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Existence of T-$\vec{p}(\cdot)$-solutions for some quasilinear anisotropic elliptic problem

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Abstract. *The aim of this paper is to study the existence of solutions for some quasilineare anisotropic elliptic problem associated with differential inclusion. We study the two cases of $f \in L^\infty(\Omega)$ and $f \in L^1(\Omega)$. Moreover, we show the uniqueness of solution under some additional assumptions.*

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, with a Lipschitz boundary condition $\partial\Omega$. For $2 - \frac{1}{N} < p < N$, L. Boccardo and T. Gallouët [11] have treated the problem

$$\begin{cases} Au = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $Au = -\text{div}\, a(x, u, \nabla u)$ is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into its dual, and f is a bounded Radon measure on Ω. They have proved the existence and some regularity results (see also [18, 19]). M. Bendahmane and P. Wittbold in [9] have shown the existence and uniqueness of the renormalized solution for the nonlinear elliptic problem

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

in the variable exponent Sobolev spaces, where the right-hand side $f \in L^1(\Omega)$, we refer the reader also to [25] for the existence and uniqueness of entropy solution.

Recently, anisotropic variable exponent Sobolev spaces $W^{1,\vec{p}(\cdot)}(\Omega)$ have attracted the interest of many scientists and researchers, this attention come essentially from their applications in nonhomogeneous materials that behave differently on different space directions, we can refer here to the electrorheological and thermoelectric fluids (see for example [5, 23]).

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These spaces are the appropriate framework to deal with a class of problems having non-standard structural conditions, involving a variable growth exponent $\vec{p}(\cdot)$, where prototype of the differential operator considered is the $\vec{p}(\cdot)$-Laplacian

$$\Delta_{\vec{p}(\cdot)}(u) = \sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(\cdot)-2}\partial_{x_i} u),$$

which generalize the $p(\cdot)$-Laplace operator. Di Nardo, Feo and Guibé have studied in [16] the existence of renormalized solutions for some class of nonlinear anisotropic elliptic problems of the type

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}(a_i(x,u)|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u) = f - \text{div } g & \text{in} \quad \partial\Omega, \\ u \; = \; 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$ and $g \in \Pi_{i=1}^{N} L^{p_i'}(\Omega)$, the uniqueness of renormalized solution was concluded under some local Lipschitz conditions on the function $a_i(x,s)$ with respect to s (see also [1, 3, 4, 6, 15]).

In [14], Gwiazda and al. have proved the existence of renormalized solutions for the quasilinear elliptic equation

$$\begin{cases} -\text{div } A(x, \nabla u) = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

in the Musielak-Orlicz-Sobolev spaces, where $f \in L^1(\Omega)$ and $A(\cdot,\cdot)$ is a Carathéodory function verifying some non-standard growth and coercivity conditions, and without using the Δ_2-condition. For more results we refer the reader to [2, 20] and [21].

In this work, we establish the existence of T-$\vec{p}(\cdot)$-solutions for the following quasilinear anisotropic elliptic problem

$$\begin{cases} \beta(u) - \sum_{i=1}^{N} D^i a_i(x, \nabla u) \ni f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\beta(\cdot)\colon \mathbb{R} \longmapsto 2^{\mathbb{R}}$ is a set-valued maximal monotone mapping such that $0 \in \beta(0)$. We assume that $a_i(\cdot,\cdot)\colon \Omega \times \mathbb{R}^N \longmapsto \mathbb{R}$ are Carathéodory functions for $i = 1, 2, \ldots, N$ (i.e. measurable with respect to x in Ω for every ξ in $\mathbb{R}^N$ and continuous with respect to ξ in $\mathbb{R}^N$ for almost every x in Ω), which satisfies the following conditions

$$|a_i(x,\xi)| \leq K_i(x) + |\xi_i|^{p_i(x)-1} \qquad \text{for} \quad i = 1, \ldots, N, \tag{1.2}$$

$$a_i(x,\xi)\xi_i \geq \alpha|\xi_i|^{p_i(x)} \qquad \text{for} \quad i = 1, \ldots, N, \tag{1.3}$$

and the function $a_i(x,\cdot)$ have only a wide monotone, i.e.

$$(a_i(x,\xi) - a_i(x,\xi'))(\xi_i - \xi_i') \geq 0, \tag{1.4}$$

for a.e. $x \in \Omega$, and all $\xi \in \mathbb{R}^N$, where $\alpha > 0$ and $K_i(\cdot)$ is a non-negative function lying in $L^{p_i'(\cdot)}(\Omega)$ where $\dfrac{1}{p_i(x)} + \dfrac{1}{p_i'(x)} = 1$.

As a natural hypothesis on the Carathéodory function $a(x,\xi)$, we assume that

$$a(x,0) = 0\,.$$

In this paper we will extend the results of [26] to the anisotropic variable exponent case, and our main ideas and methods come from [11] and [12].

The paper is organized as follows. In section 2, we recall some definitions and results concerning the anisotropic variable exponent Sobolev Spaces. Also, we introduce some lemmas useful to prove our main results. In the section 3, we will study the existence and regularity of weak solutions for our quasilinear anisotropic elliptic problem (1.1) in the case of $f \in L^\infty(\Omega)$. The section 4 will be devoted to the study of the existence of T-$\vec{p}(\cdot)$-solution in the case of $f \in L^1(\Omega)$. In the last section, we will prove the uniqueness of T-$\vec{p}(\cdot)$-solution under some additional assumption.

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we denote

$$\mathcal{C}_+(\Omega) = \{\text{measurable function} \quad p(\cdot) : \Omega \longmapsto \mathbb{R} \quad \text{such that} \quad 1 < p^- \leq p^+ < N\},$$

where

$$p^- = ess\inf\{p(x) \,/\, x \in \Omega\} \qquad \text{and} \qquad p^+ = ess\sup\{p(x) \,/\, x \in \Omega\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \longmapsto \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e. if $p^+ < +\infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. Then $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)}+\dfrac{1}{p'(x)}=1$. Finally, we have the Hölder type inequality:

$$\left|\int_{\Omega} uv\ dx\right| \leq \left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}$$

for any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$. The Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega)=\{u\in L^{p(\cdot)}(\Omega)\quad \text{and}\quad |\nabla u|\in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space, equipped with the following norm

$$\|u\|_{1,p(\cdot)}=\|u\|_{p(\cdot)}+\|\nabla u\|_{p(\cdot)}\,.$$

The space $(W^{1,p(\cdot)}(\Omega),\|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. We define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For more details on variable exponent Lebesgue and Sobolev spaces, we refer the reader to [17].

Remark 2.1. Recall that the definition of these spaces requires only the measurability of $p(\cdot)$, while the Poincaré and the Sobolev-Poincaré inequalities are proved for $p(\cdot)-$log-Hölder continuous, (see. [17]).

Now, we present the anisotropic variable exponent Sobolev space, used in the study of our quasilinear anisotropic elliptic problem. Let $p_1(\cdot),p_2(\cdot),\dots,p_N(\cdot)$ be N variable exponents in $\mathcal{C}_+(\Omega)$. We denote

$$\vec{p}(\cdot)=(p_1(\cdot),\dots,p_N(\cdot)),\quad \text{and}\quad D^iu=\frac{\partial u}{\partial x_i}\quad \text{for}\quad i=1,\dots,N,$$

and we define

$$\underline{p}^+=\max\{p_1^-,\dots,p_N^-\}\quad\text{and}\quad \underline{p}=\min\{p_1^-,\dots,p_N^-\}\quad\text{then}\quad 1<\underline{p}\leq\underline{p}^+\,.$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follow:

$$W^{1,\vec{p}(\cdot)}(\Omega)=\{u\in W^{1,1}(\Omega)\quad\text{and}\quad D^iu\in L^{p_i(\cdot)}(\Omega)\quad\text{for}\quad i=1,2,\dots,N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)}=\|u\|_{1,1}+\sum_{i=1}^{N}\|D^iu\|_{p_i(\cdot)}. \tag{2.1}$$

We define also $W_0^{1,\vec{p}(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm (2.1). The space $(W_0^{1,\vec{p}(\cdot)}(\Omega),\|u\|_{1,\vec{p}(\cdot)})$ is a reflexive Banach space (cf. [24]).

Lemma 2.2. *We have the following continuous and compact embedding*

- *if* $\underline{p} < N$ *then* $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for* $q \in [\underline{p}, \underline{p}^*[$, *where* $\underline{p}^* = \dfrac{N\underline{p}}{N-\underline{p}}$,

- *if* $\underline{p} = N$ *then* $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ $\forall q \in [\underline{p}, +\infty[$,

- *if* $\underline{p} > N$ *then* $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Remark 2.3. In view of the continuous embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$ and the Poincaré type inequality we conclude that the two norms $\|u\|_{1,\vec{p}(\cdot)}$ and $|u|_{1,\vec{p}(\cdot)} = \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)}$ are equivalents in the anisotropic variable exponent Sobolev spaces $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Indeed, thanks to Hölder's inequality we have

$$\|\nabla u\|_1 = \sum_{i=1}^{N} \|D^i u\|_1 \leq C_1 \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)} \quad \text{for any} \quad u \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Moreover, the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$ is continuous, by using Poincaré's inequality, we have

$$\|u\|_1 \leq C_p \|\nabla u\|_1 \leq C_2 \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)} \quad \text{for any} \quad u \in W_0^{1,\vec{p}(\cdot)}(\Omega)\,.$$

It follows that

$$\|u\|_{1,1} = \|u\|_1 + \|\nabla u\|_1 \leq (C_p + 1)\|\nabla u\|_1 \leq C_3 \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)}.$$

We conclude that for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$

$$|u|_{1,\vec{p}(\cdot)} \leq \|u\|_{1,\vec{p}(\cdot)} = \|u\|_{1,1} + |u|_{1,\vec{p}(\cdot)} \leq (C_3 + 1)|u|_{1,\vec{p}(\cdot)},$$

thus, the result is concluded.

Proposition 2.4. *The dual of $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is denoted by $W^{-1,\vec{p'}(\cdot)}(\Omega)$, where*

$$\vec{p'}(\cdot) = (p'_1(\cdot), \dots, p'_N(\cdot)) \text{ and } \frac{1}{p'_i(\cdot)} + \frac{1}{p_i(\cdot)} = 1,$$

(cf. [8] for the constant exponent case). For each $F \in W^{-1,\vec{p'}(\cdot)}(\Omega)$ there exist $F_0 \in (L^{\underline{p}^+}(\Omega))'$ and $F_i \in L^{p'_i(\cdot)}(\Omega)$ for $i = 1, 2, \dots, N$, such that $F = F_0 - \sum_{i=1}^{N} D^i F_i$. Moreover, for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\langle F, u \rangle = \sum_{i=0}^{N} \int_\Omega F_i \, D^i u \, dx.$$

We define a norm on the dual space by

$$\begin{aligned} \|F\|_{-1,\vec{p'}(\cdot)} &= \inf \Big\{ \|F_i\|_{(\underline{p}^+)'} + \sum_{i=1}^{N} \|F_i\|_{p'_i(\cdot)} \quad \text{with} \quad F = F_0 - \sum_{i=1}^{N} D^i F_i \\ &\qquad \text{such that} \quad F_0 \in (L^{\underline{p}^+}(\Omega))' \quad \text{and} \quad F_i \in L^{p'_i(\cdot)}(\Omega) \Big\}. \end{aligned}$$

Definition 2.5. Let $k > 0$, the truncation function $T_k(\cdot) \colon \mathbb{R} \longmapsto \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega) := \{u \colon \Omega \mapsto \mathbb{R}$ measurable, such that $T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ for any $k > 0\}$.

Note that, a measurable function u verifying $T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ for all $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, for any $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$ it is possible to define the weak gradient of u, still denoted ∇u.

Proposition 2.6. *Let $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. For any $i \in \{1, \dots, N\}$, there exists a unique measurable function $v_i \colon \Omega \longmapsto \mathbb{R}$ such that*

$$\forall k > 0 \quad D^i T_k(u) = v_i . \chi_{\{|u|<k\}} \quad a.e. \quad x \in \Omega,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

The proof of the Proposition 2.6 follows the usual techniques developed in [10] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [8] and [16].

Lemma 2.7 (see [7])**.** *Let $g \in L^{r(\cdot)}(\Omega)$ and $g_n \in L^{r(\cdot)}(\Omega)$ with $\|g_n\|_{r(\cdot)} \le C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{r(\cdot)}(\Omega)$.*

3. Existence of weak solutions in the case of $f \in L^\infty(\Omega)$

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, and let $p_i(\cdot) \in \mathcal{C}_+(\Omega)$ for $i = 1, \ldots, N$. In this section, we will study the existence of weak solution in the case of $f \in L^\infty(\Omega)$.

Definition 3.1. Let $f \in L^\infty(\Omega)$. A weak solution of the quasilinear elliptic problem (1.1) is a pair of functions (u, b) such that $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $b \in \beta(u)$ with $b \in L^\infty(\Omega)$, satisfying

$$\int_\Omega bv\,dx + \sum_{i=1}^N \int_\Omega a_i(x, \nabla u) D^i v\,dx = \int_\Omega fv\,dx,$$

for any $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Theorem 3.2. *Let $f \in L^\infty(\Omega)$, assuming that the conditions (1.2) – (1.4) hold true. Then, there exists at least one weak solution $(u, b) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \times L^\infty(\Omega)$ of the quasilinear anisotropic elliptic problem (1.1).*

Proof. ***Step 1: Approximate problems***

Let $0 < \varepsilon \leq 1$, we consider the approximate problem

$$\begin{cases} \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + Au_\varepsilon = f & \text{in} \quad \Omega, \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where $Av = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \nabla v)$ and $\beta_\varepsilon(\cdot)\colon \mathbb{R} \longmapsto \mathbb{R}$ be the Yosida approximation of $\beta(\cdot)$, note that, for any $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $0 < \varepsilon \leq 1$ we have

$$\langle \beta_\varepsilon(v), v \rangle \geq 0, \quad |\beta_\varepsilon(v)| \leq \frac{1}{\varepsilon}|v| \quad \text{and} \quad \lim_{\varepsilon \to 0} \beta_\varepsilon(v) = \beta(v).$$

We refer the reader to [13] for more details.

We introduce the operators $G_\varepsilon\colon W_0^{1,\vec{p}(\cdot)}(\Omega) \longmapsto W^{-1,\vec{p}'(\cdot)}(\Omega)$, defined by

$$\langle G_\varepsilon u, v \rangle = \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u)) v\,dx \quad \text{for any} \quad u, v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Thanks to the generalized Hölder type inequality, we have

$$\begin{aligned} |\langle G_n u, v \rangle| &\leq \int_\Omega |\beta_\varepsilon(T_{1/\varepsilon}(u))|\,|v|\,dx \leq \frac{1}{\varepsilon} \int_\Omega |T_{\frac{1}{\varepsilon}}(u)|\,|v|\,dx \\ &\leq \frac{1}{\varepsilon^2} \int_\Omega |v|\,dx \\ &\leq \frac{1}{\varepsilon^2} \|v\|_{1,\vec{p}(\cdot)}. \end{aligned} \tag{3.2}$$

Lemma 3.3 (see [22]). *The bounded operator $B_\varepsilon = A + G_\varepsilon$ acted from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into $W^{-1,\vec{p}'(\cdot)}(\Omega)$ is pseudo-monotone. Moreover, B_ε is coercive in the following sense:*

$$\frac{\langle B_\varepsilon v, v\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \longrightarrow +\infty \quad \textit{as} \quad \|v\|_{1,\vec{p}(\cdot)} \to \infty \quad \textit{for} \quad v \in W_0^{1,\vec{p}(\cdot)}(\Omega)\,.$$

Proof. In view of the Hölder's inequality and the growth condition (1.2), it's easy to see that the operator A is bounded, and by (3.2) we conclude that B_ε is bounded. For the coercivity, we have for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$\begin{aligned}
\langle B_\varepsilon u, u\rangle &= \langle Au, u\rangle + \langle G_\varepsilon u, u\rangle \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u)\; D^i u\, dx + \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u))u\, dx \\
&\geq \alpha \sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)}\, dx \\
&\geq \alpha \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)}^{\underline{p}} - \alpha N \\
&\geq C_0\; \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \alpha N,
\end{aligned}$$

it follows that

$$\frac{\langle B_\varepsilon u, u\rangle}{\|u\|_{1,\vec{p}(\cdot)}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,\vec{p}(\cdot)} \to \infty\,.$$

It remains to show that B_ε is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } \; W_0^{1,\vec{p}(\cdot)}(\Omega), \\ B_\varepsilon u_k \rightharpoonup \chi_\varepsilon & \text{in } \; W^{-1,\vec{p}'(\cdot)}(\Omega), \\ \limsup\limits_{k\to\infty} \langle B_\varepsilon u_k, u_k\rangle \leq \langle \chi_\varepsilon, u\rangle. \end{cases} \tag{3.3}$$

We will prove that

$$\chi_\varepsilon = B_\varepsilon u \quad \text{and} \quad \langle B_\varepsilon u_k, u_k\rangle \longrightarrow \langle \chi_\varepsilon, u\rangle \quad \text{as} \quad k \to +\infty.$$

In view of the compact embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, we have $u_k \to u$ in $L^{\underline{p}}(\Omega)$ for a subsequence still denoted $(u_k)_{k\in\mathbb{N}}$. As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, using the growth condition (1.2) it's clear that the sequence $(a_i(x, \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a function $\psi_i \in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x, \nabla u_k) \rightharpoonup \psi_i \;\; \text{in} \;\; L^{p_i'(\cdot)}(\Omega) \text{ as } k \to \infty, \quad \text{for any} \quad i = 1, \ldots, N. \tag{3.4}$$

Moreover, we have $|\beta_\varepsilon(T_{1/\varepsilon}(u_k))| \leq \dfrac{1}{\varepsilon^2}$, and since $u_k \to u$ almost everywhere in Ω. In view of Lebesgue's dominated convergence theorem we obtain

$$\beta_\varepsilon(T_{1/\varepsilon}(u_k)) \longrightarrow \beta_\varepsilon(T_{1/\varepsilon}(u)) \quad \text{in} \quad L^{\underline{p}'}(\Omega). \tag{3.5}$$

For any $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}
\langle \chi_\varepsilon, v\rangle &= \lim_{k\to\infty} \langle B_\varepsilon u_k, v\rangle \\
&= \lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_k) D^i v \, dx + \lim_{k\to\infty} \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_k)) v \, dx \\
&= \sum_{i=1}^{N} \int_\Omega \psi_i \; D^i v \, dx + \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u)) v \, dx.
\end{aligned} \tag{3.6}$$

Having in mind (3.3) and (3.6), we obtain

$$\begin{aligned}
\limsup_{k\to\infty} \langle B_\varepsilon(u_k), u_k\rangle &= \limsup_{k\to\infty} \Big\{ \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_k) D^i u_k \, dx \\
&\qquad + \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_k)) u_k \, dx \Big\} \\
&\leq \sum_{i=1}^{N} \int_\Omega \psi_i \; D^i u \, dx + \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u)) u \, dx.
\end{aligned}$$

Thanks to (3.5), we have

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_k)) u_k \, dx \longrightarrow \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u)) u \, dx, \tag{3.7}$$

it follows that

$$\limsup_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_k) D^i u_k \, dx \leq \sum_{i=1}^{N} \int_\Omega \psi_i \; D^i u \, dx. \tag{3.8}$$

On the other hand, using (1.4) we have

$$\sum_{i=1}^{N} \int_\Omega (a_i(x, \nabla u_k) - a_i(x, \nabla u))(D^i u_k - D^i u) \, dx \geq 0,$$

then

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_k) D^i u_k \, dx &\geq \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_k) D^i u \, dx \\
&\qquad + \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u)(D^i u_k - D^i u) \, dx.
\end{aligned}$$

Since $D^i u_k \rightharpoonup D^i u$ in $L^{p_i(\cdot)}(\Omega)$ for $i = 1, \ldots, N$, and using (3.4) we get

$$\liminf_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_k) D^i u_k \, dx \geq \sum_{i=1}^{N} \int_\Omega \psi_i \; D^i u \, dx.$$

Having in mind (3.8), we conclude that

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_k) D^i u_k \, dx = \sum_{i=1}^{N} \int_\Omega \psi_i \; D^i u \, dx. \tag{3.9}$$

Therefore, by combining (3.6), (3.7) and (3.9), we obtain

$$\langle B_\varepsilon u_k, u_k \rangle \longrightarrow \langle \chi_\varepsilon, u \rangle \quad \text{as} \quad k \to \infty.$$

It remain to show that $a_i(x, \nabla u_k) \rightharpoonup a_i(x, \nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$. In view of (3.9) we can prove that

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega (a_i(x, \nabla u_k) - a_i(x, \nabla u))(D^i u_k - D^i u) \, dx = 0,$$

by virtue of (3.4), we obtain

$$\lim_{k\to\infty} \int_\Omega a_i(x, \nabla u_k) D^i u_k \, dx = \int_\Omega \psi_i D^i u \, dx \quad \text{for} \quad i = 1, \ldots, N. \tag{3.10}$$

Now, thanks to (1.4) we have any $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$

$$(a_i(x, \nabla u_k)) - a_i(x, \nabla v)) \left(D^i u_k - D^i v \right) \geq 0 \quad \text{for} \quad i = 1, \ldots, N,$$

then

$$\lim_{k\to\infty} \int_\Omega a_i(x, \nabla u_k)(D^i u_k - D^i v) dx \geq \lim_{k\to\infty} \int_\Omega a_i(x, \nabla v)(D^i u_k - D^i v) \, dx,$$

thanks to (3.10) we conclude that

$$\int_\Omega (\psi_i - a_i(x, \nabla v) \, (D^i u - D^i v) \, dx \geq 0 \quad \text{for any } v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Let $\omega \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, taking $v = u - t\omega$, for $t > 0$ we have

$$t \int_\Omega (\psi_i - a_i(x, \nabla(u - t\omega))) D^i \omega \, dx \geq 0 \, .$$

Dividing by t, then letting t tends to 0 we obtain

$$\int_\Omega (\psi_i - a_i(x, \nabla u)) D^i \omega \, dx \geq 0 \, .$$

Similarly, by taking $v = u - t\omega$ with $t < 0$, then letting t tends to 0, we get

$$\int_\Omega (\psi_i - a_i(x, \nabla u)) D^i\omega\, dx \le 0.$$

It follows that

$$\int_\Omega (\psi_i - a_i(x, \nabla u)) D^i\omega\, dx = 0 \quad \forall \omega \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Consequently, we have $\psi_i = a(x, \nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$, and we deduce that

$$a_i(x, \nabla u_k) \rightharpoonup a_i(x, \nabla u) \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega) \quad \text{for} \quad i = 1, \dots, N. \tag{3.11}$$

Thanks to (3.5) we obtain $\chi_\varepsilon = B_\varepsilon u$, which conclude the proof of Lemma 3.3. □

In view of Lemma 3.3, there exists at least one weak solution $u_\varepsilon \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ of the problem (3.1) (cf. [22, Theorem 8.2]).

Step 2: A priori estimates.

In this step, we will give some estimates on weak solutions of approximate problems. By taking u_ε as a test function in (3.1), we obtain

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) u_\varepsilon dx + \sum_{i=1}^N \int_\Omega a_i(x, \nabla u_\varepsilon) D^i u_\varepsilon\, dx = \int_\Omega f u_\varepsilon dx,$$

since $\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))$ has the same sign as u_ε, and thanks to (1.3) we obtain

$$\alpha \sum_{i=1}^N \int_\Omega |D^i u_\varepsilon|^{p_i(x)} dx \le \int_\Omega f u_\varepsilon\, dx.$$

We have $f \in L^\infty(\Omega)$, and in view of Young's inequality we obtain

$$\begin{aligned} \left| \int_\Omega f u_\varepsilon\, dx \right| &\le C_0 \int_\Omega |f|^{\underline{p}'}\, dx + \frac{\alpha}{2C_p^{\underline{p}}} \int_\Omega |u_\varepsilon|^{\underline{p}}\, dx \\ &\le C_0 \|f\|_{L^\infty(\Omega)}^{\underline{p}'} \text{meas}(\Omega) + \frac{\alpha}{2} \int_\Omega |D^i u_\varepsilon|^{\underline{p}}\, dx \\ &\le C_1 + \frac{\alpha}{2} \sum_{i=1}^N \int_\Omega |D^i u_\varepsilon|^{p_i(x)}\, dx + \frac{\alpha}{2} \text{meas}(\Omega), \end{aligned}$$

it follows that

$$\frac{\alpha}{2} \sum_{i=1}^N \int_\Omega |D^i u_\varepsilon|^{p_i(x)} dx \le C_2\,.$$

Thanks to Remark 2.3, the two norms $\|\cdot\|_{1,\vec{p}(\cdot)}$ and $\sum_{i=1}^{N}|\cdot|_{p_i(\cdot)}$ are equivalent in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, we conclude that

$$\begin{aligned}\|u_\varepsilon\|_{1,\vec{p}(\cdot)}^{\underline{p}} &\leq C_3\sum_{i=1}^{N}\|D^i u_\varepsilon\|_{p_i(\cdot)}^{\underline{p}}\\ &\leq C_3\sum_{i=1}^{N}\Big(\int_\Omega |D^i u_\varepsilon|^{p_i(x)}dx+1\Big)\\ &\leq C_4.\end{aligned}$$

Consequently,

$$\|u_\varepsilon\|_{1,\vec{p}(\cdot)}\leq C_5,$$

with C_5 is a constant that don't depend on ε. It follows that there exists a subsequence still denoted $(u_\varepsilon)_\varepsilon$ such that

$$\left\{\begin{array}{llll} u_\varepsilon \rightharpoonup u & \text{in} & W_0^{1,\vec{p}(\cdot)}(\Omega), & \\ u_\varepsilon \to u & \text{in} & L^{\underline{p}}(\Omega) \quad \text{and} & \text{a.e. in} \quad \Omega. \end{array}\right. \tag{3.12}$$

On the other hand, by taking $v_{\delta,\varepsilon}=\frac{1}{\delta}\big(T_{k+\delta}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))-T_k(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))\big)$ as a test function in the approximate problem (3.1) where $\delta>0$, we have

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))v_{\delta,\varepsilon}dx+\sum_{i=1}^{N}\int_\Omega a_i(x,\nabla u_\varepsilon)D^i v_{\delta,\varepsilon}dx=\int_\Omega f v_{\delta,\varepsilon}dx,$$

and it's clear that $|v_{\delta,\varepsilon}|\leq 1$, then

$$\begin{aligned}&\frac{1}{\delta}\int_{\{k+\delta\leq|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\}}\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))T_{k+\delta}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))-T_k(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))\,dx\\ &\quad+\frac{1}{\delta}\sum_{i=1}^{N}\int_{\{k\leq|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|<k+\delta\}}a_i(x,\nabla u_\varepsilon)\beta'_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))D^iT_{1/\varepsilon}(u_\varepsilon)\,dx\\ &\leq\int_{\{k\leq|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\}}|f|\,dx.\end{aligned}\tag{3.13}$$

In view of (1.3), the second term on the left-hand side of (3.13) is positive. Having in mind that $v_{\delta,\varepsilon}$ has the same sign as u_ε, and using the monotonicity of the operator $\beta_\varepsilon(\cdot)$ we conclude that

$$\begin{aligned}
&k \text{ meas}\{k+\delta \le |\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\}\\
&\le \int_{\{k+\delta\le|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\}} |\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\, dx\\
&\le \frac{1}{\delta}\int_{\{k\le|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\}} \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\left(T_{k+\delta}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))\right)\, dx\\
&\le \int_{\{k\le|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\}} |f|\, dx\\
&\le \|f\|_{L^\infty(\Omega)} \text{ meas}\{k \le |\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\}.
\end{aligned}$$

By passing to the limit with $\delta \to 0$ and choosing $k > \|f\|_{L^\infty(\Omega)}$ we obtain

$$k \text{ meas}\{k \le |\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\} \le \|f\|_{L^\infty(\Omega)} \text{ meas}\{k \le |\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\},$$

it follows necessary that $\text{meas}\{k \le |\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))|\} = 0$ for any $k > \|f\|_{L^\infty(\Omega)}$. Therefore

$$\|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)},$$

and there exists $b \in L^\infty(\Omega)$ such that

$$\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \rightharpoonup b \quad \text{weak}-* \text{ in } \quad L^\infty(\Omega).$$

Step 3: Weak convergence of $(a_i(x,\nabla u_\varepsilon))_\varepsilon$ ***in*** $L^{p_i'(\cdot)}(\Omega)$

In the sequel, we denote by $\eta_i(n)$, $i = 1, 2, \ldots,$ various real-valued functions of real variable that converge to 0 as n tends to infinity. We will show that

$$a_i(x,\nabla u_\varepsilon) \rightharpoonup a_i(x,\nabla u) \quad \text{weakly in} \quad L^{p_i'(\cdot)}(\Omega) \quad \text{for} \quad i = 1,\ldots,N. \tag{3.14}$$

Indeed, by taking $w_\varepsilon = u_\varepsilon - u$ as a test function in (3.1), we obtain

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))(u_\varepsilon - u)dx + \sum_{i=1}^N \int_\Omega a_i(x,\nabla u_\varepsilon)(D^i u_\varepsilon - D^i u)dx = \int_\Omega f(u_\varepsilon - u)dx. \tag{3.15}$$

For the first term on the left-hand side of (3.15). In view of (3.12) we have $u_\varepsilon \to u$ in $L^1(\Omega)$, and thanks to $\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \rightharpoonup b$ weak$-*$ in $L^\infty(\Omega)$, then

$$\eta_1(\varepsilon) = \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))(u_\varepsilon - u)\, dx \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.16}$$

Similarly, we have f belongs to $L^\infty(\Omega)$ then

$$\eta_2(\varepsilon) = \int_\Omega f\ (u_\varepsilon - u)\, dx \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.17}$$

By combining (3.15) – (3.17), we deduce that

$$\sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_\varepsilon)(D^i u_\varepsilon - D^i u)\, dx = \eta_3(\varepsilon),$$

then

$$\sum_{i=1}^{N} \int_\Omega \left(a_i(x, \nabla u_\varepsilon) - a_i(x, \nabla u)\right)\left(D^i u_\varepsilon - D^i u\right) dx + \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u)(D^i u_\varepsilon - D^i u)\, dx = \eta_3(\varepsilon).$$

Thanks to (1.2) we have $a_i(x, \nabla u) \in L^{p_i'(\cdot)}(\Omega)$, and since $D^i u_\varepsilon \rightharpoonup D^i u$ weakly in $L^{p_i(\cdot)}(\Omega)$, then

$$\eta_4(\varepsilon) = \int_\Omega a_i(x, \nabla u)(D^i u_\varepsilon - D^i u)\, dx \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for any} \quad i = 1, \dots, N.$$

It follows that

$$\sum_{i=1}^{N} \int_\Omega \left(a_i(x, \nabla u_\varepsilon) - a_i(x, \nabla u)\right)\left(D^i u_\varepsilon - D^i u\right) dx = \eta_5(\varepsilon).$$

Therefore, by letting ε goes to zero we conclude that

$$\lim_{\varepsilon \to 0} \int_\Omega (a_i(x, \nabla u_\varepsilon) - a_i(x, \nabla u))(D^i u_\varepsilon - D^i u)\, dx = 0 \quad \text{for} \quad i = 1, \dots, N.$$

We have $(a_i(x, \nabla u_\varepsilon))_\varepsilon$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a function $\psi_i \in L^{p_i'(\cdot)}(\Omega)$, such that $a_i(x, \nabla u_\varepsilon) \rightharpoonup \psi_i$ in $L^{p_i'(\cdot)}(\Omega)$, we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega a_i(x, \nabla u_\varepsilon) D^i u_\varepsilon\, dx = \int_\Omega \psi_i D^i u\, dx.$$

On the other hand, thanks to (1.4), we have for any $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$

$$(a_i(x, \nabla u_\varepsilon)) - a_i(x, \nabla v))\left(D^i u_\varepsilon - D^i v\right) \geq 0 \quad \text{for} \quad i = 1, \dots, N.$$

Following the same way used in the proof of (3.11), we can show that

$$\int_\Omega (\psi_i - a(x, \nabla u)) D^i \omega\, dx = 0 \quad \text{for any} \quad \omega \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Consequently,

$$\psi_i = a(x, \nabla u) \quad \text{in} \quad L^{p_i'(\cdot)}(\Omega) \quad \text{for} \quad i = 1, \dots, N,$$

which conclude the proof of the convergence (3.14).

Step 4: Passage to the limit.

By taking $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ as a test function in the approximate problem (3.1), we have

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))v\,dx + \sum_{i=1}^N \int_\Omega a_i(x,\nabla u_\varepsilon)D^i v\,dx = \int_\Omega f\ v\,dx. \tag{3.18}$$

Since $\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \rightharpoonup b$ weak$-*$ in $L^\infty(\Omega)$ for $i = 1,\dots,N$, then

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))v\,dx \longrightarrow \int_\Omega bv\,dx \qquad \text{as} \quad \varepsilon \to 0.$$

Also, we have $a_i(x,\nabla u_\varepsilon) \rightharpoonup a_i(x,\nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$ then

$$\sum_{i=1}^N \int_\Omega a_i(x,\nabla u_\varepsilon)D^i v\,dx \longrightarrow \sum_{i=1}^N \int_\Omega a_i(x,\nabla u)D^i v\,dx \quad \text{as} \quad \varepsilon \to 0.$$

Therefore, by letting ε goes to zero in (3.18), we conclude that

$$\int_\Omega bv\,dx + \sum_{i=1}^N \int_\Omega a_i(x,\nabla u)D^i v\,dx = \int_\Omega fv\,dx \quad \text{for any} \quad v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Step 5: Subdifferential argument.

Firstly, since $\beta(\cdot)$ a is maximal monotone graph, there exists a convex lower semi-continuous and proper function $j\colon \mathbb{R} \longmapsto [0,\infty]$, such that

$$\beta(r) = \partial j(r) \quad \text{for all} \quad r \in \mathbb{R}.$$

According to [13], we have the following result.

Proposition 3.4. *For any $0 < \varepsilon \le 1$, the mapping $j_\varepsilon\colon \mathbb{R} \longmapsto \mathbb{R}$ defined by: $j_\varepsilon(r) = \displaystyle\int_0^r \beta_\varepsilon(s)ds$, has the following properties:*

(i) *The mapping j_ε is convex and differentiable for all $r \in \mathbb{R}$, such that:*

$$j_\varepsilon^{'}(r) = \beta_\varepsilon(r) \quad \textit{for any} \quad 0 < \varepsilon \le 1$$

(ii) *For all $r \in \mathbb{R}$ we have: $j_\varepsilon(r) \longrightarrow j(r)$ as $\varepsilon \to 0$.*

It remain to show that $u(x) \in D(\beta(\cdot))$ and $b(x) \in \beta(u(x))$ for a.e $x \in \Omega$. We have $\beta(\cdot)$ is a maximal monotone operator, and in view of **(i)** for any $0 < \varepsilon \le 1$, we have

$$j_\varepsilon(r) \ge j_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + (r - T_{1/\varepsilon}(u_\varepsilon))\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)), \tag{3.19}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω.
Let E be an arbitrary measurable subset of Ω and χ_E its characteristic function. Let $h, \varepsilon_0 > 0$ and we set $v_{h,\varepsilon} = 1 - |T_1(u_\varepsilon - T_h(u_\varepsilon))|$. By multiplying (3.19) by the test function $v_{h,\varepsilon}\chi_E$, then integrating over Ω, we obtain

$$\int_E j_\varepsilon(r) v_{h,\varepsilon}\, dx \geq \int_E j_{\varepsilon_0}(T_{h+1}(u_\varepsilon)) v_{h,\varepsilon}\, dx + \int_E (r - T_{h+1}(u_\varepsilon)) v_{h,\varepsilon} \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\, dx,$$

for all $r \in \mathbb{R}$ and all $0 < \varepsilon \leq min(\varepsilon_0, \dfrac{1}{h+1})$, we have $v_{h,\varepsilon} = 0$ on the set $\{|u_\varepsilon| \geq h+1\}$. By letting ε tends to 0, we have $v_{h,\varepsilon} \to v_h = 1 - |T_1(u - T_h(u))|$, having in mind **(ii)** we obtain

$$\int_E j(r) v_h\, dx \geq \int_E j_{\varepsilon_0}(T_{h+1}(u)) v_h\, dx + \int_E (r - T_{h+1}(u)) v_h b\, dx\,.$$

Taking into account that E is arbitrary we obtain

$$j(r) v_h \geq j_{\varepsilon_0}(T_{h+1}(u)) v_h + (r - T_{h+1}(u)) v_h b \tag{3.20}$$

for all $r \in \mathbb{R}$ almost everywhere in Ω. By letting h tends to infinity, then ε_0 goes to zero in (3.20) we deduce that

$$j(r) \geq j(u(x)) + b(x)(r - u(x)) \quad \text{a.e. in} \quad \Omega, \quad \text{for any} \quad r \in \mathbb{R}.$$

Hence $u \in D(\beta)$ and $b \in \beta(u)$ almost everywhere in Ω. which conclude the proof of the Theorem 3.2. □

4. The existence of T-$\vec{p}(\cdot)$-solution in the case of $f \in L^1(\Omega)$.

Definition 4.1. Let $f \in L^1(\Omega)$ and $\beta(\cdot)$ a maximal monotone mapping, the pair of measurable functions (u, b) is called T-$\vec{p}(\cdot)$-solution of the quasilinear elliptic problem (1.1), if this pair satisfying the following conditions:

(C1) The function $u\colon \Omega \longmapsto \mathbb{R}$ is measurable and $b \in L^1(\Omega)$, such that $u(x) \in D(\beta)$ and $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$.

(C2) For each $k > 0$, we have $T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ and

$$\int_\Omega b T_k(u-\varphi)\, dx + \sum_{i=1}^N \int_\Omega a_i(x, \nabla u) D^i T_k(u-\varphi)\, dx = \int_\Omega f T_k(u-\varphi)\, dx, \tag{4.1}$$

for every $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Theorem 4.2. *Let $f \in L^1(\Omega)$, assuming that (1.2)–(1.4) hold true, then the quasilinear anisotropic elliptic problem (1.1) has at least one T-$\vec{p}(\cdot)$-solution. Moreover, if $\underline{p} \geq 2 - \dfrac{1}{N}$, then the solution belongs to $W_0^{1,q}(\Omega)$ for any $1 \leq q < \dfrac{N(\underline{p}-1)}{N-1}$.*

Proof. ***Step 1: The approximate problems.***

Let $(f_n)_{n\in\mathbb{N}^*}$ be a sequence of measurable function in $L^\infty(\Omega)\cap L^1(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and $|f_m| \leq |f_n|$ for any $m \leq n$. Let $\beta_{\frac{1}{n}}(\cdot)$ be the Yosida approximation of $\beta(\cdot)$, note that

$$\langle \beta_{\frac{1}{n}}(v), v\rangle \geq 0, \quad |\beta_{\frac{1}{n}}(v)| \leq n|v| \quad \text{and} \quad \lim_{n\to\infty} \beta_{\frac{1}{n}}(v) = \beta(v).$$

We consider the approximate problem

$$\begin{cases} \beta_{\frac{1}{n}}(u_n) + Au_n = f_n \quad \text{in} \quad \Omega, \\ u_n \in W_0^{1,\vec{p}(\cdot)}(\Omega). \end{cases}$$

In view of Theorem 3.2, there exists at least one pair of functions $(u_n, b_n) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \times L^\infty(\Omega)$ satisfying $u_n \in D(\beta_{\frac{1}{n}})$ and $b_n \in \beta_{\frac{1}{n}}(u_n)$ almost everywhere in Ω such that

$$\int_\Omega b_n w dx + \sum_{i=1}^N \int_\Omega a_i(x, \nabla u_n) D^i w dx = \int_\Omega f_n w dx \quad \text{for any } w \in W_0^{1,\vec{p}(\cdot)}(\Omega). \tag{4.2}$$

Now, let $m \in \mathbb{N}^*$ with $m \leq n$. Similarly, we have the existence of $(u_m, b_m) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \times L^\infty(\Omega)$ satisfying $u_m \in D(\beta_{\frac{1}{m}})$ and $b_m \in \beta_{\frac{1}{m}}(u_m)$ such that

$$\int_\Omega b_m w dx + \sum_{i=1}^N \int_\Omega a_i(x, \nabla u_m) D^i w dx = \int_\Omega f_m w dx \text{ for any } w \in W_0^{1,\vec{p}(\cdot)}(\Omega). \tag{4.3}$$

Let E be a measurable subset of Ω, by taking $w = (u_n - v_n)\cdot \chi_E$ in the two equations (4.2) and (4.3), and then subtracting the two equations we obtain

$$\int_E (b_n - b_m)(u_n - u_m)\, dx + \sum_{i=1}^N \int_E (a_i(x, \nabla u_n) - a_i(x, \nabla u_m))(D^i u_n - D^i u_m)\, dx$$
$$= \int_E (f_n - f_m)(u_n - u_m)\, dx.$$

We have $b_n \in \beta_{\frac{1}{n}}(u_n)$ and $b_m \in \beta_{\frac{1}{m}}(u_m)$, then thanks to (1.4) we deduce that

$$0 \leq \int_E (b_n - b_m)(u_n - u_m)\, dx \leq \int_E (f_n - f_m)(u_n - u_m)\, dx \quad \text{for any} \quad E \in \Omega.$$

It follows necessary that the two sequences $(u_n)_n$ and $(b_n)_n$ are increasing.

Step 2: Weak convergence of $T_k(u_n)$ ***in*** $W_0^{1,\vec{p}(\cdot)}(\Omega)$***.***

By taking $T_k(u_n)$ as a test function in the approximate problem (4.2), we have

$$\int_\Omega b_n T_k(u_n) dx + \sum_{i=1}^N \int_\Omega a_i(x, \nabla u_n) D^i T_k(u_n)\, dx = \int_\Omega f_n T_k(u_n)\, dx, \tag{4.4}$$

since b_n has the same sign as u_n, and thanks to (1.3) we obtain

$$\alpha \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i(x)}\, dx \leq \int_\Omega b_n T_k(u_n) dx + \sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n) D^i T_k(u_n)\, dx$$
$$\leq k\|f\|_1.$$

It follows that

$$\begin{aligned} \|T_k(u_n)\|^{\underline{p}}_{1,\vec{p}(\cdot)} &\leq C \sum_{i=1}^{N} \|D^i T_k(u_n)\|^{\underline{p}}_{p_i(\cdot)} \\ &\leq C \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i(x)}\, dx + CN \\ &\leq C\frac{k}{\alpha}\|f\|_1 + CN. \end{aligned}$$

We conclude that there exists a constant C_1 that does not depend on n and k, such that

$$\|T_k(u_n)\|_{1,\vec{p}(\cdot)} \leq C_1 k^{\frac{1}{\underline{p}}} \quad \text{for any} \quad k \geq 1. \tag{4.5}$$

It follows that the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Therefore, there exists a subsequence still denoted $(T_k(u_n))_n$, and a measurable function $v_k \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k & \text{in} \quad W_0^{1,\vec{p}(\cdot)}(\Omega), \\ T_k(u_n) \to v_k & \text{in} \quad L^{\underline{p}}(\Omega) \quad \text{and a.e. in } \Omega. \end{cases} \tag{4.6}$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure in Ω.
Firstly, according to (4.5) we have

$$\begin{aligned} k\ \text{meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\, dx \leq \int_\Omega |T_k(u_n)|\, dx \\ &\leq \|T_k(u_n)\|_{1,\vec{p}(\cdot)} \\ &\leq C_2 k^{\frac{1}{\underline{p}}}. \end{aligned}$$

Consequently,

$$\text{meas}\big\{|u_n| > k\big\} \leq C_2 \frac{1}{k^{1-\frac{1}{\underline{p}}}} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{4.7}$$

Taking $\lambda > 0$, it's clear that

$$\begin{aligned} \text{meas}\big\{|u_n - u_m| > \lambda\big\} &\leq \text{meas}\big\{|u_n| > k\big\} + \text{meas}\big\{|u_m| > k\big\} \\ &\quad + \text{meas}\big\{|T_k(u_n) - T_k(u_m)| > \lambda\big\}. \end{aligned}$$

Let $\sigma > 0$, using (4.7) we can choose $k = k(\sigma)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\sigma}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \leq \frac{\sigma}{3}. \tag{4.8}$$

On the other hand, thanks to (4.6) we can assume that $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus, for any $k > 0$ and $\lambda, \sigma > 0$, there exists $n_0 = n_0(k, \lambda, \sigma)$ such that

$$\text{meas}\big\{|T_k(u_n) - T_k(u_m)| > \lambda\big\} \leq \frac{\sigma}{3} \quad \text{for any } n, m \geq n_0(k, \lambda, \sigma). \tag{4.9}$$

In view of (4.8) and (4.9), we deduce that for any $\lambda, \sigma > 0$, there exists $n_0 = n_0(\lambda, \sigma)$ such that

$$\text{meas}\{|u_n - u_m| > \lambda\} \leq \sigma \quad \text{for any} \quad n, m \geq n_0(\lambda, \sigma),$$

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Consequently, we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad W_0^{1,\vec{p}(\cdot)}(\Omega),$$

and using Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \to T_k(u) \quad \text{in} \quad L^{p_i(\cdot)}(\Omega) \quad \text{for any} \quad i = 1, \ldots, N.$$

Moreover, thanks to (4.4) we have

$$\int_\Omega |b_n|\ |T_k(u_n)|\, dx \leq \int_\Omega f_n T_k(u_n)\, dx \leq k\|f\|_{L^1(\Omega)} \quad \text{for any} \quad k > 0,$$

it follows that

$$\|b_n\|_{L^1(\Omega)} = \lim_{k\to 0} \int_\Omega |b_n|\ \frac{|T_k(u_n)|}{k}\, dx \leq \|f\|_{L^1(\Omega)},$$

we have $(b_n)_n$ is increasing and uniformly bounded sequence in $L^1(\Omega)$, then, there exists a measurable function $b \in L^1(\Omega)$ such that

$$b_n \longrightarrow b \quad \text{strongly in} \quad L^1(\Omega). \tag{4.10}$$

Step 3: Some regularity results.

Assume that $\underline{p} \geq 2 - \dfrac{1}{N}$ and $1 < \theta < \underline{p}$. By taking $\omega = \left(1 - \dfrac{1}{(1 + |u_n|)^{\theta-1}}\right)\text{sign}(u_n)$ as a test function in the approximate problem (4.2), we have

$$\int_\Omega b_n \omega\, dx + (\theta - 1)\sum_{i=1}^N \int_\Omega \frac{a_i(x, \nabla u_n) D^i u_n}{(1 + |u_n|)^\theta}\, dx = \int_\Omega f_n \omega\, dx,$$

since b_n has the same sign as u_n and $|\omega| \leq 1$, in view of (1.3) we get

$$\alpha(\theta-1)\sum_{i=1}^{N}\int_{\Omega}\frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}\,dx \leq \|f\|_1.$$

By choosing $q = \dfrac{N(\underline{p}-\theta)}{N-\theta}$ we have $q^* = \dfrac{Nq}{N-q} = \dfrac{\theta q}{\underline{p}-q}$. Thus, in view of Hölder's and Sobolev inequalities we deduce that

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\Omega}|D^i u_n|^q\,dx &= \sum_{i=1}^{N}\int_{\Omega}\frac{|D^i u_n|^q}{(1+|u_n|)^{\frac{\theta q}{\underline{p}}}}(1+|u_n|)^{\frac{\theta q}{\underline{p}}}\,dx\\
&\leq \sum_{i=1}^{N}\Big\|\frac{|D^i u_n|^q}{(1+|u_n|)^{\frac{\theta q}{\underline{p}}}}\Big\|_{\frac{\underline{p}}{q}}\|(1+|u_n|)^{\frac{\theta q}{\underline{p}}}\|_{\frac{\underline{p}}{\underline{p}-q}}\\
&\leq C_0\sum_{i=1}^{N}\Big(\int_{\Omega}\frac{|D^i u_n|^{\underline{p}}}{(1+|u_n|)^{\theta}}\,dx\Big)^{\frac{q}{\underline{p}}}\Big(\int_{\Omega}(1+|u_n|)^{\frac{\theta q}{\underline{p}-q}}\,dx\Big)^{\frac{\underline{p}-q}{\underline{p}}}\\
&\leq C_1\sum_{i=1}^{N}\Big(\int_{\Omega}\frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}\,dx+|\Omega|\Big)^{\frac{q}{\underline{p}}}\Big(\int_{\Omega}|u_n|^{q^*}\,dx+|\Omega|\Big)^{\frac{\theta q}{q^*\underline{p}}}\\
&\leq C_2\sum_{i=1}^{N}\Big(\frac{\|f\|_1}{\alpha(\theta-1)}+|\Omega|\Big)^{\frac{q}{\underline{p}}}\Big(\|D^i u_n\|_q^{\frac{\theta q}{\underline{p}}}+C_3\Big)\\
&\leq C_4\sum_{i=1}^{N}\Big(\int_{\Omega}|D^i u_n|^q\,dx\Big)^{\frac{\theta}{\underline{p}}}+C_5.
\end{aligned}$$

Since $\dfrac{\theta}{\underline{p}} < 1$, it follows that there exists a positive constant C_6 that does not depend on n, such that

$$\sum_{i=1}^{N}\int_{\Omega}|D^i u_n|^q\,dx \leq C_6,$$

then, there exists a subsequence still denoted $(u_n)_n$ such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,q}(\Omega).$$

We refer the reader to [11] for more details.

Step 4: Passage to the limit.

Let $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^{\infty}(\Omega)$, by taking $T_k(u_n-\varphi)$ as a test function in (4.2), we obtain

$$\int_{\Omega} b_n T_k(u_n-\varphi)\,dx + \sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla u_n)D^i T_k(u_n-\varphi)\,dx = \int_{\Omega} f_n T_k(u_n-\varphi)\,dx.$$

Choosing $M = k + \|\varphi\|_\infty$ then $\{|u_n - \varphi| \leq k\} \subseteq \{|u_n| \leq M\}$. In view of (1.4) we obtain

$$\sum_{i=1}^{N} \int_{\{|u_n-\varphi|\leq k\}} \big(a_i(x,\nabla u_n) - a_i(x,\nabla\varphi)\big)(D^i u_n - D^i\varphi)\ dx \geq 0,$$

then

$$\sum_{i=1}^{N} \int_\Omega a_i(x,\nabla\varphi) D^i T_k(u_n-\varphi)\ dx \leq \sum_{i=1}^{N} \int_\Omega a_i(x,\nabla u_n) D^i T_k(u_n-\varphi)\ dx$$

it follows that

$$\int_\Omega b_n T_k(u_n-\varphi)\ dx + \sum_{i=1}^{N} \int_\Omega a_i(x,\nabla\varphi) D^i T_k(u_n-\varphi)\ dx \leq \int_\Omega f_n T_k(u_n-\varphi)\ dx. \tag{4.11}$$

Now, we pass to the limit on each terms of (4.11), we have

$$\sum_{i=1}^{N} \int_\Omega a_i(x,\nabla\varphi) D^i T_k(u_n-\varphi) dx = \sum_{i=1}^{N} \int_{\{|u_n-\varphi|\leq k\}} a_i(x,\nabla\varphi)(D^i T_M(u_n) - D^i\varphi) dx,$$

and since $D^i T_M(u_n) \rightharpoonup D^i T_M(u)$ in $L^{p_i(\cdot)}(\Omega)$, then

$$\begin{aligned} &\lim_{n\to\infty} \int_{\{|u_n-\varphi|\leq k\}} a_i(x,\nabla\varphi)\ (D^i T_M(u_n) - D^i\varphi)\ dx \\ &= \int_{\{|u-\varphi|\leq k\}} a_i(x,\nabla\varphi)\ (D^i T_M(u) - D^i\varphi)\ dx \\ &= \int_\Omega a_i(x,\nabla\varphi) D^i T_k(u-\varphi)\ dx. \end{aligned} \tag{4.12}$$

Moreover, thanks to (4.10) and since $T_k(u_n-\varphi) \rightharpoonup T_k(u-\varphi)$ weak-$*$ in $L^\infty(\Omega)$, then

$$\int_\Omega b_n T_k(u_n-\varphi)\, dx \longrightarrow \int_\Omega b T_k(u-\varphi)\, dx, \tag{4.13}$$

and

$$\int_\Omega f_n T_k(u_n-\varphi)\, dx \longrightarrow \int_\Omega f\ T_k(u-\varphi)\, dx. \tag{4.14}$$

By combining (4.12) – (4.14), we conclude that

$$\int_\Omega b T_k(u-\varphi)\ dx + \sum_{i=1}^{N} \int_\Omega a_i(x,\nabla\varphi) D^i T_k(u-\varphi)\ dx \leq \int_\Omega f T_k(u-\varphi)\ dx, \tag{4.15}$$

for any $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Step 5: The Minty lemma.

Now, we will introduce the following lemma considered as an L^1-version of the Minty's lemma.

Lemma 4.3. *Let u be a measurable function such that $T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ for every $k > 0$. Then, for any $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, the following assertions are equivalent:*
Assertion 1:

$$\int_\Omega bT_k(u-\varphi)\,dx + \sum_{i=1}^N \int_\Omega a_i(x,\nabla\varphi)D^iT_k(u-\varphi)\,dx \le \int_\Omega fT_k(u-\varphi)\,dx,$$

for any $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.
Assertion 2:

$$\int_\Omega bT_k(u-\varphi)\,dx + \sum_{i=1}^N \int_\Omega a_i(x,\nabla u)D^iT_k(u-\varphi)\,dx = \int_\Omega fT_k(u-\varphi)\,dx,$$

for any $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Proof. **(Assertion 2) $\Longrightarrow$ (Assertion 1).** In view of (1.4), we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega a_i(x,\nabla u)D^iT_k(u-\varphi)\,dx \\
&\qquad = \sum_{i=1}^N \int_\Omega a_i(x,\nabla\varphi)D^iT_k(u-\varphi)\,dx \\
&\qquad\qquad + \sum_{i=1}^N \int_\Omega (a_i(x,\nabla u)-a_i(x,\nabla\varphi))D^iT_k(u-\varphi)\,dx \\
&\qquad \ge \sum_{i=1}^N \int_\Omega a_i(x,\nabla\varphi)D^iT_k(u-\varphi)\,dx,
\end{aligned}$$

The assertion 1 is concluded.

(Assertion 1) $\Longrightarrow$ (Assertion 2)
Let h and k be two positive real numbers and $\lambda \in [-1,1]$.
Let $\psi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, choosing $\varphi = T_h(u-\lambda T_k(u-\psi)) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$

as test function in the (Assertion 1), we have

$$\begin{aligned}
&\int_\Omega bT_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx \\
&+\sum_{i=1}^{N}\int_\Omega a_i(x,\nabla T_h(u-\lambda T_k(u-\psi)))D^iT_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx \\
&\leq \int_\Omega fT_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx\,.
\end{aligned}\tag{4.16}$$

Concerning the second term on the left-hand side of (4.16), we have $a_i(x,0)=0$ then

$$\begin{aligned}
&\int_\Omega a_i(x,\nabla T_h(u-\lambda T_k(u-\psi)))D^iT_k(u-T_h(u-\lambda T_k(u-\psi)))\ dx \\
&=\lambda\int_{\{|u-\varphi|\leq k\}\cap\{|u-\lambda T_k(u-\psi)|\leq h\}} a_i(x,\nabla T_h(u-\lambda T_k(u-\psi)))D^iT_k(u-\psi)\ dx,
\end{aligned}$$

and since $\{|u-\lambda T_k(u-\psi)|\leq h\}\to\Omega$ as $h\to\infty$, it follows that

$$\begin{aligned}
&\lim_{h\to\infty}\int_\Omega a_i(x,\nabla T_h(u-\lambda T_k(u-\psi)))D^iT_k(u-T_h(u-\lambda T_k(u-\psi)))\ dx \\
&\qquad=\lambda\int_\Omega a_i(x,\nabla(u-\lambda T_k(u-\psi))D^iT_k(u-\psi)\ dx.
\end{aligned}\tag{4.17}$$

Moreover, it is easy to see that,

$$\lim_{h\to\infty}\int_\Omega bT_k(u-T_h(u-\lambda T_k(u-\psi)))\ dx=\lambda\int_\Omega bT_k(u-\psi)\ dx,\tag{4.18}$$

and

$$\lim_{h\to\infty}\int_\Omega fT_k(u-T_h(u-\lambda T_k(u-\psi)))\ dx=\lambda\int_\Omega fT_k(u-\psi)\ dx.\tag{4.19}$$

By combining (4.16) – (4.19), we deduce that

$$\begin{aligned}
&\lambda\int_\Omega bT_k(u-\psi)\ dx+\lambda\sum_{i=1}^{N}\int_\Omega a_i(x,\nabla(u-\lambda T_k(u-\psi)))D^iT_k(u-\psi)\ dx \\
&\qquad\leq\lambda\int_\Omega fT_k(u-\psi)\ dx.
\end{aligned}$$

Choosing $\lambda>0$, dividing both sides by λ, then letting λ tend to zero, we obtain

$$\int_\Omega bT_k(u-\psi)\ dx+\sum_{i=1}^{N}\int_\Omega a_i(x,\nabla u)D^iT_k(u-\psi)\ dx\leq\int_\Omega fT_k(u-\psi)\ dx.\tag{4.20}$$

Doing the same for the case of $\lambda < 0$, we obtain

$$\int_\Omega bT_k(u-\psi)\ dx + \sum_{i=1}^{N} \int_\Omega a_i(x,\nabla u) D^i T_k(u-\psi)\ dx \geq \int_\Omega fT_k(u-\psi)\ dx. \tag{4.21}$$

By combining (4.20) and (4.21), we conclude the following equality:

$$\int_\Omega bT_k(u-\psi)\ dx + \sum_{i=1}^{N} \int_\Omega a_i(x,\nabla u) D^i T_k(u-\psi)\ dx = \int_\Omega fT_k(u-\psi)\ dx$$

for any $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, which completes the proof of Lemma 4.3. □

By using the subdifferential argument (as in the proof of Theorem 3.2) we show that $u \in D(\beta)$ and $b \in \beta(u)$ a.e. in Ω. Thus, in view of (4.15) and Lemma 4.3, we conclude the proof of the Theorem 4.2. □

5. Uniqueness of T-$\vec{p}(\cdot)$-solution solution

Theorem 5.1. *Let $f \in L^1(\Omega)$, assuming that (1.2) − (1.4) hold true. If one of the following conditions is verified:*

- *If $\beta(\cdot)$ is a strictly increasing, continuous function,*
- *If $\beta(\cdot)$ is a monotone graph, and there exists $i_0 \in \{1, 2, \ldots, N\}$ such that $a_{i_0}(x,\cdot)$ is strictly monotone.*

Then, the T-$\vec{p}(\cdot)$-solution of the quasilinear anisotropic elliptic problem (1.1) is unique.

Proof. Let $h > k > 0$. Assuming that there exists two T-$\vec{p}(\cdot)$-solutions (u, b) and (v, d) of the problem (1.1), and we will show that $u = v$.

We consider u as a T-$\vec{p}(\cdot)$-solution of the elliptic problem (1.1) and by taking $\varphi = T_h(v)$ in (4.1), we have

$$\int_\Omega bT_k(u-T_h(v))\ dx + \sum_{i=1}^{N} \int_\Omega a_i(x,\nabla u) D^i T_k(u-T_h(v))\ dx = \int_\Omega fT_k(u-T_h(v))\ dx,$$

it follows that

$$\begin{aligned} &\int_{\{|v|\leq h\}} bT_k(u-v)\ dx - k\int_{\{|v|>h\}} |b|\ dx \\ &\quad + \sum_{i=1}^{N} \int_{\{|u-v|\leq k\}\cap\{|v|\leq h\}} a_i(x,\nabla u)(D^i u - D^i v)\ dx \\ &\leq \int_{\{|v|\leq h\}} fT_k(u-v)\ dx + k\int_{\{|v|>h\}} |f|\ dx. \end{aligned} \tag{5.1}$$

For the second term on the left-hand side of (5.1), we have $b(\cdot)$ belong to $L^1(\Omega)$, and since $\text{meas}\{|v| > h\} \to 0$ as h tends to infinity, we obtain

$$\varepsilon_0(h) = k \int_{\{|v|>h\}} |b| \, dx \longrightarrow 0 \quad \text{as} \quad h \to 0. \tag{5.2}$$

Similarly, we have $f \in L^1(\Omega)$ then

$$\varepsilon_1(h) = k \int_{\{|v|>h\}} |f| \, dx \longrightarrow 0 \quad \text{as} \quad h \to 0. \tag{5.3}$$

By combining (5.1)–(5.3) we conclude that

$$\begin{aligned} &\int_{\{|v|\le h\}} bT_k(u-v) \, dx + \sum_{i=1}^{N} \int_{\{|u-v|\le k\}\cap\{|v|\le h\}} a_i(x, \nabla u)(D^i u - D^i v) \, dx \\ &\le \int_{\{|v|\le h\}} fT_k(u-v) \, dx + \varepsilon_2(h). \end{aligned}$$

By letting h goes to infinity, we get

$$\int_\Omega bT_k(u-v) \, dx + \sum_{i=1}^{N} \int_{\{|u-v|\le k\}} a_i(x, \nabla u)(D^i u - D^i v) \, dx \le \int_\Omega fT_k(u-v) \, dx.$$

Similarly, by taking (v, d) as a T-$\vec{p}(\cdot)$-solution of the elliptic problem (1.1) and using $\varphi = T_h(v)$ in (4.1), we obtain

$$\int_\Omega dT_k(v-u) \, dx + \sum_{i=1}^{N} \int_{\{|u-v|\le k\}} a_i(x, \nabla v)(D^i v - D^i u) \, dx \le \int_\Omega fT_k(v-u) \, dx.$$

By adding the two previous inequalities, we conclude that

$$\int_\Omega (b-d)T_k(u-v) \, dx + \sum_{i=1}^{N} \int_{\{|u-v|\le k\}} \Big(a_i(x, \nabla u) - a_i(x, \nabla v)\Big)(D^i u - D^i v) \, dx \le 0,$$

We have $b \in \beta(u)$ and $d \in \beta(v)$, and thanks to (1.4) we deduce that

$$\int_\Omega (b-d)T_k(u-v) \, dx = 0,$$

and

$$\int_{\{|u-v|\le k\}} \Big(a_i(x, \nabla u) - a_i(x, \nabla v)\Big)(D^i u - D^i v) \, dx = 0 \quad \text{for} \quad i = 1, \ldots, N.$$

- If the maximal monotone operator $\beta(\cdot)$ is a strictly increasing, continuous function, then

$$\int_\Omega (b-d)T_k(u-v) \, dx = 0 \quad \Longrightarrow \qquad u = v \quad \text{a.e. in} \quad \Omega.$$

- If there exists $i_0 \in \{1, 2, \ldots, N\}$ such that $a_{i_0}(x, \cdot)$ is strictly monotone, then

$$\int_{\{|u-v|\leq k\}} \Big(a_{i_0}(x, \nabla u) - a_{i_0}(x, \nabla v)\Big)(D^{i_0}u - D^{i_0}v)\,dx = 0$$
$$\Longrightarrow \quad D^{i_0}u = D^{i_0}v \quad \text{a.e. in} \quad \{|u-v| \leq k\}.$$

We have $u, v \in W_0^{1,1}(\Omega)$ for $\underline{p} \geq 2 - \frac{1}{N}$. In view of Poincaré's inequality we obtain

$$\|T_k(u-v)\|_1 \leq C_p \|D^{i_0} T_k(u-v)\|_1 = 0 \quad \text{for any} \quad k > 0,$$

it follows necessary that $u = v$ a.e. in Ω.

Which conclude the proof of the Theorem 5.1. □

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