Second Hankel determinant for a class of analytic functions defined by Komatu integral operator

Ram N. Mohapatra and Trailokya Panigrahi*

Abstract. In this paper, the authors obtain an upper bound of second Hankel determinant for a new class of analytic functions defined through the Komatu integral operator. Our result extends the corresponding previously known results.

1. Introduction

Let \( \mathcal{A} \) be the class of functions analytic in the open unit disk \( \mathbb{U} := \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathcal{A}_0 \) be the family of functions \( f \) in \( \mathcal{A} \) given by the normalized power series

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}).
\]

Let \( \mathcal{S} \) denote the class of all functions in \( \mathcal{A}_0 \) which are univalent in \( \mathbb{U} \). A function \( f(z) \in \mathcal{A}_0 \) is said to be in the class \( \mathcal{S}^*(\beta) \), starlike functions of order \( \beta \) (cf. [27]) in \( \mathbb{U} \) if it satisfies

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \quad (0 \leq \beta < 1; z \in \mathbb{U}).
\]

Further, a function \( f(z) \in \mathcal{A}_0 \) is said to be in the class \( \mathcal{CV}(\beta) \), convex function of order \( \beta \) (cf. [27]) in \( \mathbb{U} \) if it satisfies

\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \quad (0 \leq \beta < 1, z \in \mathbb{U}).
\]

In particular, \( \mathcal{S}^*(0) = \mathcal{S}^* \) and \( \mathcal{CV}(0) = \mathcal{CV} \) are the familiar classes of starlike and convex functions in \( \mathbb{U} \) (cf. [7]).

Komatu [18] introduced a certain integral operator \( \mathcal{L}_a^\delta \) defined by

\[
\mathcal{L}_a^\delta f(z) = \frac{a^\delta}{\Gamma(\delta)} \int_0^1 t^{a-2} \left( \log \frac{1}{1-t} \right)^{\delta-1} f(zt) dt \quad (z \in \mathbb{U}; a > 0; \delta \geq 0; f \in \mathcal{A}_0).
\]

2010 Mathematics Subject Classification: 30C45.
Keywords: Analytic functions, Starlike functions, Convex functions, Hankel determinant, Komatu integral operator.
© The Author(s) 2019. This article is an open access publication.
*Corresponding author.
Thus, if \( f(z) \in \mathcal{A}_0 \) is of the form (1.1), then it is clear from (1.4) that
\[
\mathcal{L}_a^\delta f(z) = z + \sum_{n=2}^{\infty} \left( \frac{a}{a+n-1} \right)^\delta a_n z^n \quad (z \in \mathbb{U}; \ a > 0; \ \delta \geq 0). \tag{1.5}
\]

The operator \( \mathcal{L}_a^\delta \) unifies several previously studied operators. Namely;

- \( \mathcal{L}_a^0 f(z) = f(z) \)
- \( \mathcal{L}_a^1 f(z) = A[f](z) \) known as Alexander operator \([1]\)
- \( \mathcal{L}_a^1 f(z) = \mathcal{L}[f](z) \) known as Libera operator \([19]\)
- \( \mathcal{L}_a^{c+1} f(z) = \mathcal{L}_c[f](z) \) called generalized Libera operator or Bernardi operator \([6]\)
- for \( a = 1 \) and \( \delta = k \) (\( k \) is any integer), the multiplier transformation \( \mathcal{L}_a^k f(z) = \mathcal{I}^k f(z) \) was studied by Flett \([10]\) and Sălăgean \([28]\) (also, see \([3, 4]\))
- for \( a = 1 \) and \( \delta = -k \) (\( k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \)), the differential operator \( \mathcal{L}_a^{-k} = \mathcal{D}^k f(z) \) was studied by Sălăgean \([28]\);
- for \( a = 2 \) and \( \delta = k \) (\( k \) is an integer), the operator \( \mathcal{L}_a^k f(z) = \mathcal{L}^k f(z) \) was studied by Uralegaddi and Somanatha \([30]\);
- for \( a = 2 \), the multiplier transformation \( \mathcal{L}_a^\delta f(z) = \mathcal{I}^\delta f(z) \) was studied by Jung et al. \([16]\).

Now, we introduce a new subclass of analytic functions by making use of the Komatu integral operator \( \mathcal{L}_a^\delta \) as follows:

**Definition 1.1.** A function \( f \in \mathcal{A}_0 \) is said to be in the class \( \mathcal{R}_a^\delta(\lambda) \) if it satisfies the inequality
\[
\Re \left\{ \frac{z (\lambda z (\mathcal{L}_a^\delta f(z))' + (1 - \lambda)\mathcal{L}_a^\delta f(z)')'}{\lambda z (\mathcal{L}_a^\delta f(z))' + (1 - \lambda)\mathcal{L}_a^\delta f(z)'} \right\} > 0 \quad (z \in \mathbb{U}; \ a > 0, \ \delta \geq 0, \ 0 \leq \lambda \leq 1). \tag{1.6}
\]

Note that, by taking \( \lambda = 0, \ \delta = 0 \) and \( \lambda = 1, \ \delta = 0 \) in the relation (1.6), the class \( \mathcal{R}_a^\delta(\lambda) \) reduces to classes \( \mathcal{S}^* \) and \( \mathcal{CV} \) respectively.

**Definition 1.2 ([26]).** For a function \( f \in \mathcal{A}_0 \) given by (1.1) and \( q \in \mathbb{N} := \{1, 2, 3, \ldots\} \), the \( q \) th Hankel determinant denoted by \( H_q(n) \) is defined as
\[
H_q(n) = \begin{vmatrix}
    a_n & a_{n+1} & \cdots & a_{n+q-1} \\
    a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix} \quad (a_1 = 1).
\]
A classical theorem of Fekete and Szegő [9] considered the Hankel determinant of \( f \in S \) for \( q = 2 \) and \( n = 1 \),

\[
H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.
\]

Then, they further generalized the functional \( |a_3 - \mu a_2^2| \), where \( \mu \) is real and \( f \in S \). In this paper, we consider the Hankel determinant for the case \( q = 2 \) and \( n = 2 \),

\[
H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2|.
\]

For a given family \( F \) of the functions in \( A_0 \), the sharp upper bound for the nonlinear functional \( |H_2(2)| \) is popularly known as the second Hankel determinant.

Janteng et al. [15] (also, see [13]) have considered the functional \( |a_2 a_4 - a_3^2| \) and found the sharp bound for the function \( f \) in the subclass \( R \) of \( S \), consisting of functions whose derivative has a positive real part. They have shown that if \( f \in R \), then \( |a_2 a_4 - a_3^2|\leq \frac{4}{9} \).

Further, Janteng et al. [14] also obtained sharp bounds for Hankel determinant for functions in certain familiar subclasses of \( S \) namely; starlike and convex functions denoted by \( S^* \) and \( C \) respectively. They have shown that if \( f \in S^* \), then \( |a_2 a_4 - a_3^2|\leq 1 \) and if \( f \in C \), then \( |a_2 a_4 - a_3^2|\leq \frac{1}{8} \).

Recently, Murugusundaramoorthy and Magesh [25] have obtained the sharp upper bound for the functional \( |a_2 a_4 - a_3^2| \) for the function \( f \in R(\alpha) \), where

\[
R(\alpha) = \left\{ f(z) \in A_0 : \Re\left\{ (1 - \alpha)\frac{f(z)}{z} + \alpha f'(z) \right\} > 0, \alpha > 0, z \in \mathbb{D} \right\}.
\]

Recently, Kaharudin et al. [17] have obtained the upper bound of the second Hankel determinant \( |a_2 a_4 - a_3^2| \) for the functions in the class \( G_k(\alpha, \delta) \) defined as

\[
\Re\left\{ e^{i\alpha} \frac{f(z)}{g'(z)} \right\} > \delta \quad (z \in \mathbb{D})
\]

where \( |\alpha| \leq \pi; \cos \alpha - \delta > 0; g(z) \) is convex function and \( g'(z) = \frac{1}{1-z} \). For some more recent work on second Hankel determinant see [2, 5, 8, 11, 12, 22, 23, 24, 29]. Motivated by the aforementioned works, in this paper, we find an upper bound for the functional \( |a_2 a_4 - a_3^2| \) for the functions \( f \) belongs to the class \( R'_0(\lambda) \). We generalize the results of Janteng et al. [13].

2. Preliminaries

Let \( P \) be the family of all functions \( p \in A \) satisfying \( p(0) = 1 \) and \( \Re\{p(z)\} > 0 \), \((z \in \mathbb{U})\).

We need the following lemmas for our present investigation:
Lemma 2.1 (see [7]). Let the function \( p \in \mathcal{P} \) be given by the series
\[
p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}). \tag{2.1}
\]
Then, the sharp estimate
\[
|c_k| \leq 2 \quad (k \in \mathbb{N}) \tag{2.2}
\]
holds.

Lemma 2.2 (cf. [20], also see [21]). Let the function \( p \in \mathcal{P} \) be given by the series (2.1). Then
\[
2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.3}
\]
for some \( x, \ |x| \leq 1 \) and
\[
4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.4}
\]
for some \( z, \ |z| \leq 1 \).

3. Main Results

Unless otherwise mentioned, we assume throughout the sequel that \( a > 0, \delta \geq 0, \ 0 \leq \lambda \leq 1 \).

Theorem 3.1. Let the function \( f \in \mathcal{A}_0 \), given by (1.1) be in the class \( \mathcal{R}_a^\delta(\lambda) \). Then
\[
|a_2 a_4 - a_3^2| \leq \left[ \frac{(a + 1)(a + 3)}{a^2} \right] \delta \frac{1}{(1 + \lambda)(1 + 3\lambda)}. \tag{3.1}
\]

Proof. Let the function \( f(z) \in \mathcal{A}_0 \) represented by (1.1) be in the class \( \mathcal{R}_a^\delta(\lambda) \). By geometric interpretation, there exists a function \( p \in \mathcal{P} \) given by (2.1) such that
\[
\frac{z \lambda z (\mathcal{L}_a^\delta f(z))' + (1 - \lambda)\mathcal{L}_a^\delta f(z))'}{\lambda z (\mathcal{L}_a^\delta f(z))' + (1 - \lambda)\mathcal{L}_a^\delta f(z)} = p(z). \tag{3.2}
\]
Comparing the coefficients, we get
\[
(1 + \lambda) \left( \frac{a}{a + 1} \right)^\delta a_2 = c_1, \tag{3.3}
\]
\[
(1 + 2\lambda) \left( \frac{a}{a + 2} \right)^\delta a_3 = \frac{c_1^2 + c_2}{2}, \tag{3.4}
\]
and
\[
(1 + 3\lambda) \left( \frac{a}{a + 3} \right)^\delta a_4 = \frac{2c_3 + 3c_1 c_2 + c_1^3}{6}. \tag{3.5}
\]
Taking the values of $a_2, a_3$ and $a_4$ from (3.3), (3.4) and (3.5) we have

$$|a_2a_4 - a_3^2| = H(a, \lambda, \delta) \left| 4c_1c_3 + 6c_1^2c_2 + 2c_1^4 \right. $$

$$- \left. \frac{(a + 2)^{2\delta}}{(a + 1)(a + 3)^\delta} \frac{(1 + 3\lambda)(1 + \lambda)}{(1 + 2\lambda)^2} \left(3c_1^4 + 3c_2^2 + 6c_1^2c_2\right) \right| \quad (3.6)$$

$$= H(a, \lambda, \delta) \left| 4c_1c_3 + 6c_1^2c_2 + 2c_1^4 - q(3c_1^4 + 3c_2^2 + 6c_1^2c_2) \right| \quad (3.7)$$

where, for convenience

$$H(a, \lambda, \delta) = \frac{(a + 1)^{\delta}(a + 3)^\delta}{12a^{2\delta}(1 + \lambda)(1 + 3\lambda)} \quad (3.8)$$

and

$$q(a, \lambda, \delta) = \frac{(a + 2)^{2\delta}(1 + 3\lambda)(1 + \lambda)}{(a + 1)^{\delta}(a + 3)^\delta(1 + 2\lambda)^2} = q \ (say). \quad (3.9)$$

Since $q \in \left[\frac{8}{9}, 1\right]$ for $0 \leq \lambda \leq 1, \ \delta = 0,$ the equation (3.6) can be written as

$$|a_2a_4 - a_3^2| = H(a, \lambda, \delta)|e_1c_1c_3 + e_2c_1^2c_2 + e_3c_1^4 + e_4c_2^2|, \quad (3.10)$$

where

$$e_1 = 4; \quad e_2 = 6(1 - q); \quad e_3 = 2 - 3q; \quad e_4 = -3q. \quad (3.11)$$

Since the functions $p(z)$ and $p(e^{i\theta}z)$ ($\theta \in \mathbb{R}$) are members of the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c \ (c \in [0, 2] \text{ see (2.2)})$. Using (2.3) and (2.4) in (3.10), we have

$$|a_2a_4 - a_3^2| = \frac{H(a, \lambda, \delta)}{4} \left| c^4(e_1 + 2e_2 + 4e_3 + e_4) + 2c^2x(4 - c^2)(e_1 + e_2 + e_4) \right.$$

$$+ (4 - c^2)x^2((4 - c^2)e_4 - e_1c^2) + 2ce_1(4 - c^2)(1 - |x|^2)z \right| \quad (3.12)$$

Upon substitute the values of $e_1, e_2, e_3$ and $e_4$ from (3.11) in resulting equation (3.12), we obtain

$$|a_2a_4 - a_3^2| = \frac{H(a, \lambda, \delta)}{4} \left| -(27q - 24)c^4 + 2c^2(10 - 9q)x(4 - c^2) \right.$$

$$- (4 - c^2)x^2(4c^2 + 3q(4 - c^2)) + 8c(4 - c^2)(1 - |x|^2)z \right| \quad (3.13)$$
An application of triangle inequality and replacement of $|x|$ by $\rho$ give
\[ |a_2a_4 - a_3^2| \leq \frac{H(a, \lambda, \delta)}{4} \left[ (27q - 24)c^4 + 8c(4 - c^2) + 2\rho c^2(4 - c^2)(10 - 9q) 
+ \rho^2(4 - c^2)\{4c^2 + 3q(4 - c^2) - 8c\} \right] 
= G(c, \rho) \text{(say)}, \quad (0 \leq c \leq 2, 0 \leq \rho \leq 1). \] (3.14)

Next, we maximize the function $G(c, \rho)$ on the closed square $[0, 2] \times [0, 1]$. Since
\[ \frac{\partial G}{\partial \rho} = \frac{H(a, \lambda, \delta)}{4} \left[ 2c^2(4 - c^2)(10 - 9q) + 2\rho(4 - c^2)(4c^2 + 3q(4 - c^2) - 8c) \right], \] (3.15)
for $0 < c < 2$ and $0 < \rho < 1$, we have $\frac{\partial G}{\partial \rho} > 0$. Thus, $G(c, \rho)$ is an increasing function of $\rho$, which implies that $G(c, \rho)$ cannot have maximum in the interior of the closed rectangle $[0, 2] \times [0, 1]$. Moreover, for fixed $c \in [0, 2]$,
\[ \max_{0 \leq \rho \leq 1} G(c, 1) = F(c) \text{ (say)}, \]
where,
\[ F(c) = 12H(a, \lambda, \delta) \left[ -(1 - q)c^4 + 2(1 - q)c^2 + q \right]. \] (3.16)

Now, we have
\[ F'(c) = 48cH(a, \lambda, \delta)[-\delta(1 - q)c^2 + (1 - q)]. \]
Setting $F'(c) = 0$ we obtain that $c = 0, -1, 1$. Since
\[ F''(c) = -48H(a, \lambda, \delta)[3(1 - q)c^2 - (1 - q)] \]
and $c \in [0, 2]$, we find that $F$ has a maximum value at $c = 1$. Thus, the upper bound for (3.14) corresponds to $\rho = 1$ and $c = 1$. Hence
\[ |a_2a_4 - a_3^2| \leq 12H(a, \lambda, \delta) = \frac{[(a + 1)(a + 3)]^\delta}{a^{2b}(1 + \lambda)(1 + 3\lambda)}. \]

This completes the proof of Theorem 3.1. \qed

**Remark 3.2.** Taking $\delta = 0, \lambda = 0$ and $\delta = 0, \lambda = 1$ we get the result due to Janteng et al. [14] as in the following corollary.

**Corollary 3.3.** (i) If $f \in S^\ast$, then
\[ |a_2a_4 - a_3^2| \leq 1. \]
(ii) If $f \in CV$, then
\[ |a_2a_4 - a_3^2| \leq \frac{1}{8}. \]

**Acknowledgements.** The authors would like to thank the esteemed referee(s) for their careful reading, valuable suggestions and comments, which helped to improve the presentation of the paper.
Second Hankel determinant for a class of analytic functions defined by Komatu . . . 57

References


Received: 16 August 2016/Accepted: 4 October 2019/Published online: 25 October 2019

Ram N. Mohapatra
Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA.
ram.mohapatra@ucf.edu

Trailokya Panigrahi
Department of Mathematics, School of Applied Sciences, KIIT Deemed to be University, Bhubaneswar-751024, Orissa, India.
trailokyap6@gmail.com

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.