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# A Phragmén-Lindelöf property of viscosity solutions to a class of nonlinear parabolic equations with growth conditions

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**Abstract.** *We study Phragmén-Lindelöf properties of viscosity solutions to a class of doubly nonlinear parabolic equations in* $\mathbb{R}^n \times (0,T)$*. We include an application to some doubly nonlinear equations. We address also the optimality of some our results.*

## 1. Introduction

In this work, we discuss Phragmén-Lindelöf type results for a class of nonlinear parabolic equations. This is a follow-up of the work in [3] where we stated similar results for viscosity solutions of Trudinger's equation in infinite strips $\mathbb{R}^n \times (0,T)$, where $n \geq 2$ and $0 < T < \infty$. The classical references [11, 13, 14, 16] contain a detailed discussion of the importance of this property and its connections to other questions. The main question of interest is: under what conditions do solutions of elliptic and parabolic equations satisfy a maximum principle on unbounded domains? Our work considers infinite strips of $\mathbb{R}^{n+1}$ and presents some results in this direction. Our results apply to a fairly large class of parabolic equations and, in many instances, appear to be optimal. Further discussion and connections to other questions can be found in, for instance, [1, 7, 12, 14].

Many of the references, cited above, address primarily linear uniformly elliptic and parabolic equations. Our current work, on the other hand, studies nonlinear, possibly degenerate, parabolic equations and includes in it a certain class of doubly nonlinear equations. The case of Trudinger's equation is an instance of such equations, see [3, 15]. Moreover, the class we study here does include some linear uniformly parabolic equations, as examples.

Our work is in the context of viscosity solutions and it is important to point out that [7] appears to be the earliest work done on this question for nonlinear elliptic operators. Our work addresses similar questions for the parabolic versions of the operators considered in [7].

We introduce notation for our discussion. Let $n \geq 2$ and $0 < T < \infty$. Define $\mathbb{R}^n_T = \mathbb{R}^n \times (0,T)$. Let $g\colon \mathbb{R}^n \to (0,\infty)$ and $h\colon \mathbb{R}^n \to \mathbb{R}$ be two continuous functions

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satisfying

$$\max\left(\sup_{x\in\mathbb{R}^n}|\log g(x)|,\ \sup_{x\in\mathbb{R}^n}|h(x)|\right)<\infty\,. \tag{1.1}$$

Our motivation for the work arises from the study of viscosity solutions of doubly nonlinear equations of the kind

$$H(Du,D^2u)-f(u)u_t=0,\ \text{in } \mathbb{R}^n_T,\ \ u(x,t)>0 \text{ and } u(x,0)=g(x),\ \forall x \text{ in } \mathbb{R}^n, \tag{1.2}$$

where $H$ satisfies certain homogeneity conditions and $f\colon \mathbb{R}^+\to\mathbb{R}^+$ is a non-decreasing continuous function, see Section 2 for more details. As shown in [5], if $f$ satisfies certain conditions then there exists a function $\phi$ such that the change of variable $u=\phi(v)$ transforms (1.2) to

$$H(Dv,D^2v+Z(v)Dv\otimes Dv)-v_t=0,\ \ \text{in } \mathbb{R}^n_T, \text{ and } v(x,0)=\phi^{-1}(g(x)),\ \forall x \text{ in } \mathbb{R}^n, \tag{1.3}$$

where $Z\colon\mathbb{R}\to\mathbb{R}^+$ is a non-increasing function. As observed in [2, 5], one can conclude a comparison principle for (1.3), and hence, for (1.2).

Consider the well-known Trudinger's equation [2, 15]:

$$\operatorname{div}(|Du|^{p-2}Du)-(p-1)u^{p-2}u_t=0,\ \ \text{in } \mathbb{R}^n_T, \text{ and } u>0.$$

Writing $u=e^v$ (see [2]), we obtain an instance of (1.3), i.e.,

$$\operatorname{div}(|Dv|^{p-2}Dv)+(p-1)|Dv|^p-(p-1)v_t=0,\ \ \text{in } \mathbb{R}^n_T\,.$$

Setting $H(Dw,D^2w)=\operatorname{div}(|Dw|^{p-2}Dw)$, the above may be written as

$$H(Dv,D^2v+Dv\otimes Dv)-(p-1)v_t=0,\ \ \text{in } \mathbb{R}^n_T\,.$$

A related and somewhat more general equation is

$$\begin{aligned}&H(Du,D^2u)+\chi(t)|Du|^\sigma-(p-1)u^{p-2}u_t=0,\ \ \text{in } \mathbb{R}^n_T,\ u>0,\\ &\qquad \text{with } u(x,0)=g(x),\ \forall x \text{ in } \mathbb{R}^n,\end{aligned}$$

where $\sigma\ge 0$ and $\chi(t)$ is continuous on $[0,T]$. Using $u=e^v$ we get that

$$\begin{aligned}&H(Dv,D^2v+Dv\otimes Dv)+\chi(t)e^{(\sigma-(p-1))v}|Dv|^\sigma-(p-1)v_t=0,\ \ \text{in } \mathbb{R}^n_T,\\ &\qquad \text{and } v(x,0)=\log g(x),\ \forall x \text{ in } \mathbb{R}^n.\end{aligned}$$

At this time, it is not clear to us as to how to address the above equation. Nonetheless, this provides motivation for addressing the following related question of studying Phragmén-Lindelöf results for equations of the kind

$$\begin{aligned}&H(Dv,D^2v+Z(v)Dv\otimes Dv)+\chi(t)|Dv|^\sigma-v_t=0,\\ &\qquad v(x,0)=h(x),\ \text{for all } x \text{ in } \mathbb{R}^n.\end{aligned} \tag{1.4}$$

Here $\chi$ and $h$ are continuous, bounded and can have any sign. The function $Z$ is non-increasing and continuous.

In this work, the operators $H$ satisfy certain monotonicity and homogeneity conditions, see Section 2 for a more precise formulation. Our goal then is to consider equations such as (1.4) and show that if $v$ satisfies certain growth conditions, for large $|x|$, then $v$ satisfies a maximum principle. A similar conclusion then follows for the equation in (1.2).

The role of the function $Z$ is important. We assume that $Z$ is non-increasing and $\inf_s Z(s) > 0$. This greatly influences our results as $Z(v)Dv \otimes Dv$ and $\chi(t)|Dv|^\sigma$ could be dueling terms and this is reflected in the nature of the imposed growth rates. Included in the work is also the role of the sign of $\chi$ in deriving a maximum principle.

The second assumption we make is the following. Let $e \in \mathbb{R}^n$ denote any unit vector, $I$ be the $n \times n$ identity matrix and $\lambda \in \mathbb{R}$ be a parameter. Set

$$\Lambda_{\min}(\lambda) = \min_{|e|=1} H(e, \lambda e \otimes e - I).$$

We require that $\Lambda_{\min}(\lambda) > 0$ for some $\lambda > 1$. As shown in Section 2, this implies that $\sup_{\lambda>0} \Lambda(\lambda) = \infty$. The $p$-Laplacian, the Infinity-Laplacian, the pseudo $p$-Laplacian and the Pucci operators all satisfy this condition, see [5, 10]. Another operator of interest included here is as follows. Let $H(X) = \sum_{i=1}^{\ell} \mu_i(X)$, $\ell < n$, where $X$ is any symmetric $n \times n$ matrix and $\mu_1(X) \ge \mu_2(X) \ge \cdots \ge \mu_n(X)$ are its eigenvalues. Such partial scalar curvature operators are of great interest and have been considered in many works, see [17] for a detailed discussion.

The current work, in a sense, complements the work in [6] wherein we study the case $\sup_\lambda[\max_{|e|=1} H(e, \lambda e \otimes e + I) < \infty$. As an example, the operator $H(X) = \sum_{i=m}^{n} \mu_i(X)$, $m > 1$ (see the above paragraph) is included in this work. Another example would be $|Du|^2 \Delta u - \langle D|Du|^2, Du\rangle/2$. Our results in the current work do not apply to these instances.

As shown in [6] and the current work, the behaviour of $H(e, \lambda e \otimes e \pm I)$ for large $\lambda$ influences greatly the nature of the imposed growth rates. A comparison of the main results shows that for smaller values of $\sigma$, the maximum principle discussed in [6] appears to hold under growth rates at infinity which are greater than ones imposed in the current work (shown to be optimal in many cases, see Section 8) even allowing for exponential growth rates in some cases. In the current work, however, for small $\sigma$, the minimum principle holds without requiring any lower bound. A lower bound is needed in all instances in [6].

Another point of contrast is that, in the current work, $0 < \inf Z \le \sup Z < \infty$ and the two bounds play an important role. In [6], $Z \ge 0$ can vanish and its upper bound does not appear to have a significant role in the work. Also, unlike the present work, a lower bound on $H(e, \lambda e \otimes e - I)$ is imposed, in some instances, to derive a minimum principle. We should also point out that we do not address optimality in [6]. It is not clear to us if rates greater than stated in the results could apply.

In both the works, the influence of the sign of $\chi$ is important and some differences are seen even here. If $\chi < 0$ then the maximum principle in [6] holds without any restrictions for a greater range of $\sigma$ than in the current work. The converse, however, appears to be true of the minimum principle if $\chi > 0$, in particular, this holds for any $\sigma \geq 0$ in the current work. It is also to be noted that for certain range of values of $\sigma$, the results turn out to be quite similar.

We have divided our work as follows. In Section 2, we present notation, assumptions and the main results. In Sections 3 and 4, we present comparison principles, a change of variables result and calculations for some of the auxiliary functions. Sections 5 and 6 address the super-solutions and sub-solutions respectively. Section 7 presents proofs of the main results. Finally, Section 8 addresses the matter of optimality.

We do not address existence and uniqueness issues in this work. It would be interesting to know if the growth rates stated in this work would imply such results. We do address the issue of optimality of the various growth rates imposed on the solutions, although, some of the results are partial in nature.

For additional discussion and motivation, we direct the reader to the works [1, 7, 8, 9, 12, 14].

## 2. Notation, definitions, assumptions and main results

We employ the notion of viscosity solutions and sub-solutions, super-solutions and solutions are all understood in the viscosity sense, see [5, 8] for definitions. We use the notation $usc(lsc)$ for upper(lower) semicontinuous functions. Throughout this work, we assume that the functions $g$ and $h$ will always satisfy (1.1).

By $o$, we denote the origin in $\mathbb{R}^n$ and $e$ denotes a unit vector in $\mathbb{R}^n$. The letters $x$, $y$ will denote points in $\mathbb{R}^n$. Let $S^{n\times n}$ be the set of all symmetric $n \times n$ real matrices, $I$ be the $n \times n$ identity matrix and $O$ the $n \times n$ zero matrix.

We now state the conditions $H$ satisfies.

**Condition A (Monotonicity):** The operator $H\colon \mathbb{R}^n \times S^n \to \mathbb{R}$ is continuous for any $(q, X) \in \mathbb{R}^n \times S^{n\times n}$. We assume that

$$\begin{aligned} &\text{(i) } H(q, X) \leq H(q, Y), \text{ for any } q \in \mathbb{R}^n \text{ and for any } X,\ Y \text{ in } S^{n\times n} \text{ with } X \leq Y, \\ &\text{(ii) } H(q, O) = 0, \text{ for any } q \in \mathbb{R}^n. \end{aligned} \tag{2.1}$$

Clearly, for any $q \in \mathbb{R}^n$ and $X \in S^{n\times n}$ with $X \geq O$, $H(q, X) \geq 0$.

**Condition B (Homogeneity):** There is a constant $k_1 \geq 0$ such that for any $(q, X) \in \mathbb{R}^n \times S^{n\times n}$,

$$\begin{aligned} &\text{(i)} \quad H(\theta q, X) = |\theta|^{k_1} H(q, X), \quad \forall \theta \in \mathbb{R}, \text{ and} \\ &\text{(ii)} \quad H(q, \theta X) = \theta H(q, X), \quad \forall \theta > 0. \end{aligned} \tag{2.2}$$

Note that if $k_1 = 0$ then $H(q, X) = H(q/\theta, X), \forall \theta > 0$. Hence, $H(q, X) = H(X)$.

Our results in this work can be adapted to include the case $H(q, \theta X) = \theta^{k_2} H(q, X)$ where $k_2 > 1$ is a natural number such that Condition A holds. For this work, however, we take $k_2 = 1$.

Before stating the next condition, we introduce additional notation. For a vector $\xi \in \mathbb{R}^n$, we write its component form as $(\xi_1, \xi_2, \ldots, \xi_n)$. Recall that $(\xi \otimes \xi)_{ij} = \xi_i \xi_j$, $i, j = 1, \ldots, n$. Clearly, $\xi \otimes \xi \in S^{n\times n}$ and $\xi \otimes \xi \geq O$.

Recalling that $e \in \mathbb{R}^n$ is a unit vector, define, for every $\lambda \in \mathbb{R}$,

$$\Lambda_{\min}(\lambda) = \min_e H(e, \lambda e \otimes e - I) \text{ and } \Lambda_{\max}(\lambda) = \max_e H(e, \lambda e \otimes e + I). \quad (2.3)$$

By Condition A, $\Lambda_{\min}(\lambda)$ and $\Lambda_{\max}(\lambda)$ are both non decreasing functions of $\lambda$.

**Condition C(Growth at Infinity):** We require that

(i) $\max_{|e|=1} H(e, -I) < 0 < \min_{|e|=1} H(e, I - \lambda e \otimes e), \ \forall \lambda < 1.$

(ii) There exists a $\lambda_0 > 1$ such that $\Lambda_{\min}(\lambda_0) = \min_{|e|=1} H(e, \lambda_0 e \otimes e - I) > 0. \quad (2.4)$

We require $\lambda_0 > 1$ since, $\lambda e \otimes e - I \leq O$, if $\lambda \leq 1$. See Condition A.

**Remark 2.1.** We state some implications of Conditions A, B and C.

By Condition A, $\Lambda_{\min}(\lambda) \geq \Lambda_{\min}(\lambda_0) > 0, \ \forall \lambda \geq \lambda_0$. By Condition B,

$$\Lambda_{\min}(\lambda) = \left(\frac{\lambda}{\lambda_0}\right) \min_{|e|=1} H\left(e, \lambda_0 e \otimes e - \frac{\lambda_0}{\lambda} I\right) \geq \frac{\lambda \Lambda_{\min}(\lambda_0)}{\lambda_0}, \quad \forall \lambda \geq \lambda_0,$$

since $(\lambda_0/\lambda) I \leq I$. Noting that $\lambda e \otimes e - I = \lambda(e \otimes e - (\lambda)^{-1} I)$, we get that

$$H(e, e \otimes e) \geq \frac{\Lambda_{\min}(\lambda)}{\lambda} \geq \frac{\Lambda_{\min}(\lambda_0)}{\lambda_0} > 0 \text{ and } \sup_{\lambda > 0} \Lambda_{\min}(\lambda) = \infty. \quad (2.5)$$

Thus, (2.4) implies that (2.5) holds. If $\min_{|e|=1} H(e, e \otimes e) > 0$ then by the continuity of $H$, Conditions A and B, $\min_{|e|=1} H(e, \lambda_0 e \otimes e - I) > 0$ for some $\lambda_0 > 1$. $\square$

We discuss some examples of $H$. We record the following: for $\lambda \in \mathbb{R}$, the eigenvalues of $\lambda e \otimes e - I$ are (i) $-1$ with multiplicity $n-1$, and (ii) $\lambda - 1$.

**Examples of $H$:** Let $q \in \mathbb{R}^n$ and $X \in S^{n\times n}$. We set

$$H_\lambda = \min_{|e|=1} H(e, \lambda e \otimes e - I).$$

The following operators satisfy Conditions A, B and C.

**(i) $p$-Laplacian:** Define

$$H(q, X) = |q|^{p-2} Tr(X) + (p-2)|q|^{p-2} q_i q_j X_{ij}, \ \ p \geq 2,$$

where $Tr$ denotes the trace. Then $H_\lambda = \lambda(p-1) - (n+p-2)$.

**(ii) Pseudo $p$-Laplacian:** Define $H(q,X) = \sum_{i=1}^n |q_i|^{p-2} X_{ii}$, $p \geq 2$. Then $H_\lambda = \sum_{i=1}^n \lambda |e_i|^p - |e_i|^{p-2} \geq \lambda n^{p/2-1} - n$.

**(iii) Infinity Laplacian:** Define $H(q,X) = \sum_{i,j=1}^n q_i q_j X_{ij}$ then $H_\lambda = \lambda - 1$.

**(iv) Pucci Operators:** Define $H^+(X) = \alpha Tr(X) + (1-n\alpha)\mu_1(X)$ and $H^-(X) = \alpha Tr(X) + (1-n\alpha)\mu_n(X)$, where $0 < \alpha < 1/n$, $\mu_1$ and $\mu_n$ are the largest eigenvalue and the smallest eigenvalue of $X$ respectively, see [10]. Then for $\lambda \geq 0$,

$$H^+(\lambda e \otimes e - I) \geq \lambda[1-(n-1)\alpha] - 1 \quad \text{and} \quad H^-(\lambda e \otimes e - I) = \alpha\lambda - 1.$$

**(v) Partial Scalar Curvature:** Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of $X$. Define for any $1 < \ell \leq n$, $H(X) = \sum_{i=1}^{\ell} \mu_i(X)$. Hence, $H_\lambda = \lambda - \ell$, $\forall \lambda > 0$. As an example, if for $n \geq 2$, $H(x) = \mu_1(X) + \mu_n(X)$, then $H_\lambda = \lambda - 2$. See [17] for more discussion and related works.

Note that Conditions A, B and C are also satisfied by operators $|q|^s H(q,x)$, $s \geq 0$, where $H$ is any of the operators listed above, and are included in this work. □

For the rest of this work, we set

$$k = k_1 + 1 \quad \text{and} \quad \gamma = k_1 + 2 = k + 1. \tag{2.6}$$

Also, $\chi\colon (0,T) \to \mathbb{R}$ is a bounded continuous function and, for some interval $I \subset \mathbb{R}$, $Z\colon I \to \mathbb{R}^+$ is a non-increasing continuous function. We set and require the following.

$$\begin{aligned} &\ell_m = \inf_{s\in I} Z(s), \quad \ell_M = \sup_{s \in I} Z(s) \quad \text{and} \quad 0 < \ell_m \leq \ell_M < \infty, \\ &\alpha_m = \inf_{0<t<T} \chi(t), \quad \alpha_M = \sup_{0<t<T} \chi(t) \quad \text{and} \quad -\infty < \alpha_m \leq \alpha_M < \infty. \end{aligned} \tag{2.7}$$

Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be bounded and continuous, see (1.1). Furthermore, we set and impose that

$$\begin{aligned} &\mu_m = \inf_{x\in\mathbb{R}^n} h(x), \quad \mu_M = \sup_{x \in \mathbb{R}^n} h(x) \ \text{ satisfy } \ -\infty < \mu_m \leq \mu_M < \infty, \\ &\mathcal{H}_m = \min_{|e|=1} H(e, e\otimes e), \ \mathcal{H}_M = \max_{|e|=1} H(e, e \otimes e) \ \text{ satisfy } \ 0 < \mathcal{H}_m \leq \mathcal{H}_M < \infty. \end{aligned} \tag{2.8}$$

We use the following notation wherever there is little possibility for confusion. From here on, set

$$G(u) \equiv H(Du, D^2u + Z(u)Du \otimes Du).$$

We now state the main results of this work. For Theorems 2.2 and 2.3, we assume that Conditions A, B and C hold. We also note that if $Z$ is defined on some interval $[\mu, \infty)$ then, if need be, one can extend $Z$ to

$$Z(s) = \begin{cases} \ell_M\,, & -\infty < s \le \mu, \\ Z(s), & s \ge \mu\,. \end{cases}$$

**Theorem 2.2.** *(Maximum Principle) Let $0 < T < \infty$ and recall (2.6), (2.7) and (2.8). Let $Z\colon [\mu, \infty) \to \mathbb{R}^+$, for some $\mu \le \inf_x h(x)$. Let $u \in usc(\mathbb{R}^n_T)$, $u \ge \mu$, solve*

$$G(u) + \chi(t)|Du|^\sigma - u_t \ge 0, \text{ in } \mathbb{R}^n_T, \text{ and } u(x,0) \le h(x),\ \forall x \in \mathbb{R}^n.$$

*Suppose that there is $\delta > 0$ such that*

$$\sup_{0 \le |x| \le R,\ 0 \le t \le T} u(x,t) = o(R^\delta), \quad \text{as } R \to \infty.$$

*The following hold.*

*In parts (a) and (b), we assume that either (i) $k = 1$ i.e., $\gamma = 2$ and $\delta = 2 - \varepsilon$, for any fixed and small $\varepsilon > 0$, or (ii) $k > 1$ and $\delta = \gamma/k$.*

**(a)** *Let $\sigma = 0$. In both (i) and (ii), we get that*

$$\sup_{\mathbb{R}^n_T} u(x,t) \le \sup_x h(x) + t(\sup_t \chi(t)).$$

**(b)** *Let $0 < \sigma \le \gamma$. In both (i) and (ii), we get that*

$$\sup_{\mathbb{R}^n_T} u(x,t) \le \sup_x h(x).$$

**(c)** *Let $\sigma > \gamma$ and $\delta = \sigma/(\sigma - 1)$. Then*

$$\sup_{\mathbb{R}^n_T} u(x,t) \le \sup_x h(x), \quad \forall k \ge 1.$$

*Moreover, if $\sup_t \chi(t) < 0$ and $\sigma \ge \gamma$ then the following hold for any $k \ge 1$.*

**(d)** *Suppose that $\sigma = \gamma$.*

*If $|\sup_t \chi(t)| > (\sup Z)(\max_{|e|=1} H(e, e \otimes e))$ then part (b) holds without imposing any upper bound.*

*The conclusion in part (b) holds, if $|\sup_t \chi(t)| \le (\sup Z)(\max_{|e|=1} H(e, e \otimes e))$. The upper bound in part (b) is needed.*

**(e)** *Let $\sigma > \gamma$. No upper bound is needed and*

$$\sup_{\mathbb{R}^n_T} u \le \sup_x h(x) + t \left[ \frac{\{(\sup Z)(\max_{|e|=1} H(e, e \otimes e))\}^\sigma}{|\sup_t \chi(t)|^\gamma} \right]^{1/(\sigma - \gamma)}. \qquad \square$$

In part (e), the term $\chi|Du|^\sigma$ dominates the term $Z(u)Du\otimes Du$ thus requiring no upper bound. In Section 8, we show that the growth rates are optimal.

**Theorem 2.3.** *(Minimum Principle) Let $0<T<\infty$ and recall (2.6), (2.7) and (2.8). Suppose that $Z\colon(-\infty,\infty)\to\mathbb{R}^+$. Let $u\in lsc(\mathbb{R}^n_T)$ solve*

$$G(u)+\chi(t)|Du|^\sigma-u_t\le 0,\ \textit{in}\ \mathbb{R}^n_T,\ \textit{and}\ u(x,0)\ge h(x),\ \forall x\in\mathbb{R}^n.$$

*Then the following hold. We impose no restrictions on $u$ in parts (a)-(c), For parts (a)-(e), assume that $\inf_{(0,T)}\chi(t)\le 0$, see part (f) below.*

**(a)** *If $\sigma=0$ then*

$$\inf_{\mathbb{R}^n_T}u(x,t)\ge\inf_x h(x)-t|\inf_t\chi(t)|.$$

**(b)** *If $0<\sigma<\gamma$ then*

$$\inf_{\mathbb{R}^n_T}u(x,t)\ge\inf_x h(x)-t\left[\frac{|\inf_t\chi(t)|^\gamma}{((\inf Z)(\min_{|e|=1}H(e,e\otimes e))^\sigma}\right]^{1/(\gamma-\sigma)}.$$

**(c)** *If $\sigma=\gamma$ and $|\inf_t\chi(t)|<(\inf Z)(\min_{|e|=1}H(e,e\otimes e))$ then $\inf_{\mathbb{R}^n_T}u(x,t)\ge\mu_m$.*

*For parts (d) and (e), assume that there is a $\delta>0$ such that*

$$\sup_{0\le|x|\le R,\ 0\le t\le T}(-u(x,t))=o(R^\delta),\quad \textit{as}\ R\to\infty,$$

**(d)** *Let $\sigma=\gamma$ and $|\inf_t\chi(t)|\ge(\inf Z)(\min_{|e|=1}H(e,e\otimes e))$. Either (i) $k=1$ ($\gamma=2$) and $\delta=2-\varepsilon$ for a fixed, small $\varepsilon>0$, or (ii) $k>1$ ($\gamma>2$) and $\delta=\gamma/k$, then*

$$\inf_{\mathbb{R}^n_T}u(x,t)\ge\inf_x h(x).$$

**(e)** *If $\sigma>\gamma$ and $\delta=\sigma/(\sigma-1)$ then $\inf_{\mathbb{R}^n_T}u(x,t)\ge\inf_x h(x)$.*

**(f)** *If $\inf_t\chi(t)>0$, i.e., $\chi>0$ then no lower bound is needed and*

$$u(x,t)\ge\inf_x h(x),\quad\forall\,0\le\sigma<\infty\,.\qquad\square$$

As seen from parts (a)-(c) and (f), the minimum principle holds, without any restrictions, as $\chi|Du|^\sigma$ is dominated by $Z(u)Du\otimes Du$.

**Remark 2.4.** If $H$ is quasilinear then Conditions A and B imply

$$H(Dw,D^2w+Z(w)Dw\otimes Dw)=H(Dw,D^2w)+Z(w)|Dw|^\gamma H(e,e\otimes e),$$

since $\gamma = k + 1$ and $\sigma = \gamma$.

Take $Z(s) \equiv 1$. Clearly, $\ell_m = \ell_M = 1$. Set $\mathcal{H}_m = \min_{|e|=1} H(e, e \otimes e)$ and $\mathcal{H}_M = \max_{|e|=1} H(e, e \otimes e)$.

We observe the following. If $H(Dw, D^2w) - w_t \geq 0$ then

$$H_\gamma(w) \equiv H(Dw, D^2w + Dw \otimes Dw) - \mathcal{H}_m |Dw|^\gamma - w_t \geq 0,$$

Thus, Theorem 2.2 (b) and (d) apply with $\chi = -\mathcal{H}_m$ and $|\alpha_m| = |\alpha_M| = \mathcal{H}_m \leq \mathcal{H}_M$.

If $H(Dw, D^2w) - w_t \leq 0$ then we take $\chi = -\mathcal{H}_M$ and

$$H_\gamma(w) \equiv H(Dw, D^2w + Dw \otimes Dw) - \mathcal{H}_M |Dw|^\gamma - w_t \leq 0.$$

Thus, Theorem 2.3 (d) applies as $|\alpha_m| = \mathcal{H}_M \geq \mathcal{H}_m$.

The growth rate is $\gamma/k$ for both results. Clearly, our results apply to the case of the parabolic $p$-Laplacian i.e., $\Delta_p u - u_t = 0$, $p \geq 2$ and $\gamma/k = p/(p-1)$. A stronger result for the heat equation is obtained from Theorem 2.5 viewing it as a case of Trudinger's equation. □

Before stating results for a class of doubly nonlinear equations, we introduce a change of variables, see Lemma 2.3 in [5]. See also [3].

Recall that $k = k_1 + 1$ and $\gamma = k_1 + 2$. Let $f\colon [0, \infty) \to [0, \infty)$ be a $C^1$ increasing function. For $k > 1$, we assume that $f^{1/(k-1)}$ is a concave function. Set $\eta(s) = (f(s))^{-1/(k-1)}$. Let $F$ be a primitive of $\eta$ i.e.,

$$F(\tau) - F(\tau_0) = \int_{\tau_0}^{\tau} \eta(s)ds, \quad 0 \leq \tau_0 \leq \tau < \infty. \tag{2.9}$$

Clearly, $F$ is increasing. Let $0 < \varepsilon < 1$. Either

$$\text{(i)} \quad \lim_{\varepsilon \to 0^+} F(1) - F(\varepsilon) < \infty \quad \text{or} \quad \text{(ii)} \quad \lim_{\varepsilon \to 0^+} F(1) - F(\varepsilon) = \infty. \tag{2.10}$$

If (2.10)(i) holds then we take $\tau_0 = 0$ and $F(0) = 0$ in (2.9) and define

$$F(\tau) = \int_0^{\tau} \eta(s)ds, \quad \tau \geq 0. \tag{2.11}$$

If (2.10)(ii) holds then $F(\varepsilon) \to -\infty$, as $\varepsilon \to 0^+$. We take $F$ as

$$F(\tau) = \int^{\tau} \eta(s)ds, \quad \tau > 0. \tag{2.12}$$

Suppose that (2.10)(i) holds then we use (2.11) and define $\phi$ by

$$F(\phi(\tau)) = \tau \text{ and the domain of } \phi \text{ is } [0, \infty). \tag{2.13}$$

Since $F(0) = 0$ and $F$ is increasing, $\phi$ is increasing and $\phi(0) = 0$.

If (2.10)(ii) holds then we use $F$ in (2.12) and define $\phi$ by

$$F(\phi(\tau)) = \tau \text{ and the domain of } \phi \text{ is } (-\infty, \infty). \tag{2.14}$$

Clearly, $\phi$ is increasing.

Moreover, the definition of $\phi$ and (2.9) lead to

$$\begin{cases} \phi'(\tau) = [(f \circ \phi)(\tau)]^{1/(k-1)} \quad \text{and} \quad \phi''(\tau)/\phi'(\tau) = \left\{ f(\theta)^{1/(k-1)} \right\}' \Big|_{\phi(\tau)}. \\ \text{Set } Z(s) = Z(\phi(s)) := \phi''(s)/\phi'(s). \end{cases} \tag{2.15}$$

Since $f^{1/(k-1)}$ is concave and $\phi$ is increasing (2.15) shows that $Z$ is non-increasing in $s$ and the domain of $Z$ contains $(0, \infty)$. By Lemma 2.3 in [5], if $u > 0$ solves

$$H(Du, D^2u) - f(u)u_t \geq (\leq) 0, \quad \text{in } \mathbb{R}^n_T,$$

then $u = \phi(v)$ solves

$$H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t \geq (\leq) 0, \quad \text{where} \quad Z(v) = \phi''(v)/\phi'(v).$$

Since $Z$ is non-increasing, a comparison principle (see Lemma 3.3 and Corollary 3.4 in Section 3) holds.

Next, (2.7) together with (2.15) implies that

$$\begin{cases} (\ell_m \theta + a)^{k-1} \leq f(\theta) \leq (\ell_M \theta + b)^{k-1}, & \theta \geq 0 \\ \hat{A} \exp(\ell_m s) - \bar{A} \leq \phi(s) \leq \hat{B} \exp(\ell_M s) - \bar{B}, \end{cases}$$

for some $a \geq 0$, $b \geq 0$, $\hat{A} > 0$, $\hat{B} > 0$. Here, either $\hat{A} = \bar{A}$ or $\bar{A} = 0$ and $\hat{B} = \bar{B}$ or $\bar{B} = 0$. An example is $f(s) = (s + h(s))^{k-1}$, $s \geq 0$, where $h(s) \geq 0$ and $h$ is concave such as $s^\alpha$, $0 < \alpha < 1$, $\log(s+1)$ and $\tan^{-1}(s)$.

We now state the final result of the work. Some of the claims follow from Theorems 2.2(a) and 2.3(a) with $\alpha = \sigma = 0$. The domain of $Z$ is either (i) $(0, \infty)$ or $[0, \infty)$, or (ii) $(-\infty, \infty)$.

**Theorem 2.5.** *Let $k \geq 1$, $f\colon [0, \infty) \to [0, \infty)$ be a $C^1$ non-decreasing function, and $g\colon \mathbb{R}^n \to (0, \infty)$ be such that $0 < \inf_{x \in \mathbb{R}^n} g(x) \leq \sup_{x \in \mathbb{R}^n} g(x) < \infty$.*

*Recall (2.7), (2.13), (2.14) and (2.15).*

**(i)** *Let $k > 1$. We assume that $f^{1/(k-1)}$ is concave and*

$$0 < \inf_{0 \leq s < \infty} \frac{d}{ds} f^{1/(k-1)}(s) \leq \sup_{0 \leq s < \infty} \frac{d}{ds} f^{1/(k-1)}(s) < \infty.$$

*Select $\phi\colon \mathbb{R} \to [0, \infty)$, a $C^2$ increasing function such that*

$$\phi'(\tau) = f(\phi(\tau))^{1/(k-1)}.$$

**(a)** *Let* $u \in usc(\overline{\mathbb{R}^n_T})$, $u > 0$, *solve*

$$H(Du, D^2u) - f(u)u_t \geq 0, \text{ in } \mathbb{R}^n_T \text{ and } u(x,0) \leq g(x),\ \forall x \in \mathbb{R}^n.$$

*Suppose that* $\sup_{|x|\leq R,\ 0\leq t\leq T} u(x,t) \leq \phi(o(R^{\gamma/k}))$, *as* $R \to \infty$.

*Then* $\sup_{\mathbb{R}^n_T} u(x,t) \leq \sup_{x\in\mathbb{R}^n} g(x)$.

**(b)** *Let* $u \in lsc(\overline{\mathbb{R}^n_T})$, $u > 0$, *solve*

$$H(Du, D^2u) - f(u)u_t \leq 0, \text{ in } \mathbb{R}^n_T \text{ and } u(x,0) \geq g(x),\ \forall x \in \mathbb{R}^n.$$

*Then* $\inf_{\mathbb{R}^n_T} u(x,t) \geq \inf_{x\in\mathbb{R}^n} g(x)$. *No lower bound is needed.*

**(ii)** *If* $k = 1$, *we take* $f \equiv 1$ *and* $\phi(\tau) = e^\tau$. *The conclusion in part (i)(a) holds provided that we assume that, for any* $\varepsilon > 0$, $\sup_{|x|\leq R,\ 0\leq t\leq T} u(x,t) \leq \exp(o(R^{2-\varepsilon}))$, *as* $R \to \infty$. *The conclusion in part (i)(b) holds without any modifications.*

We provide proofs of the main results in Section 7. Section 8 addresses optimality.

## 3. Preliminaries

In this section, we present some calculations important for our work, a comparison principle and a change of variable result useful for our work. Some additional discussion about the condition in (2.4) is also included.

For definitions and a discussion of viscosity solutions, we direct the reader to [8] and Section 2 in [3].

Recall that for some $\mu \in \mathbb{R}$, $Z\colon [\mu,\infty) \to \mathbb{R}^+$ is continuous, non-increasing and (2.7) holds, i.e.,

$$0 < \ell_m = \inf_{\mathbb{R}} Z \leq \ell_M = \sup_{\mathbb{R}} Z < \infty. \tag{3.1}$$

We present some elementary calculations. Let $z \in \mathbb{R}^n$ and $r = |x - z|$. For $0 < R \leq \infty$, let $B_R(z)$ be the ball of radius $R$ with center $z$. We define $B_\infty(z) = \mathbb{R}^n$. Also, set $B^R_T = B_R(z) \times (0,T)$ and $P^R_T$ be its parabolic boundary.

Suppose that $v(x) = v(r)$ is a $C^2$ function. Set $e = (e_1, e_2, \ldots, e_n)$ where $e_i = (x-z)_i/r$, $\forall i = 1,2,\ldots,n$. For $x \neq z$,

$$\begin{cases} Dv = v'(r)e, \quad Dv \otimes Dv = (v'(r))^2 e \otimes e, \quad \text{and} \\ D^2v = (v'(r)/r)\,(I - e\otimes e) + v''(r)e\otimes e. \end{cases} \tag{3.2}$$

**Remark 3.1.** Let $\kappa\colon [0,T] \to (0,\infty)$ be a $C^1$ function and $Z$ be as in (3.1). Suppose that $D \subset \mathbb{R}^{n+1}$ is a domain and $(z,t) \in D$. Let $w\colon D \to \mathbb{R}$ be $C^1$ in $x$ and $t$, in $D$, and $C^2$ in $x$ in $D \setminus \{(z,t)\}$. Set $r = |x-z|$, $w(r,t) = w(x,t)$ and assume that $w_r \neq 0$ for $r \neq 0$ and $w \geq \mu$.

Using (3.2) in $r > 0$, we get that

$$G(w) = H\left(w_r e,\ \frac{w_r}{r}(I - e\otimes e) + \left(w_{rr} + Z(w)(w_r)^2\right) e\otimes e\right). \tag{3.3}$$

We recall Condition B in (2.2), (2.6) i.e, $k = k_1 + 1$ and $\gamma = k_1 + 2$, and derive two versions for $G(w)$ from (3.3). For the first, we factor $w_r$ from the first entry, $|w_r|/r$ from the second. For the second version, we factor $w_r$ from the first entry and $w_r^2$ from the second. Thus,

$$G(w) = \frac{|w_r|^k}{r} H\left(e, \frac{w_r}{|w_r|}(I - e\otimes e) + \left( r|w_r|Z(w) \ + \frac{rw_{rr}}{|w_r|}\right) e\otimes e\right) \quad \text{and}$$
$$G(w) = |w_r|^\gamma H\left(e, \ \frac{I - e\otimes e}{rw_r} + \left(\frac{w_{rr}}{w_r^2} + Z(w)\right) e\otimes e\ \right). \tag{3.4}$$

**Case (i)** $w_r > 0$**:** Let $a$ be any scalar and $b \geq 0$. Suppose that $w(x,t) = (a + bv(r))\kappa(t)$, where $v'(r) > 0$ and $\kappa \geq 0$. The first version in (3.4) yields, in $r > 0$,

$$G(w) = \frac{(bv'(r)\kappa(t))^k}{r} H\left(e, \ I + \left(\frac{rv''(r)}{v'(r)} - 1 \ + \ b\kappa(t)(rv'(r))Z(w) \ \right) e\otimes e\ \right). \tag{3.5}$$

This version will be used for small values of $r$.

Apply the second version in (3.4) to obtain in $r > 0$,

$$G(w) = (bv'(r)\kappa(t))^\gamma H\left(e, \ \frac{I - e\otimes e}{b\kappa(t)(rv'(r))} \ + \left(\frac{v''(r)}{b\kappa(t)(v'(r))^2} + Z(w)\right) e\otimes e\right). \tag{3.6}$$

We use this version for large values of $r$.

In this work, we take $0 < b < 1$. By factoring $1/b$ from the second entry in $H$ (in (3.6)), using Condition B and $\gamma = k + 1$, the above may be rewritten as

$$G(w) = b^k(\kappa(t)v'(r))^\gamma H\left(e, \ \frac{I - e\otimes e}{\kappa(t)rv'(r)} \ + \left(\frac{v''(r)}{\kappa(t)(v'(r))^2} + bZ(w)\right) e\otimes e\right). \tag{3.7}$$

**Case (ii)** $w_r < 0$**:** Set $w(x,t) = v(r) - \kappa(t)$, where $v'(r) < 0$. We use (3.4) and argue as in part (i). We obtain

$$G(w) = \frac{|v'(r)|^k}{r} H\left(e, \ \left( r|v'(r)|Z(w) \ + 1 - \frac{rv''(r)}{v'(r)}\right) e\otimes e - I\right), \quad r \text{ small}$$
$$G(w) = |v'(r)|^\gamma H\left(e, \ \frac{I - e\otimes e}{rv'(r)} + \left(\frac{v''(r)}{(v'(r))^2} + Z(w)\right) e\otimes e\ \right), \quad r \text{ large}. \tag{3.8}$$

□

We now state a comparison principle which is a slight adaptation of Theorem 8.2 in [8]. See also Section 4 in [5].

Let $F\colon \mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^n\times S^{n\times n}\to\mathbb{R}$ be continuous. Suppose that $\forall X,\ Y\in S^{n\times n}$, with $X\le Y$, $F$ satisfies

$$F(t,r_1,p,X)\le F(t,r_2,p,Y),\ \forall(t,p)\in\mathbb{R}^+\times\mathbb{R}^n \text{ and } r_1\ge r_2. \tag{3.9}$$

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and $T>0$. Set $\Omega_T=\Omega\times(0,T)$ and $P_T$ its parabolic boundary.

**Lemma 3.2** (Comparison principle)**.** *Let $F$ satisfy (3.9), $\zeta\colon[\mu,\infty)\to\mathbb{R}$, for some $\mu\in\mathbb{R}$, be a bounded non-increasing continuous function and $\hat f\colon\mathbb{R}^+\to\mathbb{R}^+$ be continuous. Let $u\in usc(\Omega_T\cup P_T)$ and $v\in lsc(\Omega_T\cup P_T)$ satisfy in $\Omega_T$,*

$$F(t,u,Du,D^2u+\zeta(u)Du\otimes Du)-\hat f(t)u_t\ge 0 \ \text{ and}$$
$$F(t,v,Dv,D^2v+\zeta(v)Dv\otimes Dv)-\hat f(t)v_t\le 0.$$

*If $\inf(u,v)\ge\mu$, $\sup_{P_T}v<\infty$ and $u\le v$ on $P_T$ then $u\le v$ in $\Omega_T$.* □

Lemma 3.2 leads to a comparison principle for

$$H(Du,D^2u)-f(u)u_t=0,\quad \text{where } u>0.$$

This is shown in Lemma 2.3 in [5]. An earlier version appears in [3].

We employ a change of variables $u=\phi(v)$ for our purpose, where $\phi$ is defined in (2.13) and (2.14). Recall from (2.15) that

$$Z(v)=\phi''(v)/\phi'(v). \tag{3.10}$$

Then $Z$ is non-increasing and the domain of $Z$ is $(0,\infty)$ or $[0,\infty)$, or $(-\infty,\infty)$

We state the following change of variables lemma which is a simplified version of Lemma 2.3 in [5].

**Lemma 3.3.** *Let $H$ satisfy Conditions A and B, see (2.1) and (2.2) and let $f\colon[0,\infty)\to[0,\infty)$ be a $C^1$ increasing function. Assume that $f^{1/(k-1)}$ is concave if $k>1$, and $f\equiv 1$ if $k=1$. Suppose that $\phi$ is defined either as in (2.13) or as in (2.14).*

**Case (i):** *Let $k>1$ and $Z$ be as in (3.10). We assume that $f$ is non-constant, $u>0$ and $v=\phi^{-1}(u)$.*

*Then $u\in usc(lsc)(\Omega_T)$ solves $H(Du,D^2u)-f(u)u_t\ge(\le)0$ in $\Omega_T$ if and only if $v\in usc(lsc)(\Omega_T)$ and*

$$H\left(Dv,D^2v+Z(v)Dv\otimes Dv\right)-v_t\ge(\le)0 \ \text{ in } \Omega_T.$$

**Case (ii):** *Let $k=1$, i.e., $k_1=0$. If $f\equiv 1$ and $\phi(\tau)$ is any increasing positive $C^2$ function then the claim in (i) holds with $u=\phi(v)$. In particular, if $\phi(\tau)=e^\tau$ and $u\in usc(lsc)(\Omega_T)$, $u>0$, then $H(D^2u)-u_t\ge(\le)0$ if and only if $v\in usc(lsc)(\Omega_T)$ and $H(D^2v+Dv\otimes Dv)-v_t\ge(\le)0$.*

**Corollary 3.4.** *(Comparison principle) Let $f$ and $\phi$ be as described in Lemma 3.3. Suppose that $u \in usc(\Omega_T)$, $u > 0$, and $v \in lsc(\Omega_T)$, $v > 0$, satisfy*

$$H(Du, D^2u) - f(u)u_t \geq 0 \ \ \text{and} \ \ H(Dv, D^2v) - f(v)v_t \geq 0 \ \ \text{in } \Omega_T.$$

*If $u \leq v$ in $P_T$ then $u \leq v$ in $\Omega_T$.*

*Proof.* We assume that $k > 1$. If $k = 1$ and $f \equiv 1$ then Lemma 3.2 applies directly. For $k > 1$, we apply the change of variables $u = \phi(\bar{u})$ and $v = \phi(\bar{v})$ and Lemma 3.3 shows that

$$H(D\bar{u}, D^2\bar{u} + Z(\bar{u})D\bar{u} \otimes D\bar{u}) - \bar{u}_t \geq 0 \ \ \text{and} \ \ H(D\bar{v}, D^2\bar{v} + Z(\bar{v})D\bar{v} \otimes D\bar{v}) - \bar{v}_t \geq 0$$

in $\Omega_T$. By (2.13) and (2.14), the domain of $\phi$ is either (i) $(-\infty, \infty)$, the domain of $Z$ is $(-\infty, \infty)$ and $-\infty < \bar{u}, \ \bar{v} < \infty$, or (ii) $[0, \infty)$, the domain of $Z$ is at least $(0, \infty)$ and $0 < \bar{u}, \ \bar{v} < \infty$. Since $\bar{u} \leq \bar{v}$ in $P_T$, Lemma 3.2 implies that $\bar{u} \leq \bar{v}$ in $\Omega_T$ and the claim holds. □

## 4. Auxiliary Functions

In this section, we construct auxiliary functions that are used in this work. Recall that $k = k_1 + 1$ and $\gamma = k_1 + 2 = k + 1$. Through out this section, $z \in \mathbb{R}^n$ is a fixed point and $r = |x - z|$. We begin with

**Lemma 4.1.** *Let $1 < \bar{\beta} < \beta$. For $r \geq 0$, define*

$$v(r) = \int_0^{r^\beta} (1 + \tau^p)^{-1} \, d\tau, \quad \text{where } \ p = \frac{\beta - \bar{\beta}}{\beta}.$$

*Then:*

***(i)*** $0 < p < 1$, ***(ii)*** $(1 - p)\beta = \bar{\beta}$, *and*

***(iii)*** $\min(r^\beta, \ r^{\bar{\beta}})/2 \leq r^\beta/(1 + r^{\beta p}) \leq v(r) \leq \beta \min(r^\beta, \ r^{\bar{\beta}})/\bar{\beta}, \ \ \forall \, r \geq 0,$

***(iv)*** *If $R \geq 1$ then* $\left(\dfrac{\beta}{2\bar{\beta}}\right)(r^{\bar{\beta}} - R^{\bar{\beta}}) \leq v(r) - v(R) \leq \left(\dfrac{\beta}{\bar{\beta}}\right)(r^{\bar{\beta}} - R^{\bar{\beta}}), \ \ \forall \, r \geq R.$

*Moreover, in $r > 0$, we have*

***(v)*** $\dfrac{\beta}{2}\min(r^{\beta-1}, r^{\bar{\beta}-1}) \leq v'(r) = \dfrac{\beta r^{\beta-1}}{1+r^{p\beta}} \leq \beta \min\left(r^{\bar{\beta}-1}, r^{\beta-1}\right),$

***(vi)*** $\dfrac{\beta}{2}\min(r^{\beta}, r^{\bar{\beta}}) \leq rv'(r) \leq \beta \min\left(r^{\bar{\beta}},\ r^{\beta}\right),$

***(vii)*** $\dfrac{(v'(r))^k}{r} = \dfrac{\beta^k r^{k\beta-\gamma}}{(1+r^{p\beta})^k}, \quad v''(r) = \beta r^{\beta-2}\left[\dfrac{\beta-1+(\bar{\beta}-1)r^{p\beta}}{(1+r^{p\beta})^2}\right],$

***(viii)*** $\left(\dfrac{\beta}{2}\right)^k \min\left(r^{k\beta-\gamma},\ r^{k\bar{\beta}-\gamma}\right) \leq \dfrac{(v'(r))^k}{r} \leq \beta^k \min\left(r^{k\beta-\gamma},\ r^{k\bar{\beta}-\gamma}\right),$

***(ix)*** $\bar{\beta}-1 \leq \dfrac{rv''(r)}{v'(r)} = \dfrac{\beta-1+(\bar{\beta}-1)r^{p\beta}}{1+r^{p\beta}} \leq \beta - 1,$

***(x)*** $\dfrac{v''(r)}{(v'(r))^2} = \left(\dfrac{\beta-1}{\beta}\right) r^{-\beta} + \left(\dfrac{\bar{\beta}-1}{\beta}\right) r^{-\bar{\beta}},$ *and*

***(xi)*** $\dfrac{(\bar{\beta}-1)r^{-\bar{\beta}}}{\beta} \leq \dfrac{v''(r)}{(v'(r))^2} \leq \dfrac{2(\beta-1)r^{-\bar{\beta}}}{\beta},\ \forall r \geq 1.$

*Proof.* Parts (i) and (ii) follow easily. Part (iii) is a consequence of the bounds $1+\tau^p \geq \tau^p$ and $1+\tau^p \leq 1+r^{p\beta}$, $\forall \tau \leq r^{\beta}$. Part (iv) follows by noting part (ii), that $\tau^p \leq 1+\tau^p \leq 2\tau^p$, $\tau \geq 1$, and writing

$$v(r) = v(R) + \int_{R^\beta}^{r^\beta} (1+\tau^p)^{-1} d\tau.$$

Parts (v), (vi) and (viii) are easily obtained from the estimate $1 + r^{p\beta} \geq \max(1,\ r^{p\beta})$ and noting that $\gamma = k+1$ and $\beta - \bar{\beta} = p\beta$.

To see (vii), we differentiate (v) and use (ii) to find

$$v''(r) = \beta\left[\frac{(\beta-1)r^{\beta-2}}{1+r^{p\beta}} - \frac{p\beta r^{p\beta+\beta-2}}{(1+r^{p\beta})^2}\right] = \beta r^{\beta-2}\left[\frac{(\beta-1)(1+r^{p\beta}) - p\beta r^{p\beta}}{(1+r^{p\beta})^2}\right]$$
$$= \beta r^{\beta-2}\left[\frac{\beta-1+(\bar{\beta}-1)r^{p\beta}}{(1+r^{p\beta})^2}\right].$$

Applying (v), (vii) and using $\bar{\beta} < \beta$, (ix) follows. To see (x) and (xi), use (ii), (v) and (vii) to get

$$\frac{v''(r)}{(v'(r))^2} = \frac{\beta-1+(\bar{\beta}-1)r^{p\beta}}{\beta r^{\beta}} = \frac{\beta-1}{\beta r^{\beta}} + \frac{\bar{\beta}-1}{\beta r^{\bar{\beta}}}.$$

Since $\bar{\beta} < \beta$, (xi) holds in $r \geq 1$. □

**Remark 4.2.** We list some consequences of Lemma 4.1. These are used in the proofs of Theorems 2.2 and 2.3. The functions of $r$ are $C^1$ in $x$, in $\mathbb{R}^n$, and $C^2$ in $x$ in $r > 0$.

Recall that $k = k_1 + 1$, $\gamma = k + 1 = k_1 + 2$ and $\sigma$ is as in Theorems 2.2 and 2.3. Set

$$\gamma^* = \gamma/k \quad \text{and} \quad \sigma^* = \sigma/(\sigma - 1),\ \forall \sigma > 1.$$

We have divided our work into three cases.

**Case (A) ($k = 1$):** Take $\beta = 2$, $\bar{\beta} = 2 - \varepsilon$, $0 < \varepsilon < 1$. From Lemma 4.1, $p = \varepsilon/2$ and

$$v(r) = \int_0^{r^2} (1 + \tau^{\varepsilon/2})^{-1}\, d\tau, \ \ 0 < \varepsilon < 1.$$

We apply Lemma 4.1 (iii), (iv), (vi), (vii), (viii), (ix) and (xi). Thus,

$$\textbf{(iii)} \quad \frac{\min\left(r^{2-\varepsilon},\ r^2\right)}{2} \le v(r) \le 2\min(r^{2-\varepsilon},\ r^2),\ \forall r \ge 0,$$

$$\textbf{(iv)} \quad \frac{r^{2-\varepsilon} - R^{2-\varepsilon}}{2} \le v(r) - v(R) \le 2\left(r^{2-\varepsilon} - R^{2-\varepsilon}\right),\ \forall r \ge R \ge 1.$$

Noting that $\gamma = 2$ and $k\beta - \gamma = 0$, we find that, in $r > 0$,

$$\textbf{(vi)}\ \ 1 \le \frac{rv'(r)}{\min\left(r^{2-\varepsilon},\ r^2\right)} \le 2, \quad \textbf{(viii)}\ \min(1, r^{-\varepsilon}) \le \frac{v'(r)}{r} \le 2\min(1,\ r^{-\varepsilon}),$$

$$\textbf{(ix)}\ \ 1 - \varepsilon \le \frac{rv''(r)}{v'(r)} \le 1, \quad \textbf{(xi)}\ \ \frac{1-\varepsilon}{2r^{2-\varepsilon}} \le \frac{v''(r)}{(v'(r))^2} \le \frac{1}{r^{2-\varepsilon}},\ \ \forall r \ge 1.$$

**Case (B) ($k > 1$):** Set $\beta = \bar{\beta} = \gamma^*$ and $v(r) = r^{\gamma^*}$. Using that $\gamma = k + 1$ and $k(\gamma^* - 1) = 1$, we have

$$\textbf{(vi)}\quad rv'(r) = \gamma^* r^{\gamma^*}, \quad \textbf{(viii)}\quad \frac{(v'(r))^k}{r} = (\gamma^*)^k, \quad \textbf{(ix)}\quad \frac{rv''(r)}{v'(r)} = \gamma^* - 1 = \frac{1}{k},$$

$$\textbf{(xi)}\quad \frac{v''(r)}{(v'(r))^2} = \left(\frac{\gamma^* - 1}{\gamma^*}\right) r^{-\gamma^*} = \frac{1}{\gamma r^{\gamma^*}}.$$

**Case (C) ($k \ge 1$):** Set $\beta = \gamma^*$ and $\bar{\beta} = \sigma^*$, where $\sigma > \gamma$.

Since $\sigma > \gamma$, we have that $\beta > \bar{\beta}$. We get that

$$p = \frac{\beta - \bar{\beta}}{\beta} = \frac{\gamma(\sigma - 1) - k\sigma}{\gamma(\sigma - 1)} = \frac{\sigma - \gamma}{\gamma(\sigma - 1)} > 0 \ \text{ and } \ k\beta - \gamma = 0.$$

Set

$$v(r) = \int_0^{r^{\gamma^*}} (1 + \tau^p)^{-1}\, d\tau.$$

We apply parts (iii), (iv), (vii), (viii), (ix) and (xi) of Lemma 4.1.
In $r > 0$, parts (iii) and (iv) read

$$\textbf{(iii)}\ \ \frac{1}{2} \le \frac{v(r)}{\min(r^{\gamma^*},\ r^{\sigma^*})} \le \frac{\gamma^*}{\sigma^*}, \quad \textbf{(iv)}\ \ \frac{\gamma^*}{2\sigma^*} \le \frac{v(r) - v(R)}{r^{\sigma^*} - R^{\sigma^*}} \le \frac{\gamma^*}{\sigma^*},\ \forall r > R \ge 1.$$

Next, in $r > 0$,

$$\textbf{(vi)}\ \frac{\gamma^* \min\left(r^{\sigma^*},\ r^{\gamma^*}\right)}{2} \le rv'(r) \le \gamma^* \min\left(r^{\sigma^*},\ r^{\gamma^*}\right),$$

$$\textbf{(viii)}\ \left(\frac{\gamma^*}{2}\right)^k \min\left(1,\ \frac{1}{r^{(\sigma-\gamma)/(\sigma-1)}}\right) \le \frac{(v'(r))^k}{r} \le (\gamma^*)^k \min\left(1,\ \frac{1}{r^{(\sigma-\gamma)/(\sigma-1)}}\right).$$

The versions in (iii) and (iv) may be rewritten so that (iii) holds in $r \ge 0$ and (iv) in $r \ge R$.

Since $\sigma > \gamma \ge 2$ and $\gamma^* - 1 = 1/k$, Lemma 4.1 (ix) and (xi) read

$$\textbf{(ix)}\ \frac{1}{\sigma} \le \frac{rv''(r)}{v'(r)} \le \frac{1}{k}, \qquad \textbf{(xi)}\ \frac{1}{\gamma^* \sigma r^{\sigma^*}} \le \frac{v''(r)}{(v'(r))^2} \le \frac{2}{r^{\sigma^*}}, \quad \forall\, r \ge 1.$$

Since $k\beta - \gamma = 0$, parts (vii) and (ix) of Lemma 4.1 imply that

$$\lim_{r\to 0} \frac{(v'(r))^k}{r} = (\gamma^*)^k \quad \text{and} \quad \lim_{r\to 0} \frac{rv''(r)}{v'(r)} = \frac{1}{k}.$$

Some of the versions versions stated in Cases A, B and C may be rewritten so that they hold in $r \ge 0$ and in $r \ge R$. □

Next, we study a second auxiliary function that is used in this work.

**Lemma 4.3.** *Let $0 < R < \infty$, $0 \le r < R$ and $p > 0$. Set $r = |x|$, $x \in \mathbb{R}^n$, define $\omega = r/R$ and*

$$v_E(\omega) = v_E(r) = E \int_0^{\omega^2} (1 - \tau^p)^{-1} d\tau = ER^{2(p-1)} \int_0^{r^2} (R^{2p} - s^p)^{-1} ds,$$

*where $E \ne 0$. Set $sgn(E)$ =sign of $E$ and*

$$L_p(\omega) = \frac{2E}{1 - \omega^{2p}}, \quad \forall\, 0 \le \omega < 1.$$

*Then* **(i)** *$v_E(0) = 0$, $sgn(v_E) = sgn(E)$ and $|v_E(r)| \to \infty$ as $r \to R$. Also, for $r < R$, $v_E(r) \to 0$ as $R \to \infty$.*

$$\textbf{(ii)}\ v_E'(r) = \frac{L_p(\omega)\omega}{R} = \frac{L_p(\omega) r}{R^2}, \qquad \textbf{(iii)}\ v_E''(r) = \frac{L_p(\omega)}{R^2}\left(\frac{1 + (2p-1)\omega^{2p}}{1 - \omega^{2p}}\right),$$

$$\textbf{(iv)}\ \frac{rv_E''(r)}{v_E'(r)} = \frac{1 + (2p-1)\omega^{2p}}{1 - \omega^{2p}}, \qquad \textbf{(v)}\ \frac{rv_E''(r)}{v_E'(r)} - 1 = \frac{2p\omega^{2p}}{1 - \omega^{2p}},$$

$$\textbf{(vi)}\ \frac{rv_E''(r)}{v_E'(r)} - 1 + rZ(\cdot)|v_E'(r)| = \frac{2|E|Z(\cdot)\omega^2 + 2p\omega^{2p}}{1 - \omega^{2p}}, \ \forall E > 0,$$

$$\textbf{(vii)}\ 1 - \frac{rv_E''(r)}{v_E'(r)} + rZ(\cdot)|v_E'(r)| = \frac{2|E|Z(\cdot)\omega^2 - 2p\omega^{2p}}{1 - \omega^{2p}}, \ \forall E < 0,$$

$$\textbf{(viii)}\ \frac{|v_E'(r)|^k}{r} = \frac{|L_p(\omega)|^k \omega^k}{R^k \omega R} = \frac{|L_p(\omega)|^k \omega^{k_1}}{R^\gamma}.$$

*Moreover, $v_E$ is $C^2$ in $x$, in $0 \le r < R$.*

*Proof.* Parts (ii) and (iii) follow from a differentiation. The rest follow from (ii) and (iii). In part (viii), we use $r = \omega R$, $k = k_1 + 1$ and $\gamma = k_1 + 2$. □

**Remark 4.4.** The sub-solutions and super-solutions in this work involve a $C^1$ function of $x$ and $t$, and $C^2$ in $x$, except at $x = z$. See Remark 4.2. We verify that the expressions for the operator $H$ hold in the sense of viscosity at $r = 0$ and any $0 < s < T$.

By Lemma 4.1, $v'(r) \neq 0$ in $r \neq 0$. Also, $v(0) = v'(0) = 0$. Let $\kappa(t) \ge 0$ be a $C^1$ function in $t \ge 0$. Recall that $\gamma^* = \gamma/k$ and $\sigma^* = \sigma/(\sigma - 1)$, for $\sigma > 1$.

We now refer to Remark 4.2. Note that in Case (A) and in the sub-case $k = 1$ of Case C, $\gamma^* = 2$ and $v$ is $C^2$ everywhere.

We address $k > 1$ in Cases B and C, i.e., $1 < \gamma^* < 2$. Set $r = |x|$ and

$$w(x,t) = (a + bv(r))\kappa(t), \quad \text{where } b > 0 \text{ and}$$

$$v(r) = \begin{cases} r^{\gamma^*}, & \bar{\beta} = \beta = \gamma^*, \\ \int_0^{r^{\gamma^*}} (1 + \tau^p)^{-1} d\tau, & \beta = \gamma^*,\ \bar{\beta} = \sigma^*. \end{cases}$$

We apply (3.5) in Remark 3.1. Taking $r > 0$ and setting $e = x/r$ and $w = \kappa(t)v(r)$, we get after a slight rearrangement that

$$\begin{aligned} G(w) + \chi(t)|Dw|^\sigma - w_t &= \chi(t)(\kappa(t))^\sigma |bv'(r)|^\sigma - \kappa'(t)(a + bv(r)) \\ &+ \frac{(bv'(r)\kappa(t))^k}{r} H\left(e,\ I + \left(\frac{rv''(r)}{v'(r)} - 1 \ + \ b\kappa(t)(rv'(r))Z(w)\right) e \otimes e\right). \end{aligned} \tag{4.1}$$

From Cases (B) and (C) in Remark 4.2 $rv''(r)/v'(r) \to 1/k$ and $(v'(r))^k/r \to (\gamma^*)^k$ as $r \to 0$. It is clear that the right hand side of (4.1) may be extended continuously to $r = 0$. Set the limit $r \to 0$ of the right hand side of (4.1) as

$$\hat{H}(0) + \chi(t)L(\sigma) - a\kappa'(t), \quad \text{where } \hat{H}(0) = (\gamma^* b\kappa(t))^k H\left(e,\ I - [(k-1)/k] e \otimes e\ \right),$$

and $L(\sigma) = 1$ if $\sigma = 0$, and $L(\sigma) = 0$ if $\sigma \neq 0$.

Note that $\hat{H}(0) \ge 0$ since $(k-1)/k < 1$.

We show that

$$G(w) + \chi(s)|Dw|^\sigma - w_t = \hat{H}(0) + \chi(s)L(\sigma) - a\kappa'(s), \tag{4.2}$$

holds at points $(o, s)$, i,e., at $r = 0$ and $0 < s < T$, in the viscosity sense.

Let $s > 0$. Suppose that $\psi$, $C^1$ in $t$ and $C^2$ in $x$, is such that $(w - \psi)(x,t) \le (w - \psi)(o,s)$, for $(x,t)$ near $(o,s)$. Since $w(o,s) = a\kappa(s)$, we have that

$$a(\kappa(t) - \kappa(s)) + bv(r)\kappa(t) \le \langle D\psi(o,s), x\rangle + \psi_t(o,s)(t - s) + o(|x| + |t - s|),$$

as $(x, t) \to (o, s)$. Since $v'(0) = 0$, it follows that $D\psi(o, s) = 0$ and $\psi_t(o, s) = a\kappa'(s)$. Using that

$$bv(r)\kappa(t) \leq \langle D^2\psi(0, s)x, x\rangle/2 + o(|t-s| + |x|^2), \quad \text{as } (x,t) \to (o,s),$$

we see that $D^2\psi(o, s)$ does not exist, since $v(r) \approx r^{\gamma^*}$ ($\gamma^* < 2$) near $r = 0$. Thus, $w$ is a sub-solution.

Now, let $\psi$, $C^1$ in $t$ and $C^2$ in $x$, be such that $(w - \psi)(x, t) \geq (w - \psi)(o, s)$, for $(x, t)$ near $(o, s)$. Thus, $w(x, t) - w(o, s) \geq \langle D\psi(o, s), x\rangle + \psi_t(o, s)(t - s) + o(|x| + |t-s|)$, as $(x, t) \to (o, s)$. Clearly, $D\psi(o, s) = 0$ and $\psi_t(o, s) = a\kappa'(s)$. Since $k > 1$, i.e., $k_1 > 0$, by Condition B, $H(0, X) = 0$. Hence,

$$\begin{aligned} H\left(D\psi, D^2\psi + Z(w)D\psi \otimes D\psi\right)(o,s) + \chi(s)|D\psi|^\sigma(o,s) - \psi_t(o,s) \\ \leq \hat{H}(0) + \chi(s)L(\sigma) - a\kappa'(s). \end{aligned}$$

Thus, $w$ is a super-solution. □

From here on, we include $r = 0$ in applying the expressions in (3.4), (3.5), (3.7) and (3.8).

## 5. Super-solutions

In this section, we construct super-solutions for Theorem 2.2. To achieve this, we employ the auxiliary functions discussed in Remark 4.2. For small $r$, (3.5) is used and, for large $r$, we use (3.6). See Remark 3.1. The two situations are treated separately.

The section has been divided into two parts: (I) $0 \leq \sigma \leq \gamma$ and (II) $\sigma > \gamma$. The work in Part I is further divided into two sub-parts (i) $k = 1$ and (ii) $k > 1$. Part (II) provides a unified treatment for $k \geq 1$.

Fix $z \in \mathbb{R}^n$ and set $r = |x - z|$. Recall that $\mu_M = \sup_{\mathbb{R}^n} h$ with $-\infty < \mu_M < \infty$. Define

$$w(x,t) = \mu_M + at + b(1+t)v(r), \quad \text{in } \mathbb{R}^n_T, \quad \text{where } a \geq 0,\ 0 < b < 1, \tag{5.1}$$

and

$$\forall r \geq 0, \quad v(r) = \int_0^{r^\beta} \frac{1}{1+\tau^p}\, d\tau \qquad \text{or} \qquad v(r) = r^{\bar{\beta}},$$

for an appropriate $\beta$ and $p$ (or $\bar{\beta}$), see Lemma 4.1.

We show that $w$ is a super-solution for an appropriate $b$ small enough, and an $a$ that may depend on $b$. This aids the calculation of $\lim_{b\to 0^+} a$, wherever applicable. Moreover, $w$ is a super-solution for any $0 < b < b_0$ and corresponding $a$, where $b_0 < 1$ is small enough. This is important in showing the claims in Theorem 2.2.

Throughout this section $\beta = \gamma/k = \gamma^*$, see (5.1) and Remark 4.2. The quantity $\bar{\beta}$ varies with $\sigma$, see (5.3) below.

To make our presentation more compact, we often use the notation

$$H_\sigma(w) := H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t.$$

**Preliminary Estimates:** These will apply to both Parts I and II.
In what follows, set

$$\alpha_m = \inf \chi(t),\ \alpha_M = \sup \chi(t),\ \ell_m = \inf_{\mathbb{R}} Z,\ \ell_M = \sup_{\mathbb{R}} Z,\ \gamma^* = \gamma/k \text{ and } \sigma^* = \frac{\sigma}{\sigma - 1}. \tag{5.2}$$

We assume that $0 < \ell_m \leq \ell_M < \infty$. In Lemma 4.1 (see also Remark 4.2) we take

$$\beta = \gamma^* = \begin{cases} 2, & k = 1, \\ \gamma/k, & k > 1, \end{cases} \quad \text{and} \quad \bar{\beta} = \begin{cases} 2 - \varepsilon, & k = 1,\ 0 \leq \sigma \leq \gamma, \\ \gamma^*, & k > 1,\ 0 \leq \sigma \leq \gamma, \\ \sigma^*, & k \geq 1, \gamma < \sigma < \infty. \end{cases} \tag{5.3}$$

Moreover, we require that for $k = 1$,

$$\text{(i)}\ \ 0 < \varepsilon < 1/8 \ \text{ if } \sigma = 0, \quad \text{and} \quad \text{(ii)}\ \ 0 < \varepsilon < \frac{\min\{1, \sigma\}}{8} \ \text{ if } \sigma > 0. \tag{5.4}$$

Next, we state bounds for $H$. Recall that $k = k_1 + 1$, $\gamma = k + 1$, $\gamma^* = \gamma/k$, $a \geq 0$ and $0 < b < 1$. We use $w$ as in (5.1) and note that $w \geq \mu_M$.

In the following the constants $E$, $F_k$ and $G_\gamma$, to be defined later, are positive and do not depend on $r$.

**Step 1:** For small $r$, we use (3.5) with $\kappa(t) = 1 + t$ to obtain that

$$H_\sigma(w) = \frac{[b(1+t)v'(r)]^k}{r} H\left(e, I + \left(\frac{rv''(r)}{v'(r)} - 1 + b(1+t)(rv'(r))Z(w)\right) e \otimes e\right) + \chi(t)[b(1+t)v'(r)]^\sigma - a - bv(r). \tag{5.5}$$

For large $r$, we use (3.6)(or (3.7)) to obtain that

$$H_\sigma(w) = b^k[(1+t)v'(r)]^\gamma H\left(e, \frac{I - e \otimes e}{(1+t)rv'(r)} + \left(\frac{v''(r)}{(1+t)(v'(r))^2} + bZ(w)\right) e \otimes e\right) + \chi(t)(b(1+t)v'(r))^\sigma - a - bv(r). \tag{5.6}$$

**Step 2 Bounds for $H$:** We employ Remark 4.2 and use estimates for $v(r)$ from (5.5) and (5.6) to obtain upper bounds for $H$. Let $R \geq 1$, to be chosen later.

**(i)** $0 \leq r \leq R$**:** Since $\ell_m \leq Z(w) \leq Z(\mu_M) \leq \ell_M$(see (5.2)), define

$$M(b, r) = \max_{|e|=1} H\left(e, I + b\gamma^*(1+T)\ell_M r^{\gamma^*} e \otimes e\right). \tag{5.7}$$

By using Condition A (see (2.1)(i)), Condition B (see (2.2)) and that $M(b,r)$ is non-decreasing in $r$ and $b$, we have that

$$0 < \max_{|e|=1} H(e,I) \le M(b,r) \le M(1,R) \le R^{\gamma^*} M(1,1), \quad \forall R \ge 1.$$

Recall parts (vi), (viii) and (ix) of Cases A, B and C in Remark 4.2. Since $1 < \gamma^* \le 2$,

$$rv' \le \gamma^* r^{\gamma^*}, \quad \frac{(v'(r))^k}{r} \le (\gamma^*)^k \quad \text{and} \quad \frac{rv''(r)}{v'(r)} - 1 \le \gamma^* - 2 \le 0, \quad \forall k \ge 1. \quad (5.8)$$

Applying (5.8) in (5.5) and using Condition A, we get that

$$H\left(e, I + \left(\frac{rv''(r)}{v'(r)} - 1 + b(1+t)Z(w)rv'(r)\right) e \otimes e\right) \le H\left(e, I + \gamma^* b(1+T)\ell_M r^{\gamma^*} e \otimes e\right).$$

Since $0 < b \le 1$, using (5.7) and the bound for $M(b,r)$ we obtain that for $0 \le r \le R$,

$$H\left(e, I + \left(\frac{rv''(r)}{v'(r)} - 1 + b(1+t)Z(w)rv'(r)\right) e \otimes e\right) \le R^{\gamma^*} M(1,1), \quad \forall R \ge 1. \quad (5.9)$$

Next we set

$$E = (1+T) \quad \text{and} \quad F_k = E^k M(1,1), \ k \ge 1,$$

and use the estimate for $(v'(r))^k/r$ from (5.8) to get

$$\frac{[b(1+t)v'(r)]^k}{r} \le (b\gamma^*)^k (1+T)^k = (b\gamma^*)^k E^k.$$

Thus, (5.2), (5.5) and (5.9) lead to the estimate

$$\forall R \ge 1, \quad H_\sigma(w) \le (b\gamma^*)^k F_k R^{\gamma^*} + \alpha_M [bEv'(r)]^\sigma - a - bv(r), \quad \forall r \le R. \quad (5.10)$$

**(ii)** $1 \le R \le r$**:** Recall parts (vi) and (xi) of Cases A, B and C in Remark 4.2. Then

$$rv'(r) \ge \begin{cases} r^{2-\varepsilon}, & k = 1, \ 0 \le \sigma \le \gamma, \\ \gamma^* r^{\gamma^*}, & k > 1, \ 0 \le \sigma \le \gamma, \\ (\gamma^*/2) r^{\sigma^*}, & k \ge 1, \ \sigma > \gamma. \end{cases}$$

Note that, $\sigma^* < \gamma^* \le 2$, if $\sigma > \gamma$.

Using the above bounds and part (xi) of the Cases A, B and C, we obtain that

$$\max\left(\frac{1}{rv'(r)},\ \frac{v''(r)}{(v'(r))^2}\right) \le \begin{cases} 2r^{-(2-\varepsilon)}, & k=1,\quad 0\le\sigma\le\gamma,\\ 2r^{-\gamma^*}, & k>1,\quad 0\le\sigma\le\gamma,\\ 2r^{-\sigma^*}, & k\ge 1,\quad \sigma>\gamma. \end{cases}$$

Thus,

$$\forall\sigma\ge 0 \text{ and } \forall k\ge 1,\quad 0\le \max\left(\frac{1}{rv'(r)},\ \frac{v''(r)}{(v'(r))^2}\right)\le 2,\quad \text{in } r\ge 1. \tag{5.11}$$

Using Condition A and (5.11) in (5.6), we see that, in $t\ge 0$ and $r\ge 1$,

$$\begin{aligned} &H\left(e, \frac{I-e\otimes e}{(1+t)rv'(r)} + \left(\frac{v''(r)}{(1+t)(v'(r))^2} + bZ(w)\right)e\otimes e\right)\\ &\quad\le H\left(e, \frac{I-e\otimes e}{rv'(r)} + \left(\frac{v''(r)}{(v'(r))^2} + bZ(w)\right)e\otimes e\right)\\ &\quad\le H\left(e, 2(I-e\otimes e) + 2e\otimes e + bZ(w)I\right) \le H(e,(2+\ell_M)I), \end{aligned} \tag{5.12}$$

since $0<b<1$ and $0<Z\le \ell_M$.

Observing that $H(e,2I)=2H(e,I)>0$, we define

$$\bar M = \max_{|e|=1} H\left(e,(2+\ell_M)I\right). \tag{5.13}$$

Using (5.13) in (5.12) we get

$$H\left(e, \frac{I-e\otimes e}{(1+t)rv'(r)} + \left(\frac{v''(r)}{(1+t)(v'(r))^2} + Z(w)b\right)e\otimes e\right)\le \bar M.$$

Set

$$E = 1+T \quad\text{and}\quad G_\gamma = E^\gamma \bar M,\qquad \forall\gamma\ge 2.$$

Use (5.2) and the above in (5.6) to get that

$$H_\sigma(w)\le b^k G_\gamma[v'(r)]^\gamma + \alpha_M[bEv'(r)]^\sigma - a - bv(r),\qquad \forall r\ge R\ge 1. \tag{5.14}$$

**Step 3: Additional bounds:** Refer to part (vi) of Cases A, B and C in Remark 4.2. In $r\ge 0$,

$$v'(r)\le \begin{cases} 2\min(r^{1-\varepsilon}, r), & k=1,\quad 0\le\sigma\le\gamma,\\ \gamma^* r^{\gamma^*-1}, & k>1,\quad 0\le\sigma\le\gamma,\\ \gamma^*\min(r^{1/(\sigma-1)}, r^{\gamma^*-1}), & k\ge 1,\quad \sigma>\gamma. \end{cases} \tag{5.15}$$

**Construction of Super-Solutions:**

**Part I ($0\le\sigma\le\gamma$):** We take $R\ge 1$. A value of $R$ is determined in what follows.

**Sub-part (i) ($k = 1$ or $k_1 = 0$):** Let $\varepsilon > 0$ be as in (5.4). Also, $\gamma = \gamma^* = 2$ and the interval $[0, \gamma] = [0, 2]$.

Take $p = \varepsilon/2$; using (5.1) we get that $\forall (x,t) \in \mathbb{R}^n_T$,

$$w(x,t) = \mu_M + at + b(1+t)v(r), \quad \text{where} \quad v(r) = \int_0^{r^2} (1+\tau^{\varepsilon/2})^{-1}\, d\tau, \qquad (5.16)$$

and $a \geq 0$ and $0 < b < 1$ are to be determined.

• Consider $0 \leq r \leq R$. Using (5.15), we have $v'(r) \leq 2R$. Employing this in the second term on the right hand side of (5.10)($F_k = F_1$, $\gamma^* = 2$), we get

$$H_\sigma(w) \leq 2bF_1R^2 + \alpha_M(2bE)^\sigma R^\sigma - a.$$

Thus, $w$ is a super-solution in $0 \leq r \leq R$, if

$$a = \begin{cases} 2F_1(bR^2) + \alpha_M + (b/2)R^{2-\varepsilon}, & \sigma = 0, \\ 2F_1(bR^2) + \alpha_M(2E)^\sigma(bR)^\sigma + (b/2)R^{2-\varepsilon}, & 0 < \sigma \leq 2. \end{cases} \qquad (5.17)$$

• Let $r \geq R$. Use the estimate $v'(r) \leq 2r^{1-\varepsilon}$ (see (5.15)) in (5.14) and $G_\gamma = G_2$ to obtain that

$$H_\sigma(w) \leq 4bG_2r^{2(1-\varepsilon)} + \alpha_M(2bEr^{1-\varepsilon})^\sigma - a - bv(r). \qquad (5.18)$$

The bound in part (iv) of Case A in Remark 4.2 reads

$$v(r) \geq (r^{2-\varepsilon} - R^{2-\varepsilon})/2, \quad \forall r \geq R \geq 1.$$

Calling $\hat{a} = a - bR^{2-\varepsilon}/2 \geq 0$ (see (5.17)), (5.18) yields that

$$\begin{aligned} H_\sigma(w) &\leq 4bG_2r^{2-2\varepsilon} + \alpha_M(2bE)^\sigma r^{\sigma(1-\varepsilon)} - a - b(r^{2-\varepsilon} - R^{2-\varepsilon})/2 \\ &\leq 4bG_2r^{2-2\varepsilon} + \alpha_M(2bE)^\sigma r^{\sigma(1-\varepsilon)} - \hat{a} - br^{2-\varepsilon}/2. \end{aligned} \qquad (5.19)$$

**(a) $\sigma = 0$:** Using (5.17) and that $\hat{a} = 2F_1(bR^2) + \alpha_M \geq 0$, the right hand side in (5.19) yields

$$4bG_2r^{2-2\varepsilon} + \alpha_M - \hat{a} - (b/2)r^{2-\varepsilon} \leq br^{2-2\varepsilon}\left[4G_2 - r^\varepsilon/2\right].$$

Let $R$ be such that $R^\varepsilon = \max(1,\ 8G_2)$. Clearly, $w$ is a super-solution in $\mathbb{R}^n_T$ for any small $b > 0$. The choice of $R$ and (5.17) yield that

$$\lim_{b\to 0^+} a = \alpha_M \quad \text{if } \sigma = 0. \qquad (5.20)$$

**(b) $0 < \sigma \le \gamma = 2$:** Set $P = 4G_2$ and $Q = \alpha_M(2E)^\sigma$ in (5.19) to obtain that $\forall r \ge R \ge 1$,

$$H_\sigma(w) \le Pbr^{2-2\varepsilon} + Qb^\sigma r^{\sigma(1-\varepsilon)} - \frac{br^{2-\varepsilon}}{2} \le br^{2-2\varepsilon}\left(P + \frac{b^{\sigma-1}Q}{r^{(2-\sigma)(1-\varepsilon)}} - \frac{R^\varepsilon}{2}\right). \tag{5.21}$$

Noting that $(2-\sigma)(1-\varepsilon) \ge 0$, select

$$R = \begin{cases} \max\left\{(2(1+P))^{1/\varepsilon},\ (2Qb^{\sigma-1})^{1/(2-\sigma)(1-\varepsilon)}\right\}, & 0 < \sigma < 1, \\ \max\left\{1,\ (2P+2Q)^{1/\varepsilon}\right\}, & 1 \le \sigma \le 2. \end{cases} \tag{5.22}$$

For $1 \le \sigma \le 2$, take $r = b = 1$ in the second term of (5.21).

Using (5.22) in (5.21), (5.19) shows that $w$ is a super-solution in $\mathbb{R}^n_T$ for any small enough $b > 0$.

We recall the expression for $a$ in (5.17) and claim that $a \to 0$ as $b \to 0$. This is clear for $1 \le \sigma \le 2$ because of the choice of $R$ in (5.22). The case of interest is $0 < \sigma < 1$ since $R \to \infty$ as $b \to 0$. We show that $\lim_{b\to 0} bR^2 = 0$ and this would imply the same of $bR$ and $bR^{2-\varepsilon}$. From (5.22), one obtains

$$R = Kb^{(\sigma-1)/(2-\sigma)(1-\varepsilon)} \qquad \text{and} \qquad bR^2 = K^2 b^{1+2(\sigma-1)/(2-\sigma)(1-\varepsilon)},$$

for an appropriate $K$ that is independent of $b$. A simple calculation shows that

$$1 + \frac{2(\sigma-1)}{(2-\sigma)(1-\varepsilon)} = \frac{\sigma(1+\varepsilon) - 2\varepsilon}{(2-\sigma)(1-\varepsilon)}.$$

From (5.4), $\sigma(1+\varepsilon) - 2\varepsilon > 0$.

Summarizing, (5.16), (5.17) and (5.20) imply that

$$\lim_{b\to 0} a = \begin{cases} \alpha_M, & \sigma = 0, \\ 0, & 0 < \sigma \le \gamma = 2, \end{cases} \quad \text{for any small } \varepsilon > 0. \tag{5.23}$$

**Sub-part (ii) ($k > 1$ or $\gamma > 2$):** Note that $\gamma^* < 2$. We take $v(r) = r^{\gamma^*}$ and

$$w(x,t) = \mu_M + at + b(1+t)r^{\gamma^*}, \ \text{ in } \mathbb{R}^n_T. \tag{5.24}$$

Consider $0 \le r \le R$. We use $v'(r) = \gamma^* r^{\gamma^*-1} = \gamma^* r^{1/k}$ in (5.10) to obtain that

$$H_\sigma(w) \le (\gamma^*)^k F_k(b^k R^{\gamma^*}) + \alpha_M[\gamma^* E(bR^{1/k})]^\sigma - a - bv(r).$$

Hence, $w$ is super-solution in $B^R_T$ if

$$a = (\gamma^*)^k F_k(b^k R^{\gamma^*}) + \alpha_M(\gamma^* E)^\sigma (bR^{1/k})^\sigma. \tag{5.25}$$

Let $r \ge R$. Using (5.14) and $\gamma^* = \gamma/k$, we get that

$$H_\sigma(w) \le G_\gamma(b^k {\gamma^*}^\gamma r^{\gamma^*}) + \alpha_M(\gamma^* E)^\sigma (br^{1/k})^\sigma - a - br^{\gamma^*}. \tag{5.26}$$

Our main focus is (5.26) as $w$ is a super-solution in $B_T^R$ by using (5.25).

We analyze separately: **(1)** $\sigma = 0$, **(2)** $0 < \sigma \le 1$, and **(3)** $1 < \sigma \le \gamma$. The choice of $R$ differs in each situation.

**(1)** $\sigma = 0$**:** Setting $R = 1$, (5.25) and (5.26) yield that

$$H_\sigma(w) \le G_\gamma(b^k {\gamma^*}^\gamma r^{\gamma^*}) + \alpha_M - a - br^{\gamma^*} \le br^{\gamma^*}\left(G_\gamma b^{k-1} - 1\right) \le 0,$$

if we choose $0 < b \le \min\left(1,\ (G_\gamma {\gamma^*}^\gamma)^{-1/(k-1)}\right)$. Thus, $w$ is a super-solution in $\mathbb{R}_T^n$ for any small enough $b > 0$, and from (5.25)

$$\lim_{b\to 0} a = \lim_{b\to 0}[(\gamma^*)^k F_k b^k + \alpha_M] = \alpha_M. \tag{5.27}$$

**(2)** $0 < \sigma \le 1$**:** We ignore $a$ in the right side of (5.26) and factor $br^{\gamma^*}$ to obtain that

$$H_\sigma(w) \le br^{\gamma^*}\left[G_\gamma {\gamma^*}^\gamma b^{k-1} + \frac{\alpha_M(\gamma^* E)^\sigma b^{\sigma-1}}{R^{(\gamma-\sigma)/k}} - 1\right], \quad \forall r \ge R.$$

Noting that $\sigma \le 1 < \gamma$, we choose

$$b < \min\left[1,\ (4G_\gamma {\gamma^*}^\gamma)^{-1/(k-1)}\right] \quad \text{and} \quad R = \max\left[1,\ \left\{4\alpha_M(\gamma^* E)^\sigma b^{\sigma-1}\right\}^{k/(\gamma-\sigma)}\right].$$

Then $w$ is a super-solution in $\mathbb{R}_T^n$ for any $b > 0$, small enough.

We show next that $\lim_{b\to 0} a = 0$. We recall (5.25) and note that $\sigma/k < \gamma^*$. For $\sigma = 1$, $R$ does not depend on $b$ and hence, $\lim_{b\to 0} a = 0$.

For $0 < \sigma < 1$, $R \to \infty$, as $b \to 0$. Using the choice of $R$ in (5.25), we see that (use $\gamma^* = \gamma/k$)

$$b^k R^{\gamma^*} = K b^k \left[b^{k(\sigma-1)/(\gamma-\sigma)}\right]^{\gamma^*} = K b^{k+\gamma(\sigma-1)/(\gamma-\sigma)} \ \text{ and } \ bR^{1/k} = \hat{K} b^{1+(\sigma-1)/(\gamma-\sigma)},$$

for some $K$ and $\hat{K}$ independent of $b$ and $r$. Since $\gamma = k + 1 = k_1 + 2$, we see that

$$k + \frac{\gamma(\sigma-1)}{\gamma-\sigma} = \frac{k_1\gamma + \sigma}{\gamma-\sigma} > 0 \ \text{ and } \ 1 + \frac{\sigma-1}{\gamma-\sigma} = \frac{\gamma-1}{\gamma-\sigma} > 0.$$

Thus, $\lim_{b\to 0} a = 0$.

**(3)** $1 < \sigma \le \gamma$**:** We bound (5.26) (see also **(2)** above), in $r \ge R$, by

$$H_\sigma(w) \le br^{\gamma^*}\left[G_\gamma {\gamma^*}^\gamma b^{k-1} + \frac{\alpha_M(\gamma^* E)^\sigma b^{\sigma-1}}{R^{(\gamma-\sigma)/k}} - 1\right]. \tag{5.28}$$

Setting $R = 1$ in (5.28), we get from (5.26) that

$$H_\sigma(w) \le br^{\gamma^*}\left[G_\gamma {\gamma^*}^\gamma b^{k-1} + \alpha_M(\gamma^* E)^\sigma b^{\sigma-1} - 1\right].$$

Choosing $0 < b < 1$, small enough, we get that $w$ is super-solution in $\mathbb{R}^n_T$. Moreover, using (5.25) $\lim_{b\to 0} a = 0$.

Summarizing from Sub-Parts (i) (see (5.23)) and (ii) (see **(1), (2)** and **(3)**), we get

$$\lim_{b\to 0} a = \begin{cases} \alpha_M, & \sigma = 0, \\ 0, & 0 < \sigma \le \gamma, \end{cases} \qquad \forall k \ge 1. \tag{5.29}$$

**Part II** $\sigma > \gamma$, $k \ge 1$**:** We use Case C of Remark 4.2 and take $\beta = \gamma^*$ and $\bar\beta = \sigma^* = \sigma/(\sigma-1)$. Set $p = (\beta - \bar\beta)/\beta = (\sigma-\gamma)/\gamma(\sigma-1)$ and

$$w(x,t) = \mu_M + at + b(1+t)v(r), \quad \text{where} \ \ v(r) = \int_0^{r^{\gamma^*}} \frac{1}{1+\tau^p}\, d\tau. \tag{5.30}$$

Recall from Part I that $E = 1+T$, $F_k = E^k M(1,1)$ and $G_\gamma = E^\gamma \bar M$.

- Take $R \ge 1$ and consider $0 \le r \le R$. We employ (5.10) i.e.,

$$H_\sigma(w) \le (\gamma^*)^k F_k(b^k R^{\gamma^*}) + \alpha_M [bEv'(r)]^\sigma - a - bv(r).$$

Since $\sigma > \gamma$, using the bound $v'(r) \le \gamma^* r^{1/(\sigma-1)}$ from (5.15), we obtain that

$$H_\sigma(w) \le (\gamma^*)^k F_k(b^k R^{\gamma^*}) + \alpha_M(\gamma^* E)^\sigma (b^\sigma R^{\sigma^*}) - a.$$

Select

$$a = (\gamma^*)^k F_k(b^k R^{\gamma^*}) + \alpha_M(\gamma^* E)^\sigma (b^\sigma R^{\sigma^*}) + \gamma^*(bR^{\sigma^*})/(2\sigma^*). \tag{5.31}$$

Thus, $w$ is a super-solution in $0 \le r \le R$.

- In $r \ge R$, we use (5.14) i.e.,

$$H_\sigma(w) \le b^k G_\gamma(v'(r))^\gamma + \alpha_M(bEv'(r))^\sigma - (a + bv(r)). \tag{5.32}$$

From part (iv) of Case C in Remark 4.2 and (5.31) we have that

$$v(r) \ge \gamma^* \left(r^{\sigma^*} - R^{\sigma^*}\right)/(2\sigma^*), \ \ \forall r \ge R, \ \ \text{and} \ \ a + bv(r) \ge \gamma^* b r^{\sigma^*}/(2\sigma^*).$$

Using $v'(r) \le \gamma^* r^{1/(\sigma-1)}$ from (5.15), the lower bound for $a + bv(r)$ stated above and (5.31) in (5.32), we get that

$$\begin{aligned} H_\sigma(w) &\le {\gamma^*}^\gamma G_\gamma(b^k r^{\gamma/(\sigma-1)}) + \alpha_M(\gamma^* E)^\sigma b^\sigma r^{\sigma^*} - \frac{\gamma^* b r^{\sigma^*}}{2\sigma^*} \\ &\le b r^{\sigma^*}\left[\frac{{\gamma^*}^\gamma G_\gamma b^{k-1}}{R^{(\sigma-\gamma)/(\sigma-1)}} + \alpha_M(\gamma^* E)^\sigma b^{\sigma-1} - \frac{\gamma^*}{2\sigma^*}\right]. \end{aligned} \tag{5.33}$$

The second inequality holds since $1 < \gamma < \sigma$ and $r \ge R \ge 1$.

If $k > 1$ we take $R = 1$ and $0 < b < b_0$, where $b_0 = b_0(\gamma^*,\ \sigma,\ \alpha_M,\ E,\ G_\gamma)$ is small enough. Hence, (5.33) implies that $H_\sigma(w) \le 0$. If $k = 1$ we take

$$R = \max\left[1,\ \left(\frac{8\sigma^*{\gamma^*}^\gamma G_\gamma}{\gamma^*}\right)^{\frac{\sigma-1}{\sigma-\gamma}}\right] \quad \text{and} \quad b \le \min\left[1,\ \left(\frac{\gamma^*}{8\sigma^*\alpha_M(\gamma^* E)^\sigma}\right)^{\frac{1}{\sigma-1}}\right]. \tag{5.34}$$

Using these selections in (5.33), $H_\sigma(w) \le 0$. Thus, $w$ is super-solution in $\mathbb{R}^n_T$ for any small enough $b > 0$. Recalling (5.29) and (5.31), we see that

$$\lim_{b\to 0} a = \begin{cases} \alpha_M, & \sigma = 0, \\ 0, & \sigma > 0, \end{cases} \qquad \forall k \ge 1. \tag{5.35}$$

**Remark 5.1.** Parts I and II apply to any $\chi$. However, if $\chi < 0$ and $\sigma > \gamma$ then the maximum principle holds without imposing any upper bound. For $\sigma = \gamma$ the issue is unclear. See below. □

**Case $\sigma \ge \gamma$ and $\chi < 0$:** We provide a complete result for $\sigma > \gamma$. For $\sigma = \gamma$, our method fails in some situations and it is not clear to us if an upper bound is really needed.

Recall from (2.8) that

$$\alpha_m = \inf \chi,\ \ \alpha_M = \sup \chi,\ \ \ell_m = \inf Z,\ \ \ell_M = \sup Z,\ \ \mathcal{H}_m = \min_{|e|=1} H(e, e\otimes e)$$
$$\text{and}\ \ \mathcal{H}_M = \max_{|e|=1} H(e, e\otimes e). \tag{5.36}$$

The definitions of $E$ and $F$ used here differ from the ones used in the work prior to Remark 5.1.

We assume $\alpha_M < 0$ and use Lemma 4.3. Let $E > 0$ and $R > 0$. Define

$$\omega = \frac{r}{R},\quad v_E(r) = v_E(\omega) = E\int_0^{\omega^2} (1-\tau^p)^{-1} d\tau,\quad \forall 0 \le r < R. \tag{5.37}$$

We set $v(r) = v_E(r)$; clearly, $v$ is defined in $0 \le \omega < 1$.

Using (5.36) and parts (ii), (vi) and (viii) of Lemma 4.3, we have

$$L_p(\omega) = \frac{2E}{1-\omega^{2p}},\quad v'(r) = \frac{L_p(\omega)\omega}{R},\quad \frac{(v'(r))^k}{r} = \frac{L_p(\omega)^k\omega^{k_1}}{R^\gamma}, \tag{5.38}$$
$$\text{and}\quad \frac{rv''(r)}{v'(r)} - 1 + rZ(\cdot)|v'(r)| \le 2\omega^2\left(\frac{\ell_M E + p\omega^{2(p-1)}}{1-\omega^{2p}}\right),$$

**Sub-Case ($\sigma = \gamma$):** Recall $\ell_M$ and $\mathcal{H}_M$ from (5.36). Assume that

$$\alpha_M = \sup \chi < 0 \ \text{ and } \ |\alpha_M| > \ell_M \mathcal{H}_M.$$

Let $\mu_M = \sup_{\mathbb{R}^n} h(x)$ and $R > 0$. Employing $v(r)$ from (5.37), set in $0 \le r < R$,

$$\bar{w}(x,t) = \mu_M + v(r) + Ft, \tag{5.39}$$

where $E$, $F = F(R)$ and $p \ge 2$ are to be determined. Of importance is the limit $\lim_{R\to\infty} F$.

Employing (3.5), (5.37), (5.38) and (5.39), we see that in $0 \le r < R$,

$$H_\gamma(\bar{w}) \le \frac{L_p(\omega)^k \omega^{k_1}}{R^\gamma} H\left(e, I + 2\omega^2 \left(\frac{\ell_M E + p\omega^{2(p-1)}}{1-\omega^{2p}}\right) e\otimes e\right) + \chi \left(\frac{L_p(\omega)\omega}{R}\right)^\gamma - F. \tag{5.40}$$

Select

$$E = \frac{p^2-2p}{2\ell_M} \quad \text{and} \quad L_p(\omega) = \frac{2E}{1-\omega^{2p}} = \frac{p^2-2p}{\ell_M(1-\omega^{2p})}. \tag{5.41}$$

As $0 \le \omega < 1$ and $p \ge 2$, we get that $2\omega^2\left(\ell_M E + p\omega^{2(p-1)}\right) \le p^2\omega^2$ and

$$2\omega^2\left(\ell_M E + p\omega^{2(p-1)}\right)/(1-\omega^{2p}) \le p^2\omega^2/(1-\omega^{2p}).$$

Using (5.41), we set

$$J_p(\omega) = \frac{p^2\omega^2}{1-\omega^{2p}} = \left(\frac{p^2}{p^2-2p}\right)\ell_M L_p(\omega)\omega^2. \tag{5.42}$$

From (5.40) we see that in $0 \le \omega < 1$ or in $0 \le r < R$,

$$H_\gamma(\bar{w}) \le \left(\frac{L_p(\omega)^k\omega^{k_1}}{R^\gamma}\right) H\left(e, I + J_p(\omega)\, e\otimes e\right) - |\alpha_M| \left(\frac{L_p(\omega)\omega}{R}\right)^\gamma - F. \tag{5.43}$$

Set $\omega_0 = 1/\sqrt{2}$. We consider separately: (i) $0 \le \omega \le \omega_0$, and (ii) $\omega_0 \le \omega < 1$. We employ (5.43) in both cases.

Observe that $J_p(\omega)$ is increasing in $\omega$ and, since $p \ge 2$, we note from (5.42) that

$$p^2/2 \le J_p(\omega_0) \le p^2, \tag{5.44}$$

Let $\varepsilon > 0$ be such that $|\alpha_M| > (1+\varepsilon)\ell_M \mathcal{H}_M$. Choose $p > 2$, large enough (see (5.36)) so that

$$\begin{cases} H\left(e, e\otimes e + I/J_p(\omega_0)\right) \le (1+\varepsilon)\mathcal{H}_M \\ \text{and} \\ p^2(1+\varepsilon)\ell_M\mathcal{H}_M/(p^2-2p) < |\alpha_M|. \end{cases} \tag{5.45}$$

Call $p_0$ such a value of $p$ and fix $p_0$. Note that (5.45) continues to hold, if $\omega_0$ is replaced by any $\omega \ge \omega_0$ and any $p \ge p_0$.

**(i)** $0 \le \omega \le \omega_0$**:** Set

$$N = N(p_0) = \max_{|e|=1} H\left(e, I + J_{p_0}(\omega_0) e\otimes e\right).$$

Recalling (5.41), (5.43) yields that

$$H_\gamma(\bar{w}) \le \left[L_{p_0}^k N\omega_0^{k_1}\right]/R^\gamma - F, \quad \forall 0 \le \omega \le \omega_0.$$

Thus $\bar{w}$ is a super-solution in $B_T^{\omega_0 R}$ if

$$F = \left[L_{p_0}^k N\omega_0^{k_1}\right]/R^\gamma. \tag{5.46}$$

**(ii) $\omega_0 \le \omega < 1$:** Let $p_0$ be as in (5.45). Note that $J_{p_0}(\omega)$ is increasing in $\omega$. Factoring $J_{p_0}(\omega)$ in (5.43) and then using (5.42) and (5.45), we obtain that

$$\begin{aligned}
H_\gamma(\bar{w}) &\le \frac{L_{p_0}(\omega)^k \omega^{k_1} J_{p_0}(\omega)(1+\varepsilon)\mathcal{H}_M}{R^\gamma} - |\alpha_M|\left(\frac{L_{p_0}(\omega)\omega}{R}\right)^\gamma - F \\
&= \left(\frac{p_0^2}{p_0^2 - 2p_0}\right)(1+\varepsilon)\ell_M\mathcal{H}_M\left(\frac{L_{p_0}(\omega)\omega}{R}\right)^\gamma - |\alpha_M|\left(\frac{L_{p_0}(\omega)\omega}{R}\right)^\gamma \\
&= \left(\frac{L_{p_0}(\omega)\omega}{R}\right)^\gamma\left(\left(\frac{p_0^2}{p_0^2 - 2p_0}\right)(1+\varepsilon)\ell_M\mathcal{H}_M - |\alpha_M|\right) \le 0.
\end{aligned}$$

In the last inequality, we have used $\gamma = k + 1 = k_1 + 2$.

Hence, $\bar{w}$ is a super-solution in any $R > 0$. Moreover,

$$\lim_{R\to\infty} F = \lim_{R\to\infty} \frac{N\omega_0^{k_1}}{R}\left(\frac{L_{p_0}(\omega_0)}{R}\right)^k = 0. \tag{5.47}$$

**Sub-Case ($\sigma > \gamma$):** We use the same approach. The inequality in (5.43) reads

$$H_\sigma(\bar{w}) \le \left(\frac{L_p(\omega)^k\omega^{k_1}}{R^\gamma}\right) H\left(e, I + J_p(\omega)e\otimes e\right) - |\alpha_M|\left(\frac{L_p(\omega)\omega}{R}\right)^\sigma - F. \tag{5.48}$$

Recall from (5.41) and (5.42) that

$$L_p(\omega) = \frac{p^2 - 2p}{\ell_M(1-\omega^{2p})} \quad \text{and} \quad J_p(\omega) = \frac{p^2\omega^2}{1-\omega^{2p}} = \left(\frac{p^2}{p^2-2p}\right)\ell_M L_p(\omega)\omega^2. \tag{5.49}$$

Set $\omega_0 = 1/\sqrt{2}$ and consider $0 \le \omega \le \omega_0$. Then $J_p(\omega) \le J_p(\omega_0)$ and (5.48) implies that

$$\begin{aligned}
H_\sigma(\bar{w}) &\le \left(\frac{L_p(\omega_0)^k\omega_0^{k_1}}{R^\gamma}\right) H\left(e, I + J_p(\omega_0)e\otimes e\right) - |\alpha_M|\left(\frac{L_p(\omega)\omega}{R}\right)^\sigma - F \\
&\le \left(\frac{L_p(\omega_0)^k\omega_0^{k_1}J_p(\omega_0)}{R^\gamma}\right) H\left(e, (J_p(\omega_0))^{-1}I + e\otimes e\right) - F
\end{aligned} \tag{5.50}$$

We choose $p_0 > 2$ be such that $J_{p_0}(\omega_0) > 2$. Set

$$\hat{N}_p(\omega) = \max_{|e|=1} H(e, J_p(\omega)^{-1}I + e \otimes e).$$

Applying (5.49) in (5.50), we get that

$$H_\sigma(\bar{w}) \le \left(\frac{L_p(\omega_0)\omega_0}{R}\right)^\gamma \left(\frac{p^2 \ell_M \hat{N}_p(\omega_0)}{p^2 - 2p}\right) - F, \quad \forall p \ge p_0 \text{ and } \forall \omega \le \omega_0.$$

We select

$$F = F_p = \left(\frac{p^2 \ell_M \hat{N}_p(\omega_0)}{p^2 - 2p}\right) \left(\frac{L_p(\omega_0)\omega_0}{R}\right)^\gamma. \tag{5.51}$$

We consider $\omega_0 \le \omega < 1$. Since, for each $p \ge p_0$, $J_p(\omega) \ge J_p(\omega_0)$, we have that $\hat{N}_p(\omega_0) \ge \hat{N}_p(\omega)$. Disregarding $F$ in (5.48), factoring $J_p(\omega)$ and then using (5.49) and recalling $\hat{N}_p(\omega)$, we see that

$$\begin{aligned} H_\sigma(\bar{w}) &\le \left(\frac{p^2 \ell_M \hat{N}_p(\omega_0)}{p^2 - 2p}\right) \left(\frac{L_p(\omega)\omega}{R}\right)^\gamma - |\alpha_M| \left(\frac{L_p(\omega)\omega}{R}\right)^\sigma \\ &\le \left(\frac{L_p(\omega)\omega}{R}\right)^\gamma \left[\left(\frac{p^2 \ell_M \hat{N}_p(\omega_0)}{p^2 - 2p}\right) - |\alpha_M| \left(\frac{L_p(\omega_0)\omega_0}{R}\right)^{\sigma-\gamma}\right] \end{aligned}$$

Choose $R > 0$ such that

$$R = L_p(\omega_0)\omega_0 \left(\frac{|\alpha_M|(p^2 - 2p)}{p^2 \ell_M \hat{N}_p(\omega_0)}\right)^{1/(\sigma-\gamma)} = \frac{(p^2 - 2p)\omega_0}{1 - \omega_0^{2p}} \left(\frac{|\alpha_M|(p^2 - 2p)}{p^2 \ell_M \hat{N}_p(\omega_0)}\right)^{1/(\sigma-\gamma)}, \tag{5.52}$$

where we have used (5.49). Thus, $\bar{w}$ is a super-solution in $B_T^R$ for $R$ large enough.

From (5.52), $R \to \infty$ if and only if $p \to \infty$. Since, $\lim_{p\to\infty} \hat{N}_p(\omega_0) = \mathcal{H}_M$, (5.51) leads to

$$\lim_{R\to\infty} F = \lim_{p\to\infty} |\alpha_M|^{-\gamma/(\sigma-\gamma)} \left(\frac{\ell_M \hat{N}_p(\omega_0) p^2}{p^2 - 2p}\right)^{\sigma/(\sigma-\gamma)} = \left(\frac{(\ell_M \mathcal{H}_M)^\sigma}{|\alpha_M|^\gamma}\right)^{1/(\sigma-\gamma)}. \tag{5.53}$$

## 6. Sub-solutions

In this section, we construct sub-solutions. It will follow from the work that if $0 \le \sigma \le \gamma$ then a minimum principle holds without any restrictions on the growth rate. However, a lower bound is needed if $\sigma > \gamma$. Our work is quite similar to that in Section 5.

To achieve our goal, we use (3.4) and (3.8) in Remark 3.1. Thus, setting $w(x,t) = v(r) - \kappa(t)$ and assuming that $v'(r) < 0$, we get that

$$H_\sigma(w) = \frac{|v'(r)|^k}{r} H\left(e, \left(r|v'(r)|Z(w) + 1 - \frac{rv''(r)}{v'(r)}\right) e \otimes e - I\right) \\ + \chi(t)|v'(r)|^\sigma + \kappa'(t). \tag{6.1}$$

We recall Condition C (see (2.4)), Remark 2.1, (2.5) and (3.1) and set

$$\aleph = \min_{|e|=1} H(e, -I), \quad K_0 = \frac{\Lambda_{\min}(\lambda_0)}{\lambda_0} \quad \ell_m = \inf_s Z(s) \quad \text{and} \quad \alpha_m = \inf \chi, \tag{6.2}$$

where $\Lambda_{\min}(\lambda) = \min_{|e|=1} H(e, \lambda e \otimes -I)$.

Set $\mathcal{H}(\lambda) = \min_{|e|=1} H(e, e \otimes e - \lambda^{-1} I)$. Let $\lambda_0 > 0$ be large enough so that $K_0 > 0$. We record that

$$0 < \ell_m < \infty, \quad \aleph < 0, \quad 0 < K_0 \leq \mathcal{H}(\lambda) \leq \mathcal{H}_M, \ \forall \lambda \geq \lambda_0, \ \text{and} \ \lim_{\lambda \to \infty} \mathcal{H}(\lambda) = \mathcal{H}_m. \tag{6.3}$$

**Sub-solutions.** We treat separately the cases $0 \leq \sigma \leq \gamma$ and $\sigma \geq \gamma$. The case $\sigma = \gamma$ is addressed in both the situations. Recall that $\mu_m = \inf_{\mathbb{R}^n} h$.

**Case I $0 \leq \sigma \leq \gamma$:** Our work utilizes Lemma 4.3. Set in $0 \leq r < R$, $\omega = r/R$ and

$$\bar{w}(x,t) = \mu_m + v(r) - Ft, \ \text{where} \ v(r) = E \int_0^{\omega^2} (\tau^p - 1)^{-1} d\tau, \ \forall 0 \leq r < R. \tag{6.4}$$

We observe that $v(r) \leq 0$ and $v'(r) \leq 0$. Here $E > 0$, $F = F(R)$ and $p \geq 2$ are to be determined. We calculate also $\lim_{R \to \infty} F(R)$.

Note that the definition of $E$ differs from the one in Lemma 4.3 but is consistent with Case (ii) in Remark 3.1.

Set

$$L_p(\omega) = \frac{2E}{1 - \omega^{2p}}. \tag{6.5}$$

We recall part (vii) of Lemma 4.3, $k = k_1 + 1$, $\gamma = k + 1$ and $\ell_m = \inf Z$ and observe that

$$1 - \frac{rv''(r)}{v'(r)} + rZ(\cdot)|v'(r)| = \frac{2EZ(\cdot)\omega^2}{1 - \omega^{2p}} - \frac{2p\omega^{2p}}{1 - \omega^{2p}} \geq 2\omega^2 \left(\frac{\ell_m E - p\omega^{2(p-1)}}{1 - \omega^{2p}}\right).$$

Employing the above lower bound, (5.38), (6.2), parts (ii) and (vii) of Lemma 4.3 in (6.1) leads to

$$H_\sigma(\bar{w}) \geq \frac{L_p(\omega)^k \omega^{k_1}}{R^\gamma} H\left(e, 2\omega^2 \left(\frac{\ell_m E - p\omega^{2(p-1)}}{1 - \omega^{2p}}\right) e \otimes e - I\right) + \alpha_m \left(\frac{L_p(\omega)\omega}{R}\right)^\sigma + F. \tag{6.6}$$

Select

$$E = \frac{p(p+1)}{\ell_m} \quad \text{and} \quad L_p(\omega) = \frac{2p(p+1)}{\ell_m(1-\omega^{2p})}. \tag{6.7}$$

As $0 \le \omega < 1$ and $p \ge 2$, we get that $2\omega^2 \left(\ell_m E - p\omega^{2(p-1)}\right) \ge 2p^2\omega^2$. Set

$$J_p(\omega) = \frac{2p^2\omega^2}{1-\omega^{2p}} = \left(\frac{p}{p+1}\right) \ell_m L_p(\omega)\omega^2, \tag{6.8}$$

where we have used (6.5). Thus,

$$J_p(\omega) \le 2\omega^2 \left(\frac{\ell_m E - p\omega^{2(p-1)}}{1-\omega^{2p}}\right).$$

Recalling (6.6) and (6.8) we see that

$$H_\sigma(\bar{w}) \ge \left(\frac{L_p(\omega)^k \omega^{k_1}}{R^\gamma}\right) H\left(e, J_p(\omega)\, e \otimes e - I\right) + \alpha_m \left(\frac{L_p(\omega)\omega}{R}\right)^\sigma + F. \tag{6.9}$$

Set $\omega_0 = 1/\sqrt{2}$. We consider separately the cases: (i) $0 \le \omega \le \omega_0$, and (ii) $\omega_0 \le \omega < 1$.

**(i)** $0 \le \omega \le \omega_0$**:** Recall (6.2), (6.3) and (6.8). We bound

$$H\left(e, J_p(\omega) e \otimes e - I\right) \ge H(e, -I) \ge -|\aleph|.$$

Using the above in (6.9) we get that

$$H_\sigma(\bar{w}) \ge F - \left(\frac{L_p(\omega)^k |\aleph| \omega^{k_1}}{R^\gamma} - \alpha_m \left(\frac{L_p(\omega)\omega}{R}\right)^\sigma\right). \tag{6.10}$$

From (6.7), $L_p(\omega)$ is increasing in $\omega$. Since $0 \le \omega \le \omega_0$, we choose

$$F = \frac{L_p(\omega_0)^k |\aleph| \omega_0^{k_1}}{R^\gamma} - \alpha_m \left(\frac{L_p(\omega_0)\omega_0}{R}\right)^\sigma, \quad \forall p > 2. \tag{6.11}$$

Thus (6.10) implies that $\bar{w}$ is a sub-solution in $B_T^{\omega_0 R}$.

**(ii)** $\omega_0 \le \omega < 1$**:** This leads to a determination of $p$. Since $J_p$ is increasing in $\omega$, using (6.8) we get that

$$J_p(\omega) \ge J_p(\omega_0) = \frac{2p^2\omega_0^2}{1-\omega_0^{2p}} \ge p^2 \quad \text{and} \quad \lim_{p\to\infty} J_p(\omega_0) = \infty. \tag{6.12}$$

Using Conditions A and B (see Section 2) and (6.12), we have that for $\omega \ge \omega_0$,

$$\begin{aligned} \min_{|e|=1} H\left(e, J_p(\omega)\, e \otimes e - I\right) &\ge J_p(\omega) \min_{|e|=1} H\left(e, e \otimes e - \frac{I}{J_p(\omega_0))}\right) \\ &\ge J_p(\omega)\, \mathcal{H}(p^2) \ge K_0 J_p(\omega_0) \ge K_0 p^2 > 0. \end{aligned} \tag{6.13}$$

Here we have used (6.3) and chosen $p \geq p_0$, where $p_0 \geq 2$ is large enough such that $\mathcal{H}(p^2) \geq K_0$.

From here on we take $p \geq p_0$ such that (6.13) holds (see also (6.12)). Using (6.8) and (6.13) in (6.9), we obtain

$$\begin{aligned} H_\sigma(\bar{w}) &\geq \frac{L_p(\omega)^k \omega^{k_1} J_p(\omega)\mathcal{H}(p^2)}{R^\gamma} + \alpha_m \left(\frac{L_p(\omega)\omega}{R}\right)^\sigma + F \\ &\geq \ell_m \mathcal{H}(p^2)\left(\frac{p}{p+1}\right)\left(\frac{L_p(\omega)\omega}{R}\right)^\gamma + \alpha_m \left(\frac{L_p(\omega)\omega}{R}\right)^\sigma + F. \qquad (6.14) \end{aligned}$$

In the last inequality, we have used $\gamma = k+1 = k_1+2$.

We factor $(\omega L_p(\omega)/R)^\sigma$ from (6.14) and use $\omega_0 \leq \omega < 1$ to obtain that

$$H_\sigma(\bar{w}) \geq \left(\frac{L_p(\omega)\omega}{R}\right)^\sigma \left[\ell_m \mathcal{H}(p^2)\left(\frac{p}{p+1}\right)\left(\frac{L_p(\omega_0)\omega_0}{R}\right)^{\gamma-\sigma} + \alpha_m\right] + F. \quad (6.15)$$

**Sub-Case (a)** $0 \leq \sigma < \gamma$**:** As noted earlier, $\bar{w}$ is a sub-solution in $B_T^{\omega_0 R}$, if $F$ is chosen as in (6.11). We assume that $\alpha_m < 0$. For $\alpha_m \geq 0$, see Sub-Case (b).

We refer to (6.15) and select $R$ such that

$$\frac{L_p(\omega_0)\omega_0}{R} = \left[\left(\frac{|\alpha_m|}{\ell_m \mathcal{H}(p^2)}\right)\left(\frac{1+p}{p}\right)\right]^{1/(\gamma-\sigma)}. \qquad (6.16)$$

With this choice, $\bar{w}$ is a sub-solution in $B_T^R$.

From (6.3) and (6.7), $\mathcal{H}(p^2) = O(1)$ and $L_p(\omega_0) = O(p^2)$ as $p \to \infty$. Thus, (6.16) yields that for some $K_1 = K_1(\alpha, \gamma, \ell_m, \omega_0, K_0) > 0$,

$$R \sim K_1 p^2 \ \text{ as } p \to \infty.$$

Thus, $R \to \infty$ if and only if $p \to \infty$.

We calculate $\lim_{R\to\infty} F$. We write $F$ in (6.11) as the sum of two terms $X$ and $Y$ as follows:

$$F = \frac{|\aleph| L_p(\omega_0)^k \omega_0^{k_1}}{R^\gamma} + |\alpha_m| \left(\frac{L_p(\omega_0)\omega_0}{R}\right)^\sigma = X + Y.$$

We use (6.16), $\gamma = k+1$ and $k = k_1+1$ to observe that

$$\lim_{p\to\infty} X = \lim_{R\to\infty} X = \frac{|\aleph|\omega_0^{k_1}}{R}\left(\frac{L_p(\omega_0)}{R}\right)^k = 0.$$

Next, using (6.16), we get

$$Y = |\alpha_m| \left(\frac{L_p(\omega_0)\omega_0}{R}\right)^\sigma = \left[\frac{|\alpha_m|^\gamma}{(\ell_m \mathcal{H}(p^2))^\sigma}\left(\frac{p+1}{p}\right)^\sigma\right]^{1/(\gamma-\sigma)}.$$

Referring to (6.3), we see that

$$\lim_{R\to\infty} F = \lim_{R\to\infty} Y = \lim_{p\to\infty} Y = \left( \frac{|\alpha_m|^\gamma}{(\ell_m \mathcal{H}_m)^\sigma} \right)^{1/(\gamma-\sigma)}, \quad 0 \le \sigma < \gamma. \tag{6.17}$$

From (6.17), if $\alpha_m < 0$ then

$$\lim_{R\to\infty} F = \begin{cases} |\alpha_m|, & \sigma = 0, \\ (|\alpha_m|^\gamma/(\ell_m \mathcal{H}_m)^\sigma)^{1/(\gamma-\sigma)}, & 0 < \sigma < \gamma. \end{cases} \tag{6.18}$$

**Sub-Case (b)** $\chi \ge 0$**:** We may choose $F = X$ as seen in (6.9), (6.11) and (6.15)(since $\alpha_m \ge 0$). For a large enough value of $p$, $\bar{w}$ is a sub-solution in $B_T^R$ for any $\sigma \ge 0$ and any $R > 0$. Clearly, $\lim_{R\to\infty} F = 0$. No lower bound is needed.

**Sub-Case (c)** $\sigma = \gamma$**:** An inspection of (6.15) shows that if

$$|\alpha_m| < \ell_m \mathcal{H}_m = [\inf_s Z(s)][\min_{|e|=1} H(e, e \otimes e)],$$

by selecting $p$, large enough, (6.15) may be written as

$$H_\gamma(\bar{w}) \ge \left( \frac{\omega L(\omega)}{R} \right)^\gamma \left[ \ell_m \mathcal{H}(p^2) \left( \frac{p}{p+1} \right) - |\alpha_m| \right] + F \ge 0.$$

For the chosen $p$, $\bar{w}$ is a sub-solution in $B_T^R$ for any $R > 0$. Moreover, $R$ is independent of $p$ and $F(R) \to 0$ as $R \to \infty$. However, if $|\alpha_m| \ge \ell_m \mathcal{H}_m$ then it is not clear if a lower bound is needed. See Case II below. In Section 7, a minimum principle is proven by imposing a lower bound.

**Case II** $\gamma \le \sigma < \infty$**:** We adapt the work in Section 5, see Step 2, in particular.

Recall that $k_2 = 1$, $\gamma = k + 1 = k_1 + 2$, $\sigma^* = \sigma/(\sigma - 1)$ and $\gamma^* = \gamma/k$.

First, we describe the two sub-cases of interest and then present the work that addresses them. Set

$$\kappa(t) = 1 + t, \; 0 \le t \le T.$$

**Sub-Case (i)** $\sigma = \gamma$**:** We take $|\alpha_m| \ge \ell_m \mathcal{H}_m$ and refer to Sub-Parts (i) and (ii) of Part I in Section 5.

**(i1)** $k = 1$**:** Thus, $\gamma = \gamma^* = 2$. Assume that for any $\varepsilon > 0$, small, $\sup_{|x|\le r}(-u(x,t)) \le o(|r|^{2-\varepsilon})$ as $r \to \infty$. We take

$$\bar{w}(x,t) = \mu_m - at - b\kappa(t)v(r), \quad \text{where} \quad v(r) = \int_0^{r^2} (1 + \tau^{\varepsilon/2})^{-1} d\tau.$$

**(i2)** $k > 1$**:** Thus, $1 < \gamma^* < 2 < \gamma$. We assume that $\sup_{|x|\le r}(-u(x,t)) \le o(|r|^{\gamma^*})$ as $r \to \infty$. We take

$$\bar{w} = \mu_m - at - b\kappa(t) r^{\gamma^*}.$$

**Sub-Case (ii) $\sigma > \gamma$:** We allow $k \geq 1$ and refer to Part II of Section 5. We assume that $\sup_{|x|\leq r}(-u(x,t)) \leq o(|r|^{\sigma^*})$ as $r \to \infty$. We take

$$\bar{w} = \mu_m - at - b\kappa(t)v(r), \quad \text{where} \quad v(r) = \int_0^{r^{\gamma^*}} (1+\tau^p)^{-1} d\tau \quad \text{and} \quad p = \frac{\sigma - \gamma}{\gamma(\sigma - 1)}.$$

We present the calculations that apply to both the sub-cases. Since $\bar{w}_r < 0$, except at $r = 0$, we use the two versions in (3.4). For $R > 0$, to be determined,

$$\begin{aligned} G(\bar{w}) &= \frac{|\bar{w}_r|^k}{r} H\left(e, \left(r|\bar{w}_r|Z(\bar{w}) + 1 - \frac{r\bar{w}_{rr}}{\bar{w}_r}\right) e\otimes e - I\right), \quad \forall 0 \leq r \leq R, \\ G(\bar{w}) &= |\bar{w}_r|^\gamma H\left(e, \frac{I - e\otimes e}{r\bar{w}_r} + \left(\frac{\bar{w}_{rr}}{\bar{w}_r^2} + Z(\bar{w})\right) e\otimes e\right), \quad \forall r \geq R. \end{aligned} \tag{6.19}$$

From parts (ix) of Cases A, B and C of Remark 4.2, we have that

$$\frac{r\bar{w}_{rr}}{\bar{w}_r} = \frac{rv''(r)}{v'(r)} \leq \begin{cases} 1, & \sigma = \gamma = 2, \quad k = 1, \\ \gamma^* - 1, & \sigma = \gamma > 2, \quad k > 1, \\ \gamma^* - 1, & \sigma > \gamma \geq 2, \quad k \geq 1. \end{cases}$$

• Using the first version in (6.19), noting that $\gamma^* \leq 2$ and $1 - r\bar{w}_{rr}/\bar{w}_r \geq 0$, one estimates (see (6.2))

$$H\left(e, \left(r|\bar{w}_r|Z(\bar{w}) + 1 - \frac{r\bar{w}_{rr}}{\bar{w}_r}\right) e\otimes e - I\right) \geq H(e, -I) \geq -|\aleph|, \quad 0 \leq r \leq R.$$

Hence, in $0 \leq r \leq R$,

$$H_\sigma(\bar{w}) \geq -\left(\frac{[b\kappa(T)v'(r)]^k|\aleph|}{r} + |\alpha_m|\,[b\kappa(T)v'(r)]^\sigma\right) + a + bv(r).$$

We employ the estimate in (5.8), i.e., $v'(r) \leq \gamma^* r^{\gamma^*-1} = \gamma^* r^{1/k}$ we get, in $0 \leq r \leq R$,

$$\begin{aligned} H_\sigma(\bar{w}) &\geq -\left(\frac{[\gamma^* b\kappa(T)r^{\gamma^*-1}]^k|\aleph|}{r} + |\alpha_m|[\gamma^* b\kappa(T)r^{\gamma^*-1}]^\sigma - a\right) \\ &\geq -\left([\gamma^* b\kappa(T)]^k|\aleph| + |\alpha_m|[\gamma^* b\kappa(T)]^\sigma R^{\sigma/k} - a\right). \end{aligned}$$

As done in (5.31), we select an appropriate $a$. Thus, $\bar{w}$ is a sub-solution in $|x| \leq R$.

• Next, in $r \geq R$, one finds that (see (5.6) and (6.19))

$$\begin{aligned} &|\bar{w}_r|^\gamma H\left(e, \frac{I - e\otimes e}{r\bar{w}_r} + \left(\frac{\bar{w}_{rr}}{\bar{w}_r^2} + Z(\bar{w})\right) e\otimes e\right) \\ &= [b\kappa(t)v'(r)]^\gamma H\left(e, \frac{e\otimes e - I}{b\kappa(t)rv'(r)} + \left(Z(\bar{w}) - \frac{v''(r)}{b\kappa(t)(v'(r))^2}\right) e\otimes e\right) \\ &\geq b^k[\kappa(t)v'(r)]^\gamma H\left(e, \frac{e\otimes e - I}{rv'(r)} + \left(bZ(\bar{w}) - \frac{v''(r)}{(v'(r))^2}\right) e\otimes e\right), \end{aligned} \tag{6.20}$$

where we have factored out $1/b$, used that $\gamma = k+1$, $\kappa(t) \geq 1$ and $e \otimes e - I \leq 0$.

We now recall (5.11) i.e.,

$$0 < \max\left(\frac{1}{rv'(r)}, \frac{v''(r)}{(v'(r))^2}\right) \leq 2, \quad \text{in } r \geq R \geq 1.$$

Employing this estimate in (6.20) and disregarding the term with $Z$, we get

$$|\bar{w}_r|^\gamma H\left(e, \frac{I - e\otimes e}{r\bar{w}_r} + \left(\frac{\bar{w}_{rr}}{\bar{w}_r^2} + Z(\bar{w})\right) e \otimes e\right) \geq b^k[\kappa(T)v'(r)]^\gamma S,$$

where $S = \min_{|e|=1} H(e, -2(I + e\otimes e))$. Clearly, by (6.2), $-\infty < S \leq \aleph < 0$ and we get that

$$H_\sigma(\bar{w}) \geq -\left\{b^k[\kappa(T)v'(r)]^\gamma |S| + \alpha(b\kappa(T))^\sigma (v'(r))^\sigma - a - bv(r)\right\},$$

which is analogous to (5.14). As done in Section 5, a choice for $b$ (see (5.34)) can now be made. Thus, $\bar{w}$ is a sub-solution for any small enough $b > 0$ and

$$\lim_{b\to 0} a = 0. \tag{6.21}$$

□

## 7. Proofs of Theorems 2.2-2.5

Let $T > 0$ and we take $(x,t) \in \mathbb{R}^n_T$, $n \geq 2$. Set

- **(i)** $\mu_m = \inf_{\mathbb{R}^n} h$, $\mu_M = \sup_{\mathbb{R}^n} h$, **(ii)** $\alpha_m = \inf \chi(t)$, $\alpha_M = \sup \chi(t)$,
- **(iii)** $\ell_m = \inf Z$, $\ell_M = \sup Z$, and assume that
- **(iv)** $-\infty < \mu_m \leq \mu_M < \infty$, $-\infty < \alpha_m \leq \alpha_M < \infty$ and $0 < \ell_m \leq \ell_M < \infty$.

Recall that $k = k_1 + 1$, $\gamma = k+1$, $\gamma^* = \gamma/k$ and $\sigma^* = \sigma/(\sigma - 1)$, $\forall \sigma > 1$. Let $z \in \mathbb{R}^n$ be a fixed point and $r = |x - z|$. Define the cylinder $B^\rho_T = B_\rho(z) \times (0,T)$ and $P^\rho_T$ be its parabolic boundary.

**Proof of Theorem 2.2.** Let $u$ be a sub-solution as described in the theorem. By the hypotheses, for a fixed small $\eta > 0$, let $\rho > \rho_0$, where $\rho_0$ is large enough so that

$$\sup_{B^\rho_T} u(x,t) \leq \eta\rho^\delta, \quad \forall \rho \geq \rho_0, \tag{7.1}$$

where $\delta$ is as in Theorem 2.2.

**Proof of Theorem 2.2(a) ($\sigma = 0$):** Recall from (5.1) the super-solution

$$w(x,t) = \mu_M + at + bv(r), \quad \forall (x,t) \in \mathbb{R}^n_T, \tag{7.2}$$

where

$$(1)\ \text{ if } k=1 \text{ then } v(r)=\int_0^{r^2}(1+\tau^{\varepsilon/2})^{-1}d\tau, \text{ and } (2) \text{ if } k>1 \text{ then } \ v(r)=r^{\gamma^*}, \tag{7.3}$$

where $\varepsilon>0$, small, is as in the theorem. See (5.16) and (5.24). Also, in (7.1)

$$(1)\ \ \delta=2-\varepsilon \quad \text{if } k=1, \text{ and } (2)\ \ \delta=\gamma^* \quad \text{if } k>1. \tag{7.4}$$

For details, see Part I in Section 5, (5.20) in Sub-Part (i), (5.27) in Sub-Part (ii) and (5.29).

Thus, $w$ is a super-solution in $\mathbb{R}^n_T$ for any $0<b<b_0$, for $b_0$ small enough, and an appropriate $a$ that depends on $b$. Moreover, $\lim_{b\to 0} a=\alpha_M$. See (5.29).

By part (iv) of Cases A and B of Remark 4.2, $v(r)\ge r^{\delta}/4$, for $r\ge\rho_1$, where $\rho_1$ is large enough. We choose $b=8\eta$. If needed, choose $\eta$ smaller and $\rho_0$ in (7.1) larger so that $\eta<b_0/8$.

Set $\rho_2=\max(\rho_0,\rho_1)$ and consider a cylinder $B^{\rho}_T$, where $\rho>\rho_2$. Then $u(x,0)\le h(x)\le\mu_M,\ \forall x\in\mathbb{R}^n$. Clearly, $w(x,0)=\mu_M+bv(r)\ge u(x,0)$, for $|x|\le\rho$. On $|x|=\rho$, we have by (7.1),

$$w(x,t)\ge bv(\rho)\ge\frac{8\eta\rho^{\delta}}{4}=2\eta\rho^{\delta}\ge u(x,t).$$

Thus, $w\ge u$ on $P^{\rho}_T,\ \forall\rho\ge\rho_2$. We use Lemma 3.2 to conclude that $u(x,t)\le w(x,t)$ in $B^{\rho}_T$,

$$\forall\rho>\rho_2,\quad u(x,t)\le\mu_M+at+bv(r),\ \ \forall|x|\le\rho\ \text{ and }\ 0<t<T.$$

Fixing $(x,t)$ and letting $\rho\to\infty$, we see that $u(x,t)\le\mu_M+at+bv(r)$ in $\mathbb{R}^n_T$. Since this holds for any small $b$, we obtain $u(x,t)\le\mu_M+\alpha_M t$, see (5.29). □

**Proof of Theorem 2.2(b) ($0<\sigma\le\gamma$):** The quantities $w$, $v$ and $\delta$ are as in (7.2), (7.3) and (7.4). Refer to Part I in Section 5 and see Sub-Parts (i) and (ii). Arguing as in the proof of Theorem 2.2(a) above, we see that $u(x,t)\le\mu_M+at+bv(r)$, in $\mathbb{R}^n_T$, for any $b>0$ small enough. Recalling (5.23) and (5.29), i.e., $\lim_{b\to 0} a=0$, we get that $u(x,t)\le\mu_M$ and the claim holds.

**Proof of Theorem 2.2(c) ($\sigma>\gamma$):** Refer to Part II in Section 5. The quantity $\delta=\sigma^*$ in (7.1). From (5.30) $w(x,t)=\mu_M+at+b(1+t)v(r)$, where

$$v(r)=\int_0^{r^{\gamma^*}}(1+\tau^p)^{-1}d\tau\ \text{ and }\ p=\frac{\sigma-\gamma}{\gamma(\sigma-1)},$$

where $a>0$ and $b>0$. Then $w$ is a super-solution in $\mathbb{R}^n_T$ for any $0<b<b_0$, where $b_0$ is small enough, and an appropriate $a$ that depends on $b$. Moreover, by (5.35),

$$\lim_{b\to 0} a=0.$$

The rest of the proof is similar to the proof of Theorem 2.2(a). □

**Proof of Theorem 2.2(d) and (e)** ($\sigma \geq \gamma$)**:** We take $\bar{w}(x,t) = \mu_M + v(r) + Ft$, where $v(r)$ and $F$ are as in (5.37) and (5.39). See also (5.47) and (5.53). Observe that $v(r) \to \infty$ as $r \to R$.

Clearly, $\bar{w}(x,0) \geq \mu_M \geq u(x,0)$, $\forall |x| < R$. Since $\sup_{|x|\leq R}|u| < \infty$, select $\bar{R} < R$, close to $R$, such that $\bar{w}(x,t) \geq u(x,t)$ on $|x| = \bar{R}$. Thus, $\bar{w}(x,t) \geq u(x,t)$ in $B_T^{\bar{R}}$ and $u(z,t) \leq \bar{w}(z,t) = \mu_M + Ft$. Letting $R \to \infty$ and noting the limits in (5.47) and (5.53) the claims in parts (d) and (e) follow. Note that for $\sigma = \gamma$, we require $|\alpha_M| > \ell_M \mathcal{H}_M$ for the argument to apply. □

**Proof of Theorem 2.3:** We start with the proofs of parts (a)-(c). Recall that $\mu_m = \inf h$ and assume that $\alpha_m = \inf \chi < 0$.

**Proofs of parts (a), (b) and (c)** ($0 \leq \sigma \leq \gamma$)**:** Recall from (6.4) that for $R > 0$,

$$\bar{w}_R(x,t) = \mu_m + v(r) - Ft, \text{ where } v(r) = E\int_0^{\omega^2} (\tau^p - 1)^{-1} d\tau, \quad \omega = r/R,$$

and $E > 0$ and $F = F(R) > 0$. Note that $v(r) \leq 0$. See Sub-Cases (a), (b) and (c) of Case I in Section 6.

From (6.11) and (6.18) we see that

$$\lim_{R\to\infty} F = \lim_{R\to\infty} F(R) = \begin{cases} |\alpha_m|, & \sigma = 0, \\ (|\alpha_m|^\gamma/(\ell_m \mathcal{H}_m)^\sigma)^{1/(\gamma-\sigma)}, & 0 < \sigma < \gamma, \\ 0, & \sigma = \gamma,\ |\alpha_m| < \ell_m \mathcal{H}_m. \end{cases} \tag{7.5}$$

Let $u$ be as in the theorem. Since $v(r) \leq 0$, clearly, $\bar{w}(x,0) = \mu_m + v(r) \leq h(x) \leq u(x,0)$ in $|x| < R$. Since $\sup |u| < \infty$ in $B_T^R$, $\bar{w}(x,t) \leq u(x,t)$ on $r = R'$, for any $R' < R$, close to $R$. By Lemma 3.2, $\bar{w}_R \leq u$ in $B_T^{R'}$ and hence, in $B_R^T$.

Thus, $w(z,t) \leq u(z,t)$ and since $v(0) = 0$, $u(z,t) \geq \mu_m - Ft$. Letting $R \to \infty$, we get,

$$u(z,t) \geq \begin{cases} \mu_m - |\alpha_m| t, & \sigma = 0, \\ \mu_m - t(|\alpha_m|^\gamma/(\ell_m \mathcal{H}_m)^\sigma)^{1/(\gamma-\sigma)}, & 0 < \sigma < \gamma, \\ \mu_m, & \sigma = \gamma,\ |\alpha_m| < \ell_m \mathcal{H}_m. \end{cases}$$

**Proof of part (f)** ($\sigma \geq 0$)**:** If $\chi \geq 0$, take $\alpha_m \geq 0$ and refer to Case I in Section 6. The claim $u(x,t) \geq \mu_m$, $\forall (x,t) \in \mathbb{R}_T^n$, holds for any $\sigma \geq 0$.

**Proofs of parts (d) and (e)** ($\sigma \geq \gamma$)**:**

- Let $\sigma = \gamma$. We take $|\alpha_m| \geq \ell_m \mathcal{H}_m$. Assume that

$$\sup_{B_T^\rho}(-u(x,t)) \leq o(\rho^\delta), \quad \text{as } \rho \to \infty. \tag{7.6}$$

Recall Sub-Cases (i) and (ii) in Case II in Section 6. Take $a > 0$ and $0 < b < 1$ and set

$$\bar{w}(x,t) = \mu_m - at - b(1+t)v(r), \quad \text{in } \mathbb{R}^n_T.$$

If $k = 1$ i.e., $\gamma = 2$ then $\delta = 2 - \varepsilon$, for a small and fixed $\varepsilon > 0$, in (7.6), and

$$v(r) = \int_0^{r^2} (1 + \tau^{\varepsilon/2})^{-1} \, d\tau.$$

If $k > 1$ and $\gamma > 2$ then $\delta = \gamma^*$, in (7.6), and $v(r) = r^{\gamma^*}$.

- If $\sigma > \gamma$ and $k \geq 1$ then $\delta = \sigma^*$ in (7.6), and

$$v(r) = \int_0^{r^{\gamma^*}} (1 + \tau^p)^{-1} \, d\tau \quad \text{where } p = \frac{\sigma - \gamma}{\gamma(\sigma - 1)}.$$

It follows that $\lim_{b \to 0} a = 0$ in all the above situations, see (5.35) and (6.21). The rest of the proof is similar to that of Theorem 2.2. $\square$

**Proof of Theorem 2.5.** We take $\chi = \sigma = 0$ in Theorem 2.2. Let $u > 0$ be as in the statement of the theorem.

Let $k > 1$. Set $\hat{u} = \phi^{-1}(u)$ and $h = \phi^{-1}(g)$, see Lemma 3.3.

We recall (2.10), (2.13) and (2.14). If $\eta(s) = f^{-1/(k-1)}(s)$ then, as noted before,

$$\text{either (i)} \quad \int_0^\infty \eta(s) ds < \infty \quad \text{or (ii)} \quad \int_0^\infty \eta(s) ds = \infty. \tag{7.7}$$

The domain of $\phi$ in (7.7) (i) is $[0, \infty)$, and in (7.7) (ii) it is $(-\infty, \infty)$. Also,

$$Z(\hat{u}) = \phi''(\hat{u})/\phi'(\hat{u}) = (d\eta(s)/ds)|_{s=\hat{u}}$$

is non-increasing and $0 < \ell_m \leq Z(\hat{u}) \leq \ell_M < \infty$. Moreover, the domain of $Z$ in (i) is $(0, \infty)$ or $[0, \infty)$, and in (ii) it is $(-\infty, \infty)$. Set

$$\mu_m = \inf_x \phi^{-1}(g(x)) \quad \text{and} \quad \mu_M = \sup_x \phi^{-1}(g(x)).$$

We employ Lemma 3.3 and Corollary 3.4. Set $r = |x - z|$.

**Proof of part (a):** Since $\hat{u}$ is a sub-solution we have that $\sup_{B^R_T} \hat{u}(x,t) \leq o(R^{\gamma^*})$ as $R \to \infty$. In (7.7) (i) $\hat{u} > 0$ and $\mu_M > 0$, and in (7.7) (ii) $-\infty < \hat{u} < \infty$ and $-\infty < \mu_M < \infty$.

By Lemma 3.3, $\hat{u} \in usc(\overline{\mathbb{R}^n_T})$ solves

$$H(D\hat{u}, D^2\hat{u} + Z(\hat{u}) D\hat{u} \otimes D\hat{u}) - \hat{u}_t \geq 0, \text{ in } \mathbb{R}^n_T,$$
$$\text{and } \hat{u}(x,0) \leq \phi^{-1}(g(x)), \text{ for all } x \in \mathbb{R}^n.$$

By Theorem 2.2(a), $\sup_{\mathbb{R}^n_T} \hat{u} \le \mu_M$, and thus, $\sup_{\mathbb{R}^n_T} u \le \sup_{\mathbb{R}^n} g$.

**Proof of part (b):** In this case, $\hat{u} \in lsc(\overline{\mathbb{R}^n_T})$ solves

$$H(D\hat{u}, D^2\hat{u} + Z(\hat{u})D\hat{u} \otimes D\hat{u}) - \hat{u}_t \le 0, \text{ in } \mathbb{R}^n_T,$$
$$\text{and } \hat{u}(x,0) \ge \phi^{-1}(g(x)), \text{ for all } x \in \mathbb{R}^n.$$

We first discuss (7.7) (ii). Since the domain of $\phi$ is $(-\infty, \infty)$, we apply Theorem 2.3 (a), with $\chi \equiv 0$ and $\sigma = 0$, to obtain $\inf_{\mathbb{R}^n_T} \hat{u} \ge \mu_m$ and hence, $\inf_{\mathbb{R}^n_T} u \ge \inf_{\mathbb{R}^n} g$. No lower bound is needed.

In (7.7) (i), the domains of $\phi$ and $Z$ are $[0, \infty)$ and at least $(0, \infty)$ respectively. Thus, $\hat{u} > 0$ and $\mu_m > 0$. Let $\bar{w}_R$ be as in the proof of Theorem 2.3(a), see also (6.4), that is,

$$\bar{w}_R(x,t) = \mu_m + v(r) - Ft, \text{ where } v(r) = E\int_0^{\omega^2} (\tau^p - 1)^{-1} d\tau, \quad \omega = r/R,$$

where $E > 0$. Here, $v(0) = 0$, $v(r) \le 0$ and $\bar{w}_R(r) \to -\infty$ as $r \to R$. Extend $Z$ to $(-\infty, 0)$ by $\ell_M$ thus defining $Z$ on $(-\infty, \infty)$.

For all $\rho > 0$, set

$$\varepsilon(\rho) = \inf_{B^\rho_T} \hat{u}.$$

Also, recall from (7.5) that $\lim_{R\to\infty} F = \lim_{R\to\infty} F(R) = 0$, since $\chi = 0$. Select $R > 0$, large enough, such that $FT < \mu_m/4$. This ensures that $\bar{w}_R(z,t) = \mu_m + v(0) - Ft \ge \mu_m/2 > 0$. Fix $R$.

Clearly, as $\varepsilon(\rho) > -\infty$ and is decreasing in $\rho$, $\varepsilon(R) \le \varepsilon(\rho)$ for $\rho \le R$.

Since $v(r) \le 0$, $w_R(x,0) = \mu_m + v(r) \le h(x) \le \hat{u}(x,0)$, $0 \le r < R$. Since $v(r) \to -\infty$ as $r \to R$, we choose $R' < R$, close to $R$, such that $\bar{w}_R(R', t) \le \varepsilon(R)/4 \le \varepsilon(R')/4$. Applying the comparison principle, $w_R \le \hat{u}$ in $B^{R'}_T$ and, hence, in $B^R_T$.

Taking $r = 0$,

$$\mu_m + v(0) - Ft = \bar{w}_R(z,t) \le \hat{u}(z,t).$$

Since, $v(0) = 0$ and $\lim_{R\to\infty} F = 0$, the claim holds.

Note that a proof can also be worked by considering the sub-solution $\psi_R(x,t) = \max\{\bar{w}_R(x,t), \varepsilon(R)/2\} > 0$ in $B^T_R$. Clearly, $\psi_R(x,t) \le \hat{u}(x,t)$ on $r = R$ and $\psi_R(x,0) \le \hat{u}(x,0)$. This follows as $\bar{w}_R(x,0) \le \mu_m$. This leads to $\hat{u} \ge \psi_R \ge \bar{w}_R$ in $B^R_T$. This does not require extending $Z$.

For $k = 1$, set $u = \phi(\hat{u}) = e^{\hat{u}}$. Then, $Z(\hat{u}) \equiv 1$ and a proof follows analogously. □

## 8. Optimality

In this section, we address optimality for Theorems 2.2 (a), (b) and (c) and 2.5 (i) (a). We point out that some of our results discussed here are partial in nature.

Recall the the assumption $-\infty < \mu_m = \inf h \le \sup h = \mu_M < \infty$, and the notation

$$G(v) := H(Dv, D^2v + Z(v)Dv \otimes Dv) \;\text{ and }\; H_\sigma(v) = G(v) + \chi(t)|Dv|^\sigma - v_t. \quad (8.1)$$

To address optimality of Theorem 2.2, we construct sub-solutions $\phi(x,t)$ which tend to $-\infty$ as $t \to 0^+$ and grow at the rate indicated in theorem. We then take $\max\{\mu_M, \phi\}$ to show optimality for the maximum principle. We construct $\phi = \phi(r)$ where

$$r = |x|, \; \forall x \in \mathbb{R}^n, \quad \phi_r > 0 \quad \text{and} \quad \phi_{rr} > 0 \; \text{ in } r \neq 0.$$

Also, $o$ stands for the origin in $\mathbb{R}^n$.

Thus, (3.3) and Condition A(see Section 2) lead to

$$\begin{aligned} G(\phi) &= H\left(\phi_r e, \left(\frac{\phi_r}{r}(I - e\otimes e) + \phi_{rr} e \otimes e\right) + Z(\phi)\phi_r^2 e \otimes e\right) \\ &\ge H\left(\phi_r e, \left(\frac{\phi_r}{r}(I - e \otimes e)\right)\right) = \frac{\phi_r^k}{r} H\left(e, I + \left(\frac{r\phi_{rr}}{\phi_r} - 1\right) e \otimes e\right). \end{aligned} \quad (8.2)$$

Another version follows from the first equation in (8.2) by using that $I - e\otimes e \ge 0$, $\ell_m = \inf Z$, $\mathcal{H}_m = \min_{|e|=1} H(e, e\otimes e)$ and writing

$$G(\phi) \ge H\left(\phi_r e, Z(\phi)\phi_r^2 e \otimes e\right) \ge (\phi_r)^\gamma Z(\phi) H(e, e\otimes e) \ge (\phi_r)^\gamma \ell_m \mathcal{H}_m. \quad (8.3)$$

We may combine (8.2) and (8.3) as follows. For $0 \le \tau \le 1$,

$$G(\phi) \ge \tau \left[\frac{\phi_r^k}{r} H\left(e, I + \left(\frac{r\phi_{rr}}{\phi_r} - 1\right) e \otimes e\right)\right] + (1-\tau)(\phi_r)^\gamma \ell_m \mathcal{H}_m. \quad (8.4)$$

Note that to show that the functions $\phi$ satisfy the inequalities at $r = 0$, we use Remark 4.4.

**Sub-solutions: Optimality for the maximum principle**

The discussion here refers to Theorem 2.2.

**Part I:** $k > 1$**.** Recall that $k_1 > 0$, $k = k_1 + 1$, $\gamma = k+1$ and $\gamma^* = \gamma/k$. Note that $\gamma = k + 1 > 2$ and $1 < \gamma^* < 2$.

**Case** $(0 \le \sigma < \gamma)$ : We take

$$\phi = \frac{a r^{\gamma^*}}{t^{\theta_1}} - \frac{b}{t^{\theta_2}}, \quad \text{where } \theta_1 = \frac{2}{k-1} \; \text{ and } \; \theta_2 = \frac{k+1}{k-1} = \frac{\gamma\theta_1}{2}. \quad (8.5)$$

Here, $a > 0$ and $b > 0$ are to be determined. Set $E = a\gamma^*$. Then,

$$\phi_r = \frac{Er^{\gamma^*-1}}{t^{\theta_1}}, \quad \phi_{rr} = \frac{E(\gamma^*-1)r^{\gamma^*-2}}{t^{\theta_1}}, \quad \frac{r\phi_{rr}}{\phi_r} - 1 = \gamma^* - 2 = \frac{1-k}{k}. \tag{8.6}$$

Taking $\tau = 1/2$ in (8.4) and using (8.6), we obtain that

$$G(\phi) \geq \frac{1}{2}\left[\frac{E^k r^{\gamma^* k-\gamma}}{t^{k\theta_1}} H\left(e, I + (\gamma^*-2)e\otimes e\right) + \frac{E^\gamma r^{(\gamma^*-1)\gamma}}{t^{\gamma\theta_1}}\ell_m\mathcal{H}_m\right]. \tag{8.7}$$

Observe that, $\gamma^* - 1 = 1/k$, $\gamma^* k - \gamma = 0$ and $I + (\gamma^*-2)e\otimes e \geq (e\otimes e)/k$. Thus,

$$H(e, I - \gamma^* e\otimes e) \geq H(e, e\otimes e)/k \geq \mathcal{H}_m/k.$$

We get from (8.7) that

$$G(\phi) \geq \frac{1}{2}\left[\frac{E^k\mathcal{H}_m}{kt^{k\theta_1}} + \frac{E^\gamma r^{\gamma/k}\ell_m\mathcal{H}_m}{t^{\gamma\theta_1}}\right]. \tag{8.8}$$

Recall that $\alpha_m = \inf_t \chi(t)$. Using (8.8) and assuming that $\alpha_m \leq 0$ (otherwise disregard $\chi(t)|Du|^\sigma$), we get that

$$\begin{aligned}
H_\sigma(\phi) &= G(\phi) + \chi(t)\phi_r^\sigma - \phi_t \\
&\geq \frac{1}{2}\left[\frac{E^k\mathcal{H}_m}{kt^{k\theta_1}} + \frac{E^\gamma r^{\gamma/k}\ell_m\mathcal{H}_m}{t^{\gamma\theta_1}}\right] + \frac{\alpha_m E^\sigma r^{\sigma/k}}{t^{\sigma\theta_1}} + \frac{E\theta_1 r^{\gamma/k}}{\gamma^* t^{\theta_1+1}} - \frac{b\theta_2}{t^{\theta_2+1}} \\
&= \left[\frac{E^k\mathcal{H}_m}{2kt^{k\theta_1}} - \frac{b\theta_2}{t^{\theta_2+1}}\right] + \left[\frac{E^\gamma r^{\gamma/k}\ell_m\mathcal{H}_m}{2t^{\gamma\theta_1}} - \frac{|\alpha_m|E^\sigma r^{\sigma/k}}{t^{\sigma\theta_1}}\right] + \frac{E\theta_1 r^{\gamma/k}}{\gamma^* t^{\theta_1+1}}.
\end{aligned} \tag{8.9}$$

Apply Young's inequality to obtain

$$\frac{|\alpha_m|E^\sigma r^{\sigma/k}}{t^{\sigma\theta_1}} \leq \left(\frac{\sigma}{\gamma}\right)\frac{E^\gamma r^{\gamma/k}\ell_m\mathcal{H}_m}{2t^{\gamma\theta_1}} + \left(\frac{\gamma-\sigma}{\gamma}\right)\left(\frac{2}{\ell_m\mathcal{H}_m}\right)^{\sigma/(\gamma-\sigma)}|\alpha_m|^{\gamma/(\gamma-\sigma)}.$$

Employing this in the right side of (8.9), we get

$$H_\sigma(\phi) \geq \frac{E^k\mathcal{H}_m}{2kt^{k\theta_1}} - \frac{b\theta_2}{t^{\theta_2+1}} - \left(\frac{\gamma-\sigma}{\gamma}\right)\left(\frac{2}{\ell_m\mathcal{H}_m}\right)^{\sigma/(\gamma-\sigma)}|\alpha_m|^{\gamma/(\gamma-\sigma)}$$

By (8.5), $k\theta_1 = \theta_2 + 1$. Hence,

$$H_\sigma(\phi) \geq \frac{1}{t^{k\theta_1}}\left[\frac{E^k\mathcal{H}_m}{2k} - b\theta_2 - \left(\frac{\gamma-\sigma}{\gamma}\right)\left(\frac{2}{\ell_m\mathcal{H}_m}\right)^{\sigma/(\gamma-\sigma)}|\alpha_m|^{\gamma/(\gamma-\sigma)}T^{k\theta_1}\right]. \tag{8.10}$$

Select $b = 1$ and $E$ large enough to get a sub-solution for $0 < \sigma < \gamma$. For $\sigma = 0$, we use (8.9) to see that

$$H_0(\phi) \geq \frac{E^k\mathcal{H}_m}{2t^{k\theta_1}} - |\alpha_m| - \frac{\theta_2}{t^{\theta_2+1}} = \frac{1}{2t^{k\theta_1}}\left[E^k\mathcal{H}_m - 2|\alpha_m|T^{k\theta_1} - 2\theta_2\right]. \tag{8.11}$$

Choosing $E$ large enough we get a sub-solution. □

**Case $\sigma = \gamma$ and $|\alpha_m| < \ell_m \mathcal{H}_m$:** Let $0 < \varepsilon < 1$ be such that $|\alpha_m| \le (1-\varepsilon)\ell_m \mathcal{H}_m$. Setting $\tau = \varepsilon$ in (8.4), using (8.5) and arguing as in (8.7) and (8.8), we get that

$$G(\phi) \ge \varepsilon \frac{E^k \mathcal{H}_m}{k t^{k\theta_1}} + (1-\varepsilon)\frac{E^\gamma r^{\gamma/k}\ell_m \mathcal{H}_m}{t^{\gamma\theta_1}}. \tag{8.12}$$

Selecting $\theta_1$ and $\theta_2$ as in (8.5), arguing as in (8.9) and replacing $|\alpha_m|$ by $(1-\varepsilon)\ell_m\mathcal{H}_m$, we get that

$$\begin{aligned} H_\gamma(\phi) &\ge \left[\frac{\varepsilon E^k \mathcal{H}_m}{k t^{k\theta_1}} - \frac{b\theta_2}{t^{\theta_2+1}}\right] \\ &\quad + \left[\frac{(1-\varepsilon)E^\gamma r^{\gamma/k}\ell_m\mathcal{H}_m}{t^{\gamma\theta_1}} - \frac{(1-\varepsilon)\ell_m\mathcal{H}_m E^\sigma r^{\sigma/k}}{t^{\gamma\theta_1}}\right] + \frac{E\theta_1 r^{\gamma/k}}{\gamma^* t^{\theta_1+1}} \\ &= \frac{1}{t^{k\theta_1}}\left[\frac{\varepsilon E^k \mathcal{H}_m}{k} - b\theta_2\right] + \frac{E\theta_1 r^{\gamma/k}}{\gamma^* t^{\theta_1+1}}. \end{aligned}$$

Choose $E = 1$ and $b = \varepsilon\mathcal{H}_m/k\theta_2$ and conclude that $\phi$ is a sub-solution in $\mathbb{R}^n_T$. □

**Sub-case $\sigma = \gamma$ and $|\alpha_m| = \ell_m\mathcal{H}_m$:** This is not clear to us for general $H$. However, if $H$ is quasilinear, i.e, if $G$ is quasilinear (such as the parabolic $p$-Laplacian) then $r^{\gamma/k}$ is optimal. We assume that $\alpha_m \le 0$.

Observe that (8.2), (8.3), (8.7) and (8.8) lead to

$$\begin{aligned} G(\phi) &= H\left(D\phi, D^2\phi + Z(\phi)D\phi\otimes D\phi\right) = H(D\phi, D^2\phi) + H(D\phi, Z(\phi)D\phi\otimes D\phi) \\ &= H\left(\phi_r e, \frac{\phi_r}{r}(I - e\otimes e) + \phi_{rr} e\otimes e\right) + H\left(\phi_r e, Z(\phi)\phi_r^2 e\otimes e\right) \\ &\ge \frac{E^k\mathcal{H}_m}{k t^{k\theta_1}} + \frac{E^\gamma r^{\gamma/k}\ell_m\mathcal{H}_m}{t^{\gamma\theta_1}}. \end{aligned}$$

As done in (8.9), we get that

$$\begin{aligned} H_\gamma(\phi) &\ge \frac{E^k\mathcal{H}_m}{k t^{k\theta_1}} + \frac{E^\gamma r^{\gamma/k}\ell_m\mathcal{H}_m}{t^{\gamma\theta_1}} - \frac{|\alpha_m|E^\gamma r^{\gamma/k}}{t^{\gamma\theta_1}} - \frac{b\theta_2}{t^{\theta_2+1}} + \frac{E\theta_1 r^{\gamma/k}}{\gamma^* t^{\theta_1+1}} \\ &\ge \frac{E^k\mathcal{H}_m}{k t^{k\theta_1}} - \frac{b\theta_2}{t^{\theta_2+1}} + \frac{E\theta_1 r^{\gamma/k}}{\gamma^* t^{\theta_1+1}}. \end{aligned}$$

Since $k\theta_1 = \theta_2 + 1$, we may now conclude optimality. □

**Case $\sigma > \gamma$:** We assume that $\chi \ge 0$. Choose

$$\phi(r,t) = \frac{af(r)}{t^{\theta_1}} - \frac{b}{t^{\theta_2}}, \quad \text{where} \quad \theta_1 = \frac{2}{k-1}, \quad \theta_2 = \frac{k+1}{k-1} = \frac{\gamma\theta_1}{2},$$

$$f(r) = \int_0^{r^{\gamma^*}} \frac{1}{1+\tau^p}d\tau, \quad p = \frac{\sigma-\gamma}{\gamma(\sigma-1)} \quad \text{and} \quad \sigma^* = \frac{\sigma}{\sigma-1}.$$

See (5.30) in this context. We apply the two versions (8.2) and (8.3) as follows:

$$G(\phi) \geq \frac{a^k (f')^k}{r t^{k\theta_1}} H\left(e, I + \left(\frac{r f''}{f'} - 1\right) e \otimes e\right), \quad 0 \leq r \leq 1,$$
$$G(\phi) \geq \frac{a^\gamma (f')^\gamma}{t^{\gamma\theta_1}} \ell_m \mathcal{H}_m, \qquad 1 \leq r < \infty.$$

Refer to (vi), (viii) and (ix) in Case C of Remark 4.2. Then

$$f'(r) \geq (\gamma^*/2) \min(r^{\gamma^*-1}, r^{\sigma^*-1}) \quad \text{and} \quad r f''/f' \geq 1/\sigma.$$

Set

$$E = a\gamma^*/2 \leq a.$$

Since $I + (\sigma^{-1} - 1) e \otimes e \geq e \otimes e/\sigma$, we obtain from above that

$$G(\phi) \geq \frac{E^k}{t^{k\theta_1}} H\left(e, I + \left(\sigma^{-1} - 1\right) e \otimes e\right) \geq \frac{E^k \mathcal{H}_m}{\sigma t^{k\theta_1}}, \quad 0 \leq r \leq 1,$$
$$G(\phi) \geq \frac{a^\gamma (f')^\gamma}{t^{\gamma\theta_1}} \ell_m \mathcal{H}_m \geq \frac{E^\gamma r^{\gamma/(\sigma-1)} \ell_m \mathcal{H}_m}{t^{\gamma\theta_1}} \geq \frac{E^\gamma \ell_m \mathcal{H}_m}{t^{\gamma\theta_1}}, \quad 1 \leq r < \infty. \quad (8.13)$$

Since $\chi \geq 0$, we see from (8.1) that $H_\sigma(\phi) \geq G(\phi) - \phi_t$ and

$$H_\sigma(\phi) \geq \frac{E^k \mathcal{H}_m}{\sigma t^{k\theta_1}} - \frac{b\theta_2}{t^{\theta_2+1}}, \ 0 \leq r \leq 1, \ \text{ and } \ H_\sigma(\phi) \geq \frac{E^\gamma \ell_m \mathcal{H}_m}{2^\gamma t^{\gamma\theta_1}} - \frac{b\theta_2}{t^{\theta_2+1}}, \ 1 \leq r < \infty. \quad (8.14)$$

Since $k\theta_1 = \theta_2 + 1$ and $\gamma\theta_1 - (\theta_2 + 1) = \theta_1$, we get that $\phi$ is a sub-solution if we select $E = 1$ and

$$0 < b \leq \min\left(\frac{\mathcal{H}_m}{\sigma\theta_2}, \ \frac{\ell_m \mathcal{H}_m}{\theta_2 T^{\theta_1}}\right).$$

To show optimality for $\sigma > \gamma$, we observe that for any $\chi$, $\sup_{\mathbb{R}^n_T} u \leq \mu_M$ if we impose that $\sup_{B^R_T} u = o(R^{\sigma^*})$ as $R \to \infty$.

If $\sup \chi < 0$ then the maximum principle holds without any restrictions, see Theorem 2.2(e). If $\chi \geq 0$ somewhere in $(0, T)$ then the above shows that the growth rate of $o(R^{\sigma^*})$ is optimal, see Theorem 2.2(c).

The above also applies to part (a) of Theorem 2.5(i). □

**Part II:** $k = 1$**.** Note that $k_1 = 0$ and $\gamma = 2$. Also, $H(p, X) = H(X)$ for all $(p, X) \in \mathbb{R}^n \times S^{n\times n}$. See Condition B in Section 2. We take $f \equiv 1$.

This part applies to Theorem 2.5 (ii). Clearly, $u > 0$ satisfies

$$H(D^2 u) - u_t \geq 0, \ \text{ in } \mathbb{R}^n_T \text{ and } u(x, 0) \leq g(x), \ \forall x \in \mathbb{R}^n.$$

We apply the change of variable $v = \log u$. Then, by Case (ii) of Lemma 3.3,

$$H(D^2 v + Dv \otimes Dv) - v_t \geq 0, \ \text{in } \mathbb{R}^n_T \text{ and } v(x, 0) \leq \log g(x), \ \forall x \in \mathbb{R}^n.$$

For $a > 0$ and $b > 0$, we define

$$w = \frac{a(r+r^2)}{t^{3/2}} - \frac{b}{t^2}, \quad \text{in } r \geq 0 \text{ and } t > 0.$$

In $r > 0$, $w_r > 0$, $w_{rr} > 0$ and hence,

$$\begin{aligned} H(D^2w + Dw \otimes Dw) &= H\left(\frac{w_r}{r}(I - e\otimes e) + w_{rr} e\otimes e + w_r^2 e\otimes e\right) \\ &\geq H(w_r^2 e\otimes e) = w_r^2 H(e\otimes e) \geq \frac{a^2(1+2r)^2}{t^3}\mathcal{H}_m. \end{aligned}$$

Thus,

$$H(D^2w + Dw\otimes Dw) - w_t \geq \frac{a^2(1+2r)^2\mathcal{H}_m}{t^3} + \frac{3a(r+r^2)}{2t^{5/2}} - \frac{2b}{t^3}.$$

Choosing $b = a^2\mathcal{H}_m/2$, we see that $w$ is sub-solution in $r > 0$.

To show that $w$ is a sub-solution in all of $\mathbb{R}^n_T$, let $\psi$, $C^2$ in $x$ and $C^1$ in $t$, be such that $(w-\psi)(x,t) \leq (w-\psi)(o,s)$ for some $0 < s < T$. Hence,

$$\frac{a(r+r^2)}{t^{3/2}} - \frac{b}{t^2} + \frac{b}{s^2} \leq \langle D\psi(o,s), x\rangle + \psi_t(t-s) \text{ as } (x,t) \to (o,s).$$

It is clear that $\psi_t(o,s) = 2b/s^3$. Taking $t = s$, we see that

$$\frac{a(r+r^2)}{s^{3/2}} \leq \langle D\psi(o,s), x\rangle \text{ as } (x,t) \to (o,s).$$

Dividing by $r = |x|$ and writing $e = x/|x|$ and letting $r \to 0$, we get

$$0 < \frac{a}{s^{3/2}} \leq \langle D\psi(o,s), e\rangle, \ \forall e.$$

Thus, $D\psi(o,s)$ does not exist contradicting that $\psi$ is $C^2$ in $x$. Hence, $w$ is a sub-solution in $\mathbb{R}^n_T$. $\square$

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