipartimento di Matematica, Sapienza Università di Roma, P.le Aldo Moro 5, 00185 Roma, Italy

danielecetrone@libero.it

Gabriele Serafini

Dipartimento di Matematica, Sapienza Università di Roma, P.le Aldo Moro 5, 00185 Roma, Italy

gabriele.serafini94@gmail.com