FAQ on the g-theorem and the hard Lefschetz theorem for face rings

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Abstract. *We discuss the hard Lefschetz theorem for simplicial spheres, as well as the theory at its core: perturbations of maps, biased Poincaré pairings and a cobordism argument that relates the Lefschetz property of a manifold to the Lefschetz property on its boundary, and sketch an alternative argument based on edge-contractions.*

1. Counting

One of the most basic problems that almost every combinatorialist asks is to count things. And when you ask a combinatorial topologist, they usually want to count things associated to topological spaces.

More precisely: many topological spaces can be triangulated, that is, there are simplicial complexes that encode them. A tetrahedron, for instance, triangulates a 3-dimensional ball. The boundary of that tetrahedron triangulates the 2-dimensional sphere. But so does the boundary of the icosahedron. Now we can count the number of faces that each of these triangulations have:

In the former case, we get one empty face, four vertices, six edges, and four triangles. Nice and symmetric (well, almost). This is usually recorded in a vector called the f-vector recording the number of i-dimensional faces f_i, which reads in this case as

$$(1, 4, 6, 4).$$

In the latter, we get one empty face, twelve vertices, thirty edges, and twenty triangles (where it gets its name), and hence an f-vector

$$(1, 12, 30, 20).$$

This is no longer so nice and symmetric. However, Sommerville [14] had the idea to define another vector

$$h_k := \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

And now, there is a small miracle: he then established[1] that for a simplicial sphere of dimension $(d-1)$, we have

$$h_k = h_{d-k}.$$

These are the so-called Dehn-Sommerville relations. This means in particular that there are nontrivial linear relations between the face numbers of simplicial polytopes, and that everything is defined from the first half of the entries. So, McMullen [11] had the idea to consider another vector:

$$g_k := h_k - h_{k-1} \text{ for } k \leq \frac{d}{2}$$

He formulated the following conjecture

Conjecture 1.1. *A vector of d integers is the f-vector of a simplicial sphere Σ if and only if the associated g-vector is an M-sequence, that is, there is a quotient Q of a polynomial ring $\mathbb{R}[\mathbf{x}]$ by a homogeneous ideal so that*

$$g_i(\Sigma) = \dim Q^i.$$

At this point, Stanley entered. He realized that there is at least always a ring that encodes the *h*-vector.

2. Rings

If Δ is an abstract simplicial complex defined on the groundset $[n] := \{1, \cdots, n\}$, let $I_\Delta := \langle \mathbf{x^a} : \text{supp}(\mathbf{a}) \notin \Delta \rangle$ denote the nonface ideal in $\mathbb{R}[\mathbf{x}]$, where $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \cdots, x_n]$. Let $\mathbb{R}^*[\Delta] := \mathbb{R}[\mathbf{x}]/I_\Delta$ denote the **face ring** of Δ. Now, we pick a sufficient number of linear forms to make sure the quotient is finite dimensional:

We may associate to the vertices of Δ the coordinates $\mathbf{V}_\Delta = (v_1, \cdots, v_n) \in \mathbb{R}^{d \times n}$, obtaining a system of linear forms by considering $\mathbf{V}_\Delta \mathbf{x} = \Theta$. With this, we obtain a **geometric simplicial complex** in $\mathbb{R}^d$.

The face ring of a geometric simplicial complex Δ is considered with respect to its natural system of parameters induced by the coordinates, that is,

$$\mathscr{A}^*(\Delta) := \mathbb{R}^*[\Delta]/\Theta\mathbb{R}^*[\Delta].$$

A geometric simplicial complex in $\mathbb{R}^d$ is **proper** if the image of every *k*-face, with $k < d$, linearly spans a subspace of dimension $k+1$. If Δ is of dimension $(d-1)$, and is given a proper coordinates in $\mathbb{R}^d$, then $\mathscr{A}^*(\Delta)$ is finite-dimensional as a vector space.

Stanley observed [15], based on a theorem of Reisner [13]:

Theorem 2.1. *For a triangulated sphere Σ of dimension $(d-1)$,*

$$h_i(\Sigma) = \dim \mathscr{A}^i(\Sigma).$$

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[1]He stated this for boundaries of simplicial polytopes, but the proof works for simplicial spheres

3. Stanley and Lefschetz

Here, Stanley observed [16] that McMullen's conjecture is true if there existed an ℓ in $\mathscr{A}^1[\Sigma]$ so that

$$\mathscr{A}^i(\Sigma) \xrightarrow{\cdot \ell} \mathscr{A}^{i+1}(\Sigma)$$

is injective for $i \leq \frac{d}{2} - 1$, or stronger if

$$\mathscr{A}^k(\Sigma) \xrightarrow{\cdot \ell^{d-2k}} \mathscr{A}^{d-k}(\Sigma).$$

is an isomorphism for every $k \leq \frac{d}{2}$. The former is known as the **weak Lefschetz property**, the latter as the **hard Lefschetz property**. He needed this for *some* geometric realization of the simplicial complex.

And amazingly, Stanley then observed that the hard Lefschetz property is actually true for spheres that arise as boundaries of simplicial polytopes (with respect to their given geometric realization) using deep results in algebraic geometry. Specifically, if Σ is realized as the boundary of a polytope, then the class of a convex function acts as the desired Lefschetz element (we refer to [6] for an excellent exposition). But the general case remained open, and only recently, the hard Lefschetz theorem was proven in this generality. It is useful to state this in a more general perspective for the purposes of this paper:

A **relative simplicial complex** $\Psi = (\Delta, \Gamma)$ is a pair of simplicial complexes Δ, Γ with $\Gamma \subset \Delta$. If $\Psi = (\Delta, \Gamma)$ is a relative simplicial complex, then we can define the **relative face module**

$$\mathbb{R}^*[\Psi] \; := \; {}^{I_\Gamma}\Big/ I_\Delta.$$

The relative face ring of a geometric complex is then defined as

$$\mathscr{A}^*(\Psi) \; := \; \mathbb{R}^*[\Psi]/\Theta\mathbb{R}^*[\Psi].$$

Theorem 3.1. *Consider a triangulated $(d-1)$-sphere or ball Δ, and the associated graded commutative face ring $\mathbb{R}[\Delta]$. Then there exists an open dense subset of the Artinian reductions $\mathscr{R}$ of $\mathbb{R}[\Delta]$ and an open dense subset $\mathscr{L} \subset \mathscr{A}^1(\Delta)$, where $\mathscr{A}(\Delta) \in \mathscr{R}$, such that for every $k \leq {}^d\!/_2$, we have the hard Lefschetz property for every $\mathscr{A}(\Delta) \in \mathscr{R}$ and every $\ell \in \mathscr{L}$:*

$$\mathscr{A}^k(\Delta, \partial\Delta) \xrightarrow{\cdot \ell^{d-2k}} \mathscr{A}^{d-k}(\Delta)$$

is an isomorphism.

There are two proofs of this theorem: one is detailed in [2], the other, slightly different and relying on edge contractions, is sketched here. However, their core, the theory of biased pairings, is the same, and is what will be presented here. We want to give a brief overview over the key ideas of these proofs. Up until the final two sections, they are essentially the same, with some variations in generality.

Let us first recall a basic ingredient to the inductive structure of the proofs: Two classical lemmas that help tremendously in reducing from higher dimensional cases to lower-dimensional ones.

4. Two cone lemmas

Recall that the **star** and **link** of a face σ in Δ are the subcomplexes

$$\mathrm{st}_\sigma \Delta \; \coloneqq \; \{\tau : \exists \tau' \supset \tau, \; \sigma \subset \tau' \in \Delta\} \;\; \text{and} \;\; \mathrm{lk}_\sigma \Delta \; \coloneqq \; \{\tau \setminus \sigma : \sigma \subset \tau \in \Delta\}.$$

For geometric simplicial complexes Δ, we shall think of the star of a face as a geometric subcomplex of Δ, and the link of a face σ as the geometric simplicial complex obtained by the orthogonal projection to $\mathrm{span}(\sigma)^\perp$. Let us denote the **deletion** of σ by $\Delta - \sigma$, the maximal subcomplex of Δ that does not contain σ. Let

$$\mathrm{st}_\sigma^\circ \Delta \; \coloneqq \; (\mathrm{st}_\sigma \Delta, \mathrm{st}_\sigma \Delta - \sigma).$$

We have the following two elementary lemmas.

Lemma 4.1 (Cone lemma I, see [9, Thm. 7])**.** *For any vertex $v \in \Delta$, where Δ is a geometric simplicial complex in $\mathbb{R}^d$, and for any integer k, we have an isomorphism*

$$\mathscr{A}^k(\mathrm{lk}_v \Delta) \; \cong \; \mathscr{A}^k(\mathrm{st}_v \Delta).$$

Lemma 4.2 (Cone lemma II, see [1, Lem. 3.3])**.** *In the situation of the first cone lemma we have a natural isomorphism*

$$\mathscr{A}^k(\mathrm{st}_v \Delta) \; \xrightarrow{\; \cdot x_v \;} \; \mathscr{A}^{k+1}(\mathrm{st}_v^\circ \Delta).$$

We come to the first core principle, a key new idea for the proof of the hard Lefschetz property.

5. Generic combinations of linear maps

First is the idea to construct the map ℓ iteratively. We rely on the following principle:

Lemma 5.1 (Lemma 6.1, [2])**.** *Given two linear maps*

$$A, B \colon \mathscr{X} \; \longrightarrow \; \mathscr{Y}$$

of two vector spaces $\mathscr{X}$ and $\mathscr{Y}$ over $\mathbb{R}$. Assume that

$$B(\ker A) \; \cap \; \mathrm{im} A \; = \; 0 \; \subset \; \mathscr{Y}.$$

Then a generic linear combination $A\text{``}+\text{''}B$ of A and B has kernel

$$\ker(A \text{ ``}+\text{'' } B) \; = \; \ker A \; \cap \; \ker B.$$

This is simple linear algebra, but at first sight, may feel a little unnatural. However, it is exactly fitting for the spaces we consider. As it turns out, there is a perfect pairing [7, 17], the Poincaré pairing,

$$\mathscr{A}^k(\Delta, \partial\Delta) \ \times \ \mathscr{A}^{d-k}(\Delta) \ \longrightarrow \ \mathscr{A}^d(\Delta, \partial\Delta) \ \cong \ \mathbb{R}.$$

Now, we notice that $\ker \ell^{d-2k}$ and $\operatorname{im} \ell^{d-2k}$ are orthogonal complements for whatever ℓ we choose. Lets assume $d - 2k = 1$ for now, and look at the map

$$\mathscr{A}^k(\Delta, \partial\Delta) \ \xrightarrow{\ \cdot\ell\ } \ \mathscr{A}^{k+1}(\Delta)$$

with ℓ to be constructed.

Stay with me: Say now we wish to construct ℓ as a Lefschetz map inductively. Then we can start with a vertex v to begin with, and take the associated element x_v. This is usually not good enough to induce an isomorphism, but it has controllable kernel and image. In each and every step, we now pick another vertex (say w), take the associated element (x_w), and generically combine it with what we have (say ℓ'). Then to make sure that we have an isomorphism at the end, or equivalently an injection, it is good if the kernel of this generic combination is as small as possible.

The ideal case is, if ℓ' is supported in vertices W, that

$$\ker \ell' \ = \ \ker \text{``}\sum_{v \in W}\text{''} \, x_v \ = \ \bigcap_{v \in W} \ker x_v$$

This is called the **transversal prime property** for W. Notice that it is equivalent to saying that

$$\operatorname{im} \ell' \ = \ \operatorname{im} \text{``}\sum_{v \in W}\text{''} \, x_v \ = \ \operatorname{span}\{\ker x_v : v \in W\}.$$

If we want to prove the analogue for ℓ' "+" x_w, then the lemma tells us what to do: show that

$$B(\ker A) \ \cap \ \operatorname{im} A \ = \ 0 \ \subset \ \mathscr{Y}$$

where A is the multiplication with ℓ', and B is the multiplication with x_w.

Now comes the cool part: Multiplication with x_w is the same as pullback to the link of w, which is again a disk or a sphere, depending on whether w is in the interior of Δ or on the boundary. Hence we are trying to show that two orthogonal complements intersect trivially. Which is equivalent to saying that the Poincaré pairing is nondegenerate on one (and equivalently both) of these spaces.

Hence, we want to show that $x_w \ker \ell'$, seen as subspace in $\mathscr{A}^*(\operatorname{lk}_w \Delta, \operatorname{lk}_w \partial\Delta)$ via the cone lemmas, has a perfect Poincaré pairing induced by the Poincaré pairing

$$\mathscr{A}^k(\operatorname{lk}_w \Delta, \operatorname{lk}_w \partial\Delta) \times \mathscr{A}^{d-k-1}(\operatorname{lk}_w \Delta) \ \longrightarrow \ \mathbb{R}.$$

This is the notion of biased Poincaré duality, the second key part of the proofs. We shall introduce it after an example.

6. An example

As it turns out, this property is quite controllable again, by reducing it to Lefschetz isomorphisms. Lets explain how in an example, following [2, Section 6.5]: Assume we are in a sphere Σ of dimension $d - 1 = 2k$, and have shown that

(1) There exists a set of vertices W such that $\Sigma - W$ is a disk Δ and
(2) We have the **transversal prime property** for W in Σ, that is, the kernel of $\ell' = $ "$\sum_{v \in W}$"x_v is exactly $\mathscr{A}^k(\Delta, \partial\Delta)$, the intersection of the kernels of the $x_v, v \in W$.

Lets pick w a vertex in the boundary of Δ. We wish to establish that ℓ'" + "x_w has kernel $\mathscr{A}^k(\Delta - w, \partial(\Delta - w))$.

Then, as we observed above, we wish to understand what to do to ensure that the Poincaré pairing of $\mathscr{A}^k(\mathrm{st}_w\Sigma)$ does not degenerate when restricting to the pullback of $\mathscr{A}^k(\Delta, \partial\Delta)$.

On the other hand, we also have a short exact sequence

$$\mathscr{A}^k(\mathrm{st}_w\Delta, \mathrm{st}_w\partial\Delta) \xrightarrow{\cdot x_w} \mathscr{A}^{k+1}(\Delta, \Delta - w) \longrightarrow \mathscr{A}^{k+1}(\partial\Delta, \partial\Delta - w) \longrightarrow 0\,.$$

Hence, the image of the multiplication with x_w of the kernel of ℓ' is $\mathscr{A}^{k+1}(\Delta, \Delta - w)$, and equivalently the kernel of the multiplication with x_w is $\mathscr{A}^k(\Delta - w, \partial\Delta - w)$ if and only if

$$\mathscr{A}^{k+1}(\partial\Delta, \partial\Delta - w) = \mathscr{A}^k(\mathrm{lk}_w\partial\Delta) = 0.$$

Where is the Lefschetz theorem though?

Well π is a projection of $\mathbb{R}^d/w$ to a hyperplane, and h is the coordinate with respect to that projection, $\theta = h \cdot \mathbf{x}$ the associated linear form, then the last condition is equivalent to the Lefschetz property in degree $k - 1$: For the map

$$\mathscr{A}^{k-1}(\pi\mathrm{lk}_w\partial\Delta) \xrightarrow{\cdot\theta} \mathscr{A}^k(\pi\mathrm{lk}_w\partial\Delta)$$

to be an isomorphism is exactly equivalent to $\mathscr{A}^k(\mathrm{lk}_w\partial\Delta) = 0$.

Let us stop for a second, because without noticing, we have now the core of the entire argument: We have seen that, if we can show that restrictions of the Poincaré to certain ideals are non-degenerate, we can construct Lefschetz elements. Now we have seen also a glimpse of the other direction: that lower-dimensional cases of the Lefschetz property imply non-degeneracy of such restrictions.

Unfortunately, this does not give a complete proof yet: We need to make sure that we can remove vertices iteratively so that the deletions remain disks. Also, we need to make sure that the same happens to the $\mathrm{lk}_w\partial\Delta$. This is a rather strong restriction, and introduced as *L*-decomposability in [2] (somewhat different from what Billera and Provan introduced originally [5], which misses the condition on $\mathrm{lk}_w\partial\Delta$). Still, not all spheres can have this property, see for instance [10].

For the sake of clarity, we call a sphere or disk Δ *B*-**decomposable** if it is a simplex or there exists a vertex w such that $\Delta - w$ is *B*-decomposable of the same dimension. We call it *A*-**decomposable** if the deletion is also *A*-decomposable,

and $\mathrm{lk}_w \partial \Delta$ is **A-decomposable** as well. We also point out the relation to work of Murai and Babson-Nevo: A-decomposable spheres are strongly edge-decomposable in the sense of [3, 12].

So, to summarize, we have understood how to prove the hard Lefschetz theorem for A-decomposable spheres. Though if we knew the Lefschetz theorem for $\mathrm{lk}_w \partial \Delta$ for any other reason (say, induction on the dimension), B-decomposability would be enough.

7. Pairings

7.1. Biased pairing

We noted in the last example that the Lefschetz theorem is important for the induction. But we also noted that a property of the Poincaré pairing is important. And as we saw, these are in fact equivalent. Precisely, consider a triangulated $(d-1)$-ball or sphere Δ in $\mathbb{R}^d$. We say that it satisfies the **biased pairing property** (with respect to its boundary) if the map

$$\mathscr{A}^k(\Delta, \partial \Delta) \ \longrightarrow \ \mathscr{A}^k(\Delta)$$

is an injection for all degrees $k \leq \frac{d}{2}$, or equivalently, that the pairing

$$\mathscr{A}^k(\Delta, \partial \Delta) \ \times \ \mathscr{A}^{d-k}(\Delta, \partial \Delta) \ \longrightarrow \ \mathscr{A}^d(\Delta, \partial \Delta) \ \cong \ \mathbb{R}$$

is perfect. Note that this property is trivial for spheres. If $k = \frac{d}{2}$, then this is equivalent to the middle Lefschetz isomorphism on the boundary of the disk (see [2, Proposition 5.8]), explaining the two different perspectives above. In other words, identifying what is necessary to do to show the non-degeneracy of the Poincaré product on kernel and image is easy in the case of B-decomposability.

7.2. Biased Poincaré duality

This is a special case of biased Poincaré duality introduced in [2, Section 5]. This, in essence, has the goal of extending the reach of the proof above: Instead of requiring that we look at disks in each step, we want to understand what happens in other cases to the pullbacks of kernel and image of the previous map.

Let Σ be a sphere of dimension $(d-1)$, with Poincaré pairing

$$\mathscr{A}^k(\Sigma) \ \times \ \mathscr{A}^{d-k}(\Sigma) \ \longrightarrow \ \mathbb{R} \,.$$

We say that an ideal I in $\mathscr{A}(\Sigma)$ satisfies **biased Poincaré duality**, if this pairing, restricted to I, is non-degenerate for $k \leq \dfrac{d}{2}$. A thing to note:

If Δ is a disk in Σ of the same dimension, and Γ the complementary disk, then biased Poincaré duality for the ideal

$$\ker[\mathscr{A}(\Sigma) \ \longrightarrow \ \mathscr{A}(\Gamma)]$$

is the biased pairing property for Δ. And as we saw in the previous section, the biased pairing property boils down to a Lefschetz theorem on $\partial\Gamma$.

Such ideals, obtained as kernels $\mathscr{I}(X,\Sigma)$ of

$$\mathscr{A}(\Sigma) \ \longrightarrow \ \mathscr{A}(X)$$

where X is any subcomplex, are tremendously important. We also say therefore that biased Poincaré duality is satisfied at X if and only if it is satisfied with respect to the associated ideal. And once again, we shall see that this is related to Lefschetz properties.

7.3. Lefschetz and Hall-Laman

Let us finally come to the strongest notion: The biased pairing property is a weaker form of the hard Lefschetz property stated before, which we recall applies if with respect to some ℓ in $\mathbb{R}^1(\Delta)$

$$\mathscr{A}^k(\Delta,\partial\Delta) \ \xrightarrow{\ \cdot\ell^{d-2k}\ } \ \mathscr{A}^{d-k}(\Delta)$$

is an isomorphism for all $k \leq \frac{d}{2}$. This map is naturally factored as

$$\mathscr{A}^k(\Delta,\partial\Delta) \ \xrightarrow{\ \cdot\ell^{d-k}\ } \ \mathscr{A}^{d-k}(\Delta,\partial\Delta) \ \longrightarrow \ \mathscr{A}^{d-k}(\Delta)\,.$$

This is a special case of the Hall-Laman relations of [2], but enough for our purposes.

8. Cones and boundaries

Next, the proof of the hard Lefschetz theorem is reduced to the **middle case** when

$$k \ = \ \frac{d}{2}.$$

Note that in this case, both properties coincide.

To this end, assume we wish to prove the hard Lefschetz property for Δ, which is a simplicial $(d-1)$-sphere or ball. Let cone Δ denote the cone over Δ with apex $\mathbf{n}$, realized in $\mathbb{R}^{d+1}$. Let π denote the projection along $\mathbf{n}$, and let $\theta = h \cdot \mathbf{x}$, where h is the vector of coordinates of the individual vertices in coordinate direction $\mathbf{n}$. The cone lemmas then give:

Lemma 8.1 ([2, Lemma 7.6]). *Considering* cone Σ *realized in* $\mathbb{R}^{d+1}$, *and* $k < \frac{d}{2}$, *the following two are equivalent:*
(1) *The Hall-Laman relations for* cone Δ *with respect to* $x_{\mathbf{n}}$ *and in degree* $k+1$.
(2) *The Hall-Laman relations for* $\pi\Delta$ *with respect to* θ *in degree* k.

This iteratively reduces to the middle case.

Proof. We may assume without loss of generality that $\theta = x_{\mathbf{n}}$ in $\mathscr{A}^*(\operatorname{cone}\Delta)$. Consider then the diagram

$$
\begin{array}{ccc}
\mathscr{A}^k(\pi\Delta, \partial\pi\Delta) & \xrightarrow{\cdot\theta^{d-2k}} & \mathscr{A}^{d-k}(\pi\Delta) \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
\mathscr{A}^{k+1}(\operatorname{cone}\Delta, \partial\operatorname{cone}\Delta) & \xrightarrow{\cdot x_{\mathbf{n}}^{d-2k-1}} & \mathscr{A}^{d-k}(\operatorname{cone}\Delta)
\end{array}
$$

where the first vertical map is defined by the composition of cone lemmas

$$
\mathscr{A}^k(\pi\Delta, \pi\partial\Delta) \;\cong\; \mathscr{A}^k(\operatorname{cone}\Delta, \operatorname{cone}\partial\Delta) \;\xrightarrow{\cdot x_{\mathbf{n}}}\; \mathscr{A}^{k+1}(\operatorname{cone}\Delta, \partial\operatorname{cone}\Delta)\,.
$$

and the second vertical map is simply the cone lemma. An isomorphism on the top is then equivalent to an isomorphism of the bottom map. $\qquad\square$

Once we arrive there, another miracle happens, because now Lefschetz has become biased pairing:

Lemma 8.2 ([2, Lemma 5.6])**.** *A PL homeomorphism* $\varphi : \Delta \to \Delta'$ *of balls* Δ, Δ' *in* $\mathbb{R}^d$ *that restricts to the identity on the boundary preserves the pairing property. That is,* Δ *satisfies the biased pairing property if and only if* Δ' *does.*

This is a rather simple application of the decomposition theorem [4] in its simplest form: We want to show that passing from Δ to Δ' does not affect the non-degeneracy of the pairing

$$
\mathscr{A}^k(\Delta, \partial\Delta) \;\times\; \mathscr{A}^{d-k}(\Delta, \partial\Delta) \;\longrightarrow\; \mathbb{R}
$$

For this, we may restrict to single stellar subdivisions and inverses when passing from Δ to Δ', and the decomposition theorem shows that the contribution of such a modification splits off orthogonally.

In particular, this lemma allows us to arbitrarily refine, and therefore make simpler, the triangulation of Δ in the interior, only having to keep the boundary intact.

9. So, how do we proceed?

We now have reduced ourselves to prove the biased pairing property for a disk Δ of odd dimension $(2d-1)$, and specifically do so in dimension d. Equivalently, we can take the double of Δ (two copies of Δ, identified along the boundary). In that sphere Σ, we are now expected to prove biased Poincaré duality in degree d and with respect to the ideal generated by $\partial\Delta$. Which in turn necessitates proving the middle Lefschetz isomorphism on the same. We are caught in a loop.

Or are we?

First, notice we only care about the $(d-1)$-skeleton, say S of $\partial\Delta$. Biased Poincaré duality in degree d, with respect to the former or the latter, are equivalent, as the associated ideals coincide in degree d, because $\mathscr{A}^d(\partial\Delta) \cong \mathscr{A}^d(S)$.

Now, the following main idea comes into play: Say we find another hypersurface E, possibly with boundary, that is an envelope for S in degree d, that is, $\mathscr{A}^d(E) \cong \mathscr{A}^d(S)$. If biased Poincaré duality is implied a Lefschetz statement on E.

This is indeed the case:

Consider for this purpose the complement $\widetilde{\Sigma}$ of E in Σ, where E is a $(d-2)$-acyclic rational, orientable hypersurface (with induced boundary) in Σ, and the double $\mathrm{D}E$ of E.

The open manifold $\widetilde{\Sigma}$ can be compactified canonically to a compact manifold $\widetilde{\mathrm{D}}\Sigma$ with boundary $\mathrm{D}E$.

Let π denote the general position projection to a hyperplane H, and θ the height over that projection, so that

$$\mathscr{A}^k(X) \;=\; \mathscr{A}^k(\pi X) \Big/ \theta\, \mathscr{A}^{k-1}(\pi X)$$

for a complex X in $\mathbb{R}^d$.

Lemma 9.1. *In the situation above, assume that:*
(1) The map

$$\mathscr{A}^{d-1}(\pi E, \partial\pi E) \xrightarrow{\;\cdot\theta\;} \mathscr{A}^d(\pi E)\,.$$

is injective.
(2) The map

$$(H^{d-1})^{\binom{2d}{d}}(\widetilde{\mathrm{D}}M) \;\longrightarrow\; \mathscr{A}^d(\mathrm{D}E),$$

which factors as

$$(H^{d-1})^{\binom{2d}{d}}(\widetilde{\mathrm{D}}M) \longrightarrow \mathscr{A}^d(\mathrm{D}E)$$
$$\searrow \qquad \uparrow$$
$$(H^{d-1})^{\binom{2d}{d}}(\mathrm{D}E)$$

is an injection.
Then Σ satisfies biased Poincaré duality with respect to E.

The main work of [2] is to show that both conditions are, in fact, Lefschetz type properties. For the first property, this may jump into the reader's eye, but the second is more subtle, and in fact, it may be nonobvious initially how cohomology of triangulated manifolds is even related or mapped to the face rings, as stated in the second condition. The identification of this map can be traced back to Ishida [8], and can be understood in terms of an extension of the Čech complex of a simplicial complex by stars of its faces to face rings.

The next, simpler step is to understand that once again, we can prove these Lefschetz properties inductively, at least, provided E is nicely decomposable. Unfortunately, neither A nor B-decomposability are flexible enough.

10. Conclusion

One can show that every triangulated manifold M, after some subdivision not involving its boundary, becomes C-**decomposable**, that is, there exists an order on a subset of the interior vertices of M' such that, for any initial segment W and the next vertex w

(1) $\bigcup_{v \in W} \mathrm{st}_v M'$ is a submanifold of M', and
(2) for the final segment $\overline{W}$, $\bigcup_{v \in \overline{W}} \mathrm{st}_v M' = M'$, and
(3) $\mathrm{lk}_w M' \cap (\partial M' \cup \bigcup_{v \in W} \mathrm{st}_v M')$ and $\mathrm{lk}_w \bigcup_{v \in W} \mathrm{st}_v M'$ is C-decomposable of codimension one, or empty.

Finally, we call a set of vertices, seen as a 0-dimensional manifold, C-decomposable by default. This notion is slightly weaker than what [2] calls a métro, where additional homological assumptions are made to simplify the proof of the Lefschetz property.
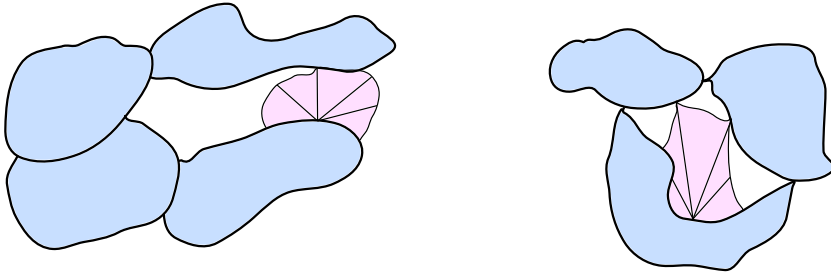
Figure 10.1: C-decomposability allows for simpler analysis of kernels and images in the pullback of a new vertex.

We now have the following basic construction, using the general position principle:

Lemma 10.1. *Assume S is a $(d-1)$-dimensional complex in a PL $(2d-1)$-sphere, or it is the $(d-1)$-skeleton of a $(2d-2)$-sphere Σ', living in the suspension Σ of the latter. Then, assuming the middle Lefschetz theorem for $(2d-4)$-spheres, there exists a hypersurface E in a subdivision of Σ that does not affect S such that*
(1) E contains S, and is an envelope for S in degree d.
(2) E is C-decomposable.

To start the reader off with the following basic intuition on how the construction works, we note that S is a $(d-1)$-dimensional complex in a PL $(2d-1)$-manifold, then S embeds into the boundary of its regular neighborhood (a hypersurface without boundary). By using this cleverly, we can easily modify this principle to obtain a hypersurface with boundary satisfying (2). Ensuring the first condition uses induction, relying on Lefschetz theorem for spheres of lower-dimensional spheres and just basic glimpses of Whitehead's notion of collapses (of neighborhoods) .

The key point of [2] is that, while not as simple as the ideals that arise in the case of B-decomposability, C-decomposability guarantees they are simple enough to control to allow for the understanding in each application step of Lemma 5.1.

The observation is that the two conditions guarantee that in each step, the kernel of the map "$\sum_{v \in W}$"x_v, pulled back along x_w, can be analyzed once again using Lefschetz theorems on $\mathrm{lk}_w M' \cap (\partial M' \cup \bigcup_{v \in W} \mathrm{st}_v M')$.

Indeed, this is even simpler to see and understand by working the orthogonal complement, that is, we need to understand the image of "$\sum_{v \in W}$"x_v, intersected with the ideal $\langle x_w \rangle \in \mathscr{A}^*(M', \partial M')$, thereby transferring us to $\mathscr{A}^*(\mathrm{st}_w M')$.

But now observe the very best thing: This intersection vanishes under multiplication with x_w as all the vertices of W are at distance at least two from w by the first condition. That condition then guarantees that the image is exactly consisting of the primitive elements under the multiplication with x_w on $\mathrm{lk}_w M' \cap (\partial M' \cup \bigcup_{v \in W} \mathrm{st}_v M')$. The hard Lefschetz property on the latter then implies that the ideal satisfies biased Poincaré duality in $\mathrm{lk}_w M'$. As this is, in turn, a (C-decomposable) manifold, it in turn closes the induction.

11. Via edge contractions

The alternative proof uses, instead of the perturbation lemma, edge contractions. Edge contractions are a well-known and used to prove Lefschetz type theorems, but a major problem arises when trying to do this in full generality. As observed first by Whiteley [19] (see also [3, 12], it is easy to ensure the Lefschetz property for a sphere Σ if, after a contraction of the edge e of Σ, the resulting complex Σ' is (1) still a sphere, satisfies (2) the Lefschetz theorem and so does (3) the link of the contracted edge. Conditions (2) and (3) are easily satisfiable by induction, but it is condition (1) that breaks our neck. Indeed, it is not clear that we can find a contractible edge. And indeed, starting from dimension 3, there exist spheres that are not the simplex, but still have no contractible edges (see again [10, 12]).

The theory we introduced offers us a way out: We apply edge contraction to a suitable envelope of the $(d-1)$-skeleton instead.

Instead of showing, for an even dimensional sphere Σ' (say, of dimension $2d-2$) the Lefschetz isomorphism between $\mathscr{A}^{d-1}(\Sigma')$ and $\mathscr{A}^d(\Sigma')$, we show the equivalent claim that in a suspension Σ of the same, the $(d-1)$-skeleton induces an ideal that satisfies biased Poincaré duality.

But once again, we can try to find a sufficiently nice hypersurface envelope E for that skeleton that is contractible. In other words, we call a hypersurface envelope E (in degree d) for a $(d-1)$-complex X then D-contractible if there exists an edge e such that

(1) the contracted complex E' is an envelope for the contracted complex X' in degree d and

(2) E' contains as a subcomplex a hypersurface $\widetilde{E}$ that contains X' and is D-contractible, or X' is of a dimension lower than $d-1$.

With this, we can use Whiteley's induction and prove the theorem via edge contractions. We only need the following lemma:

Lemma 11.1. *Assume S is a $(d-1)$-dimensional complex in a PL $(2d-1)$-sphere, or it is the $(d-1)$-skeleton of a $(2d-2)$-sphere Σ', living in the suspension Σ of the latter. Then, assuming the middle Lefschetz theorem for $(2d-4)$-spheres, there exists a D-contractible hypersurface envelope E for S in a subdivision of Σ.*

This is once again a rather simple geometric construction based on the general position principle.

12. Open questions

There are several open questions concerning the construction of Lefschetz elements, hard Lefschetz type theorems. Let us stay with the construction in detail: To prove the hard Lefschetz theorem for spheres, we needed to discuss more general manifolds:

Question 12.1. Is it possible to prove the Lefschetz theorem for spheres without the detour over manifolds and hypersurfaces?

Secondly, it remains to see whether there is a symmetric, or perhaps even homological version of the perturbation lemma, that does not rely on analyzing two vertices at a time.

Question 12.2. Is it possible to combine more than two maps at once in Lemma 5.1?

Finally, we could ask for a characterization of "good" artinian reductions of face rings of spheres, such that a Lefschetz element can be found. From the above, we can distill the following way to find the right Artinian reductions:

Theorem 12.3. *Consider a $(2d)$-sphere Σ realized in $\mathbb{R}^{2d+1}$. Assume that for every vertex v of Σ, and every $(d-1)$-dimensional subcomplex of its link S, every embedding of S into a $(2d-1)$-sphere satisfies biased Poincaré duality. Then Σ has a Lefschetz element for the middle isomorphism.*

We draw a useful corollary.

Corollary 12.4. *If Σ is a 2-sphere in $\mathbb{R}^3$, then it has a Lefschetz element if no three vertices lie in a hyperplane through the origin.*

This leaves open the search for explicit coordinates for spheres to satisfy Lefschetz thoerems. To avoid cheating via transcendental extensions, we ask:

Problem 12.5. Is there a natural set of rational coordinates/rational Artinian reduction for a sphere Σ to satisfy the Lefschetz theorem?

References

[1] Adiprasito, K.: Toric chordality. J. Math. Pures Appl. (9) **108**, no. 5, 783–807 (2017).

[2] Adiprasito, K.: Combinatorial Lefschetz theorems beyond positivity. preprint (2018), arXiv:1812.10454

[3] Babson, E., Nevo, E.: Lefschetz properties and basic constructions on simplicial spheres. J. Algebraic Combin. **31**, no. 1, 111–129 (2010)

[4] Beilinson, A., Bernstein, J., Deligne, P., Gabber, O.: Faisceaux pervers. Actes du colloque "Analyse et Topologie sur les Espaces Singuliers". Partie I., 2nd edition ed., vol. 100, Paris: Société Mathématique de France (SMF), 2018.

[5] Billera, L. J., Provan, J. S.: A decomposition property for simplicial complexes and its relation to diameters and shellings. Second International Conference on Combinatorial Mathematics (New York, 1978), Ann. New York Acad. Sci., **319**, New York Acad. Sci., New York, 82–85 (1979)

[6] Fleming, B., Karu, K.: Hard Lefschetz theorem for simple polytopes. J. Algebraic Combin. **32**, no. 2, 227–239 (2010)

[7] Gräbe, H.-G.: The canonical module of a Stanley–Reisner ring. J. Algebra **86**, 272–281 (1984).

[8] Ishida, M.-N.: Torus embeddings and dualizing complexes., Tohoku Math. J., II. Ser. **32**, 111–146 (1980)

[9] Lee, C. W.: P.L.-spheres, convex polytopes, and stress. Discrete Comput. Geom. **15**, no. 4, 389–421 (1996)

[10] Lickorish, W.: Unshellable triangulations of spheres., Eur. J. Comb. **12**, no. 6, 527–530 (1991)

[11] McMullen, P.: The numbers of faces of simplicial polytopes. Israel J. Math. **9**, 559–570 (1971)

[12] Murai, S.: Algebraic shifting of strongly edge decomposable spheres. J. Comb. Theory Ser. A, **117**, no. 1, 1–16 (2010)

[13] Reisner, G. A.: Cohen-Macaulay quotients of polynomial rings. Adv. Math. **21**, no. 1, 30–49 (1976)

[14] Sommerville, D. M. Y.: The relations connecting the angle-sums and volume of a polytope in space of n dimensions. Proc. R. Soc. Lond., Ser. A **115** 103–119 (1927)

[15] Stanley, R.: Cohen-Macaulay complexes. Higher Comb., Proc. NATO Adv. Study Inst., Berlin (West), 51-62 (1977)

[16] Stanley, R.: The number of faces of a simplicial convex polytope. Adv. Math. **35**, no. 3, 236–238 (1980)

[17] Stanley, R.: Combinatorics and commutative algebra, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston Inc., Boston, MA, 1996.
[18] Whitehead, J. H. C.: Simplicial Spaces, Nuclei and m-Groups, Proc. London Math. Soc. (2) **45**, no. 4, 243–327 (1939)
[19] Whiteley, W.: La division de sommet dans les charpentes isostatiques. Topologie Struct. **16**, 23–30 (1990)

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