Rend. Mat. Appl. (7).
Volume 41 (2020), 51 – 58

Rendiconti di Matematica
e delle sue Applicazioni

Second Hankel determinant for a class of analytic functions defined by Komatu integral operator

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Abstract. *In this paper, the authors obtain an upper bound of second Hankel determinant for a new class of analytic functions defined through the Komatu integral operator. Our result extends the corresponding previously known results.*

1. Introduction

Let $\mathcal{A}$ be the class of functions analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}_0$ be the family of functions f in $\mathcal{A}$ given by the normalized power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}). \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}_0$ which are univalent in $\mathbb{U}$. A function $f(z) \in \mathcal{A}_0$ is said to be in the class $\mathcal{S}^*(\beta)$, starlike functions of order β (cf. [27]) in $\mathbb{U}$ if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (0 \leq \beta < 1; z \in \mathbb{U}). \tag{1.2}$$

Further, a function $f(z) \in \mathcal{A}_0$ is said to be in the class $\mathcal{CV}(\beta)$, convex function of order β (cf. [27]) in $\mathbb{U}$ if it satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \quad (0 \leq \beta < 1, z \in \mathbb{U}). \tag{1.3}$$

In particular, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{CV}(0) = \mathcal{CV}$ are the familiar classes of starlike and convex functions in $\mathbb{U}$ (cf. [7]).

Komatu [18] introduced a certain integral operator $\mathcal{L}_a^\delta$ defined by

$$\mathcal{L}_a^\delta f(z) = \frac{a^\delta}{\Gamma(\delta)} \int_0^1 t^{a-2} \left(log\frac{1}{t}\right)^{\delta-1} f(zt)dt \quad (z \in \mathbb{U};\ a > 0;\ \delta \geq 0;\ f \in \mathcal{A}_0). \tag{1.4}$$

2010 Mathematics Subject Classification: 30C45.
Keywords: Analytic functions, Starlike functions, Convex functions, Hankel determinant, Komatu integral operator.

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Thus, if $f(z) \in \mathcal{A}_0$ is of the form (1.1), then it is clear from (1.4) that

$$\mathcal{L}_a^\delta f(z) = z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1} \right)^\delta a_n z^n \quad (z \in \mathbb{U};\ a > 0;\ \delta \geq 0). \tag{1.5}$$

The operator $\mathcal{L}_a^\delta$ unifies several previously studied operators. Namely;

- $\mathcal{L}_a^0 f(z) = f(z)$
- $\mathcal{L}_1^1 f(z) = A[f](z)$ known as Alexander operator [1]
- $\mathcal{L}_2^1 f(z) = \mathcal{L}[f](z)$ known as Libera operator [19]
- $\mathcal{L}_{c+1}^1 f(z) = \mathcal{L}_c[f](z)$ called generalized Libera operator or Bernardi operator [6]
- for $a = 1$ and $\delta = k$ (k is any integer), the multiplier transformation $\mathcal{L}_1^k f(z) = \mathcal{I}^k f(z)$ was studied by Flett [10] and Sălăgeăn [28] (also, see [3, 4])
- for $a = 1$ and $\delta = -k$ $(k \in \mathbb{N}_0 = \{0, 1, 2, ...\})$, the differential operator $\mathcal{L}_1^{-k} = \mathcal{D}^k f(z)$ was studied by Sălăgeăn [28];
- for $a = 2$ and $\delta = k$ (k is an integer), the operator $\mathcal{L}_2^k f(z) = \mathcal{L}^k f(z)$ was studied by Uralegaddi and Somanatha [30];
- for $a = 2$, the multiplier transformation $\mathcal{L}_2^\delta f(z) = \mathcal{I}^\delta f(z)$ was studied by Jung et al. [16].

Now, we introduce a new subclass of analytic functions by making use of the Komatu integral operator $\mathcal{L}_a^\delta$ as follows:

Definition 1.1. A function $f \in \mathcal{A}_0$ is said to be in the class $\mathcal{R}_a^\delta(\lambda)$ if it satisfies the inequality

$$\Re \left\{ \frac{z \left(\lambda z (\mathcal{L}_a^\delta f(z))' + (1-\lambda) \mathcal{L}_a^\delta f(z) \right)'}{\lambda z (\mathcal{L}_a^\delta f(z))' + (1-\lambda) \mathcal{L}_a^\delta f(z)} \right\} > 0 \quad (z \in \mathbb{U}; a > 0,\ \delta \geq 0,\ 0 \leq \lambda \leq 1). \tag{1.6}$$

Note that, by taking $\lambda = 0,\ \delta = 0$ and $\lambda = 1,\ \delta = 0$ in the relation (1.6), the class $\mathcal{R}_a^\delta(\lambda)$ reduces to classes $\mathcal{S}^*$ and $\mathcal{CV}$ respectively.

Definition 1.2 ([26])**.** For a function $f \in \mathcal{A}_0$ given by (1.1) and $q \in \mathbb{N} := \{1, 2, 3, ...\}$, the q th Hankel determinant denoted by $H_q(n)$ is defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

A classical theorem of Fekete and Szegö [9] considered the Hankel determinant of $f \in \mathcal{S}$ for $q = 2$ and $n = 1$,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.$$

Then, they further generalized the functional $|a_3 - \mu a_2^2|$, where μ is real and $f \in \mathcal{S}$. In this paper, we consider the Hankel determinant for the case $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2|.$$

For a given family $\mathcal{F}$ of the functions in $\mathcal{A}_0$, the sharp upper bound for the nonlinear functional $|H_2(2)|$ is popularly known as the second Hankel determinant.

Janteng et al. [15] (also, see [13]) have considered the functional $|a_2 a_4 - a_3^2|$ and found the sharp bound for the function f in the subclass $\mathcal{R}$ of $\mathcal{S}$, consisting of functions whose derivative has a positive real part. They have shown that if $f \in \mathcal{R}$, then $|a_2 a_4 - a_3^2| \leq \frac{4}{9}$.

Further, Janteng et al. [14] also obtained sharp bounds for Hankel determinant for functions in certain familiar subclasses of $\mathcal{S}$ namely; starlike and convex functions denoted by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. They have shown that if $f \in \mathcal{S}^*$, then $|a_2 a_4 - a_3^2| \leq 1$ and if $f \in \mathcal{C}$, then $|a_2 a_4 - a_3^2| \leq \frac{1}{8}$.

Recently, Murugusundaramoorthy and Magesh [25] have obtained the sharp upper bound for the functional $|a_2 a_4 - a_3^2|$ for the function $f \in \mathcal{R}(\alpha)$, where

$$\mathcal{R}(\alpha) = \left\{ f(z) \in \mathcal{A}_0 : \Re\left\{ (1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \right\} > 0, \alpha > 0, z \in \mathbb{U} \right\}.$$

Recently, Kaharudin et al. [17] have obtained the upper bound of the second Hankel determinant $|a_2 a_4 - a_3^2|$ for the functions in the class $G_k(\alpha, \delta)$ defined as

$$\Re\left\{ e^{i\alpha} \frac{f(z)}{g'(z)} \right\} > \delta \quad (z \in \mathbb{U})$$

where $|\alpha| \leq \pi; cos\alpha - \delta > 0$; $g(z)$ is convex function and $g'(z) = \frac{1}{1-z}$. For some more recent work on second Hankel determinant see [2, 5, 8, 11, 12, 22, 23, 24, 29]. Motivated by the aforementioned works, in this paper, we find an upper bound for the functional $|a_2 a_4 - a_3^2|$ for the functions f belongs to the class $\mathcal{R}_a^\delta(\lambda)$. We generalize the results of Janteng et al. [13].

2. Preliminaries

Let $\mathcal{P}$ be the family of all functions $p \in \mathcal{A}$ satisfying $p(0) = 1$ and $\Re\{p(z)\} > 0$, $(z \in \mathbb{U})$.

We need the following lemmas for our present investigation:

Lemma 2.1 (see [7]). *Let the function $p \in \mathcal{P}$ be given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}). \tag{2.1}$$

Then, the sharp estimate

$$|c_k| \leq 2 \quad (k \in \mathbb{N}) \tag{2.2}$$

holds.

Lemma 2.2 (cf. [20], also see [21]). *Let the function $p \in \mathcal{P}$ be given by the series* (2.1). *Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.3}$$

for some x, $|x| \leq 1$ and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.4}$$

for some z, $|z| \leq 1$.

3. Main Results

Unless otherwise mentioned, we assume throughout the sequel that $a > 0$, $\delta \geq 0$, $0 \leq \lambda \leq 1$.

Theorem 3.1. *Let the function $f \in \mathcal{A}_0$, given by (1.1) be in the class $\mathcal{R}_a^\delta(\lambda)$. Then*

$$|a_2 a_4 - a_3^2| \leq \left[\frac{(a+1)(a+3)}{a^2}\right]^\delta \frac{1}{(1+\lambda)(1+3\lambda)}. \tag{3.1}$$

Proof. Let the function $f(z) \in \mathcal{A}_0$ represented by (1.1) be in the class $\mathcal{R}_a^\delta(\lambda)$. By geometric interpretation, there exists a function $p \in \mathcal{P}$ given by (2.1) such that

$$\frac{z\left(\lambda z(\mathcal{L}_a^\delta f(z))' + (1-\lambda)\mathcal{L}_a^\delta f(z)\right)'}{\lambda z(\mathcal{L}_a^\delta f(z))' + (1-\lambda)\mathcal{L}_a^\delta f(z)} = p(z). \tag{3.2}$$

Comparing the coefficients, we get

$$(1+\lambda)\left(\frac{a}{a+1}\right)^\delta a_2 = c_1\,, \tag{3.3}$$

$$(1+2\lambda)\left(\frac{a}{a+2}\right)^\delta a_3 = \frac{c_1^2 + c_2}{2}\,, \tag{3.4}$$

and

$$(1+3\lambda)\left(\frac{a}{a+3}\right)^\delta a_4 = \frac{2c_3 + 3c_1 c_2 + c_1^3}{6}\,. \tag{3.5}$$

Taking the values of $a_2,\ a_3$ and a_4 from (3.3), (3.4) and (3.5) we have

$$|a_2a_4 - a_3^2| = H(a,\lambda,\delta)\Big|4c_1c_3 + 6c_1^2c_2 + 2c_1^4 - \frac{(a+2)^{2\delta}}{(a+1)(a+3)^{\delta}}\frac{(1+3\lambda)(1+\lambda)}{(1+2\lambda)^2}(3c_1^4 + 3c_2^2 + 6c_1^2c_2)\Big| \quad (3.6)$$

$$= H(a,\lambda,\delta)\Big|4c_1c_3 + 6c_1^2c_2 + 2c_1^4 - q(3c_1^4 + 3c_2^2 + 6c_1^2c_2)\Big|$$

$$= H(a,\lambda,\delta)\Big|4c_1c_3 + 6(1-q)c_1^2c_2 + (2-3q)c_1^4 - 3qc_2^2\Big| \quad (3.7)$$

where, for convenience

$$H(a,\lambda,\delta) = \frac{(a+1)^{\delta}(a+3)^{\delta}}{12a^{2\delta}(1+\lambda)(1+3\lambda)} \quad (3.8)$$

and

$$q(a,\lambda,\delta) = \frac{(a+2)^{2\delta}(1+3\lambda)(1+\lambda)}{(a+1)^{\delta}(a+3)^{\delta}(1+2\lambda)^2} = q\ (say). \quad (3.9)$$

Since $q \in [\frac{8}{9}, 1]$ for $0 \le \lambda \le 1,\quad \delta = 0$, the equation (3.6) can be written as

$$|a_2a_4 - a_3^2| = H(a,\lambda,\delta)|e_1c_1c_3 + e_2c_1^2c_2 + e_3c_1^4 + e_4c_2^2|, \quad (3.10)$$

where

$$e_1 = 4;\quad e_2 = 6(1-q);\quad e_3 = 2-3q;\quad e_4 = -3q\,. \quad (3.11)$$

Since the functions $p(z)$ and $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ are members of the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ ($c \in [0,2]$ see (2.2)). Using (2.3) and (2.4) in (3.10), we have

$$|a_2a_4 - a_3^2| = \frac{H(a,\lambda,\delta)}{4}\Big|c^4(e_1+2e_2+4e_3+e_4) + 2c^2x(4-c^2)(e_1+e_2+e_4) + (4-c^2)x^2\left((4-c^2)e_4 - e_1c^2\right) + 2ce_1(4-c^2)(1-|x|^2)z\Big| \quad (3.12)$$

Upon substitute the values of $e_1,\ e_2,\ e_3$ and e_4 from (3.11) in resulting equation (3.12), we obtain

$$|a_2a_4 - a_3^2| = \frac{H(a,\lambda,\delta)}{4}\Big| - (27q-24)c^4 + 2c^2(10-9q)x(4-c^2) - (4-c^2)x^2(4c^2 + 3q(4-c^2)) + 8c(4-c^2)(1-|x|^2)z\Big| \quad (3.13)$$

An application of triangle inequality and replacement of $|x|$ by ρ give

$$\begin{aligned}|a_2a_4 - a_3^2| \leq & \frac{H(a,\lambda,\delta)}{4}\Big[(27q-24)c^4 + 8c(4-c^2) + 2\rho c^2(4-c^2)(10-9q) \\ & + \rho^2(4-c^2)\left\{4c^2 + 3q(4-c^2) - 8c\right\}\Big] \\ = & G(c,\rho)(say), \quad (0 \leq c \leq 2, 0 \leq \rho \leq 1). \end{aligned} \tag{3.14}$$

Next, we maximize the function $G(c,\rho)$ on the closed square $[0,2]\times[0,1]$. Since

$$\frac{\partial G}{\partial \rho} = \frac{H(a,\lambda,\delta)}{4}\big[2c^2(4-c^2)(10-9q) + 2\rho(4-c^2)(4c^2+3q(4-c^2)-8c)\big], \tag{3.15}$$

for $0 < c < 2$ and $0 < \rho < 1$, we have $\frac{\partial G}{\partial \rho} > 0$. Thus, $G(c,\rho)$ is an increasing function of ρ, which implies that $G(c,\rho)$ cannot have maximum in the interior of the closed rectangle $[0,2]\times[0,1]$. Moreover, for fixed $c \in [0,2]$,

$$\max_{0\leq\rho\leq1} G(c,1) = F(c) \quad \text{(say)},$$

where,

$$F(c) = 12H(a,\lambda,\delta)\Big[-(1-q)c^4 + 2(1-q)c^2 + q\Big]. \tag{3.16}$$

Now, we have

$$F'(c) = 48cH(a,\lambda,\delta)[-(1-q)c^2 + (1-q)].$$

Setting $F'(c) = 0$ we obtain that $c = 0, \; -1, \; 1$. Since

$$F''(c) = -48H(a,\lambda,\delta)\big[3(1-q)c^2 - (1-q)\big]$$

and $c \in [0,2]$, we find that F has a maximum value at $c = 1$. Thus, the upper bound for (3.14) corresponds to $\rho = 1$ and $c = 1$. Hence

$$|a_2a_4 - a_3^2| \leq 12H(a,\lambda,\delta) = \frac{[(a+1)(a+3)]^\delta}{a^{2\delta}(1+\lambda)(1+3\lambda)}.$$

This completes the proof of Theorem 3.1. □

Remark 3.2. Taking $\delta = 0, \lambda = 0$ and $\delta = 0, \lambda = 1$ we get the result due to Janteng et al. [14] as in the following corollary.

Corollary 3.3. *(i) If $f \in \mathcal{S}^*$, then*

$$|a_2a_4 - a_3^2| \leq 1.$$

(ii) If $f \in \mathcal{CV}$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}.$$

Acknowledgements. The authors would like to thank the esteemed referee(s) for their careful reading, valuable suggestions and comments, which helped to improve the presentation of the paper.

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Received: 16 August 2016/Accepted: 4 October 2019/Published online: 25 october 2019

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