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Existence and Ulam stability for two orders delay fractional differential equations

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Abstract. *In this paper, we study the existence and uniqueness for nonlinear delay fractional differential equations with two orders of Caputo's fractional derivative using the Banach fixed point theorem. Also, we establish the Ulam stability of solutions. Finally, we give an example to illustrate the results.*

1. Introduction

The original motivation of the area of fractional calculus has started when L'Hôspital in 1695 wrote a letter to Leibniz related to the generalization of differentiation and raised the question about fractional derivative. After, Leibniz, Euler, Laplace, Lacroix and Fourier made mention of fractional derivatives or arbitrary order but the first use of fractional operations can be found in Abel's 1823 paper [4] that was considered as chapter II in his posthumous Euvres completes de Niels Henrik Abel [5] compiled by L. Sylow and S. Lie in 1881. Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone (isochrone) problem. However a rigorous investigation was first carried out by Liouville in a series of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration. Later investigations and further developments by among others Riemann led to the construction of the integral-based Riemann-Liouville fractional integral operator, which has been a valuable cornerstone in fractional calculus ever since. To learn more about the chronological progress of fractional calculus from 1695 to 1900 (see [38]). Along side with Riemann-Liouville, Professor Michele Caputo introduced an alternative definition in his paper in 1967 [14] and in his book [15] in 1969, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Fractional differential equations ($\mathrm{FDE_s}$) have gained considerable attention in various fields of applied mathematics and engineering such as physics, polymer

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rheology, regular vibration in thermodynamics and biophysics, etc. For more details, see the monographs of Kilbas et al. [28], Podlubny [32] and Samko et.al. [39]. The theory of (FDE_s) has attracted the attention of many mathematicians and many works has been released in this area (for example [3, 6, 7, 11, 22, 30]). On the other hand, delay differential equations arise in many processes and describe a lot of phenomena emanates from physics and life sciences (populations biology, physiology, economics and epidemiology). In [41] we can find many applications of delay differential equations in biological science, economic and physiology. In [10] Belair et al. obtained a system of delay differential equation concerning the production of red blood cells by the stem cells in the bone marrow. Also we can find more application of delay differential equations on the regulation of blood cell in [19, 20]. Recently, the topic of delay fractional differential equations ($DFDE_s$) is growing interest among mathematicians and physicists, which explains the emergence of a large number of papers in this area. Among the first investigations about ($DFDE_s$) we cite the work of Y. Chen and K. L. Moore [18] where they established the analytical stability bound for a special class of delayed fractional-order dynamic systems by using Lambert function. Also, the asymptotic stability of linear ($DFDE_s$) has studied by using different methods such as the final-value theorem of the Laplace transform [21] and a Gronwall inequality approach [31]. Other investigations has appeared regarding the existence of solutions by exploiting the fixed point theorems (see [2, 13, 16, 25, 29, 30, 42]) and the references therein.

In 1940, Ulam [44] proposed a general stability problem in the talk before the Mathematics Club of University of Wisconsin in which he discussed a number of important unsolved problems: "Under what conditions does there exist an additive mapping near an approximately additive mapping?" (for more details see [43, 44]). In the following year, Hyers [24] gave the first answer to the question of Ulam in the case of Banach spaces. In fact, let E_1, E_2 be two real Banach spaces and $\epsilon > 0$. Then for every mapping $f\colon E_1 \longrightarrow E_2$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon,$$

for all $x, y \in E_1$, there exists a unique additive mapping $g\colon E_1 \longrightarrow E_2$ with the property

$$\|f(x) - g(x)\| \leq \epsilon, \text{ for all } x \in E_1.$$

Thereafter, this type of stability is called the Ulam-Hyers stability and it means that one does not seek the exact solution for an Ulam-Hyers stable system but it is required to find a function which satisfies a suitable approximation inequality. This approach can guarantee that there exists a close exact solution useful in many applications. Further in 1978, Rassias [33, 35, 36, 43] provided an extension of Ulam-Hyers stability by introducing new function variables. As a result, another new stability concept, Ulam-Hyers-Rassias stability, was named by mathematicians. For more details on the recent advances on the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of differential equations, one can see the books

[17, 33, 34] and the research papers ([23, 26, 27]). Moreover, many investigations were realized concerning Ulam-Hyers stability and Ulam-Hyers-Rassias stability of ($\mathrm{FDE_s}$) such as [1, 9, 12, 16, 37, 42] and the references therein.

For example in [9], Atmania and Bouzitouna discussed the existence of the unique solution and the Ulam stability for the following nonlinear fractional differential equation with two orders

$$\begin{cases} {}^{C}D^{\beta}_{0+}\left(p\left(t\right){}^{C}D^{\alpha}_{0+}u\left(t\right)\right)+h\left(t\right)u\left(t\right)=f\left(t,u\left(t\right)\right),\ t\in\left[0,T\right], \\ u\left(t\right)=\phi\left(t\right),\ t\in\left[-r,0\right], \end{cases}$$

where ${}^{C}D^{\beta}_{0+}$ is the Caputo fractional derivative, $\alpha,\beta\in(0,1)$ such that $0<\alpha+\beta<1$.

In [2], Abbas established the existence of solutions using Krasnoselskii's fixed point theorem for the following delay fractional differential equation

$$\begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}}u\left(t\right)=f\left(t,u\left(t\right),u\left(t-\tau\right)\right),\ t\in\left[0,T\right], \\ u\left(t\right)=\phi\left(t\right),\ t\in\left[-\tau,0\right],\ 0<\alpha<1, \end{cases}$$

where $\dfrac{d^{\alpha}}{dt^{\alpha}}$ denotes the Riemann-Liouville fractional derivative.

In [8], Ardjouni, Boulares and Djoudi showed the asymptotic stability of the zero solution for the following nonlinear fractional differential equation with varying delay

$$\begin{cases} {}^{C}D^{\alpha}_{0+}u\left(t\right)=ku\left(t\right)+f\left(t,u\left(t\right),u\left(t-\tau\left(t\right)\right)\right)+{}^{C}D^{\alpha-1}_{0+}g\left(t,u\left(t-\tau\left(t\right)\right)\right),\ t\geq 0, \\ x'\left(0\right)=0,\ x\left(t\right)=\phi\left(t\right),\ t\in\left[m_0,0\right], \end{cases}$$

where ${}^{C}D^{\beta}_{0+}$ is the Caputo fractional derivative, $k\in\mathbb{R}$, $1<\alpha<2$, ϕ is real function defined on $[m_0,0]$ where $m_0=\inf\limits_{t\in[0,T]}\{t-\tau(t)\}$, and $\tau:\mathbb{R}^{+}\longrightarrow\mathbb{R}^{+}$is the delay variable.

Motivated to the above problems, we consider the following nonlinear delay fractional differential equation with two-orders

$$ {}^{C}D^{\beta}_{0+}\left(p\left(t\right){}^{C}D^{\alpha}_{0+}u\left(t\right)-g\left(t,u\left(t-\tau\left(t\right)\right)\right)\right)=f\left(t,u\left(t\right),u\left(t-\tau\left(t\right)\right)\right),\ t\in\left[0,T\right], \tag{1.1}$$

subject to the initial history condition

$$u\left(t\right)=\phi\left(t\right),\ t\in\left[m_0,0\right], \tag{1.2}$$

where $\alpha,\beta\in(0,1)$ such that $0<\alpha+\beta<1$, p and ϕ are real functions defined respectively on $[0,T]$ and $[m_0,0]$, $m_0=\inf\limits_{t\in[0,T]}\{t-\tau(t)\}$, $\tau:[0,T]\rightarrow\mathbb{R}_{+}$ represent the delay term, $f\colon\ [0,T]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ and $g\colon\ [0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ are given real functions.

In this paper we study the existence of the solution of (1.1)-(1.2) using the Banach fixed point theorem and we establish the four types Ulam stability for this problem.

The paper is organized as follows. In Section 2, we introduce some preliminaries concerning the hypothesis and several lemmas needed throughout this work. In Section 3, we prove the result of existence and uniqueness of solutions by using the Banach fixed point theorem. In section 4, we give and prove our main results on stability. Finally, we give an example to illustrate our results.

2. Preliminaries

In this section, we present some definitions and properties from fractional calculus that used throughout this paper. For more details see [28, 32, 39, 45].

Definition 2.1 ([28])**.** The Riemann-Liouville fractional (arbitrary) integral of order $\alpha > 0$ of the function $f \in L^1([0,T],\mathbb{R})$ is formally defined by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

where Γ is the classical Gamma function.

Definition 2.2 ([28])**.** The Caputo fractional derivative of order $\alpha > 0$ for a given function f on $[0,T]$ is defined by

$$^{C}D_{0+}^{\alpha} f(t) = D_{0+}^{\alpha}\left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \right], \tag{2.1}$$

where $n = [\alpha]+1$, $[\alpha]$ means the integer part of α and D_{0+}^{α} is the Riemann-Liouville fractional derivative operator of order α defined by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds = D^n I_{0+}^{n-\alpha} f(t) \text{ for } t > 0.$$

The Caputo fractional derivative $^{c}D_{0+}^{\alpha} f$ exists for f belonging to $AC^n([0,T],\mathbb{R})$ the space of functions which have continuous derivatives up to order $(n-1)$ on $[0,T]$ such that $f^{(n-1)} \in AC^1([0,T],\mathbb{R})$. $AC^1([0,T],\mathbb{R})$ also denoted $AC([0,T],\mathbb{R})$ is the space of absolutely continuous functions. In this case, Caputo's fractional derivative is defined by

$$^{C}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds = I_{0+}^{n-\alpha} D^n f(t) \text{ for } t > 0.$$

Remark that when $\alpha = n$, we have $^{C}D_{0+}^{n} f(t) = D^n f(t)$.

Lemma 2.3 ([28])**.** *The fractional integration operator is bounded on* $C([0,T],\mathbb{R})$*, in the sense that for each* $f \in C([0,T],\mathbb{R})$ *there exists a positive constant* a *such that*

$$\|I_{0+}^{\alpha} f\|_{\infty} \leq a \|f\|_{\infty}.$$

Furthermore,

$$I_{0+}^{\alpha} t^{\mu} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\mu}, \ \mu > -1, \ \alpha > 0. \tag{2.2}$$

Lemma 2.4 ([28])**.** *Let* $f \in AC^n([0,T],\mathbb{R})$*, then the Caputo fractional derivative of order* $\alpha > 0$ *such that* $n = [\alpha]+1$ *is continuous on* $[0,T]$ *and*

$${}^{C}D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t), \ I_{0+}^{\alpha} {}^{C}D_{0+}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k. \tag{2.3}$$

In particular, when $0 < \alpha \leq 1$ *we have* $I_{0+}^{\alpha} {}^{C}D_{0+}^{\alpha} f(t) = f(t) - f(0)$.

To define four types of Ulam stability, we consider the following fractional differential equation

$${}^{C}D_{0+}^{\alpha} u(t) = f(t, u(t)), \ 0 < \alpha < 1, \ t \in [0,T]. \tag{2.4}$$

Definition 2.5 ([45])**.** The equation (2.4) is said to be Ulam-Hyers stable if there exists a real number $K_f > 0$ such that for each $\epsilon > 0$ and for each $y \in AC([0,T],\mathbb{R})$ solution of the inequality

$$\left|{}^{C}D_{0+}^{\alpha} y(t) - f(t, y(t))\right| \leq \epsilon, \ t \in [0,T], \tag{2.5}$$

there exists a solution $u \in AC([0,T],\mathbb{R})$ of the equation (2.4) with

$$|y(t) - u(t)| \leq K_f \epsilon, \quad t \in [0,T].$$

Definition 2.6 ([45])**.** The equation (2.4) is generalized Ulam-Hyers stable if there exists $\psi \in C([0,T],\mathbb{R}_+)$ with $\psi(0) = 0$ such that for each $\epsilon > 0$ and for each solution $y \in AC([0,T],\mathbb{R})$ of the inequality

$$\left|{}^{C}D_{0+}^{\alpha} y(t) - f(t, y(t))\right| \leq \epsilon, \quad t \in [0,T],$$

there exists a solution $u \in AC([0,T],\mathbb{R})$ of the equation (2.4) with

$$|y(t) - u(t)| \leq \psi(\epsilon), \quad t \in [0,T].$$

Definition 2.7 ([45])**.** The equation (2.4) is Ulam-Hyers-Rassias stable with respect to $\psi \in C([0,T],\mathbb{R}_+)$ if there exists a real number $J_{f,\psi} > 0$ such that for each $\epsilon > 0$ and for each $y \in AC([0,T],\mathbb{R})$ solution of the inequality

$$\left|{}^{C}D_{0+}^{\alpha} y(t) - f(t, y(t))\right| \leq \epsilon\psi(t), \quad t \in [0,T],$$

there exists a solution $u \in AC([0,T],\mathbb{R})$ of the equation (2.4) with

$$|y(t) - u(t)| \leq J_{f,\psi} \epsilon \psi(t), \quad t \in [0,T].$$

Definition 2.8 ([45])**.** The equation (2.4) is generalized Ulam-Hyers-Rassias stable with respect to $\psi \in C([0,T],\mathbb{R}_+)$ if there exists $J_{f,\psi} > 0$ such that for each solution $y \in AC([0,T],\mathbb{R})$ of the inequality

$$\left|{}^{C}D_{0+}^{\alpha} y(t) - f(t, y(t))\right| \leq \psi(t), \quad t \in [0,T],$$

there exists a solution $u \in AC([0,T],\mathbb{R})$ of the equation (2.4) with

$$|y(t) - u(t)| \leq J_{f,\psi}\psi(t), \quad t \in [0,T].$$

Definition 2.9. A function $y \in AC([0,T],\mathbb{R})$ is a solution of the inequality (2.5) if and only if there exists a function $h \in AC([0,T],\mathbb{R})$ such that for every $t \in [0,T]$, $|h(t)| \leq \epsilon$ and ${}^{C}D_{0+}^{\alpha} y(t) = f(t, y(t)) + h(t)$.

Lastly in this section, we state the Banach fixed point theorem which enable us to prove the existence and uniqueness of a solution of (1.1)-(1.2).

Definition 2.10. Let $(X, \|\cdot\|)$ be a Banach space and $A : X \to X$. The operator A is a contraction operator if there is an $\lambda \in (0,1)$ such that $x, y \in X$ imply

$$\|Ax - Ay\| \leq \lambda \|x - y\|.$$

Theorem 2.11 (Banach [40])**.** *Let $\mathcal{K}$ be a nonempty closed convex subset of a Banach space X and $A\colon \mathcal{K} \to \mathcal{K}$ be a contraction operator. Then there is a unique $x \in \mathcal{K}$ with $Ax = x$.*

3. Existence and uniqueness

In this section, we are concerned with the existence of a unique solution for the problem (1.1)-(1.2). Let us start by recalling what we mean by a solution.

Definition 3.1. A function $u \in AC([m_0,T],\mathbb{R})$ is said to be a solution of the initial value problem (1.1)-(1.2) if it satisfies the equation (1.1) on $[0,T]$ and the initial condition (1.2) on the small interval $[m_0, 0]$.

In the sequel, we introduce the following assumptions

(H1) $p \in AC([0,T],\mathbb{R})$ such that $p(t) \neq 0$, $t \in [0,T]$, $p(0) = p_0$, $\phi \in C^1([m_0,0],\mathbb{R})$ with

$$\phi(0) = \phi_0 \text{ and } {}^{C}D_{0+}^{\alpha}\, \phi(0) = \phi_\alpha, \tag{3.1}$$

where ϕ_0, p_0 and ϕ_α are real constants.

(H2) $f\colon [0,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g\colon [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

(H3) There exist positive constants L_1, L_2, L_3 such that for any $t \in [0,T]$ and $u_1, v_1, u_2, v_2 \in \mathbb{R}$, we have

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_1 |u_1 - v_1| + L_2 |u_2 - v_2|,$$

and

$$|g(t, u_1) - g(t, v_1)| \leq L_3 |u_1 - v_1|.$$

(H4) For $q = \inf_{t\in[0,T]} |p(t)|$ with $q \neq 0$, we have

$$k := \left(\frac{L_3}{\Gamma(\alpha+1)} + \frac{L_1 + L_2}{\Gamma(\alpha+\beta+1)} T^\beta \right) \frac{T^\alpha}{q} < 1. \tag{3.2}$$

Now we convert the initial value problem to an integral equation which is also used in the existence and the stability studies. Indeed, we are interested in the solution of the problem (1.1) with (3.1) on the interval $[0,T]$ in view of the supplementary data of u on the interval $[m_0, 0]$.

Lemma 3.2. *Assume that* $(H1)$ *and* $(H2)$ *hold. A function* $u \in C([m_0,T],\mathbb{R})$ *is a solution of the following fractional integral equation for* $t \in [0,T]$

$$\begin{aligned} u(t) &= \phi_0 + (p_0\phi_\alpha - g_0) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)\, p(s)} ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)\, p(s)} g(s, u(s-\tau(s)))\, ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)\, p(s)} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} f(\sigma, u(\sigma), u(\sigma-\tau(\sigma)))\, d\sigma ds, \end{aligned} \tag{3.3}$$

with $g_0 = g(0, u(-\tau(0)))$ *and* $u(t) = \phi(t)$ *for* $t \in [m_0, 0]$ *if and only if* u *is a solution of the two-orders delay fractional initial value problem (1.1)-(1.2).*

Proof. First, we apply ${}^C D^\alpha_{0+}$ to (3.3) and obtain with ${}^C D^\alpha_{0+}\phi_0 = 0$

$${}^C D^\alpha_{0+} u(t) = \frac{p_0\phi_\alpha - g_0}{p(t)} + \frac{g(t, u(t-\tau(t)))}{p(t)} + \frac{1}{p(t)} I^\beta_{0+} f(t, u(t), u(t-\tau(t))).$$

Then, we apply ${}^C D^\beta_{0+}$ to $p(t)^C D^\alpha_{0+} u(t)$ to get (1.1). For $t = 0$, $u(0) = \phi_0$. Furthermore, under $(H1)$ and $(H2)$ we conclude that $u \in AC([m_0,T],\mathbb{R})$.

Conversely, we apply the fractional integral I^β_{0+} to (1.1) to obtain, in view of Lemma 2.4,

$$\begin{aligned} {}^C D^\alpha_{0+} u(t) &= \frac{p(0)^C D^\alpha_{0+} u(0) - g(0, u(-\tau(0)))}{p(t)} + \frac{g(t, u(t-\tau(t)))}{p(t)} \\ &\quad + \frac{1}{p(t)} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s, u(s), u(s-\tau(s)))\, ds. \end{aligned} \tag{3.4}$$

Using the fact that $u \in C([m_0,T],\mathbb{R})$ and $\phi \in C^1([m_0,0],\mathbb{R})$, we obtain

$${}^C D^\alpha_{0+} u(0) = {}^C D^\alpha_{0+} u(t)\big|_{t=0} = {}^C D^\alpha_{0+} \phi(t)\big|_{t=0} = \phi_\alpha.$$

We apply the fractional integral I^α_{0+} to (3.4), we get (3.3). This completes the proof. □

Now, we give the existence and uniqueness result based on the Banach fixed point theorem.

Theorem 3.3. *Assume that* $(H1)-(H4)$ *are satisfied. Then the problem (1.1)-(1.2) has a unique solution.*

Proof. First, we denote by $X = C([m_0, T], \mathbb{R})$ the Banach space of all continuous functions from $[m_0, T]$ into $\mathbb{R}$ with the sup norm $\|u\|_\infty = \sup_{t\in[m_0,T]} |u(t)|$.

Define the operator $A : X \longrightarrow X$ for all $t \in [0, T]$

$$\begin{aligned}(Au)(t) &= \phi_0 + (p_0\phi_\alpha - g_0)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} g(s, u(s-\tau(s)))\,ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} f(\sigma, u(\sigma), u(\sigma-\tau(\sigma)))\,d\sigma ds,\end{aligned}$$

and $(Au)(t) = \phi(t)$ for $t \in [m_0, 0]$. It is clear that the fixed points of the operator A are solutions of the problem (1.1)-(1.2).

Now, we define the nonempty convex closed set of X as follows

$$B_R = \{u \in X : \|u - \phi_0\|_\infty \le R\},$$

such that

$$R \ge \left(\frac{|p_0\phi_\alpha| + |g_0| + c_g}{\Gamma(\alpha+1)} + \frac{c_f}{\Gamma(\alpha+\beta+1)}T^\beta\right)\frac{T^\alpha}{q(1-k)}, \tag{3.5}$$

where $c_f = \sup_{t\in[0,T]} |f(t, \phi_0, \phi_0)|$ and $c_g = \sup_{t\in[0,T]} |g(t, \phi_0)|$.

To show that $AB_R \subset B_R$ for each $u \in B_R$

$$\begin{aligned}&|(Au)(t) - \phi_0| \\ &\le (|p_0\phi_\alpha| + |g_0|)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)|p(s)|}ds \\ &\quad+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)|p(s)|} |g(s, u(s-\tau(s)))|\,ds \\ &\quad+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)|p(s)|} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} |f(\sigma, u(\sigma), u(\sigma-\tau(\sigma)))|\,d\sigma ds. \end{aligned} \tag{3.6}$$

On the other hand, we have

$$
\begin{aligned}
&\left|f\left(t,u\left(t\right),u\left(t-\tau\left(t\right)\right)\right)\right| \\
&\quad\leq \left|f\left(t,u\left(t\right),u\left(t-\tau\left(t\right)\right)\right)-f\left(t,\phi_0,\phi_0\right)\right|+\left|f\left(t,\phi_0,\phi_0\right)\right| \\
&\quad\leq L_1\left|u\left(t\right)-\phi_0\right|+L_2\left|u\left(t-\tau\left(t\right)\right)-\phi_0\right|+\left|f\left(t,\phi_0,\phi_0\right)\right| \\
&\quad\leq L_1\left\|u-\phi_0\right\|_\infty+L_2\left\|u-\phi_0\right\|_\infty+c_f \\
&\quad\leq \left(L_1+L_2\right)R+c_f,
\end{aligned}
$$

By the same technique, we obtain the following estimation

$$
\left|g\left(t,u\left(t-\tau\left(t\right)\right)\right)\right|\leq L_3R+c_g.
$$

Therefore the estimate (3.6) becomes

$$
\begin{aligned}
&\left|\left(Au\right)\left(t\right)-\phi_0\right| \\
&\quad\leq \frac{\left(\left|p_0\phi_\alpha\right|+\left|g_0\right|\right)+L_3R+c_g}{q}\int_0^t\frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}ds \\
&\qquad+\frac{\left(L_1+L_2\right)R+c_f}{q}\int_0^t\frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\int_0^s\frac{\left(s-\sigma\right)^{\beta-1}}{\Gamma\left(\beta\right)}d\sigma ds \\
&\quad\leq \frac{\left(\left|p_0\phi_\alpha\right|+\left|g_0\right|\right)+L_3R+c_g}{\alpha\Gamma\left(\alpha\right)q}t^\alpha+\frac{\left(L_1+L_2\right)R+c_f}{q\Gamma\left(\alpha+\beta+1\right)}t^{\alpha+\beta} \\
&\quad\leq \frac{\left(\left|p_0\phi_\alpha\right|+\left|g_0\right|\right)+L_3R+c_g}{q\Gamma\left(\alpha+1\right)}T^\alpha+\frac{\left(L_1+L_2\right)R+c_f}{q\Gamma\left(\alpha+\beta+1\right)}T^{\alpha+\beta}.
\end{aligned}
$$

We conclude from (3.5) that

$$
\left\|Au-\phi_0\right\|_\infty\leq R.
$$

Then B_R is stable by A. We proceed to prove that A is a contraction mapping. For each $u,v\in B_R$ and for all $t\in[m_0,T]$ we have

$$
\begin{aligned}
&\left|\left(Au\right)\left(t\right)-\left(Av\right)\left(t\right)\right| \\
&\leq \int_0^t\frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)\left|p\left(s\right)\right|}\left|g\left(s,u\left(s-\tau\left(s\right)\right)\right)-g\left(s,v\left(s-\tau\left(s\right)\right)\right)\right|ds \\
&\quad+\int_0^t\frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)\left|p\left(s\right)\right|}\int_0^s\frac{\left(s-\sigma\right)^{\beta-1}}{\Gamma\left(\beta\right)} \\
&\quad\times\left|f\left(\sigma,u\left(\sigma\right),u\left(\sigma-\tau\left(\sigma\right)\right)\right)-f\left(\sigma,v\left(\sigma\right),v\left(\sigma-\tau\left(\sigma\right)\right)\right)\right|d\sigma ds.
\end{aligned}
$$

By using $(H3)$, $(H4)$ and (2.2), we get

$$\begin{aligned}
&\left|(Au)(t)-(Av)(t)\right| \\
&\leq \frac{L_3}{q\Gamma(\alpha)}\|u-v\|_\infty \int_0^t (t-s)^{\alpha-1}\,ds \\
&\quad + \left(\frac{L_1+L_2}{q\Gamma(\beta)}\right)\|u-v\|_\infty \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s (s-\sigma)^{\beta-1}\,d\sigma ds \\
&= \left(\frac{L_3}{q\Gamma(\alpha+1)}t^\alpha + \left(\frac{L_1+L_2}{q\Gamma(\beta+1)}\right) I_{0+}^{\alpha}\left(t^\beta\right)\right)\|u-v\|_\infty \\
&= \left(\frac{L_3}{q\Gamma(\alpha+1)}t^\alpha + \frac{L_1+L_2}{q\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}\right)\|u-v\|_\infty \\
&\leq \left(\frac{L_3}{\Gamma(\alpha+1)} + \frac{L_1+L_2}{\Gamma(\alpha+\beta+1)}T^\beta\right)\frac{T^\alpha}{q}\|u-v\|_\infty .
\end{aligned}$$

Thus,

$$\|Au-Av\|_\infty \leq k\|u-v\|_\infty .$$

Then, A is a contraction by (3.2). The conclusion of the theorem follows by the Banach fixed point theorem. This completes the proof. □

4. Ulam stability

In this section, we study four types of Ulam stability of the problem (1.1)-(1.2) which are Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stabilities.

Lemma 4.1. *If $y \in AC([m_0,T],\mathbb{R})$ is a solution of the fractional differential inequality for each $\epsilon > 0$*

$$\left|{}^C D_{0+}^{\beta}\left(p(t)^C D_{0+}^{\alpha} y(t) - g(t,y(t-\tau(t)))\right) - f(t,y(t),y(t-\tau(t)))\right| \leq \epsilon, \quad (4.1)$$

and the initial condition (1.2) then y is a solution of the following integral inequality

$$\begin{aligned}
&\left| y(t) - \phi_0 - (p_0\phi_\alpha - g_0)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}ds \right.\\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} g(s,y(s-\tau(s)))\,ds \\
&\left. - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} f(\sigma,y(\sigma),y(\sigma-\tau(\sigma)))\,d\sigma ds \right| \\
&\leq \frac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)}\epsilon.
\end{aligned}$$

Proof. Let $y \in AC([m_0, T], \mathbb{R})$ be a solution of the inequality (4.1) and (1.2) for each $\epsilon > 0$. Then, from Definition 2.9 and Lemma 3.2 for some continuous function h such that $|h(t)| \leq \epsilon$, $t \in [0, T]$, we have

$$y(t) = \phi_0 + (p_0\phi_\alpha - g_0)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}ds$$

$$+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} g(s, y(s-\tau(s)))\,ds$$

$$+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} [f(\sigma, y(\sigma), y(\sigma-\tau(\sigma))) + h(\sigma)]\,d\sigma ds.$$

Then, we use the properties of I_{0+}^{α} to get

$$\left| I_{0+}^{\alpha}\left(\frac{1}{p(t)} I_{0+}^{\beta} h(t)\right)\right| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)|p(s)|} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} |h(\sigma)|\,d\sigma ds$$

$$\leq \epsilon \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} d\sigma ds$$

$$\leq \frac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)}\epsilon.$$

The proof is complete. □

Theorem 4.2. *Assume that the assumptions* $(H1)-(H4)$ *hold. Then the problem (1.1)-(1.2) is Ulam-Hyers stable.*

Proof. Under $(H1)-(H4)$, (1.1)-(1.2) has a unique solution in $AC([m_0, T], \mathbb{R})$. Let $y \in AC([m_0, T], \mathbb{R})$ be a solution of the inequality (4.1) and (1.2), then for each $t \in [m_0, T]$

$$
\begin{aligned}
&|y(t) - u(t)| \\
&\leq \left| y(t) - \phi_0 - (p_0\phi_\alpha - g_0) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) p(s)} ds \right. \\
&\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) p(s)} g(s, y(s - \tau(s))) ds \\
&\quad \left. - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) p(s)} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} f(\sigma, y(\sigma), y(\sigma - \tau(\sigma))) d\sigma ds \right| \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) |p(s)|} |g(s, y(s - \tau(s))) - g(s, u(s - \tau(s)))| ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) |p(s)|} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} \\
&\quad \times |f(\sigma, y(\sigma), y(\sigma - \tau(\sigma))) - f(\sigma, u(\sigma), u(\sigma - \tau(\sigma)))| d\sigma ds \\
&\leq \frac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)} \epsilon + \left(\frac{L_3}{\Gamma(\alpha+1)} + \frac{L_1 + L_2}{\Gamma(\alpha+\beta+1)} T^\beta \right) \frac{T^\alpha}{q} \|y - u\|_\infty .
\end{aligned}
$$

Thus, in view of $(H4)$

$$
\|y - u\|_\infty \leq \frac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)(1-k)} \epsilon.
$$

Then, there exists a real number $K_f = \dfrac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)(1-k)} > 0$ such that

$$
|y(t) - u(t)| \leq K_f \epsilon. \tag{4.2}
$$

Thus (1.1)-(1.2) has the Ulam-Hyers stability, which completes the proof. □

Corollary 4.3. *Suppose that all the assumptions of Theorem 4.2 are satisfied. Then the problem (1.1)-(1.2) is generalized Ulam-Hyers stable.*

Proof. Let $\psi(\epsilon) = K_f \epsilon = \dfrac{T^{\alpha+\beta}}{q\Gamma(\alpha+\beta+1)(1-k)} \epsilon$ in (4.2) then $\psi(0) = 0$ and problem (1.1)-(1.2) is generalized Ulam-Hyers stable. □

In the next, we introduce the following hypothesis

(H5) $\psi \in C([0, T], \mathbb{R})$ an increasing function which satisfies the property

$$
I_{0^+}^\gamma \psi(t) \leq \lambda_{\psi,\gamma} \psi(t), \ 0 < \gamma < 1,
$$

for some constant $\lambda_{\psi,\gamma} > 0$.

Lemma 4.4. *Assume that ψ satisfies $(H5)$. If $y \in AC([m_0, T], \mathbb{R})$ is a solution of the inequality for each $\epsilon > 0$*

$$\left| {}^C D_{0+}^{\beta} \left(p(t) {}^C D_{0+}^{\alpha} y(t) - g(t, y(t - \tau(t))) \right) - f(t, y(t), y(t - \tau(t))) \right| \leq \epsilon \psi(t), \tag{4.3}$$

and the initial condition (1.2) for $t \in [m_{0,} 0]$, then y is a solution of the following integral inequality

$$\begin{aligned} &\left| y(t) - \phi_0 - (p_0 \phi_\alpha - g_0) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) p(s)} ds \right. \\ &\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) p(s)} g(s, y(s - \tau(s))) ds \\ &\quad \left. - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) p(s)} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} f(\sigma, y(\sigma), y(\sigma - \tau(\sigma))) d\sigma ds \right| \\ &\leq \frac{\lambda_\psi^2}{q} \epsilon \psi(t). \end{aligned} \tag{4.4}$$

Proof. Let $y \in AC([m_0, T], \mathbb{R})$ be a solution of the inequality (4.3) and (1.2) for each $\epsilon > 0$. From Definition 2.9 and Lemma 3.2, for some continuous function h such that $|h(t)| \leq \epsilon \psi(t)$ for each $\epsilon > 0$, $t \in [0, T]$, we have

$$\begin{aligned} &\left| y(t) - \phi_0 - (p_0 \phi_\alpha - g_0) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) p(s)} ds \right. \\ &\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) p(s)} g(s, y(s - \tau(s))) ds \\ &\quad \left. - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) p(s)} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} f(\sigma, y(\sigma), y(\sigma - \tau(\sigma))) d\sigma ds \right| \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) |p(s)|} \int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} |h(\sigma)| d\sigma ds \\ &\leq \epsilon \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) |p(s)|} I_{0+}^{\beta} (\psi(s)) ds \leq \epsilon \lambda_{\psi,\beta} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha) |p(s)|} \psi(s) ds \\ &\leq \frac{\lambda_{\psi,\beta} \lambda_{\psi,\alpha}}{q} \epsilon \psi(t) \leq \frac{\lambda_\psi^2}{q} \epsilon \psi(t), \end{aligned}$$

where $\lambda_\psi = \max(\lambda_{\psi,\beta}, \lambda_{\psi,\alpha})$. This completes the proof. □

Theorem 4.5. *Assume that the assumptions $(H1) - (H5)$ hold, then the problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to ψ.*

Proof. Under $(H1)-(H4)$, (1.1)-(1.2) has a unique solution in $AC([m_0,T],\mathbb{R})$. Let $y\in AC([m_0,T],\mathbb{R})$ be a solution of the inequality (4.3) and (1.2), then for each $t\in[m_0,T]$

$$\begin{aligned}
&\left|y(t)-u(t)\right| \\
&\leq \Bigg| y(t)-\phi_0-(p_0\phi_\alpha-g_0)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}ds \\
&\quad -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)} g(s,u(s-\tau(s)))\,ds \\
&\quad -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} f(\sigma,u(\sigma),u(\sigma-\tau(\sigma)))\,d\sigma ds\Bigg| \\
&\leq \frac{\lambda_\psi^2}{q}\epsilon\psi(t)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)\left|p(s)\right|}\left|g(s,y(s-\tau(s)))-g(s,u(s-\tau(s)))\right|ds \\
&\quad +\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)\left|p(s)\right|}\int_0^s \frac{(s-\sigma)^{\beta-1}}{\Gamma(\beta)} \\
&\quad \times\left|f(\sigma,y(\sigma),y(\sigma-\tau(\sigma)))-f(\sigma,u(\sigma),u(\sigma-\tau(\sigma)))\right|d\sigma ds \\
&\leq \frac{\lambda_\psi^2}{q}\epsilon\psi(t)+\left(\frac{L_3}{\Gamma(\alpha+1)}+\frac{L_1+L_2}{\Gamma(\alpha+\beta+1)}T^\beta\right)\frac{T^\alpha}{q}\left\|y-u\right\|_\infty.
\end{aligned}$$

Hence, it follows that there exists a real number $J_{f,\psi}=\dfrac{\lambda_\psi^2}{q(1-k)}>0$ such that

$$\left|y(t)-u(t)\right|\leq \frac{\lambda_\psi^2}{q(1-k)}\epsilon\psi(t)=J_{f,\psi}\epsilon\psi(t),\ t\in[m_0,T].$$

This gives the wanted result and completes the proof. □

Corollary 4.6. *Under the hypothesis of Theorem 4.5, the problem (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable with respect to* $\psi\in C([0,T],\mathbb{R})$.

Proof. Set $\epsilon=1$ and $J_{f,\psi}=\dfrac{\lambda_\psi^2}{q(1-k)}$ it directly follows that the problem (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable. □

5. Example

Consider the following nonlinear delay problem

$$\begin{cases} {}^CD_{0^+}^{\frac{1}{3}}\left(\frac{1}{t+1}\,{}^CD_{0^+}^{\frac{1}{2}}u(t)-\frac{t}{10}\sin\left(u\left(t-t^2-1/4\right)\right)\right) \\ =\dfrac{t}{30(t+1)}\cos u\left(t-t^2-1/4\right)+\dfrac{1}{20}\sin u(t),\ t\in[0,1], \\ u(t)=e^t,\ t\in[-0.25,0], \end{cases} \tag{5.1}$$

where

$$\alpha = \frac{1}{2},\ \beta = \frac{1}{3},\ p(t) = \frac{1}{t+1},\ \tau(t) = t^2 + 1/4,$$
$$g\left(t, u\left(t - t^2 - 1/4\right)\right) = \frac{t}{10}\sin\left(u\left(t - t^2 - 1/4\right)\right),\ \phi(t) = e^t,$$
$$f\left(t, u(t), u\left(t - t^2 - 1/4\right)\right) = \frac{t}{30(t+1)}\cos u\left(t - t^2 - 1/4\right) + \frac{1}{20}\sin u(t).$$

The unique solution exists for

$$L_1 = \frac{1}{20}, \quad L_2 = \frac{1}{30}, \quad L_3 = \frac{1}{10}, \quad q = \frac{1}{2},$$

satisfying the condition (3.2)

$$k = \left(\frac{L_3}{\Gamma(\alpha+1)} + \frac{(L_1 + L_2)}{\Gamma(\alpha+\beta+1)}T^{\beta}\right)\frac{T^{\alpha}}{q}$$
$$= 0.4028 < 1.$$

It follows from Theorem 4.2 that the problem (5.1) is Ulam-Hyers stable on $[-0.25, 1]$. Also, by Corollary 4.3, the generalized Ulam-Hyers stability is obtained.

Now, we choose $\psi(t) = t$ which satisfies $(H5)$ and in view of (2.2) we have

$$I_{0^+}^{\gamma}\psi(t) = \frac{\Gamma(2)}{\Gamma(\gamma+2)}t^{\gamma+1} \leq \frac{1}{(\gamma+1)\Gamma(\gamma+1)}t,\ \ 0 < \gamma < 1.$$

For $\alpha = \dfrac{1}{2}$ and $\beta = \dfrac{1}{3}$ we have

$$\lambda_{\psi,\alpha} = \frac{1}{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)} = 0.7447, \quad \lambda_{\psi,\beta} = \frac{1}{\frac{4}{3}\Gamma\left(\frac{1}{3}+1\right)} = 0.8398,$$

then we take $\lambda_{\psi} = 0.8398$ to get (4.4) satisfied. Hence, by Theorem 4.5, the problem (5.1) is Ulam-Hyers-Rassias stable with respect to ψ and by Corollary 4.6, it is generalized Ulam-Hyers-Rassias stable with respect to ψ.

Remark 5.1. In the case where $\alpha + \beta \geq 1$, we distinguish four cases ($\alpha \geq 1$ and $0 < \beta < 1$), ($\beta \geq 1$ and $0 < \alpha < 1$), ($\alpha \geq 1$ and $\beta \geq 1$), ($\alpha \leq 1$ and $\beta \leq 1$), which requires to equipped the problem by two initials history conditions. Furthermore, according to (2.3), the integral equation will be changed but the Lipshitz constant k remains the same and we will have the same results concerning the Ulam types stability.

6. Discussion

We can consider the following neutral (DFDE)

$$^{C}D_{0^{+}}^{\beta}\left(p\left(t\right)^{C}D_{0^{+}}^{\alpha}\left[u\left(t\right)-g\left(t,u\left(t-\tau\left(t\right)\right)\right)\right]\right)=f\left(t,u\left(t\right),u\left(t-\tau\left(t\right)\right)\right),t\in\left[0,T\right],$$

subject to the initial history condition

$$u\left(t\right)=\phi\left(t\right),\ t\in\left[m_{0},0\right].$$

The theory of neutral ($\mathrm{DFDE_s}$) is even more complicated than the theory of non-neutral ($\mathrm{DFDE_s}$). In the case of the above neutral ($\mathrm{DFDE_s}$), we can use another technique to establish the existence of solution, for example Krasnoselskii's fixed point theorem. Also, we can consider the same problem of (1.1)-(1.2) involving Hadamard fractional derivatives or Prabhakar derivatives which might be the interesting object of research.

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