# Expansion of the resolvent in a Feshbach model

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*Dedicated to Gianfausto Dell'Antonio on the occasion of his 85th birthday*

**Abstract.** *In this paper we extend the results proved in [7] about Feshbach resonances in a multichannel Hamiltonian $\mathcal{H}$, proving a low energy expansion of the resolvent $(\mathcal{H} - k^2)^{-1}$ as $k \to 0$ in the resonant case.*

## 1 Introduction

The physics of ultracold quantum gases and molecular quantum gases is a research which had had a steep growth in the last twenty years induced by incredible progress, from both the experimental point of view and from the theoretical one. The wide range of applications covers atomic and molecular physics, condensed matter and few and many-body physics.

A major breakthrough in this field was the experimental observation of a dilute Bose gas condensation in 1995 [2] as in the Einstein predictions of 1925 . After this fundamental experimental realization, a significant effort was made for a deep understanding of the physical processes involved. A crucial point was to go beyond the mean field physics, to study interactions between ultra cold atoms and observe long and short range correlation phenomena [18]. The extraordinary degree of control needed on such systems in order to reach these extreme conditions, was challenging.

From the experimental point of view, besides all the different techniques to control the physical properties of condensate, two of them in particular had greater success: one is the realization of optical lattices of different spatial dimensions ([3],[9]), while the second is the tunability of the scattering length using Feshbach resonances (see [8] and [15] for a review).

In a dilute quantum gas the density and mobility conditions are such that only the two-body scattering processes are relevant. Moreover in many situations, since the typical energies allow only elastic scattering, the most relevant parameter is the scattering length. The possibility to tune it provides an effective mean to control the interaction.
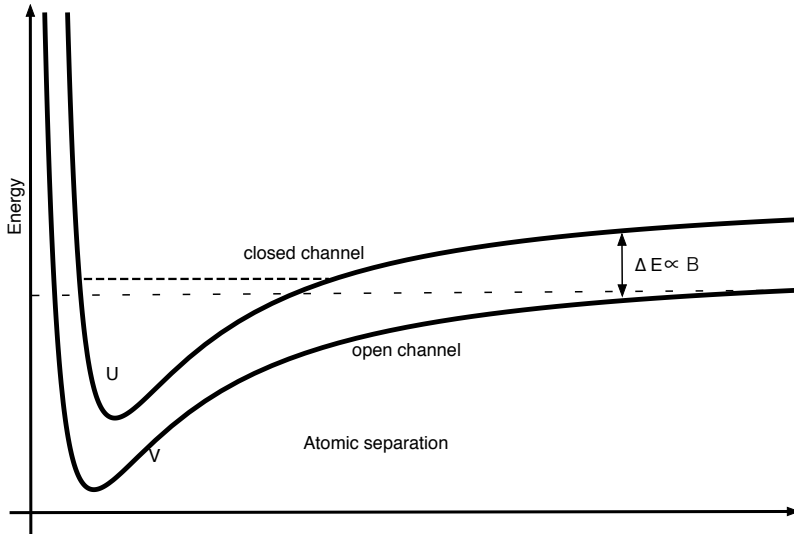
Figure 1: In the figure are plotted the potentials for an open channel and a closed channel with a bound state as function of atomic separation for alkali atoms.

A key feature of Feshbach resonances is the presence of an open channel (where scattering processes are allowed) and a closed channel (where scattering processes are forbidden) with bound states. It may happen that a bound state of the closed channel crosses the ionization threshold of the open channel, that is the bottom of the continuous spectrum of the hamiltonian of the open channel, and strongly interacts deeply influencing the scattering process, see [11], even when the interaction between channel is weak. This is often understood in the physical literature as virtual scattering process with a metastable state at the bottom of the spectrum.

In modern realizations the splitting is realized by applying an external magnetic field $B$ to, for instance, alkali atoms and it is due to the coupling of states with different spin quantum numbers with the magnetic field. In this way the energy splitting $\Delta E$ and energy levels of the bound states of the closed channel can be externally controlled and it is possible to realize the conditions for a Feshbach resonance.

In the physics literature it was obtained a formula relating the scattering length with the magnetic field around the value of the Feshbach resonance ([19])

$$a(B) = a \left( 1 - \frac{\Delta}{B - B_0} \right)$$

where $a$ is the scattering length far from the resonant values, $B_0$ is the value of the magnetic field at the resonance and $\Delta$ is the resonance width.

The rigorous analysis of resonances has a very long story [10]: the first math-

ematical model for multichannel hamiltonian traces back to the original paper of Friedrichs [12] in 1948. Many mathematical features of resonances were studied using different strategies, as the dilation-analytic technique [13] or simplified models using point interactions ([4, 5] and [6]).

Recently in [7] the formation of Feshbach resonances and the dependence of the scattering length from the magnetic field were rigorously studied, proving results about the existence and localization of resonances and the behavior of the scattering length near a resonance.

In Section 2, we fix the notation and we recall the main results of [7]. In Section 3 we prove a low energy expansion of the resolvent of a matrix hamiltonian when a Feshbach resonance is present; our main result is Theorem 3.5.

## 2   Existence of Feshbach Resonances

In this section we fix some notational conventions used in the paper and we recall the main results of [7]. Bold face letters denote vectors in $\mathbb{R}^3$, e.g., $\mathbf{x}$, while scalars are denoted by regular letters, e.g., $E$. When there is no ambiguity we also use the notation $x := |\mathbf{x}|$ for the modulus of a vector $\mathbf{x}$. Since we always deal with functions on $\mathbb{R}^3$, we often omit the base space in Banach space notation, i.e., $L^p := L^p(\mathbb{R}^3)$; moreover when there is no possible confusion we denote $\| \cdot \|_{L^p} =: \| \cdot \|_p$.

We recall the definition of weighted Hilbert spaces: set $\langle x \rangle := \sqrt{1 + x^2}$ for short, then, for any $s \geqslant 0$, we define

$$\|f\|_{L^2_s} := \|\langle x \rangle^s f\|_2 \,. \tag{2.1}$$

The closure of $C_0^\infty$ w.r.t. the above norm is denoted by $L^2_s$. The weighted Sobolev space $H^2_s$, $s \geqslant 0$, is defined analogously as the closure of $C_0^\infty$ w.r.t. the norm

$$\|f\|_{H^2_s} := \|f\|_{L^2_s} + \|\Delta f\|_{L^2_s} \,. \tag{2.2}$$

The conventional Sobolev spaces $H^p$, $p \in \mathbb{R}$ can be defined via Fourier transform as the closure of $C_0^\infty(\mathbb{R}^3)$ w.r.t. the norms

$$\|f\|_{H^p} := \left\|\langle k \rangle^p \hat{f}\right\|_2 \,, \tag{2.3}$$

where we use the following convention for the Fourier transform

$$\hat{f}(\mathbf{k}) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d\mathbf{x} \, e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \,. \tag{2.4}$$

By the properties of the Fourier transform, and a simple exchange of the role of $\mathbf{x}$ and $\mathbf{k}$, one easily gets

$$\|f\|_{L^2_s}^2 = \int_{\mathbb{R}^3} d\mathbf{x} \, \left(1 + x^2\right)^s |f(\mathbf{x})|^2 = \int_{\mathbb{R}^3} d\mathbf{x} \, \left(1 + x^2\right)^s \left|\widehat{\hat{f}}(-\mathbf{x})\right|^2 = \left\|\hat{f}\right\|_{H^s}^2 \,. \tag{2.5}$$

Hence, we obtain the useful identity

$$\|f\|_{H_s^2} = \left\|\hat{f}\right\|_{H^s} + \left\|k^2 \hat{f}\right\|_{H^s}. \tag{2.6}$$

We recall some classical results on spectral and scattering theory mostly taken from ([1, 14]), which will be used in the proofs. We denote by $\mathscr{B}$ the Banach space of continuous functions vanishing at infinity equipped with the sup norm. We also denote by $\mathcal{B}(L^2)$ the space of bounded linear operators on $L^2$ and, more in general, $\mathcal{B}(X, Y)$ stands for the space of continuous linear transformations between two Banach spaces $X$ and $Y$. Similarly, $\mathcal{B}_0(X, Y)$ is the space of compact operators from $X$ to $Y$ and $\mathcal{B}_0(X) := \mathcal{B}_0(X, X)$.

Following [7] we now make the mathematical setting more precise. We consider a multi-channel scattering of a particle in dimension three: we assume that there are an *open channel*, where the scattering is energetically possible, and a *closed* one where any scattering process is forbidden because of an energy constraint (see Figure 1).

We describe the system by the following matrix Hamiltonian acting on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$:

$$\mathcal{H} = \begin{pmatrix} -\Delta + V & W \\ W & -\Delta + U + \lambda \end{pmatrix} = \mathcal{H}_0 + \mathcal{W}, \tag{2.7}$$

$$\mathcal{H}_0 = \begin{pmatrix} -\Delta + V & 0 \\ 0 & -\Delta + U + \lambda \end{pmatrix}, \qquad \mathcal{W} = \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix}, \tag{2.8}$$

where

$$H_V = -\Delta + V, \qquad H_U = -\Delta + U. \tag{2.9}$$

The starting point of our investigation is the eigenvalue equation for the matrix Hamiltonian $\mathcal{H}$:

$$\mathcal{H}\Psi_{\mathbf{k}} = k^2 \Psi_{\mathbf{k}} \tag{2.10}$$

Writing $\Psi_{\mathbf{k}} = (\varphi_{\mathbf{k}}, \xi_{\mathbf{k}})^t$, where $t$ denotes transposition, equation (2.10) is equivalent to the system

$$\begin{cases} (-\Delta + V)\varphi_{\mathbf{k}} + W\xi_{\mathbf{k}} = k^2\varphi_{\mathbf{k}}, \\ (-\Delta + U + \lambda)\xi_{\mathbf{k}} + W\varphi_{\mathbf{k}} = k^2\xi_{\mathbf{k}}. \end{cases} \tag{2.11}$$

Since we are interested in the low energy behavior, we can restrict to the energies

$$0 < k^2 < \lambda. \tag{2.12}$$

For $k^2 - \lambda$ in the resolvent set of $H_U$, $-\Delta + U + \lambda - k^2$ has a bounded inverse $R_U(k^2 - \lambda)$ in $\mathcal{B}(L^2)$ and the system (2.11) is equivalent to the coupled integral equations

$$\begin{cases} \varphi_{\mathbf{k}} + R_V(k^2) W \xi_{\mathbf{k}} = \phi_{V,\mathbf{k}}, \\ \xi_{\mathbf{k}} + R_U(k^2 - \lambda) W \varphi_{\mathbf{k}} = 0, \end{cases} \tag{2.13}$$

where we have denoted by $\phi_{V,\mathbf{k}}$ the generalized eigenfunctions of $H_V$. The resolvent $R_V(k^2)$ is defined as a boundary value from the upper half plane. From (2.13) one sees that the problem is reduced to find the solution $\varphi_{\mathbf{k}}$ of the equation

$$\varphi_{\mathbf{k}} - R_V(k^2)\, W\, R_U(k^2 - \lambda)\, W\, \varphi_{\mathbf{k}} = \phi_{V,\mathbf{k}}\,. \tag{2.14}$$

**Definition 2.1** (Ikebe class $I_n$). We say that a measurable function $V$ belongs to the Ikebe class $I_n(\mathbb{R}^3)$, $n \in \mathbb{N}$, if $V \in L^2(\mathbb{R}^3)$, $V$ is locally Hölder continuous except for a finite number of points and there exists $R_0 > 0$ and $\delta > 0$ such that

$$|V(\mathbf{x})| \leqslant \frac{c}{x^{n+\delta}}, \qquad \text{for } x \geqslant R_0\,. \tag{2.15}$$

**Assumption 2.2.** We assume that

a) $U \in I_2(\mathbb{R}^3)$;

b) $V \in I_4(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$;

c) $W \in I_3(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$.

Under this assumptions $H_V$ and $H_U$ are self-adjoint operators on $H^2$.

**Assumption 2.3.** We assume that

a) $H_U$ has $N \geqslant 1$ negative simple eigenvalues $E_0 < E_1 < \cdots < E_{N-1} < 0$, with corresponding eigenvectors $\eta_0, \eta_1, \ldots, \eta_{N-1} \in L^2(\mathbb{R}^3)$;

b) $H_V \geqslant 0$ and zero is neither an eigenvalue nor a resonance;

c) $\ker\left(R_V^{1/2}(0)W\right) = \{0\}$ in $L^2(\mathbb{R}^3)$.

We denote by $\mathcal{E}$ the set of eigenvectors related to positive eigenvalues of $\mathcal{H}$, i.e.,

$$\mathcal{E} := \sigma_{\mathrm{pp}}(\mathcal{H}) \cap \mathbb{R}^+\,. \tag{2.16}$$

**Proposition 2.4** (Generalized eigenfunctions). *Let Assumption 2.2 hold true and let $\lambda > 0$ be fixed. Then, for any $\mathbf{k} \in \mathbb{R}^3$ with $k^2 \in (0, \lambda) \setminus \mathcal{E}$ and $k^2 - \lambda \neq E_j$, for $j = 0, \ldots, N-1$, equation (2.14) admits a unique continuous solution $\varphi_{\mathbf{k}}$, such that $\varphi_{\mathbf{k}} - \phi_{V,\mathbf{k}} \in \mathscr{B}$. Furthermore, $\varphi_{\mathbf{k}}$ satisfies the asymptotics*

$$\varphi_{\mathbf{k}}(\mathbf{x}) \underset{x \to +\infty}{\simeq} e^{i\mathbf{k}\cdot\mathbf{x}} + A_{\mathrm{eff}}(\mathbf{k}, \mathbf{k}'; \lambda)\frac{e^{ikx}}{x}, \tag{2.17}$$

*with*

$$A_{\mathrm{eff}}(\mathbf{k}, \mathbf{k}'; \lambda) = \tfrac{1}{4\pi}\, \langle \phi_{V,\mathbf{k}'} |\, W\, R_U(k^2 - \lambda)\, W\, |\varphi_{\mathbf{k}}\rangle + A_V(\mathbf{k}, \mathbf{k}'), \tag{2.18}$$

*where $A_V(\mathbf{k}, \mathbf{k}')$ is the scattering amplitude associated to the potential $V$.*

**Definition 2.5** (Effective scattering length)**.** We define the effective scattering length in the open channel as

$$a_{\text{eff}}(\lambda) := \lim_{k \to 0} A_{\text{eff}}(\mathbf{k}, \mathbf{k}'; \lambda), \tag{2.19}$$

where $A_{\text{eff}}(\mathbf{k}, \mathbf{k}'; \lambda)$ is given by (2.18).

See [7] for a motivation of this definition.

**Theorem 2.6** (Feshbach resonances)**.** *Let Assumptions 2.2 and 2.3 hold true and let $\lambda > 0$ be fixed. Then, there are at least $N$ critical values $\lambda_j$, $j = 0, \ldots, N-1$, with $|E_j| < \lambda_j$, such that $a_{\text{eff}}(\lambda)$ is continuous for $\lambda \neq \lambda_j$ and*

$$a_{\text{eff}}(\lambda) = \frac{c_j}{\lambda - \lambda_j} + \mathcal{O}(1), \tag{2.20}$$

*as $\lambda \to \lambda_j$, where $c_j \in \mathbb{R}$.*

*Furthermore, there is $\delta_0 > 0$ such that, if $\|W\|_3 \leqslant \delta_0$, then the critical values satisfy $\lambda_0 > |E_0| > \lambda_1 > |E_1| > \cdots > \lambda_{N-1} > |E_{N-1}|$ and any further critical value $\lambda_j$, with $j \geqslant N$, is such that $|E_{N-1}| > \lambda_j > 0$.*

**Corollary 2.7** (Zero-energy equation)**.** *Under the same assumptions of Theorem 2.6, if $\lambda = \lambda_j$, then there exists a distributional solution of the zero-energy equation $\mathcal{H}\Psi = 0$.*

Clearly the interesting case is when $c_j$ is different from zero. It turns out that this is true if and only if for $\lambda = \lambda_j$, we are not in the exceptional case of the second case, according to terminology of Definition 3.4. In other words, we have a Feshbach resonance, if and only if $\mathcal{H}$ presents a zero-energy resonance; see [7] for details.

## 3  Low Energy Expansion of the Resolvent

Corollary 2.7 suggests the resolvent $(\mathcal{H} - k^2)^{-1}$ is singular as $k \to 0$ when $\lambda = \lambda_j$. In this section we give a characterization of the zero energy eigenspace of $\mathcal{H}$ and we study some properties of the low energy singularities of the resolvent of $\mathcal{H}$. In what follows we will assume always that $\lambda \neq |E_j|$.

First we recall the Schur-Grushin-Feshbach formula: suppose that $X = X_0 \dot{+} X_1$, direct sum of linear spaces, and that we have a linear operator $L$ on $X$ given by

$$L = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix} \tag{3.1}$$

with $L_{11}$ invertible. Define $C = L_{11} - L_{10} L_{00}^{-1} L_{01}$. Then $L^{-1}$ exists iff $C^{-1}$ exists and

$$L^{-1} = \begin{pmatrix} C^{-1} & -C^{-1}L_{10}L_{00}^{-1} \\ -L_{00}^{-1}L_{01}C^{-1} & L_{00}^{-1} + L_{00}^{-1}L_{01}C^{-1}L_{10}L_{00}^{-1} \end{pmatrix}. \tag{3.2}$$

We can use formula (3.2) to give a representation of the resolvent of $\mathcal{H}$. In this case we put

$$C(k) = -\Delta + V - k^2 - W\,R_U(k^2 - \lambda)\,W.$$

It is straightforward to see that $C^{-1}(k) \in \mathcal{B}(L^2)$ exists for $\operatorname{Im} k^2 \neq 0$. Let $k^2 = \alpha + i\beta$ then

$$\|(-\Delta + V - k^2 - W\,R_U(k^2 - \lambda)\,W)f\|^2$$
$$= \|(-\Delta + V - \alpha - W\,R_U(k^2 - \lambda)\,W)f\|^2 + \beta^2\|f\| + 2\beta^2\|R_U(k^2 - \lambda)\,Wf\|^2.$$

and

$$(\mathcal{H} - k^2)^{-1} =$$

$$\begin{pmatrix} C^{-1}(k) & -C^{-1}(k)\,W\,R_U(k^2 - \lambda) \\ -R_U(k^2 - \lambda)\,W\,C^{-1}(k) & R_U(k^2 - \lambda) + R_U(k^2 - \lambda)\,W\,C^{-1}(k)\,W\,R_U(k^2 - \lambda) \end{pmatrix}. \tag{3.3}$$

We can write for $\operatorname{Im} k^2 \neq 0$

$$C(k) = (-\Delta + V - k^2)\left(I - R_V(k^2)\,W\,R_U(k^2 - \lambda)\,W\right)$$
$$= (I - W\,R_U(k^2 - \lambda)\,W\,R_V(k^2))(-\Delta + V - k^2)$$

then

$$C^{-1}(k) = (I - R_V(k^2)\,W\,R_U(k^2 - \lambda)\,W)^{-1}R_V(k^2)$$
$$= R_V(k^2)\,(I - W\,R_U(k^2 - \lambda)\,W\,R_V(k^2))^{-1}.$$

For sake of notation we define

$$M(k) = R_V(k^2)\,W\,R_U(k^2 - \lambda)\,W$$
$$N(k) = W\,R_U(k^2 - \lambda)\,W\,R_V(k^2)$$

so that

$$C^{-1}(k) = (I - M(k))^{-1}R_V(k^2)$$
$$= R_V(k^2)\,(I - N(k))^{-1}.$$

Notice that, $\langle x \rangle^s W \in L^2$ for some $s > 3/2$. Therefore for $0 \leqslant k^2 < \lambda$,

$$W\,R_U(k^2 - \lambda)\,W \in \mathcal{B}_0(H^2_{-s}, L^2_s), \tag{3.4}$$

then we have

$$M(k) \in \mathcal{B}_0(H^2_{-s}, H^2_{-s}) \qquad N(k) \in \mathcal{B}_0(L^2_s, L^2_s) \qquad 1/2 < s < 3/2. \tag{3.5}$$

In [7], it was proved that $k^2 \in \mathcal{E}$ if and only if there exists $\tilde{u} \in \mathscr{B}$ such that $(I - M(k))\tilde{u} = 0$. Notice that $M(k)\tilde{u} \in H^2_{-s}$ for some $s > 1/2$ and therefore $\tilde{u} \in H^2_{-s}$.

Indeed if $\tilde{u} \in \mathscr{B}$ then $W\tilde{u} \in L^2$, $R_U(k^2 - \lambda)W\tilde{u} \in H^2$, $W R_U(k^2 - \lambda)W\tilde{u} \in L^2_s$ for some some $s > 1/2$ and finally $R_V(k^2)W R_U(k^2 - \lambda)W\tilde{u} \in H^2_{-s}$ for some some $s > 1/2$.

Then by Fredholm's alternative $(I - M(k))^{-1} \in \mathcal{B}(H^2_{-s}, H^2_{-s})$ with $s > 1/2$ for $0 < k^2 < \lambda$, $k^2 \notin \mathcal{E}$, $k^2 - \lambda \neq |E_k|$. Therefore we have for such values of $k^2$ that $C^{-1}(k) \in \mathcal{B}(L^2_s, H^2_{-s'})$ for $k^2 > 0$ if $s, s' > 1/2$ and the boundary value of the resolvent is well defined as an operator between suitable weighted spaces by (3.3).

We want to discuss the limit of $k^2 \to 0$ of (3.3). We have that the existence of $C^{-1}(0)$ is related to the existence of $(I - M(0))^{-1}$ or $(I - N(0))^{-1}$. For this reason we define

$$\mathcal{M} = \left\{ u \in H^2_{-s} \text{ s.t. } (I - M(0))u = 0 \right\}, \tag{3.6}$$

$$\mathcal{N} = \left\{ u \in L^2_s \text{ s.t. } (I - N(0))u = 0 \right\}. \tag{3.7}$$

Both $\mathcal{M}$ and $\mathcal{N}$ are finite dimensional for $1/2 < s < 3/2$ due to the compactness properties pointed out.

**Proposition 3.1.** *The sets $\mathcal{M}$ and $\mathcal{N}$ are isomorphic as vector spaces and do not depend on $s$. The linear isomorphisms are given by the restrictions of $W R_U(-\lambda)W$ to $\mathcal{M}$ and $R_V(0)$ to $\mathcal{N}$ respectively, that is:*

$$W R_U(-\lambda)W : \mathcal{M} \to \mathcal{N} \qquad R_V(0) : \mathcal{N} \to \mathcal{M}. \tag{3.8}$$

*The operator $W R_U(-\lambda)W$ can be substituted by $-\Delta + V$.*

*Proof.* Let $u \in \mathcal{M}$ then $W R_U(-\lambda)W u \in \mathcal{N}$ , indeed

$$0 = W R_U(-\lambda)W(I - M(0))u = (I - N(0))W R_U(-\lambda)W u.$$

The map is injective: if $W R_U(-\lambda)W u = 0$ then we have

$$u = R_V(0)W R_U(-\lambda)W u = 0.$$

Moreover it is clear that $R_V(0)$ is the the inverse of $W R_U(-\lambda)W$ on the image of $\mathcal{M}$. We prove that $W R_U(-\lambda)W$ is also onto. First notice that $R_V(0)$ on $\mathcal{N}$ is injective since $(-\Delta + V)R_V(0) = I$ on $\mathcal{N}$.

Then if $v \in \mathcal{N}$ we have $v = W R_U(-\lambda)W R_V(0)v$ and it is sufficient to prove that $R_V(0)v \in \mathcal{M}$. This is straightforward since

$$(I - R_V(0)W R_U(-\lambda)W)R_V(0)v = R_V(0)(I - W R_U(-\lambda)W R_V(0))v = 0.$$

This proves that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic and (3.8). Since the two spaces have opposite monotony in $s$, they are in facts independent. Notice that $W R_U(-\lambda)W$ and $-\Delta + V$ coincides that on $\mathcal{M}$ by the definition of $\mathcal{M}$. $\qquad\square$

**Proposition 3.2.** *We have $C(0)\mathcal{M} = 0$ and $\operatorname{Ker} C(0) = \mathcal{M}$ in $H^2_{-s}$ for $\frac{1}{2} < s < \frac{3}{2}$.*

*Proof.* Since $\mathcal{M} \subset H^2_{-s}$ then for $u \in \mathcal{M}$ we have

$$C(0)u = (-\Delta + V)(I - M(0))u = 0$$

Suppose that $u \in H^2_{-s}$ and $C(0)u = 0$. Then $(-\Delta + V)u = W R_U(-\lambda) W u \in L^2_s$ by (3.4) and $u \in \mathscr{D}(H_V)$. Hence $u = R_V(0)(-\Delta + V)u = R_V(0) W R_U(-\lambda) W u$.
$\qquad\qquad\square$

**Proposition 3.3.** *For $1/2 < s < 3/2$ there exists operators $Q, K \in \mathcal{B}(H^2_{-s})$ such that*

$$Q^2 = Q, \quad QK = KQ = 0 \tag{3.9}$$
$$Q(I - M(0)) = (I - M(0))Q = 0 \tag{3.10}$$
$$K(I - M(0)) = (I - M(0))K = I - Q \tag{3.11}$$
$$Q \text{ is of finite rank and } K - I \in \mathscr{B}_0(H^2_{-s}) \tag{3.12}$$
$$W R_U(-\lambda) W K = K W R_U(-\lambda) W \quad R_V(0) K = K R_V(0) \tag{3.13}$$

*Proof.* Define $Q$ as a spectral projection by the analytic functional calculus,

$$Q = -\frac{1}{2\pi i} \int_{|z-1|=\delta} \frac{dz}{(M(0) - z)}, \tag{3.14}$$

with $\delta$ sufficiently small that $\{|z - 1| = \delta\}$ only includes the eigenvalue. This is possible due to the compactness of $M(0)$. Then $(I - M(0) + Q)$ is invertible and we can define

$$K = (I - M(0) + Q)^{-1}(I - Q). \tag{3.15}$$

Properties (3.9), (3.10) and (3.11) follow from the separation of spectrum, see [16] pg. 178. The rank of operator $Q$ is finite since $M(0)$ si compact. Taking into account the identity $(I + B)^{-1} = I - B(I + B)^{-1}$ , we have that $K - I$ has the same regularity of $M(0)$ and therefore it is compact. The last property (3.13) can be proved as in Lemma 3.5 in [17]. $\qquad\qquad\square$

Due to Proposition 3.2 and the positivity of $-\Delta + V$ by Assumption 2.3, we have that $(W R_U(-\lambda) W u, u)$ defines a inner product on $\mathcal{M}$. At the same time we have that $(R_V(0)v, v)$ defines an inner product on $\mathcal{N}$. Notice that $\mathcal{M}$ and $\mathcal{N}$ possess a natural duality induced by $L^2$ inner product since $H^2_{-s} \subset L^2_{-s} = (L^2_s)^*$. Under this coupling, if $\{u_j\}$, $j = 1, \ldots, d$, is an orthonormal basis in $\mathcal{M}$ then $\{v_j\}$ with $v_j = W R_U(-\lambda) W u_j$ is orthonormal in $\mathcal{N}$ and it is also the dual basis with respect to the above defined inner products. The two bases are orthonormal w.r.t. the two above defined inner products. Moreover the spectral projector $Q$ reads

$$Q = \sum_{j=1}^{d} |u_j\rangle\langle v_j|. \tag{3.16}$$

Let us call $\mathcal{M}_{\mathcal{E}} = \{u \in \mathcal{M} \text{ s.t. } u \in H^2\}$. It is straightforward to check that if $u \in \mathcal{M}_{\mathcal{E}}$ then $\tilde{\Psi} = (u, -R_U(-\lambda) W u)^t$ is a zero-energy eigenvalue of $\mathcal{H}$ that is $\mathcal{H}\tilde{\Psi} = 0$ and $\tilde{\Psi} \in \mathcal{H}$. We can have different cases

**Definition 3.4.** We say that we are in the generic case if $\mathcal{M} = \{0\}$ and in the exceptional case otherwise. In the exceptional case we distinguish the following situations: in the first kind $\mathcal{M} \neq \{0\}$ and $\mathcal{M}_{\mathcal{E}} = \{0\}$, in the second case $\mathcal{M} = \mathcal{M}_{\mathcal{E}} \neq \{0\}$ and in the third case $0 \subsetneq \mathcal{M}_{\mathcal{E}} \subsetneq \mathcal{M}$.

Notice, see [7], that

$$\left(R_V(k^2)w\right)(\mathbf{x}) \underset{x \to +\infty}{=} \langle \phi_{V,\mathbf{k}'}|w\rangle \frac{e^{ikx}}{4\pi x} + O(x^{-2}). \tag{3.17}$$

Then $u \in \mathcal{M}$ belongs to $\mathcal{M}_{\mathcal{E}}$ iff

$$\langle \phi_{V,0}|W R_U(-\lambda) W u\rangle = 0$$

and in particular $\dim \mathcal{M}/\mathcal{M}_{\mathcal{E}} \leqslant 1$ that is, there is at most one resonance. In the main theorem we discuss the expansion of $C^{-1}(k)$.

**Theorem 3.5** (Low Energy Expansion)**.**
*Let Assumption 2.2 and Assumption 2.3 hold and let $\{\lambda_j\}$ be the critical values as in Theorem 2.6. Then if $\lambda \notin \{\lambda_j\}$ then we are in the generic case and $C^{-1}(0)$ exists. Otherwise if $\lambda \in \{\lambda_j\}$ we are in exceptional case and we have the following expansions:*

- *In the exceptional case of first kind, we have:*

$$C^{-1}(k) = \frac{1}{ak}|u\rangle\langle u| + O(1) \tag{3.18}$$

  *in $\mathcal{B}(L_s^2, H_{-s'}^2)$ with $s, s' > 1/2$ and $s + s' > 2$ where*

$$a = -(W R_U(-\lambda) W u, R_V'(0) W R_U(-\lambda) W u) \neq 0.$$

- *In the exceptional case of second kind assume $W \in (I)_5$, then we have:*

$$C^{-1}(k) = \frac{1}{k^2}P_0 - \frac{1}{k}P_0 W R_U(-\lambda) W T_3 P_0 + O(1). \tag{3.19}$$

  *in $\mathcal{B}(L_s^2, H_{-s'}^2)$ with $s > 1/2$, $s' > 7/2$, where $P_0$ is the orthogonal projector on the zero-eigenspace of $\mathcal{H}$ and $T_3$ is defined by (3.30).*

We omit the discussion of the exceptional case of the third kind for sake of brevity.

*Proof.* The first part of the theorem is just a rephrasing of Theorem 2.6: the critical values $\{\lambda_j\}$ are exactly the values of $\lambda$ such $\mathcal{M} \neq \{0\}$. Therefore $(I - M(0))^{-1}$ exists by Fredholm's alternative and $C^{-1}(0) = (I - M(0))^{-1} R_V(0) \in \mathcal{B}_0(L_s^2, H_{-s'}^2)$ with $s$, $s' > 1/2$ and $s + s' > 2$.

Now we discuss the exceptional case of the first kind. Let us define $\overline{Q} = I - Q$ and decompose $M(k)$ accordingly, that is, we put

$$M(k) = \left( \begin{array}{cc} \overline{Q}(I - M(k))\overline{Q} & \overline{Q}(I - M(k))Q \\ Q(I - M(k))\overline{Q} & Q(I - M(k))Q \end{array} \right) = \left( \begin{array}{cc} m_{00}(k) & m_{01}(k) \\ m_{10}(k) & m_{11}(k) \end{array} \right) \quad (3.20)$$

Remember that in this case $Q = |u\rangle\langle W R_U(-\lambda) W u|$ with $u \in \mathcal{M}$; we normalize $u$ requiring that $(u, W R_U(-\lambda) W u) = 1$. Notice that $m_{00}(k)$ is continuous and $m_{00}^{-1}(0)$ exist by construction. Then $m_{00}^{-1}(k)$ exists and it is continuous for $k$ sufficiently small by Neumann series. By (3.2) we have to discuss the invertibility of

$$m(k) = m_{11}(k) - m_{10}(k)m_{00}^{-1}(k)m_{01}(k) \quad (3.21)$$

on the range of $Q$. In facts in this case we have

$$m(k) = Q\left[(W R_U(-\lambda) W u, (I - M(k))u) + \right. \\ \left. - (W R_U(-\lambda) W u, (I - M(k))\overline{Q}m_{00}^{-1}(k)\overline{Q}(I - M(k))u)\right] .$$

The resolvent $R_V(k^2)$ has the expansion in $\mathcal{B}(L_s^2, H_{-s'}^2)$ with $s$, $s' > 1/2$ and $s + s' > 2$, see Lemma 2.3 in [17] and Lemma 2.3 in [20].

$$R_V(k^2) = R_V(0) + R_v'(0)\,k + o(k)$$

where $'$ denote the derivative w.r.t. $k$. Notice also the analiticity of $R_U(k^2 - \lambda)$ in $k^2$ in $\mathcal{B}(L^2, H^2)$. Then the following expansion in $\mathcal{B}_0(H_{-s}^2, H_{-s}^2)$, $1/2 < s < 3/2$, holds true:

$$M(k) = R_V(0) W R_U(-\lambda) W + kR_V'(0) W R_U(-\lambda) W + o(k). \quad (3.22)$$

Then we have

$$(W R_U(-\lambda) W u, (I - M(k))u) \\ = 1 - (W R_U(-\lambda) W u, M(0)u) \\ \quad - k(W R_U(-\lambda)Wu, R_V'(0) W R_U(-\lambda) W u) + o(k) \\ = a\,k + o(k),$$

where

$$a = -(W R_U(-\lambda) W u, R_V'(0) W R_U(-\lambda) W u).$$

Since

$$R_V'(k^2) = (I - R_V(k^2)V)R_0'(k^2)(I - VR_V(k^2)) \quad (3.23)$$

then we have

$$a = \frac{1}{4\pi} |(1, (I - V R_V(0)) W R_U(-\lambda) W u)|^2$$
$$= \frac{1}{4\pi} |(\phi_{V,0}, W R_U(-\lambda) W u)|^2 \neq 0$$

The constant $a$ is different from $0$ otherwise $u$ would be a 0-energy eigenstate and we would be in the exceptional case of the second kind. Using again expansion (3.22), we have also $(I - M(k))u = O(k)$ and

$$(W R_U(-\lambda) W u, (I - M(k))f) = ((I - N(k)) W R_U(-\lambda) W u, f) = O(k).$$

Therefore we obtain

$$m(k) = a\,k\,Q + o(k) \qquad m^{-1}(k) = \frac{1}{ak} Q + o(1)$$

Using (3.2) and the above remarks, we see that the only singular terms comes from the term $m^{-1}$ and we obtain

$$(I - M(k))^{-1} = \frac{1}{ak} Q + O(1) \quad \text{in } \mathcal{B}_0(H^2_{-s}, H^2_{-s}) \quad 1/2 < s < 3/2.$$

Then

$$C^{-1}(k) = \frac{1}{ak} |u\rangle\langle u| + O(1) \quad \text{in } \mathcal{B}_0(L^2_s, H^2_{-s'}) \quad s,\, s' > 1/2 \quad s + s' > 2 \quad (3.24)$$

and this proves (3.18).

   Now we consider the exceptional case of the second case. Again the main point is the inversion of $m(k)$ on the range of $Q$. In this case we have

$$Q = \sum_{j=1}^{d} |u_j\rangle\langle W R_U(-\lambda) W u_j| \qquad\qquad (3.25)$$
$$(1, (I - V R_V(0)) W R_U(-\lambda) W u_j) = 0 \qquad j = 0, \ldots, d.$$

Let us start expanding around $k = 0$

$$m_{11}(k) = Q(I - M(k))Q = \sum_{j,k=1}^{d} |u_j\rangle\langle v_k|(v_j, (I - M(k))u_k). \qquad (3.26)$$

Taking into account $W \in (I)_5$, in the following of the proof we can choose $s > 7/2$ such that $\langle x\rangle^s W \in L^2$. Then the following expansion holds true in $\mathcal{B}_0(H^2_{-s}, H^2_{-s})$

with $s > 7/2$:

$$I - M(k) = I - M(0) + \tag{3.27}$$

$$- k R_V'(0) W \, R_U(-\lambda) \, W \tag{3.28}$$

$$- \frac{k^2}{2} \left( R_V''(0) W \, R_U(-\lambda) \, W + 2 R_V(0) W \, R_U^2(-\lambda) \, W \right) \tag{3.29}$$

$$- \frac{k^3}{6} \left( R_V'''(0) W \, R_U(-\lambda) \, W + 6 R_V'(0) W \, R_U^2(-\lambda) \, W \right) \tag{3.30}$$

$$+ O(k^4)$$

Both (3.27) and (3.28) do not contribute in the expansion of (3.26): the former since $u_j \in \mathcal{M}$ and the latter due to (3.23) and the cancellation condition in (3.25) (remember that $R_0'(0) = 1$). Now we discuss the first term in (3.30): for the same reasons as in the analysis of (3.28) we have

$$\begin{aligned}
&(W \, R_U(-\lambda) \, W u_j, R_V''(0) W \, R_U(-\lambda) \, W u_k) \\
&= (W \, R_U(-\lambda) \, W u_j, (I - R_V(0)V) R_0''(0)(I - V R_V(0)) W \, R_U(-\lambda) \, W u_k) \\
&= 2(R_0(0)(I - V R_V(0)) W \, R_U(-\lambda) \, W u_j, R_0(0)(I - V R_V(0)) W \, R_U(-\lambda) \, W u_k)
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
&= 2(R_V(0) W \, R_U(-\lambda) \, W u_j, R_V(0) W \, R_U(-\lambda) \, W u_k) \\
&= 2(u_j, u_k) \, .
\end{aligned} \tag{3.32}$$

Notice that in (3.31) we have used Lemma 2.6 of [17] and the cancellation condition in (3.25). For the second term (3.30) we have

$$(W \, R_U(-\lambda) \, W u_j, R_V(0) W \, R_U^2(-\lambda) \, W u_k) = (u_j, W \, R_U^2(-\lambda) \, W u_k) \, .$$

Define the matrix $A$ by

$$A_{j,k} = (u_j, u_k) + (u_j, W \, R_U^2(-\lambda) \, W u_k) \, .$$

Since $A$ is positive definite, we can define $B = A^{-1/2}$ and $\tilde{u}_k = \sum_j B_{k,j} u_j$ . Then $\{\tilde{u}_j\}$ is orthonormal basis of $\mathcal{M}$ w.r.t. the $L^2$ inner product and

$$\left( \sum_{j,k=1}^d |u_j\rangle\langle v_k| A_{j,k} \right)^{-1} = \sum_{j=1}^d |\tilde{u}_j\rangle\langle\tilde{u}_j| W \, R_U(-\lambda) \, W = P_0 W \, R_U(-\lambda) \, W \, .$$

By the same arguments we have

$$m_{10}(k) m_{00}^{-1}(k) m_{01}(k) = O(k^4) \, .$$

Collecting the above results we have

$$\begin{aligned}
m^{-1}(k) &= \frac{1}{k^2} P_0 W \, R_U(-\lambda) \, W \left( I + k T_3 P_0 W \, R_U(-\lambda) \, W + O(k^2) \right)^{-1} \\
&= \frac{1}{k^2} P_0 W \, R_U(-\lambda) \, W - \frac{1}{k} P_0 W \, R_U(-\lambda) \, W k T_3 P_0 W \, R_U(-\lambda) \, W + O(1)
\end{aligned}$$

where we have denoted by $T_3$ the expression in (3.30). Using (3.2) and the above remarks, we see that the only singular terms comes from the term $d^{-1}$ and we obtain

$$(I - M(k))^{-1}$$
$$= \frac{1}{k^2} P_0 W\, R_U(-\lambda)\, W - \frac{1}{k} P_0 W\, R_U(-\lambda)\, W k T_3 P_0 W\, R_U(-\lambda)\, W + O(1)$$

in $\mathcal{B}_0(H_{-s}^2, H_{-s}^2)$ $s > 7/2$. Since $P_0 W\, R_U(-\lambda)\, W\, R_V(0) = P_0$ wei finally obtain

$$C^{-1}(k) = \frac{1}{k^2} P_0 - \frac{1}{k} P_0 W\, R_U(-\lambda)\, W T_3 P_0 + O(1)$$

in $\mathcal{B}_0(L_s^2, H_{-s'}^2)$ with $s > 1/2$, $s' > 7/2$. which proves (3.19). $\qquad\square$

We expect that the hypothesis on the potential are not optimal w.r.t. to the decay at infinity. We have discussed the most simple expansion of the resolvent while leaving untouched the differentiability of the remainder which is a central issue in the proof of dispersive estimates and boundedness properties of the wave operators.

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