Rend. Mat. Appl. (7).
Volume 39, (2018), 97 – 131

Rendiconti di Matematica
e delle sue Applicazioni

New invariants of ample vector bundles over smooth projective varieties

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Abstract. *Let X be a complex smooth projective variety of dimension n, and let $\mathcal{E}$ be an ample vector bundle on X. In this paper, we will introduce new invariants of generalized polarized manifolds $(X, \mathcal{E})$, and we will study their properties. As an application, we study a lower bound for $c_1(\mathcal{E})^n$ and the sectional genus $g(X, c_1(\mathcal{E}))$ of $(X, c_1(\mathcal{E}))$.*

1 Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers, and let $\mathcal{E}$ be an ample vector bundle on X. Then $(X, \mathcal{E})$ is called a *generalized polarized manifold*. Let $r := \operatorname{rank}(\mathcal{E})$. Generalized polarized manifolds $(X, \mathcal{E})$ have been studied by using some invariants of $(X, \mathcal{E})$. Here we state the history of invariants of $(X, \mathcal{E})$.

First in [4], Fujita introduced the c_1-sectional genus and the $\mathcal{O}(1)$-sectional genus of $(X, \mathcal{E})$. Next, in [1], for the case where $r = n - 1$, Ballico defined an invariant of $(X, \mathcal{E})$ which is called the *curve genus* $cg(X, \mathcal{E})$ *of* $(X, \mathcal{E})$, and several authors (in particular Lanteri, Maeda, Sommese and so on) studied this invariant (see [16], [20], [17] and [21]).

As a generalization of the curve genus, for any ample vector bundle $\mathcal{E}$ with $r \leq n-1$, Ishihara ([15, Definition 1.1]) defined an invariant $g(X, \mathcal{E})$, which is called the *c_r-sectional genus of* $(X, \mathcal{E})$, and in [10] we investigated some properties about $g(X, \mathcal{E})$. We note that if $n - r = 1$, then $g(X, \mathcal{E})$ is the curve genus. This invariant means the following: If a general element of $H^0(\mathcal{E})$ has a zero locus Z which is smooth of expected dimension $n - r$, then $g(X, \mathcal{E}) = g(Z, \det \mathcal{E}|_Z)$, that is, $g(X, \mathcal{E})$ is the sectional genus of $(Z, \det \mathcal{E}|_Z)$. In [13] Fusi and Lanteri generalized this invariant. In [8, Definition 4.1], we introduced an invariant $v(X, \mathcal{E})$ of generalized polarized manifolds $(X, \mathcal{E})$ with $r \geq n - 1$, which is defined by using a result [8, Theorem 3.2 (3.2.3)]. Here we note that $v(X, \mathcal{E})$ is equal to the curve genus if $r = n - 1$.

In this paper, we will introduce new invariants $B^i(X, \mathcal{E})$ and $\widehat{B}^i(X, \mathcal{E})$ of $(X, \mathcal{E})$ for every integer i with $0 \leq i \leq n$ and $\operatorname{rank}(\mathcal{E}) \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$ (see Defini-

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2010 Mathematics Subject Classification: Primary 14J60; Secondary 14C17, 14C20, 14C40, 14J30, 14J35, 14J40.
Keywords: Ample vector bundle, polarized manifold, classical scroll, sectional genus.

tions 3.6 and 3.11). Then the following equalities hold (see Propositions 3.7 and 3.12).

$$\begin{aligned} b_{2n-2-i}(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E})) - h^{2n-2-i}(\mathbb{P}_X(\mathcal{E}), \mathbb{C}) &= B^i(X, \mathcal{E}) - h^i(X, \mathbb{C}), \\ b_i(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E})) - h^i(\mathbb{P}_X(\mathcal{E}), \mathbb{C}) &= \widehat{B}^i(X, \mathcal{E}) - h^i(X, \mathbb{C}), \end{aligned}$$

where $H(\mathcal{E})$ denotes the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$ and $b_k(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ denotes the k-th sectional Betti number of $(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$.

We note that if $i = 1$, then $B^1(X, \mathcal{E}) = 2v(X, \mathcal{E})$. In this paper, we will study some properties of $B^i(X, \mathcal{E})$ and $\widehat{B}^i(X, \mathcal{E})$. Furthermore we will also define and study the following invariant $P_i(X, \mathcal{E})$

$$P_i(X, \mathcal{E}) = b_i(X, c_1(\mathcal{E})) - (B^i(X, \mathcal{E}) + \widehat{B}^i(X, \mathcal{E})).$$

Here $b_i(X, c_1(\mathcal{E}))$ denotes the ith sectional Betti number of $(X, c_1(\mathcal{E}))$. By studying $P_i(X, \mathcal{E})$ for $i = 0$ and 1, we get a lower bound of $c_1(\mathcal{E})^n$ and $g_1(X, c_1(\mathcal{E}))$ (see Section 5).

2 Preliminaries

Notation 2.1. Let X be a smooth projective variety of dimension $n \geq 1$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. We put $W := \mathbb{P}_X(\mathcal{E})$, $H := H(\mathcal{E})$ and $m := \dim W$, where $H(\mathcal{E})$ denotes the tautological line bundle on W. Then $m = n + r - 1$.

Definition 2.2. Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be a vector bundle of rank r on X.

(i) The *Chern polynomial* $c_t(\mathcal{E})$ is defined by the following:

$$c_t(\mathcal{E}) = \sum_{i \geq 0} c_i(\mathcal{E}) t^i,$$

where $c_i(\mathcal{E})$ is the ith Chern classes.

(ii) For every integer j with $j \geq 0$, the *jth Segre class $s_j(\mathcal{F})$ of $\mathcal{F}$* is defined by the following equation: $c_t(\mathcal{F}^\vee) s_t(\mathcal{F}) = 1$, where $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j \geq 0} s_j(\mathcal{F}) t^j$.

Remark 2.3. (i) Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Let $\tilde{s}_j(\mathcal{F})$ be the Segre class which is defined in [11, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.

(ii) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

Definition 2.4. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a finite sequence of nonnegative integers with $\lambda_1 \geq \cdots \geq \lambda_n$. Then we call this λ a *partition*. We denote by $\Lambda(n, r)$ the set of all partitions of n in nonnegative integers $\leq r$.

Definition 2.5. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n, r)$, we put

$$\Delta_\lambda(c) = \Delta_{(\lambda_1,\ldots,\lambda_n)}(c) = \det(c_{j-i+\lambda_i}),$$

where $(c_{j-i+\lambda_i})$ denotes the n by n matrix whose ij entry is $c_{j-i+\lambda_i}$. Then we call this the *Schur polynomial associated to* λ. Here c_p denotes the pth Chern class of a vector bundle.

Remark 2.6. We note that $\Delta_{(\lambda_1,\ldots,\lambda_n)}(s) = \det(s_{j-i+\lambda_i})$, where s_p denotes the pth Segre class of a vector bundle.

Remark 2.7. Let $\mu_k = (2, \underbrace{1, \ldots, 1}_{k-1})$ for every positive integer k. By [11, Lemma 14.5.1] we have

$$\begin{aligned} s_k(\mathcal{E}) &= \Delta_{(k)}(s) \\ &= \Delta_{(\underbrace{1,\ldots,1}_{k})}(c) \\ &= c_1(\mathcal{E})\Delta_{(\underbrace{1,\ldots,1}_{k-1})}(c) - \Delta_{\mu_{k-1}}(c) \\ &= c_1(\mathcal{E})s_{k-1}(\mathcal{E}) - \Delta_{\mu_{k-1}}(c). \end{aligned}$$

Theorem 2.8. *Let X be a projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle on X with* $\operatorname{rank}(\mathcal{E}) = r$. *Let $P = \sum_{\lambda\in\Lambda(n,r)} a_\lambda \Delta_\lambda(c)$. Then the polynomial P is numerically positive for ample vector bundles if and only if P is non-zero and $a_\lambda \geq 0$ for all $\lambda \in \Lambda(n, r)$. In particular, $\Delta_\lambda(c) > 0$ for every $\lambda \in \Lambda(n, r)$.*

Proof. See [12, Theorem I]. □

Definition 2.9. (See [7, Definition 3.1].) Let (X, L) be a polarized manifold of dimension n, and let i be an integer with $0 \leq i \leq n$.

(i) The *ith sectional Euler number* $e_i(X, L)$ *of* (X, L) is defined by the following:

$$e_i(X, L) := \sum_{k=0}^{i} (-1)^k \binom{n-i+k-1}{k} c_{i-k}(X) L^{n-i+k}.$$

(ii) The *ith sectional Betti number* $b_i(X, L)$ *of* (X, L) is defined by the following:

$$b_i(X, L) := \begin{cases} e_0(X, L) & \text{if } i = 0, \\ (-1)^i \left(e_i(X, L) - \displaystyle\sum_{j=0}^{i-1} 2(-1)^j h^j(X, \mathbb{C}) \right) & \text{if } 1 \leq i \leq n. \end{cases}$$

Remark 2.10. (i) If $i = 0$, then $b_0(X, L) = L^n$.

(ii) If $i = 1$, then $b_1(X, L) = 2g(X, L)$, where $g(X, L)$ denotes the sectional genus of (X, L).

Proposition 2.11. *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that L is spanned by its global sections. Then $g(X, L) \geq 2h^1(\mathcal{O}_X) - 1$ unless (X, L) is a scroll over a smooth curve.*

Proof. The nonnegativity of $g(X, L)$ shows that Proposition 2.11 is true for the case of $h^1(\mathcal{O}_X) = 0$. So we may assume that $h^1(\mathcal{O}_X) \geq 1$. Since L is spanned by its global sections, by taking $(n-2)$ general members $D_1, \ldots, D_{n-2} \in |L|$, we can get a smooth projective surface $S := D_1 \cap \cdots \cap D_{n-2}$. We consider the polarized surface (S, L_S). Since L is ample and $\mathrm{Bs}|L_S| = \emptyset$, we see from [5, Lemma 1.15] that $g(S, L_S) \geq 2h^1(\mathcal{O}_S) - 1$ holds unless (S, L_S) is a scroll over a smooth curve.

If (S, L_S) is a scroll over a smooth projective curve, then so is (X, L) by [3, Theorems 5.5.2 and 5.5.3] because $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_X) \geq 1$. Hence if (X, L) is not a scroll over a smooth curve, then

$$\begin{aligned} g(X, L) &= g(S, L_S) \\ &\geq 2h^1(\mathcal{O}_S) - 1 \\ &= 2h^1(\mathcal{O}_X) - 1. \end{aligned}$$

So we get the assertion. □

3 Definition of new invariants

Notation 3.1. Let n be a positive integer. For every integer i with $0 \leq i \leq n$, we set

$$E_i(x_0, \ldots, x_i; y_{n-i}, \ldots, y_n) := \sum_{\substack{0 \leq k, t \\ 0 \leq k+t \leq i}} (-1)^{i-t} \binom{n-t-2}{i-t-k} x_k y_{n-k-t} c_t(X). \tag{3.1}$$

Remark 3.2. Let W and H be as in Notation 2.1. We see from (3.1) and [8, Theorem 3.1] that for every integer i with $0 \leq i \leq n$

$$e_i(W, H) = E_i(c_0(\mathcal{E}), \ldots, c_i(\mathcal{E}); s_{n-i}(\mathcal{E}), \ldots, s_n(\mathcal{E})) \tag{3.2}$$

and by (3.2) we have

$$\begin{aligned} &b_i(W, H) - h^i(W, \mathbb{C}) \\ &= (-1)^i \left(E_i(c_0(\mathcal{E}), \ldots, c_i(\mathcal{E}); s_{n-i}(\mathcal{E}), \ldots, s_n(\mathcal{E})) - 2\sum_{j=0}^{i-1} (-1)^j h^j(W, \mathbb{C}) \right) \\ &\quad - h^i(W, \mathbb{C}). \end{aligned} \tag{3.3}$$

Here we note that if $i = 0$, then we regard $\sum_{j=0}^{i-1}(-1)^j h^j(W, \mathbb{C})$ as 0.

Theorem 3.3. *Let X be a smooth projective variety of dimension $n \geq 2$, and $\mathcal{E}$ an ample vector bundle of rank r on X. Let W and H be as in Notation 2.1. If $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$, then we have*

$$
\begin{aligned}
&b_{2n-2-i}(W, H) - h^{2n-2-i}(W, \mathbb{C}) \\
&\quad = (-1)^i \left(E_i(s_0(\mathcal{E}), \ldots, s_i(\mathcal{E}); c_{n-i}(\mathcal{E}), \ldots, c_n(\mathcal{E})) - 2\sum_{j=0}^{i-1}(-1)^j h^j(W, \mathbb{C}) \right) \\
&\qquad - h^i(W, \mathbb{C}).
\end{aligned}
$$

Proof. First we prove the following lemma.

Lemma 3.4. *If $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$, then*

$$
\begin{aligned}
&(-1)^{2n-2-i} \left((n-i-1)c_n(X) - 2\sum_{j=0}^{2n-2-i-1}(-1)^j h^j(W, \mathbb{C}) \right) - h^{2n-2-i}(W, \mathbb{C}) \\
&\quad = (-1)^{i+1} \left(2\sum_{j=0}^{i-1}(-1)^j h^j(W, \mathbb{C}) \right) - h^i(W, \mathbb{C}).
\end{aligned}
$$

Proof. By [19, (2.1) Proposition] and the assumption that $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$, we obtain

$$
h^j(W, \mathbb{C}) = \begin{cases} h^j(X, \mathbb{C}) + h^{j-2}(X, \mathbb{C}) + \cdots + h^0(X, \mathbb{C}), & \text{if } j \text{ is even}, \\ h^j(X, \mathbb{C}) + h^{j-2}(X, \mathbb{C}) + \cdots + h^1(X, \mathbb{C}), & \text{if } j \text{ is odd} \end{cases} \tag{3.4}
$$

for every integer j with $0 \leq j \leq i$ and $0 \leq j \leq 2n - 2 - i$.

(A) Assume that i is even. We set $i = 2l$. Then by (3.4) we have

$$
\begin{aligned}
&(n-2l-1)c_n(X) - h^{2n-2-2l}(W,\mathbb{C}) - 2\left(\sum_{j=0}^{2n-2-2l-1}(-1)^j h^j(W,\mathbb{C})\right) \qquad (3.5)\\
&= (n-2l-1)c_n(X) - \sum_{k=0}^{n-l-1} h^{2k}(X,\mathbb{C})\\
&\quad -2\left(\sum_{k=0}^{n-l-2}(n-k-l-1)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{n-l-1}(n-k-l)h^{2k-1}(X,\mathbb{C})\right)\\
&= (n-2l-1)c_n(X)\\
&\quad -\sum_{k=0}^{n-l-1}(2n-2k-2l-1)h^{2k}(X,\mathbb{C}) + \sum_{k=1}^{n-l-1}(2n-2k-2l)h^{2k-1}(X,\mathbb{C})\\
&= (n-2l)c_n(X)\\
&\quad -\left(\sum_{k=0}^{n-l-1}(2n-2k-2l)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{n-l}(2n-2k-2l+1)h^{2k-1}(X,\mathbb{C})\right)\\
&\quad -h^{2n-2l}(X,\mathbb{C}) + h^{2n-2l+1}(X,\mathbb{C}) + \cdots + (-1)h^{2n}(X,\mathbb{C})\\
&= (n-2l)c_n(X)\\
&\quad -\left(\sum_{k=0}^{n}(2n-2k-2l)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{n}(2n-2k-2l+1)h^{2k-1}(X,\mathbb{C})\right)\\
&\quad +\sum_{k=n-l}^{n}(2n-2k-2l)h^{2k}(X,\mathbb{C}) - \sum_{k=n-l+1}^{n}(2n-2k-2l+1)h^{2k-1}(X,\mathbb{C})\\
&\quad -\left(\sum_{k=n-l}^{n}h^{2k}(X,\mathbb{C}) - \sum_{k=n-l+1}^{n}h^{2k-1}(X,\mathbb{C})\right)\\
&= \sum_{k=0}^{n}(2k-n)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{n}(2k-n-1)h^{2k-1}(X,\mathbb{C})\\
&\quad +\sum_{k=n-l}^{n}(2n-2k-2l-1)h^{2k}(X,\mathbb{C}) - \sum_{k=n-l+1}^{n}(2n-2k-2l)h^{2k-1}(X,\mathbb{C}).
\end{aligned}
$$

By [8, Claim 3.1], we see that

$$\sum_{k=0}^{n}(2k-n)h^{2k}(X,\mathbb{C}) = 0$$

and

$$\sum_{k=1}^{n}(2k-n-1)h^{2k-1}(X,\mathbb{C}) = 0.$$

Hence by (3.5) and using the Poincaré duality we get

$$
\begin{aligned}
&(n-2l-1)c_n(X) - 2\left(\sum_{j=0}^{2n-2-2l-1}(-1)^j h^j(W,\mathbb{C})\right) - h^{2n-2-2l}(W,\mathbb{C}) \\
&= \sum_{k=n-l}^{n}(2n-2k-2l-1)h^{2k}(X,\mathbb{C}) - \sum_{k=n-l+1}^{n}(2n-2k-2l)h^{2k-1}(X,\mathbb{C}) \\
&= \sum_{j=0}^{l}(-2l+2j-1)h^{2n-2j}(X,\mathbb{C}) - \sum_{j=1}^{l}(2j-2l-2)h^{2n-2j+1}(X,\mathbb{C}) \\
&= -\sum_{j=0}^{l}(2l-2j+1)h^{2j}(X,\mathbb{C}) + \sum_{j=1}^{l}(2l-2j+2)h^{2j-1}(X,\mathbb{C}).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(-1)^{i+1}\left(2\sum_{j=0}^{i-1}(-1)^j h^j(W,\mathbb{C})\right) - h^i(W,\mathbb{C}) \\
&= -2\sum_{j=0}^{2l-1}(-1)^j h^j(W,\mathbb{C}) - h^{2l}(W,\mathbb{C}) \\
&= -2\left(\sum_{k=0}^{l-1} h^{2k}(W,\mathbb{C}) - \sum_{k=1}^{l} h^{2k-1}(W,\mathbb{C})\right) - h^{2l}(W,\mathbb{C}) \\
&= -2\left(\sum_{k=0}^{l-1}(l-k)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{l}(l+1-k)h^{2k-1}(X,\mathbb{C})\right) - \sum_{k=0}^{l} h^{2k}(X,\mathbb{C}) \\
&= -\sum_{k=0}^{l}(2l-2k+1)h^{2k}(X,\mathbb{C}) + \sum_{k=1}^{l}(2l-2k+2)h^{2k-1}(X,\mathbb{C}).
\end{aligned}
$$

Hence the assertion holds if i is even.

(B) Assume that i is odd. We set $i = 2l+1$. Then by (3.4) we get

$$
\begin{aligned}
&(-1)^{2n-2-i}\left((n-i-1)c_n(X)-2\sum_{j=0}^{2n-2-i-1}(-1)^j h^j(W,\mathbb{C})\right)-h^{2n-2-i}(W,\mathbb{C})\\
&=-(n-2l-2)c_n(X)+2\sum_{j=0}^{2n-2l-4}(-1)^j h^j(W,\mathbb{C})-h^{2n-2l-3}(W,\mathbb{C})\\
&=-(n-2l-2)c_n(X)-\sum_{k=1}^{n-l-1}h^{2k-1}(X,\mathbb{C})\\
&\quad+2\left(\sum_{k=0}^{n-l-2}(n-k-l-1)h^{2k}(X,\mathbb{C})-\sum_{k=1}^{n-l-2}(n-k-l-1)h^{2k-1}(X,\mathbb{C})\right)\\
&=-(n-2l-2)c_n(X)+\sum_{k=0}^{n-l-2}(2n-2k-2l-2)h^{2k}(X,\mathbb{C})\\
&\quad-\sum_{k=1}^{n-l-2}(2n-2k-2l-1)h^{2k-1}(X,\mathbb{C})-h^{2n-2l-3}(X,\mathbb{C})\\
&=-(n-2l-1)c_n(X)+\sum_{k=0}^{n-l-2}(2n-2k-2l-1)h^{2k}(X,\mathbb{C})\\
&\quad-\sum_{k=1}^{n-l-2}(2n-2k-2l)h^{2k-1}(X,\mathbb{C})+\sum_{k=n-l-1}^{n}h^{2k}(X,\mathbb{C})\\
&\quad-\sum_{k=n-l-1}^{n}h^{2k-1}(X,\mathbb{C})-h^{2n-2l-3}(X,\mathbb{C})\\
&=-(n-2l-1)c_n(X)+\sum_{k=0}^{n}(2n-2k-2l-1)h^{2k}(X,\mathbb{C})\\
&\quad-\sum_{k=1}^{n}(2n-2k-2l)h^{2k-1}(X,\mathbb{C})-\sum_{k=n-l-1}^{n}(2n-2k-2l-2)h^{2k}(X,\mathbb{C})\\
&\quad+\sum_{k=n-l-1}^{n}(2n-2k-2l-1)h^{2k-1}(X,\mathbb{C})-h^{2n-2l-3}(X,\mathbb{C})\\
&=\sum_{k=0}^{n}(n-2k)h^{2k}(X,\mathbb{C})-\sum_{k=1}^{n}(n-2k+1)h^{2k-1}(X,\mathbb{C})\\
&\quad-\sum_{k=n-l-1}^{n}(2n-2k-2l-2)h^{2k}(X,\mathbb{C})\\
&\quad+\sum_{k=n-l-1}^{n}(2n-2k-2l-1)h^{2k-1}(X,\mathbb{C})-h^{2n-2l-3}(X,\mathbb{C}).
\end{aligned}
$$

By [8, Claim 3.1], we see that

$$\sum_{k=0}^{n}(n-2k)h^{2k}(X,\mathbb{C})=0$$

and

$$\sum_{k=1}^{n}(n-2k+1)h^{2k-1}(X,\mathbb{C})=0.$$

Hence by an argument similar to that of the case where i is even, we get

$$\begin{aligned}
&\sum_{k=0}^{n}(n-2k)h^{2k}(X,\mathbb{C})-\sum_{k=1}^{n}(n-2k+1)h^{2k-1}(X,\mathbb{C})\\
&\quad-\sum_{k=n-l-1}^{n}(2n-2k-2l-2)h^{2k}(X,\mathbb{C})\\
&\quad+\sum_{k=n-l-1}^{n}(2n-2k-2l-1)h^{2k-1}(X,\mathbb{C})-h^{2n-2l-3}(X,\mathbb{C})\\
&=\sum_{k=n-l-1}^{n}(2k+2l-2n+2)h^{2k}(X,\mathbb{C})\\
&\quad-\sum_{k=n-l-1}^{n}(2k+2l-2n+1)h^{2k-1}(X,\mathbb{C})-h^{2n-2l-3}(X,\mathbb{C})\\
&=\sum_{k=n-l-1}^{n}(2k+2l-2n+2)h^{2k}(X,\mathbb{C})\\
&\quad-\sum_{k=n-l}^{n}(2k+2l-2n+1)h^{2k-1}(X,\mathbb{C})\\
&=\sum_{k=n-l-1}^{n}(2k+2l-2n+2)h^{2n-2k}(X,\mathbb{C})\\
&\quad-\sum_{k=n-l}^{n}(2k+2l-2n+1)h^{2n-2k+1}(X,\mathbb{C})\\
&=\sum_{k=n-l}^{n}(2k+2l-2n+2)h^{2n-2k}(X,\mathbb{C})\\
&\quad-\sum_{k=n-l}^{n}(2k+2l-2n+1)h^{2n-2k+1}(X,\mathbb{C})\\
&=\sum_{k=0}^{l}(2l-2k+2)h^{2k}(X,\mathbb{C})-\sum_{k=1}^{l+1}(2l-2k+3)h^{2k-1}(X,\mathbb{C}).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&(-1)^{i+1}\left(2\sum_{j=0}^{i-1}(-1)^j h^j(W,\mathbb{C})\right) - h^i(W,\mathbb{C}) \\
&= 2\sum_{j=0}^{2l}(-1)^j h^j(W,\mathbb{C}) - h^{2l+1}(W,\mathbb{C}) \\
&= 2\left(\sum_{k=0}^{l}(l-k+1)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{l}(l+1-k)h^{2k-1}(X,\mathbb{C})\right) \\
&\quad - \sum_{k=1}^{l+1} h^{2k-1}(X,\mathbb{C}) \\
&= \sum_{k=0}^{l}(2l-2k+2)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{l+1}(2l-2k+3)h^{2k-1}(X,\mathbb{C}).
\end{aligned}
$$

Hence the assertion holds if i is odd.

In any case we obtain the assertion of Lemma 3.4. $\square$

Here we note that by [9, Claim 3.1] we have

$$
\begin{aligned}
&e_{2n-2-i}(W,H) \\
&= \sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{i-t}\binom{n-t-2}{i-t-l}c_{n-t-l}(\mathcal{E})c_t(X)s_l(\mathcal{E}) + (n-i-1)c_n(X).
\end{aligned}
\tag{3.6}
$$

Hence by (3.1), Lemma 3.4 and (3.6) we have

$$
\begin{aligned}
&b_{2n-2-i}(W,H)-h^{2n-2-i}(W,\mathbb{C})\\
&=(-1)^{2n-2-i}\left(e_{2n-2-i}(W,L)-2\sum_{j=0}^{2n-2-i-1}(-1)^{j}h^{j}(W,\mathbb{C})\right)\\
&\quad -h^{2n-2-i}(W,\mathbb{C})\\
&=(-1)^{2n-2-i}\left(\sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{i-t}\binom{n-t-2}{i-t-l}c_{n-t-l}(\mathcal{E})c_t(X)s_l(\mathcal{E})\right)\\
&\quad +(-1)^{2n-2-i}(n-i-1)c_n(X)\\
&\quad -2(-1)^{2n-2-i}\left(\sum_{j=0}^{2n-2-i-1}(-1)^{j}h^{j}(W,\mathbb{C})\right)-h^{2n-2-i}(W,\mathbb{C})\\
&=(-1)^{i}\left(\sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{i-t}\binom{n-t-2}{i-t-l}c_{n-t-l}(\mathcal{E})c_t(X)s_l(\mathcal{E})\right)\\
&\quad +(-1)^{i+1}\left(2\sum_{j=0}^{i-1}(-1)^{j}h^{j}(W,\mathbb{C})\right)-h^{i}(W,\mathbb{C})\\
&=(-1)^{i}\left(E_i(s_0(\mathcal{E}),\dots,s_i(\mathcal{E});c_{n-i}(\mathcal{E}),\dots,c_n(\mathcal{E}))-2\sum_{j=0}^{i-1}(-1)^{j}h^{j}(W,\mathbb{C})\right)\\
&\quad -h^{i}(W,\mathbb{C}).
\end{aligned}
$$

Therefore we get the assertion of Theorem 3.3. $\square$

Theorem 3.5. *Let X, $\mathcal{E}$, W, H, r and n be as in Notation 2.1. Assume that $n\geq 2$ and $r\geq\max\left\{n-\frac{i+1}{2},\frac{i+1}{2}\right\}$ for every integer i with $0\leq i\leq n$. Then the following holds.*

$$
\begin{aligned}
&b_{2n-2-i}(W,H)-h^{2n-2-i}(W,\mathbb{C})\\
&=(-1)^{i}E_i(s_0(\mathcal{E}),\dots,s_i(\mathcal{E});c_{n-i}(\mathcal{E}),\dots,c_n(\mathcal{E}))\\
&\quad +(-1)^{i+1}\left(\sum_{k=0}^{\lfloor\frac{i-1}{2}\rfloor}(i+1-2k)h^{2k}(X,\mathbb{C})-\sum_{k=1}^{\lfloor\frac{i}{2}\rfloor}(i+2-2k)h^{2k-1}(X,\mathbb{C})\right)\\
&\quad -h^{i}(X,\mathbb{C}).
\end{aligned}
$$

Proof. If $i = 2l$, then by (3.4) in the proof of Lemma 3.4 we get

$$\begin{aligned}
&(-1)^{i+1}\left(2\sum_{j=0}^{i-1}(-1)^j h^j(W,\mathbb{C})\right) - h^i(W,\mathbb{C}) \\
&= -2\sum_{k=0}^{l-1}(l-k)h^{2k}(X,\mathbb{C}) + 2\sum_{k=1}^{l}(l+1-k)h^{2k-1}(X,\mathbb{C}) - \sum_{k=0}^{l} h^{2k}(X,\mathbb{C}) \\
&= -\sum_{k=0}^{l-1}(2l-2k+1)h^{2k}(X,\mathbb{C}) + \sum_{k=1}^{l}(2l+2-2k)h^{2k-1}(X,\mathbb{C}) - h^{2l}(X,\mathbb{C}) \\
&= -\sum_{k=0}^{\lfloor \frac{i}{2}\rfloor-1}(i+1-2k)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{\lfloor \frac{i}{2}\rfloor}(i+2-2k)h^{2k-1}(X,\mathbb{C}) - h^i(X,\mathbb{C}).
\end{aligned}$$

If $i = 2l+1$, then

$$\begin{aligned}
&(-1)^{i+1}\left(2\sum_{j=0}^{i-1}(-1)^j h^j(W,\mathbb{C})\right) - h^i(W,\mathbb{C}) \\
&= 2\sum_{k=0}^{l}(l+1-k)h^{2k}(X,\mathbb{C}) - 2\sum_{k=1}^{l}(l+1-k)h^{2k-1}(X,\mathbb{C}) \\
&\quad - \sum_{k=1}^{l+1} h^{2k-1}(X,\mathbb{C}) \\
&= \sum_{k=0}^{l}(2l-2k+2)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{l}(2l+3-2k)h^{2k-1}(X,\mathbb{C}) - h^{2l+1}(X,\mathbb{C}) \\
&= \sum_{k=0}^{\lfloor \frac{i}{2}\rfloor}(i+1-2k)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{\lfloor \frac{i}{2}\rfloor}(i+2-2k)h^{2k-1}(X,\mathbb{C}) - h^i(X,\mathbb{C}).
\end{aligned}$$

Hence

$$\begin{aligned}
&(-1)^{i+1}\left(2\sum_{j=0}^{i-1}(-1)^j h^j(W,\mathbb{C})\right) - h^i(W,\mathbb{C}) \\
&= (-1)^{i+1}\left(\sum_{k=0}^{\lfloor \frac{i-1}{2}\rfloor}(i+1-2k)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{\lfloor \frac{i}{2}\rfloor}(i+2-2k)h^{2k-1}(X,\mathbb{C})\right) \\
&\quad - h^i(X,\mathbb{C}).
\end{aligned}$$

(Here we note that if i is odd (resp. even), then $\lfloor \frac{i}{2}\rfloor = \lfloor \frac{i-1}{2}\rfloor$ (resp. $\lfloor \frac{i}{2}\rfloor - 1 = \lfloor \frac{i-1}{2}\rfloor$).

So by Theorem 3.3 we get the assertion. □

Definition 3.6. Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let i be an integer with $0 \leq i \leq n$. Assume $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then we define the following invariant $B^i(X, \mathcal{E})$ of $(X, \mathcal{E})$.

$$\begin{aligned} B^i(X,\mathcal{E}) &:= (-1)^i E_i(s_0(\mathcal{E}), \ldots, s_i(\mathcal{E}); c_{n-i}(\mathcal{E}), \ldots, c_n(\mathcal{E})) \\ &\quad -(-1)^i \left(\sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} (i+1-2k) h^{2k}(X, \mathbb{C}) - \sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} (i+2-2k) h^{2k-1}(X, \mathbb{C}) \right). \end{aligned}$$

We can prove the following proposition by Theorem 3.5 and Definition 3.6.

Proposition 3.7. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let i be an integer with $0 \leq i \leq n$. Assume that $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then $b_{2n-2-i}(W, H) - h^{2n-2-i}(W, \mathbb{C}) = B^i(X, \mathcal{E}) - h^i(X, \mathbb{C})$ holds.*

Moreover we get the following result.

Proposition 3.8. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let i be an integer with $0 \leq i \leq n$. Assume that $\mathcal{E}$ is spanned by its global sections and $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then $B^i(X, \mathcal{E}) \geq h^i(X, \mathbb{C})$ holds.*

Proof. Let W and H be as in Notation 2.1. By Proposition 3.7 we have

$$b_{2n-2-i}(W, H) - h^{2n-2-i}(W, \mathbb{C}) = B^i(X, \mathcal{E}) - h^i(X, \mathbb{C}).$$

We note that $b_{2n-2-i}(W, H) \geq h^{2n-2-i}(W, \mathbb{C})$ since $\mathcal{E}$ is spanned by its global sections. Hence we get the assertion. □

Remark 3.9. (i) If $i = 0$, then we have $B^0(X, \mathcal{E}) = c_n(\mathcal{E})$. Since $\mathcal{E}$ is ample with $\operatorname{rank}(\mathcal{E}) \geq n$, we see that $B^0(X, \mathcal{E}) \geq 1 = h^0(X, \mathbb{C})$.
(ii) If $i = 1$, then $B^1(X, \mathcal{E}) = 2v(X, \mathcal{E})$, where $v(X, \mathcal{E})$ denotes the invariant in [8, Definition 4.1]. (Here we note that $\sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} (i+2-2k) h^{2k-1}(X, \mathbb{C}) = 0$ if $i = 1$.) For details on the invariant $v(X, \mathcal{E})$, see [8].

Considering Proposition 3.8, we can propose the following conjeture.

Conjecture 3.10. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let i be an integer with $0 \leq i \leq n$. Assume that $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then $B^i(X, \mathcal{E}) \geq h^i(X, \mathbb{C})$ holds.*

Definition 3.11. Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let i be an integer with $0 \leq i \leq n$.

Assume $r \geq \max\left\{n-\frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then we define the following invariant $\widehat{B}^i(X,\mathcal{E})$ of $(X,\mathcal{E})$.

$$\begin{aligned}\widehat{B}^i(X,\mathcal{E}) := {} & (-1)^i E_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E}); s_{n-i}(\mathcal{E}),\dots,s_n(\mathcal{E})) \\ & -(-1)^i \left(\sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} (i+1-2k)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} (i+2-2k)h^{2k-1}(X,\mathbb{C}) \right).\end{aligned}$$

Proposition 3.12. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let i be an integer with $0 \leq i \leq n$. Assume that $r \geq \max\left\{n-\frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then $b_i(W,H) - h^i(W,\mathbb{C}) = \widehat{B}^i(X,\mathcal{E}) - h^i(X,\mathbb{C})$ holds.*

Proof. First by (3.3) in Remark 3.2 we have

$$\begin{aligned}& b_i(W,H) - h^i(W,\mathbb{C}) \\ & = (-1)^i \left(E_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E}); s_{n-i}(\mathcal{E}),\dots,s_n(\mathcal{E})) - 2\sum_{j=0}^{i-1} (-1)^j h^j(W,\mathbb{C}) \right) \\ & \quad -h^i(W,\mathbb{C}).\end{aligned}$$

On the other hand, by the same argument as the proof of Theorem 3.5, we have

$$\begin{aligned}& b_i(W,H) - h^i(W,\mathbb{C}) \\ & = (-1)^i E_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E}); s_{n-i}(\mathcal{E}),\dots,s_n(\mathcal{E})) \\ & \quad +(-1)^{i+1} \left(\sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} (i+1-2k)h^{2k}(X,\mathbb{C}) - \sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} (i+2-2k)h^{2k-1}(X,\mathbb{C}) \right) \\ & \quad -h^i(X,\mathbb{C}).\end{aligned}$$

So we get the assertion by Definition 3.11. □

Remark 3.13. If $i=0$, then we have $\widehat{B}^0(X,\mathcal{E}) = s_n(\mathcal{E})$. Since $\mathcal{E}$ is ample with $\operatorname{rank}(\mathcal{E}) \geq n$, we see $\widehat{B}^0(X,\mathcal{E}) \geq 1 = h^0(X,\mathbb{C})$.

Here we consider the case of $i=1$. If $\mathcal{E}$ is a line bundle L, then $n=2$ and $\widehat{B}^1(X,\mathcal{E}) = 2 + (K_X+L)L = 2g(X,L)$. Therefore $\widehat{B}^1(X,\mathcal{E}) \geq 0$ and the classification of $(X,\mathcal{E})$ with $\widehat{B}^1(X,\mathcal{E}) \leq 4$ is known (see [18] and [2]). So we assume that $r \geq 2$.

Theorem 3.14. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Assume that $r \geq \max\{n-1, 2\}$. Then $\widehat{B}^1(X,\mathcal{E}) \geq 0$ holds.*

Proof. Let W and H be as in Notation 2.1. By Proposition 3.12 we have

$$\widehat{B}^1(X,\mathcal{E}) - h^1(X,\mathbb{C}) = b_1(W,H) - h^1(W,\mathbb{C}).$$

On the other hand, we see that $h^1(X,\mathbb{C}) = 2q(X) = 2q(W) = h^1(W,\mathbb{C})$. So we get

$$\widehat{B}^1(X,\mathcal{E}) = b_1(W,H). \tag{3.7}$$

Since by Remark 2.10 (ii)

$$b_1(W,H) = 2g(W,H) \geq 0, \tag{3.8}$$

we have $\widehat{B}^1(X,\mathcal{E}) \geq 0$ by (3.7) and (3.8). □

Remark 3.15. Since $\widehat{B}^1(X,\mathcal{E}) = 2g(W,H)$, we see from [4, Theorems (3.2), (3.3) and (3.4)] that we can get a classification of $(X,\mathcal{E})$ with $\widehat{B}^1(X,\mathcal{E}) \leq 4$. For details, see [4, Theorems (3.2), (3.3) and (3.4)].

Here we propose the following conjecture which is the $\widehat{B}^i(X,\mathcal{E})$'s version of Conjecture 3.10.

Conjecture 3.16. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let i be an integer with $0 \leq i \leq n$. Assume that $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then $\widehat{B}^i(X,\mathcal{E}) \geq h^i(X,\mathbb{C})$ holds.*

Proposition 3.17. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X such that $\mathcal{E}$ is generated by its global sections. Let i be an integer with $0 \leq i \leq n$. Assume that $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then $\widehat{B}^i(X,\mathcal{E}) \geq h^i(X,\mathbb{C})$ holds.*

Proof. Let W and H be as in Notation 2.1. By Proposition 3.12 we have $b_i(W,H) - h^i(W,\mathbb{C}) = \widehat{B}^i(X,\mathcal{E}) - h^i(X,\mathbb{C})$. Since $\mathcal{E}$ is spanned by its global sections, we have $b_i(W,H) \geq h^i(W,\mathbb{C})$. So we get the assertion. □

Proposition 3.18. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let i be an integer with $0 \leq i \leq n$. Assume that $r \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then $B^i(X,\mathcal{E})$ and $\widehat{B}^i(X,\mathcal{E})$ are even for every odd integer i.*

Proof. Assume that i is odd. By Proposition 3.7 we have

$$b_{2n-2-i}(W,H) - b_{2n-2-i}(W,\mathbb{C}) = B^i(X,\mathcal{E}) - h^i(X,\mathbb{C}).$$

If i is odd, then $b_{2n-2-i}(W,H)$ (resp. $b_{2n-2-i}(W,\mathbb{C})$ and $h^i(X,\mathbb{C})$) is even by [7, Theorem 3.1 (3.1.2)] (resp. the Hodge theory). Hence $B^i(X,\mathcal{E})$ is even. On the other hand, By Proposition 3.12 we have $b_i(W,H) - b_i(W,\mathbb{C}) = \widehat{B}^i(X,\mathcal{E}) - h^i(X,\mathbb{C})$. If i is odd, then $b_i(W,H)$ (resp. $b_i(W,\mathbb{C})$ and $h^i(X,\mathbb{C})$) is even by [7, Theorem 3.1 (3.1.2)] (resp. the Hodge theory). Hence $\widehat{B}^i(X,\mathcal{E})$ is also even. □

By Propositions 3.7 and 3.12 we get the following.

Proposition 3.19. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Assume that $r \geq \frac{n}{2}$. Then $B^{n-1}(X,\mathcal{E}) = \widehat{B}^{n-1}(X,\mathcal{E})$.*

Similarily we can get the following relation between $B^n(X,\mathcal{E})$ and $\widehat{B}^{n-2}(X,\mathcal{E})$ by Propositions 3.7 and 3.12.

Theorem 3.20. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Assume $r \geq \frac{n+1}{2}$. Then*

$$B^n(X,\mathcal{E}) - h^n(X,\mathbb{C}) = \widehat{B}^{n-2}(X,\mathcal{E}) - h^{n-2}(X,\mathbb{C})$$

holds.

4 On $B^2(X,\mathcal{E})$ and $\widehat{B}^2(X,\mathcal{E})$ for $\dim X = 2$ and 3

In this section we study $B^2(X,\mathcal{E})$ and $\widehat{B}^2(X,\mathcal{E})$ for $\dim X = 2$ and 3.

First we calculate $B^2(X,\mathcal{E})$ and $\widehat{B}^2(X,\mathcal{E})$ in general. We have

$$\begin{aligned}
&E_2(s_0(\mathcal{E}), s_1(\mathcal{E}), s_2(\mathcal{E}); c_{n-2}(\mathcal{E}), c_{n-1}(\mathcal{E}), c_n(\mathcal{E})) \\
&= \sum_{\substack{0\leq k,t \\ 0\leq k+t\leq 2}} (-1)^{2-t}\binom{n-t-2}{2-t-k} s_k(\mathcal{E})c_{n-k-t}(\mathcal{E})c_t(X) \\
&= \binom{n-2}{2} c_n(\mathcal{E}) - (n-3)c_1(X)c_{n-1}(\mathcal{E}) + (n-2)s_1(\mathcal{E})c_{n-1}(\mathcal{E}) \\
&\quad + c_2(X)c_{n-2}(\mathcal{E}) - s_1(\mathcal{E})c_{n-2}(\mathcal{E})c_1(X) + s_2(\mathcal{E})c_{n-2}(\mathcal{E}),
\end{aligned}$$

$$\begin{aligned}
&E_2(c_0(\mathcal{E}), c_1(\mathcal{E}), c_2(\mathcal{E}); s_{n-2}(\mathcal{E}), s_{n-1}(\mathcal{E}), s_n(\mathcal{E})) \\
&= \sum_{\substack{0\leq k,t \\ 0\leq k+t\leq 2}} (-1)^{2-t}\binom{n-t-2}{2-t-k} c_k(\mathcal{E})s_{n-k-t}(\mathcal{E})c_t(X) \\
&= \binom{n-2}{2} s_n(\mathcal{E}) - (n-3)c_1(X)s_{n-1}(\mathcal{E}) + (n-2)c_1(\mathcal{E})s_{n-1}(\mathcal{E}) \\
&\quad + c_2(X)s_{n-2}(\mathcal{E}) - c_1(\mathcal{E})s_{n-2}(\mathcal{E})c_1(X) + c_2(\mathcal{E})s_{n-2}(\mathcal{E}).
\end{aligned}$$

Moreover we have

$$\sum_{k=0}^{\lfloor \frac{2-1}{2} \rfloor} (2+1-2k)h^{2k}(X,\mathbb{C}) = 3h^0(X,\mathbb{C}) = 3,$$

$$\sum_{k=1}^{\lfloor \frac{2}{2} \rfloor} (2+2-2k)h^{2k-1}(X,\mathbb{C}) = 2h^1(X,\mathbb{C}).$$

So we get

$$B^2(X,\mathcal{E}) = \binom{n-2}{2}c_n(\mathcal{E}) - (n-3)c_1(X)c_{n-1}(\mathcal{E}) \\ +(n-2)s_1(\mathcal{E})c_{n-1}(\mathcal{E}) + c_2(X)c_{n-2}(\mathcal{E}) \\ -s_1(\mathcal{E})c_{n-2}(\mathcal{E})c_1(X) + s_2(\mathcal{E})c_{n-2}(\mathcal{E}) \\ -3 + 2h^1(X,\mathbb{C}), \tag{4.1}$$

$$\widehat{B}^2(X,\mathcal{E}) = \binom{n-2}{2}s_n(\mathcal{E}) - (n-3)c_1(X)s_{n-1}(\mathcal{E}) \\ +(n-2)c_1(\mathcal{E})s_{n-1}(\mathcal{E}) + c_2(X)s_{n-2}(\mathcal{E}) \\ -c_1(\mathcal{E})s_{n-2}(\mathcal{E})c_1(X) + c_2(\mathcal{E})s_{n-2}(\mathcal{E}) \\ -3 + 2h^1(X,\mathbb{C}). \tag{4.2}$$

Proposition 4.1. *If $n = 2$ and $\operatorname{rank}(\mathcal{E}) \geq 2$, then $B^2(X,\mathcal{E}) \geq h^2(X,\mathbb{C})$ and $\widehat{B}^2(X,\mathcal{E}) \geq h^2(X,\mathbb{C})$.*

Proof. If $n = 2$, then by (4.1) and (4.2) we have

$$B^2(X,\mathcal{E}) = s_2(\mathcal{E}) - 1 + h^2(X,\mathbb{C}),$$
$$\widehat{B}^2(X,\mathcal{E}) = c_2(\mathcal{E}) - 1 + h^2(X,\mathbb{C}).$$

Since $\mathcal{E}$ is ample, we have $s_2(\mathcal{E}) > 0$ and $c_2(\mathcal{E}) > 0$ hold. Hence we get the assertion. □

Next we consider the case $n = 3$. In particular we treat the case of $\kappa(X) \geq 0$.

Theorem 4.2. *Let X be a smooth projective variety of dimension 3 and let $\mathcal{E}$ be an ample vector bundle on X with $\operatorname{rank}(\mathcal{E}) \geq 2$. If $\kappa(X) \geq 0$, then $B^2(X,\mathcal{E}) \geq 2h^1(X,\mathbb{C})$.*

Proof. First we note that $\operatorname{rank}(\mathcal{E}) \geq 2 \geq \max\left\{3 - \frac{2+1}{2}, \frac{2+1}{2}\right\}$ and by (4.1) $B^2(X,\mathcal{E})$ is the following in this situation.

$$B^2(X,\mathcal{E}) = c_2(X)c_1(\mathcal{E}) + (K_X + c_1(\mathcal{E}))c_1(\mathcal{E})^2 - 3 + 2h^1(X,\mathbb{C}). \tag{4.3}$$

Here we note that $e_2(X, c_1(\mathcal{E})) = c_2(X)c_1(\mathcal{E}) + (K_X + c_1(\mathcal{E}))c_1(\mathcal{E})^2$ by Definition 2.9 (i). So we find

$$B^2(X,\mathcal{E}) = e_2(X, c_1(\mathcal{E})) - 3 + 2h^1(X,\mathbb{C}). \tag{4.4}$$

Since $\kappa(X) \geq 0$, we see from [22, Theorems 1, 2 and 3] that $K_X + c_1(\mathcal{E})$ is nef. Hence by [14, 2.11 Corollary], we see that

$$c_2(X)c_1(\mathcal{E}) \geq -\frac{2}{3}K_X c_1(\mathcal{E}) - \frac{1}{3}c_1(\mathcal{E})^3. \tag{4.5}$$

Hence by (4.3) and (4.5) we have

$$\begin{aligned}B^2(X,\mathcal{E}) &= c_2(X)c_1(\mathcal{E}) + (K_X + c_1(\mathcal{E}))c_1(\mathcal{E})^2 - 3 + 2h^1(X,\mathbb{C}) \\ &\geq \frac{1}{3}(K_X + c_1(\mathcal{E}))c_1(\mathcal{E})^2 + \frac{1}{3}c_1(\mathcal{E})^3 - 3 + 2h^1(X,\mathbb{C}).\end{aligned}$$

Here we note the following.

Claim 4.3. $c_1(\mathcal{E})^3 \geq 2$.

Proof. Since $\mathcal{E}$ is ample, we have $c_1(\mathcal{E})^3 > c_1(\mathcal{E})c_2(\mathcal{E}) > 0$ by [11, Example 12.1.7]. □

(i) If $(K_X + c_1(\mathcal{E}))c_1(\mathcal{E})^2 \geq 6$, then

$$B^2(X,\mathcal{E}) \geq \left\lceil \frac{1}{3}c_1(\mathcal{E})^3 - 1 + 2h^1(X,\mathbb{C}) \right\rceil \geq 2h^1(X,\mathbb{C}).$$

(ii) If $(K_X + c_1(\mathcal{E}))c_1(\mathcal{E})^2 \leq 3$, then by [3, Proposition 2.5.1] we have $(K_X + c_1(\mathcal{E}))^2 c_1(\mathcal{E}) \leq 9$. Here we note that since $\kappa(X) \geq 0$ we have $\chi_2^H(X, c_1(\mathcal{E})) > 0$ by [6, Theorem 3.3.1]. Hence by [7, Theorem 4.3] we have

$$\begin{aligned}e_2(X, c_1(\mathcal{E})) &= 12\chi_2^H(X, c_1(\mathcal{E})) - (K_X + c_1(\mathcal{E}))^2 c_1(\mathcal{E}) \\ &\geq 12 - 9 = 3.\end{aligned}$$

So by (4.4) we have

$$B^2(X,\mathcal{E}) \geq 2h^1(X,\mathbb{C}).$$

(iii) Assume that $(K_X + c_1(\mathcal{E}))c_1(\mathcal{E})^2 = 4$. Then by [3, Proposition 2.5.1] and Claim 4.3 we have $(K_X + c_1(\mathcal{E}))^2 c_1(\mathcal{E}) \leq 8$, and by the same argument as the case (ii) we have

$$\begin{aligned}e_2(X, c_1(\mathcal{E})) &= 12\chi_2^H(X, c_1(\mathcal{E})) - (K_X + c_1(\mathcal{E}))^2 c_1(\mathcal{E}) \\ &\geq 12 - 8 = 4.\end{aligned}$$

So we have

$$B^2(X,\mathcal{E}) \geq 1 + 2h^1(X,\mathbb{C}).$$

(iv) Finally we assume that $(K_X + c_1(\mathcal{E}))c_1(\mathcal{E})^2 = 5$. If $c_1(\mathcal{E})^3 \geq 3$, then we see from [3, Proposition 2.5.1] that $(K_X + c_1(\mathcal{E}))^2 c_1(\mathcal{E}) \leq 8$, and by the same argument as the case (iii) we have

$$B^2(X,\mathcal{E}) \geq 1 + 2h^1(X,\mathbb{C}).$$

So we may assume that $c_1(\mathcal{E})^3 = 1$ or 2. But $c_1(\mathcal{E})^3 = 2$ is impossible because of [3, Lemma 1.1.11]. Hence we get $c_1(\mathcal{E})^3 = 1$. But this case does not occur by Claim 4.3.

These complete the proof of Theorem 4.2. □

By Proposition 3.19 we get the following.

Corollary 4.4. *Let X be a smooth projective variety of dimension 3 and let $\mathcal{E}$ be an ample vector bundle on X with $\operatorname{rank}(\mathcal{E}) \geq 2$. If $\kappa(X) \geq 0$, then $\widehat{B}^2(X,\mathcal{E}) \geq 2h^1(X,\mathbb{C})$.*

5 A relation between $b_i(X, c_1(\mathcal{E}))$ and $B^i(X,\mathcal{E}) + \widehat{B}^i(X,\mathcal{E})$

Definition 5.1. Let X be a smooth projective variety of dimension $n \geq 2$ and $\mathcal{E}$ an ample vector bundle on X. Let i be an integer with $0 \leq i \leq n$. Assume that $r = \operatorname{rank}(\mathcal{E}) \geq \max\left\{n - \frac{i+1}{2}, \frac{i+1}{2}\right\}$. Then we set

$$P_i(X,\mathcal{E}) := b_i(X, c_1(\mathcal{E})) - (B^i(X,\mathcal{E}) + \widehat{B}^i(X,\mathcal{E})).$$

5.1 The case $i = 0$.

First we consider the case $i = 0$.

Remark 5.2. Let X, $\mathcal{E}$ and r be as in Definition 5.1. If $i = 0$, then we have

$$P_0(X,\mathcal{E}) = c_1(\mathcal{E})^n - c_n(\mathcal{E}) - s_n(\mathcal{E}). \tag{5.1}$$

Here we prove the following lemma which will be used in the next subsection.

Lemma 5.3. (i) *For $p \geq 2$, we have*

$$c_1(\mathcal{E})^p - c_p(\mathcal{E}) - s_p(\mathcal{E}) = \sum_{\lambda \in \Lambda(p,r)} a_\lambda \Delta_\lambda(c)$$

where a_λ is a non-negative integer for every $\lambda \in \Lambda(p,r)$.
(ii) *We have*

$$\sum_{\lambda \in \Lambda(p,r)} a_\lambda \geq 2(2^{p-2} - 1).$$

Proof. (i) We prove (i) by induction on p.

(i.1) If $p = 2$, then

$$c_1(\mathcal{E})^2 - c_2(\mathcal{E}) - s_2(\mathcal{E}) = 0.$$

So we get the assertion for $p = 2$.

(i.2) Assume that the assertion is true for the case of $p = k - 1$. We consider the case where $p = k$. First we note that the following holds by Remark 2.7.

$$s_k(\mathcal{E}) = c_1(\mathcal{E}) s_{k-1}(\mathcal{E}) - \Delta_{\mu_{k-1}}(c). \tag{5.2}$$

Here we put $\mu_k = (2, \underbrace{1, \ldots, 1}_{k-1})$ for every integer $k \geq 2$.

By [9, Proposition 3.1] and (5.2), we have

$$c_k(\mathcal{E}) = s_1(\mathcal{E})c_{k-1}(\mathcal{E}) - \Delta_{\mu_{k-1}}(s). \tag{5.3}$$

We see from [11, Lemma 14.5.1] that

$$\Delta_{\mu_{k-1}}(s) = \Delta_{(k-1,1)}(c). \tag{5.4}$$

Noting that $s_1(\mathcal{E}) = c_1(\mathcal{E})$, we get the following by (5.3) and (5.4).

$$c_k(\mathcal{E}) = c_1(\mathcal{E})c_{k-1}(\mathcal{E}) - \Delta_{(k-1,1)}(c). \tag{5.5}$$

Therefore by (5.2) and (5.5) we get

$$\begin{aligned} & c_1(\mathcal{E})^k - c_k(\mathcal{E}) - s_k(\mathcal{E}) \\ & = c_1(\mathcal{E})^k - c_1(\mathcal{E})c_{k-1}(\mathcal{E}) - c_1(\mathcal{E})s_{k-1}(\mathcal{E}) \\ & \quad + \Delta_{(k-1,1)}(c) + \Delta_{\mu_{k-1}}(c) \\ & = c_1(\mathcal{E})(c_1(\mathcal{E})^{k-1} - c_{k-1}(\mathcal{E}) - s_{k-1}(\mathcal{E})) \\ & \quad + \Delta_{(k-1,1)}(c) + \Delta_{\mu_{k-1}}(c). \end{aligned} \tag{5.6}$$

By assumption $c_1(\mathcal{E})^{k-1} - c_{k-1}(\mathcal{E}) - s_{k-1}(\mathcal{E})$ can be written as $\sum_{\lambda \in \Lambda(k-1,r)} b_\lambda \Delta_\lambda(c)$, where $b_\lambda \geq 0$ for every $\lambda \in \Lambda(k-1,r)$. So by [11, Lemma 14.5.2] we see that $c_1(\mathcal{E})(c_1(\mathcal{E})^{k-1} - c_{k-1}(\mathcal{E}) - s_{k-1}(\mathcal{E}))$ can be written as $\sum_{\lambda \in \Lambda(k,r)} c_\lambda \Delta_\lambda(c)$ too, where $c_\lambda \geq 0$ for every $\lambda \in \Lambda(k,r)$. Therefore we get the assertion for the case of $n = k$, and we get the assertion of Lemma 5.3 (i).

(ii) We prove (ii) by induction on n.

(ii.1) If $p = 2$, then

$$c_1(\mathcal{E})^2 - c_2(\mathcal{E}) - s_2(\mathcal{E}) = 0.$$

Hence $\sum_{\lambda \in \Lambda(2,r)} a_\lambda = 0 = 2(2^{2-2} - 1)$ and we get the assertion for $p = 2$.

(ii.2) Assume that the assertion is true for the case of $p = k-1$. We consider the case where $p = k$. We set $c_1(\mathcal{E})^{k-1} - c_{k-1}(\mathcal{E}) - s_{k-1}(\mathcal{E}) = \sum_{\lambda \in \Lambda(k-1,r)} b_\lambda \Delta_\lambda(c)$. Then by assumption we have $\sum_{\lambda \in \Lambda(k-1,r)} b_\lambda \geq 2(2^{k-3} - 1)$. Here we note that $c_1(\mathcal{E})\Delta_\lambda(c)$ has at least two Schur polynomials (see [11, Lemma 14.5.2]). By (5.6) we get

$$\sum_{\lambda \in \Lambda(k,r)} a_\lambda \geq 2 + 2\left(\sum_{\lambda \in \Lambda(k-1,r)} b_\lambda\right) \geq 2 + 2^2(2^{k-3} - 1) = 2(2^{k-2} - 1).$$

Therefore we get the assertion of Lemma 5.3 (ii). $\square$

By (5.1) in Remark 5.2 and Lemma 5.3 we get the following.

Theorem 5.4. *Let X be a smooth projective variety of dimension $n \geq 2$ and $\mathcal{E}$ an ample vector bundle on X with* $\operatorname{rank}(\mathcal{E}) \geq n$. *Then*

$$P_0(X, \mathcal{E}) \geq 2(2^{n-2} - 1).$$

Corollary 5.5. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $\mathcal{E}$ be an ample vector bundle on X with* $\operatorname{rank}(\mathcal{E}) \geq n$. *Then $c_1(\mathcal{E})^n \geq 2^{n-1}$.*

Proof. By Remark 5.2, we have

$$c_1(\mathcal{E})^n = c_n(\mathcal{E}) + s_n(\mathcal{E}) + P_0(X, \mathcal{E}).$$

Since $\mathcal{E}$ is ample, we have $c_n(\mathcal{E}) \geq 1$ and $s_n(\mathcal{E}) \geq 1$. Therefore by Theorem 5.4 we get

$$c_1(\mathcal{E})^n \geq 2 + 2(2^{n-2} - 1) = 2^{n-1}.$$

□

By (5.6) in Lemma 5.3, [11, Lemma 14.5.2] and Theorem 2.8, we get the following better lower bound for $P_0(X, L)$ with small n.

Proposition 5.6. *Let X be a smooth projective variety of dimension $n \geq 2$ and $\mathcal{E}$ an ample vector bundle on X with* $\operatorname{rank}(\mathcal{E}) \geq n$. *Then the following hold.*

(i) *If $n = 2$, then $P_0(X, \mathcal{E}) = 0$.*

(ii) *If $n = 3$, then*
$$P_0(X, \mathcal{E}) = 2\Delta_{(2,1)}(c) \geq 2.$$

(iii) *If $n = 4$, then*
$$P_0(X, \mathcal{E}) = 2\Delta_{(2,2)}(c) + 3\Delta_{(3,1)}(c) + 3\Delta_{(2,1,1)}(c) \geq 8.$$

(iv) *If $n = 5$, then*
$$\begin{aligned} P_0(X, \mathcal{E}) &= 5\Delta_{(3,2)}(c) + 5\Delta_{(2,2,1)}(c) + 4\Delta_{(4,1)}(c) \\ &\quad + 6\Delta_{(3,1,1)}(c) + 4\Delta_{(2,1,1,1)}(c) \\ &\geq 24. \end{aligned}$$

(v) *If $n = 6$, then*
$$\begin{aligned} P_0(X, \mathcal{E}) &= 5\Delta_{(3,3)}(c) + 16\Delta_{(3,2,1)}(c) + 9\Delta_{(4,2)}(c) + 5\Delta_{(2,2,2)}(c) \\ &\quad + 9\Delta_{(2,2,1,1)}(c) + 5\Delta_{(5,1)}(c) + 10\Delta_{(4,1,1)}(c) \\ &\quad + 10\Delta_{(3,1,1,1)}(c) + 5\Delta_{(2,1,1,1,1)}(c) \\ &\geq 74. \end{aligned}$$

(vi) *If $n = 7$, then*

$$\begin{aligned}P_0(X,\mathcal{E}) = {} & 14\Delta_{(4,3)}(c) + 21\Delta_{(3,3,1)}(c) + 14\Delta_{(5,2)}(c) + 35\Delta_{(4,2,1)}(c)\\ & +21\Delta_{(3,2,2)}(c) + 35\Delta_{(3,2,1,1)}(c) + 14\Delta_{(2,2,2,1)}(c)\\ & +14\Delta_{(2,2,1,1,1)}(c) + 6\Delta_{(6,1)}(c)\\ & +15\Delta_{(5,1,1)}(c) + 15\Delta_{(3,1,1,1,1)}(c)\\ & +20\Delta_{(4,1,1,1)}(c) + 6\Delta_{(2,1,1,1,1,1)}(c)\\ \geq {} & 230.\end{aligned}$$

Corollary 5.7. *Let X be a smooth projective variety of dimension $n \geq 2$ and $\mathcal{E}$ an ample vector bundle on X with* $\mathrm{rank}(\mathcal{E}) \geq n$. *Then the following hold.*

$$c_1(\mathcal{E})^n \geq \begin{cases} 2, & \text{if } n = 2,\\ 4, & \text{if } n = 3,\\ 10, & \text{if } n = 4,\\ 26, & \text{if } n = 5,\\ 76, & \text{if } n = 6,\\ 232, & \text{if } n = 7.\end{cases}$$

5.2 The case $i = 1$.

Next we consider the case $i = 1$.

Remark 5.8. We have

$$\begin{aligned}P_1(X,\mathcal{E}) = {} & (n-2)(c_1(\mathcal{E})^n - c_n(\mathcal{E}) - s_n(\mathcal{E}))\\ & +(K_X + c_1(\mathcal{E}))(c_1(\mathcal{E})^{n-1} - c_{n-1}(\mathcal{E}) - s_{n-1}(\mathcal{E})) - 2.\end{aligned} \tag{5.7}$$

Remark 5.9. If $n = 2$, then $P_1(X,\mathcal{E}) = -2-(K_X+c_1(\mathcal{E}))c_1(\mathcal{E}) = -2g(X, c_1(\mathcal{E})) \leq 0$. So we assume that $n \geq 3$ from now on.

Remark 5.10. Let X be a smooth projective variety of dimension $n \geq 3$ and $\mathcal{E}$ an ample vector bundle on X. Assume that $K_X+c_1(\mathcal{E})$ is not nef and $\mathrm{rank}(\mathcal{E}) \geq n-1$. Then $(X,\mathcal{E})$ is one of the following types (see [22, Theorems 1, 2 and 3]).

(ii.1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$.

(ii.2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1})$.

(ii.3) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^n}(2))$.

(ii.4) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n-1})$.

(ii.5) $X \cong \mathbb{P}_C(\mathcal{F})$ for some vector bundle $\mathcal{F}$ of rank n on a smooth projective curve C, and $\mathcal{E} \cong H(\mathcal{F}) \otimes \pi^*\mathcal{G}$, where $\pi : X \to C$ is the bundle projection and $\mathcal{G}$ is a vector bundle on C with $\mathrm{rank}(\mathcal{G}) = n-1$.

Then we calculate $P_1(X,\mathcal{E})$. In order to do that, first we calculate $s_i(L^{\oplus r})$ for a line bundle L on X.

Lemma 5.11. *Let X be a smooth projective variety of dimension $n\geq 3$ and let L be an ample line bundle on X. Then $s_i(L^{\oplus r})=\binom{r-1+i}{i}L^i$ for every integer i with $0\leq i\leq n$.*

Proof. We set $\mathcal{E}:=L^{\oplus r}$. First we note that $c_t(\check{\mathcal{E}})=c_t((-L)^{\oplus r})=(1-Lt)^r=(c_t(-L))^r$. On the other hand, since $c_t(-L)s_t(L)=1$, we have $s_t(L)=1+Lt+L^2t^2+\cdots+L^nt^n$. Therefore

$$\begin{aligned}
s_t(\mathcal{E})=s_t(L)^r&=(1+Lt+L^2t^2+\cdots+L^nt^n)^r\\
&=\sum_{i=0}^{n}\binom{r-1+i}{i}L^it^i.
\end{aligned}$$

Therefore we get the assertion. $\square$

(ii.1) Assume that $(X,\mathcal{E})\cong(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$. Then by (5.7) in Remark 5.8, we have

$$\begin{aligned}
P_1(X,\mathcal{E})&=(n-2)\left(n^n-1-\binom{2n-1}{n}\right)+(-1)\left(n^{n-1}-n-\binom{2n-2}{n-1}\right)-2\\
&=n^{n-1}(n^2-2n-1)-(n-2)\binom{2n-1}{n}+\binom{2n-2}{n-1}\\
&=n^{n-1}(n^2-2n-1)-\frac{2n^2-6n+2}{n}\binom{2n-2}{n-1}.
\end{aligned}$$

First we note the following claim.

Claim 5.12. *Let x and y be a positive integer with $x<y$. Then the following holds.*

$$\frac{y+1}{x+1}<\frac{y}{x}.$$

By Claim 5.12, we have

$$\begin{aligned}
\binom{2n-2}{n-1}&=\frac{(2n-2)\cdots n}{(n-1)!}\\
&=\frac{2n-2}{n-1}\cdot\frac{2n-3}{n-2}\cdots\frac{n+1}{2}\cdot\frac{n}{1}\\
&<n^{n-1}
\end{aligned}$$

for $n\geq 3$.

On the other hand, we set $f(n):=n(n^2-2n-1)-(2n^2-6n+2)$. Then

$$\begin{aligned}
f(n)&=n^3-4n^2+5n-2\\
&=n^2(n-4)+5n-2.
\end{aligned}$$

If $n \geq 4$, then $f(n) > 0$. Moreover $f(3) = 4$. So we get $f(n) > 0$ for $n \geq 3$. Therefore we have

$$n^{n-1}(n^2 - 2n - 1) > \frac{2n^2 - 6n + 2}{n}\binom{2n-2}{n-1}.$$

Namely $P_1(X, \mathcal{E}) > 0$ for $n \geq 3$.

(ii.2) Assume that $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1})$. Then by (5.7) in Remark 5.8, we have

$$\begin{aligned}
P_1(X, \mathcal{E}) &= (n-2)\left((n-1)^n - \binom{2n-2}{n}\right) \\
&\quad +(-2)\left((n-1)^{n-1} - 1 - \binom{2n-3}{n-1}\right) - 2 \\
&= (n^2 - 3n)(n-1)^{n-1} - (n-2)\binom{2n-2}{n} + 2\binom{2n-3}{n-1}.
\end{aligned}$$

First we note that

$$-(n-2)\binom{2n-2}{n} + 2\binom{2n-3}{n-1} = -\frac{2n^2 - 8n + 4}{n}\binom{2n-3}{n-1}.$$

By Claim 5.12, we have

$$\begin{aligned}
\binom{2n-3}{n-1} &= \frac{(2n-3)\cdots(n-1)}{(n-1)!} \\
&= \frac{2n-3}{n-1}\cdot\frac{2n-4}{n-2}\cdots\frac{n}{2}\cdot\frac{n-1}{1} \\
&< (n-1)^{n-1}
\end{aligned} \tag{5.8}$$

for $n \geq 3$.

On the other hand, we set $f(n) := n^2(n-3) - (2n^2 - 8n + 4)$. Then

$$\begin{aligned}
f(n) &= n^3 - 5n^2 + 8n - 4 \\
&= n^2(n-5) + 8n - 4.
\end{aligned}$$

If $n \geq 5$, then $f(n) > 0$. Moreover $f(4) = 12$ and $f(3) = 2$. So we get $f(n) > 0$ for $n \geq 3$.

Therefore we have

$$n(n-3)(n-1)^{n-1} > \frac{2n^2 - 8n + 4}{n}\binom{2n-3}{n-1}.$$

Namely $P_1(X, \mathcal{E}) > 0$ for $n \geq 3$.

(ii.4) Assume that $(X,\mathcal{E}) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n-1})$. Then by (5.7) in Remark 5.8, we have

$$\begin{aligned} P_1(X,\mathcal{E}) &= (n-2)\left(2(n-1)^n - 2\binom{2n-2}{n}\right) \\ &\quad +(-2)\left((n-1)^{n-1} - 1 - \binom{2n-3}{n-1}\right) - 2 \\ &= (2n^2-6n+2)(n-1)^{n-1} - 2(n-2)\binom{2n-2}{n} + 2\binom{2n-3}{n-1}. \end{aligned}$$

First we note that

$$2(n-2)\binom{2n-2}{n} - 2\binom{2n-3}{n-1} = \frac{4n^2-14n+8}{n}\binom{2n-3}{n-1}.$$

By (5.8) we have $(n-1)^{n-1} > \binom{2n-3}{n-1}$ for $n \geq 3$. On the other hand, we set $f(n) := n(2n^2-6n+2) - (4n^2-14n+8)$. Then

$$\begin{aligned} f(n) &= 2n^3 - 10n^2 + 16n - 8 \\ &= 2n^2(n-5) + 16n - 8. \end{aligned}$$

If $n \geq 5$, then $f(n) > 0$. Moreover $f(4) = 24$ and $f(3) = 4$. So we get $f(n) > 0$ for $n \geq 3$.

Therefore we have

$$(2n^2-6n+2)(n-1)^{n-1} > \frac{4n^2-14n+8}{n}\binom{2n-3}{n-1}.$$

Namely $P_1(X,\mathcal{E}) > 0$ for $n \geq 3$.

(ii.5) Assume that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^n}(2))$. Here we need the following.

Lemma 5.13. *Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ and $\mathcal{E}_1$ be vector bundles on X and let L be a line bundle on X. Assume that $\mathcal{E} = \mathcal{E}_1 \oplus L$. Then $s_t(\mathcal{E}) = s_t(L)s_t(\mathcal{E}_1)$.*

Proof. By assumption we have $c_t(\check{\mathcal{E}}) = c_t(-L)c_t(\check{\mathcal{E}}_1)$. Since $c_t(-L)s_t(L) = 1$ and $c_t(\check{\mathcal{E}}_1)s_t(\mathcal{E}_1) = 1$, we get the assertion. □

Here we set $L := \mathcal{O}_{\mathbb{P}^n}(2)$ and $\mathcal{E}_1 := \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-2}$. By Lemma 5.13 we have

$$\begin{aligned} s_n(\mathcal{E}) &= \sum_{i=0}^{n} 2^{n-i}\binom{n-3+i}{i}, \\ s_{n-1}(\mathcal{E}) &= \sum_{i=0}^{n-1} 2^{n-1-i}\binom{n-3+i}{i}\mathcal{O}_{\mathbb{P}^n}(1)^{n-1}. \end{aligned}$$

Since $K_X + c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^n}(-1)$, we see from (5.7) in Remark 5.8 that

$$\begin{aligned}
&P_1(X,\mathcal{E}) \\
&= (n-2)\left(n^n - \sum_{k=0}^{n} 2^{n-k}\binom{n-3+k}{k}\right) \\
&\quad - \left(n^{n-1} - 2 - \sum_{k=0}^{n-1} 2^{n-1-k}\binom{n-3+k}{k}\right) - 2 \\
&= (n-2)\left(n^n - \sum_{k=0}^{n} 2^{n-k}\binom{n-3+k}{k}\right) \\
&\quad - \left(n^{n-1} - \sum_{k=0}^{n-1} 2^{n-1-k}\binom{n-3+k}{k}\right).
\end{aligned} \tag{5.9}$$

First we note that

$$\begin{aligned}
&n^n - \sum_{k=0}^{n} 2^{n-k}\binom{n-3+k}{k} \\
&= n^n - 2\sum_{k=0}^{n} 2^{n-1-k}\binom{n-3+k}{k} \\
&= n^n - 2\sum_{k=0}^{n-1} 2^{n-1-k}\binom{n-3+k}{k} - \binom{2n-3}{n} \\
&= 2n^{n-1} - 2\sum_{k=0}^{n-1} 2^{n-1-k}\binom{n-3+k}{k} + n^n - 2n^{n-1} - \binom{2n-3}{n}.
\end{aligned} \tag{5.10}$$

Here $n^n - 2n^{n-1} - \binom{2n-3}{n} = (n-2)n^{n-1} - \binom{2n-3}{n}$ and by Claim 5.12, we have

$$\begin{aligned}
\binom{2n-3}{n} &= \frac{(2n-3)\cdots(n-2)}{(n)!} \\
&= \frac{2n-3}{n}\cdot\frac{2n-4}{n-1}\cdots\frac{n-1}{2}\cdot\frac{n-2}{1} \\
&\leq (n-2)^n
\end{aligned} \tag{5.11}$$

for $n \geq 3$. Therefore by (5.11)

$$\begin{aligned}
(n-2)n^{n-1} - \binom{2n-3}{n} &\geq (n-2)n^{n-1} - (n-2)^n \\
&= (n-2)(n^{n-1} - (n-2)^{n-1}) \\
&> 0.
\end{aligned} \tag{5.12}$$

By (5.10) and (5.12) we have

$$n^n - \sum_{k=0}^{n} 2^{n-k}\binom{n-3+k}{k} > 2\left(n^{n-1} - \sum_{k=0}^{n-1} 2^{n-1-k}\binom{n-3+k}{k}\right). \tag{5.13}$$

So by (5.9) and (5.13) we get

$$P_1(X,\mathcal{E}) > (2n-5)\left(n^{n-1} - \sum_{k=0}^{n-1} 2^{n-1-k}\binom{n-3+k}{k}\right). \tag{5.14}$$

Here we prove the following.

Lemma 5.14.

$$n^{n-1} - \sum_{k=0}^{n-1} 2^{n-1-k}\binom{n-3+k}{k} \geq 0.$$

Proof. By Claim 5.12, we have

$$\begin{aligned}\binom{n-3+k}{k} &= \frac{(n-3+k)\cdots(n-2)}{k!} \\ &= \frac{n-3+k}{k}\cdot\frac{n-4+k}{k-1}\cdots\frac{n-1}{2}\cdot\frac{n-2}{1} \\ &\leq (n-2)^k\end{aligned} \tag{5.15}$$

for $n \geq 3$. Hence by (5.15)

$$\begin{aligned}\sum_{k=0}^{n-1} 2^{n-1-k}\binom{n-3+k}{k} &\leq \sum_{k=0}^{n-1} 2^{n-1-k}(n-2)^k \\ &\leq (2+(n-2))^{n-1} \\ &= n^{n-1}.\end{aligned}$$

This completes the proof of Lemma 5.14. $\square$

By (5.14) and Lemma 5.14, $P_1(X,\mathcal{E}) \geq 0$ if $n \geq 3$.

(ii.6) Assume that $X \cong \mathbb{P}_C(\mathcal{F})$ for some vector bundle $\mathcal{F}$ of rank n on a smooth projective curve C, and $\mathcal{E} \cong H(\mathcal{F}) \otimes \pi^*\mathcal{G}$, where $\pi : X \to C$ is the bundle projection and $\mathcal{G}$ is a vector bundle on C with $\mathrm{rank}(\mathcal{G}) = n-1$.

First we calculate $s_{n-1}(\mathcal{E})$. Here we use notation in Remark 2.3 (i). Then by

[11, Example 3.1.1] we have

$$\begin{aligned}
s_{n-1}(\mathcal{E}) &= s_{n-1}(\pi^*\mathcal{G} \otimes H(\mathcal{F})) \\
&= \hat{s}_{n-1}((\pi^*\mathcal{G})^* \otimes H(\mathcal{F})^{-1}) \\
&= \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-2+n-1}{n-2+i} \hat{s}_i((\pi^*\mathcal{G})^*) c_1(H(\mathcal{F})^{-1})^{n-1-i} \\
&= \binom{2n-3}{n-2} H(\mathcal{F})^{n-1} + \binom{2n-3}{n-1} s_1(\pi^*\mathcal{G}) H(\mathcal{F})^{n-2}.
\end{aligned}$$

Next we calculate $s_n(\mathcal{E})$.

$$\begin{aligned}
s_n(\mathcal{E}) &= s_n(\pi^*\mathcal{G} \otimes H(\mathcal{F})) \\
&= \hat{s}_n((\pi^*\mathcal{G})^* \otimes H(\mathcal{F})^{-1}) \\
&= \sum_{i=0}^{n} (-1)^{n-i} \binom{n-2+n}{n-2+i} \hat{s}_i((\pi^*\mathcal{G})^*) c_1(H(\mathcal{F})^{-1})^{n-i} \\
&= \binom{2n-2}{n-2} H(\mathcal{F})^{n} + \binom{2n-2}{n-1} s_1(\pi^*\mathcal{G}) H(\mathcal{F})^{n-1}.
\end{aligned}$$

We also note that

$$\begin{aligned}
c_n(\mathcal{E}) &= 0, \\
c_1(\mathcal{E})^n &= (n-1)^n \deg \mathcal{F} + n(n-1)^{n-1} \deg \mathcal{G}, \\
c_{n-1}(\mathcal{E}) &= H(\mathcal{F})^{n-1} + H(\mathcal{F})^{n-2} c_1(\pi^*\mathcal{G}) \\
&= H(\mathcal{F})^{n-1} + H(\mathcal{F})^{n-2} s_1(\pi^*\mathcal{G}), \\
c_1(\mathcal{E})^{n-1} &= (n-1)^{n-1} H(\mathcal{F})^{n-1} + (n-1)^{n-1} H(\mathcal{F})^{n-2} c_1(\pi^*\mathcal{G}) \\
&= (n-1)^{n-1} H(\mathcal{F})^{n-1} + (n-1)^{n-1} H(\mathcal{F})^{n-2} s_1(\pi^*\mathcal{G}).
\end{aligned}$$

Hence

$$\begin{aligned}
&(K_X + s_1(\mathcal{E}))(c_1(\mathcal{E})^{n-1} - c_{n-1}(\mathcal{E}) - s_{n-1}(\mathcal{E})) \qquad (5.16) \\
&\quad = (\pi^*(K_C + \det(\mathcal{G}) + \det(\mathcal{F})) - H(\mathcal{F})) \\
&\qquad \times \left\{ \left((n-1)^{n-1} - 1 - \binom{2n-3}{n-2} \right) H(\mathcal{F})^{n-1} \right. \\
&\qquad\qquad \left. + \left((n-1)^{n-1} - 1 - \binom{2n-3}{n-1} \right) s_1(\pi^*\mathcal{G}) H(\mathcal{F})^{n-2} \right\} \\
&\quad = (2g(C)-2) \left((n-1)^{n-1} - \binom{2n-3}{n-2} - 1 \right) \\
&\qquad + \left(\binom{2n-3}{n-1} - \binom{2n-3}{n-2} \right) \deg(\mathcal{G}),
\end{aligned}$$

and

$$\begin{aligned}
&c_1(\mathcal{E})^n - c_n(\mathcal{E}) - s_n(\mathcal{E}) \qquad (5.17)\\
&= (n-1)^n H(\mathcal{F})^n + n(n-1)^{n-1} H(\mathcal{F})^{n-1}\pi^*(\det \mathcal{G})\\
&\quad -\binom{2n-2}{n-2} H(\mathcal{F})^n - \binom{2n-2}{n-1} H(\mathcal{F})^{n-1}\pi^*(\det \mathcal{G})\\
&= \left((n-1)^n - \binom{2n-2}{n-2}\right)\deg(\mathcal{F}) + \left(n(n-1)^{n-1} - \binom{2n-2}{n-1}\right)\deg(\mathcal{G}).
\end{aligned}$$

Therefore by (5.7), (5.16) and (5.17) we have

$$\begin{aligned}
&P_1(X,\mathcal{E}) \qquad (5.18)\\
&= (n-2)\left((n-1)^n - \binom{2n-2}{n-2}\right)\deg(\mathcal{F})\\
&\quad +(n-2)\left(n(n-1)^{n-1} - \binom{2n-2}{n-1}\right)\deg(\mathcal{G})\\
&\quad +(2g(C)-2)\left((n-1)^{n-1} - \binom{2n-3}{n-2} - 1\right)\\
&\quad +\left(\binom{2n-3}{n-1} - \binom{2n-3}{n-2}\right)\deg(\mathcal{G}) - 2\\
&= (n-2)\left((n-1)^n - \binom{2n-2}{n-2}\right)\deg(\mathcal{F})\\
&\quad +\left(n(n-2)(n-1)^{n-1} - (n-3)\binom{2n-2}{n-1} - 2\binom{2n-3}{n-2}\right)\deg(\mathcal{G})\\
&\quad +(2g(C)-2)\left((n-1)^{n-1} - \binom{2n-3}{n-2} - 1\right) - 2.
\end{aligned}$$

First we note the following.

Claim 5.15.

$$\frac{1}{n-1}\binom{2n-2}{n-2} = \frac{n-3}{n(n-2)}\binom{2n-2}{n-1} + \frac{2}{n(n-2)}\binom{2n-3}{n-2}.$$

Proof.

$$\begin{aligned}
&\frac{n-3}{n(n-2)}\binom{2n-2}{n-1} + \frac{2}{n(n-2)}\binom{2n-3}{n-2}\\
&= \frac{n-3}{n(n-2)}\cdot\frac{(2n-2)!}{(n-1)!(n-1)!} + \frac{2}{n(n-2)}\cdot\frac{(2n-3)!}{(n-2)!(n-1)!}\\
&= \frac{n-3}{(n-1)(n-2)}\cdot\frac{(2n-2)!}{n!(n-2)!} + \frac{2}{(n-2)(2n-2)}\cdot\frac{(2n-2)!}{(n-2)!n!}\\
&= \frac{1}{n-1}\binom{2n-2}{n-2}.
\end{aligned}$$

$\square$

Lemma 5.16.

$$(n-1)\deg\mathcal{F} + n\deg\mathcal{G} \geq 1.$$

Proof. Since $\mathcal{E}$ is ample, we have $c_1(\mathcal{E})^n > 0$. On the other hand, we have $c_1(\mathcal{E})^n = (n-1)^n \deg\mathcal{F} + n(n-1)^{n-1}\deg\mathcal{G}$. Therefore we get the assertion. $\square$

We also note that

$$(n-1)^{n-1} - \binom{2n-3}{n-2} \geq 1 \tag{5.19}$$

can be proved by Claim 5.12 as follows:

$$\begin{aligned}\binom{2n-3}{n-2} &= \binom{2n-3}{n-1} \\ &= \frac{2n-3}{n-1}\cdot\frac{2n-4}{n-2}\cdots\frac{n-1}{1} \\ &< (n-1)^{n-1}.\end{aligned}$$

By (5.18), Claim 5.15, Lemma 5.16 and (5.19), we have

$$\begin{aligned}&P_1(X,\mathcal{E}) \qquad (5.20)\\ &= (n-2)(n-1)\left((n-1)^{n-1} - \frac{1}{n-1}\binom{2n-2}{n-2}\right)\deg\mathcal{F} \\ &\quad +(n-2)n\left((n-1)^{n-1} - \frac{n-3}{n(n-2)}\binom{2n-2}{n-1} - \frac{2}{n(n-2)}\binom{2n-3}{n-2}\right)\deg\mathcal{G} \\ &\quad +(2g(C)-2)\left((n-1)^{n-1} - \binom{2n-3}{n-2} - 1\right) - 2 \\ &= (n-2)\left((n-1)^{n-1} - \frac{1}{n-1}\binom{2n-2}{n-2}\right)((n-1)\deg\mathcal{F} + n\deg\mathcal{G}) \\ &\quad +(2g(C)-2)\left((n-1)^{n-1} - \binom{2n-3}{n-2} - 1\right) - 2 \\ &\geq (n-2)\left((n-1)^{n-1} - \frac{1}{n-1}\binom{2n-2}{n-2}\right) \\ &\quad -2\left((n-1)^{n-1} - \binom{2n-3}{n-2} - 1\right) - 2.\end{aligned}$$

Hence by (5.20)

$$P_1(X,\mathcal{E}) \tag{5.21}$$
$$\geq (n-2)\left((n-1)^{n-1}-\frac{1}{n-1}\binom{2n-2}{n-2}\right)$$
$$-2\left((n-1)^{n-1}-\binom{2n-3}{n-2}-1\right)-2$$
$$=(n-4)(n-1)^{n-1}-\frac{n-2}{n-1}\binom{2n-2}{n-2}+2\binom{2n-3}{n-2}.$$

On the other hand

$$-\frac{n-2}{n-1}\binom{2n-2}{n-2}+2\binom{2n-3}{n-2} \tag{5.22}$$
$$=\left(2-\frac{(n-2)(2n-2)}{(n-1)n}\right)\frac{(2n-3)!}{(n-2)!(n-1)!}$$
$$=\frac{4}{n}\binom{2n-3}{n-1}.$$

Hence by (5.21) and (5.22) we have $P_1(X,\mathcal{E})\geq 0$ if $n\geq 3$.
We see from the above that $P_1(X,\mathcal{E})\geq 0$ if $K_X+c_1(\mathcal{E})$ is not nef.

Next we consider the case where $n\geq 3$ and $K_X+c_1(\mathcal{E})$ is nef.

Theorem 5.17. *Let X be a smooth projective variety of dimension $n\geq 3$ and let $\mathcal{E}$ be an ample vector bundle on X with* $\operatorname{rank}(\mathcal{E})\geq n-1$. *Assume that $K_X+c_1(\mathcal{E})$ is nef. Then*

$$P_1(X,\mathcal{E})\geq 2(2^{n-2}-1)(n-2)-2$$

holds. In particular, $P_1(X,\mathcal{E})\geq 0$.

Proof. Since $K_X+c_1(\mathcal{E})$ is nef, we see from Lemma 5.3 (i) and [12, Corollary 3.10] that

$$(K_X+c_1(\mathcal{E}))(c_1^{n-1}(\mathcal{E})-c_{n-1}(\mathcal{E})-s_{n-1}(\mathcal{E}))\geq 0. \tag{5.23}$$

On the other hand, by Lemma 5.3 (ii) and Theorem 2.8, we have

$$c_1(\mathcal{E})^n-c_n(\mathcal{E})-s_n(\mathcal{E})=\sum_{\lambda\in\Lambda(n,r)}a_\lambda\Delta_\lambda(c) \tag{5.24}$$
$$\geq \sum_{\lambda\in\Lambda(n,r)}a_\lambda$$
$$\geq 2(2^{n-2}-1).$$

By (5.23) and (5.24) we get

$$\begin{aligned}P_1(X,\mathcal{E})&=(n-2)(c_1(\mathcal{E})^n-c_n(\mathcal{E})-s_n(\mathcal{E}))\\&\quad+(K_X+c_1(\mathcal{E}))(c_1(\mathcal{E})^{n-1}-c_{n-1}(\mathcal{E})-s_{n-1}(\mathcal{E}))-2\\&\geq 2(2^{n-2}-1)(n-2)-2.\end{aligned}$$

Therefore we get the assertion of Theorem 5.17. $\square$

Here we get the following better lower bound for $P_1(X,\mathcal{E})$ with small n.

Proposition 5.18. *Let X be a smooth projective variety of dimension $n \geq 3$ and $\mathcal{E}$ an ample vector bundle on X with* $\operatorname{rank}(\mathcal{E}) \geq n-1$*. Assume that $K_X + c_1(\mathcal{E})$ is nef. Then the following hold.*

(i) *If $n = 3$, then $P_1(X,\mathcal{E}) \geq 0$.*

(ii) *If $n = 4$, then $P_1(X,\mathcal{E}) \geq 14$.*

(iii) *If $n = 5$, then $P_1(X,\mathcal{E}) \geq 70$.*

(iv) *If $n = 6$, then $P_1(X,\mathcal{E}) \geq 294$.*

(v) *If $n = 7$, then $P_1(X,\mathcal{E}) \geq 1148$.*

Proof. Since $K_X + c_1(\mathcal{E})$ is nef, we see from Lemma 5.3 (i) that

$$(K_X + c_1(\mathcal{E}))(c_1(\mathcal{E})^{n-1} - c_{n-1}(\mathcal{E}) - s_{n-1}(\mathcal{E})) \geq 0.$$

We also note that $c_1(\mathcal{E})^n - c_n(\mathcal{E}) - s_n(\mathcal{E}) = P_0(X,\mathcal{E})$. Hence by Remark 5.8 and Proposition 5.6 we get the assertion. $\square$

As a corollary of Theorem 5.17, we get a lower bound for c_1-sectional genus of $(X,\mathcal{E})$ for the case where $\mathcal{E}$ is generated by its global sections.

Corollary 5.19. *Let X be a smooth projective variety of dimension $n \geq 3$ and $\mathcal{E}$ an ample vector bundle on X with* $\operatorname{rank}(\mathcal{E}) \geq n-1$*. Assume that $\mathcal{E}$ is generated by its global sections and $K_X + c_1(\mathcal{E})$ is nef. Then we have*

$$g(X, c_1(\mathcal{E})) \geq 2q(X) + (2^{n-2} - 1)(n-2) - 1.$$

Proof. First we note that $b_1(X, c_1(\mathcal{E})) = 2g(X, c_1(\mathcal{E}))$ (see Remark 2.10 (ii)). Moreover by Propositions 3.8 and 3.17, we have $B^1(X,\mathcal{E}) + \widehat{B}^1(X,\mathcal{E}) \geq 2h^1(X,\mathbb{C})$. By the Lefshcetz theorem, we have $h^1(X,\mathbb{C}) = 2q(X)$. Hence by Definition 5.1

$$\begin{aligned} g(X, c_1(\mathcal{E})) &= \frac{1}{2} b_1(X, c_1(\mathcal{E})) \\ &= \frac{1}{2}(B^1(X,\mathcal{E}) + \widehat{B}^1(X,\mathcal{E}) + P_1(X,\mathcal{E})) \\ &\geq 2q(X) + \frac{1}{2} P_1(X,\mathcal{E}). \end{aligned}$$

By Theorem 5.17, we get the assertion. $\square$

By the same argument as above, we get a better lower bound for the case where $3 \leq n \leq 7$ by using Proposition 5.18.

Corollary 5.20. *Let X be a smooth projective variety of dimension $n \geq 3$ and $\mathcal{E}$ an ample vector bundle on X with $\operatorname{rank}(\mathcal{E}) \geq n-1$. Assume that $\mathcal{E}$ is generated by its global sections and $K_X + c_1(\mathcal{E})$ is nef. Then the following hold.*

(i) *If $n = 3$, then $g(X, c_1(\mathcal{E})) \geq 2q(X)$.*

(ii) *If $n = 4$, then $g(X, c_1(\mathcal{E})) \geq 2q(X) + 7$.*

(iii) *If $n = 5$, then $g(X, c_1(\mathcal{E})) \geq 2q(X) + 35$.*

(iv) *If $n = 6$, then $g(X, c_1(\mathcal{E})) \geq 2q(X) + 147$.*

(v) *If $n = 7$, then $g(X, c_1(\mathcal{E})) \geq 2q(X) + 574$.*

Moreover, for the case where $h^1(\mathcal{O}_X) > 0$, we can improve a lower bound for $g(X, c_1(\mathcal{E}))$.

Proposition 5.21. *Let X be a smooth projective variety of dimension $n \geq 3$ with $h^1(\mathcal{O}_X) > 0$ and $\mathcal{E}$ an ample vector bundle on X with $\operatorname{rank}(\mathcal{E}) \geq n-1$. Assume that $\mathcal{E}$ is generated by its global sections and $K_X + c_1(\mathcal{E})$ is nef. Then we have*

$$g(X, c_1(\mathcal{E})) \geq 3q(X) + (2^{n-2} - 1)(n-2) - 2.$$

Proof. By Proposition 3.8, we have $B^1(X, \mathcal{E}) \geq h^1(X, \mathbb{C}) = 2h^1(\mathcal{O}_X)$. We note that

$$\begin{aligned} \widehat{B}^1(X, \mathcal{E}) - h^1(X, \mathbb{C}) &= b_1(W, H) - h^1(W, \mathbb{C}) \\ &= 2g_1(W, H) - 2h^1(\mathcal{O}_W). \end{aligned}$$

Here we prove the following.

Claim 5.22. (W, H) *is not a scroll over a smooth projective curve.*

Proof. Assume that (W, H) is a scroll over a smooth projective curve C. Let $p : W \to C$ be the projection. For any fiber F of p, $f(F)$ is a point or $f(F) = X$ because $F \cong \mathbb{P}^{n+r-2}$, where $f : W = \mathbb{P}_X(\mathcal{E}) \to X$ is the projection and $r = \operatorname{rank}(\mathcal{E})$. We note that the case where $f(F) = X$ is impossible because $h^1(\mathcal{O}_X) > 0$. So we see that $f(F)$ is a point for any fiber F of p. But this case cannot occur because $\dim X \geq 3$. $\square$

By Claim 5.22 and Proposition 2.11 we get $g_1(W, H) \geq 2h^1(\mathcal{O}_W) - 1$. Therefore

$$\begin{aligned} \widehat{B}^1(X, \mathcal{E}) - h^1(X, \mathbb{C}) &= 2g_1(W, H) - 2h^1(\mathcal{O}_W) \\ &\geq 4h^1(\mathcal{O}_W) - 2 - 2h^1(\mathcal{O}_W) \\ &= 2h^1(\mathcal{O}_W) - 2. \end{aligned}$$

We also note that $h^1(\mathcal{O}_W) = h^1(\mathcal{O}_X)$ and $h^1(X, \mathbb{C}) = 2h^1(\mathcal{O}_X)$. Hence we have $\widehat{B}^1(X, \mathcal{E}) \geq 4h^1(\mathcal{O}_X) - 2$. Therefore

$$\begin{aligned} g(X, c_1(\mathcal{E})) &= \frac{1}{2} b_1(X, c_1(\mathcal{E})) \\ &= \frac{1}{2}(B^1(X, \mathcal{E}) + \widehat{B}^1(X, \mathcal{E}) + P_1(X, \mathcal{E})) \\ &\geq 3h^1(\mathcal{O}_X) - 1 + \frac{1}{2} P_1(X, \mathcal{E}). \end{aligned}$$

By Theorem 5.17, we get the assertion. □

Acknowledgements. This research was partially supported by JSPS KAKENHI Grant Number 16K05103.

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Received: 15 November 2017.
Accepted: 2 October 2018.

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