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New existence results for fractional differential equations in a weighted Sobolev space

Ahmed Hallaci, Hamid Boulares, Abdelouaheb Ardjouni* and Abderrazak Chaoui

Abstract. *In this paper, we give some conditions to prove the existence of solutions for a nonlinear boundary value problem of fractional differential equations with higher order* q, $(n-1 < q < n)$, *involving Riemann-Liouville fractional derivative. The solutions are discussed in a weighted fractional Riemann-Liouville Sobolev space using Schauder's fixed point theorem. An example is given to illustrate the main results.*

1. Introduction

Let $T > 0$ be a real number and $I = [0, T]$ be a closed and bounded interval of the set of real numbers $\mathbb{R}$. Consider the following nonlinear functional boundary value problem of the higher-order fractional differential equations (FDEs) with Riemann-Liouville derivative

$$D^q u(t) = g(t, u(t), D^s u(t)), \ t \in I, \tag{1.1}$$

$$D^{q-i} u|_{t=0} = 0, \quad i = 1, \ldots, n, \quad i \neq n-1 \text{ and } u(T) = 0, \tag{1.2}$$

where $1 < n-1 < q < n$, $0 < s < 1$, $g \colon I \times \mathbb{R}^2 \to \mathbb{R}$ is given function, D^q denotes the Riemann-Liouville's fractional derivative.

Recently, the subject of fractional calculus has become a fundamental branch of applied mathematics, has been applied widely in a variety of mathematical models in science and engineering during the last three decades such as physics, chemistry, biology, engineering, viscoelasticity, signal processing, electrotechnical, electrochemistry and controllability. Many researchers have been interested in the theory of nonlinear FDEs stimulated by the extensive applicabilities mentioned previously (see [12, 15, 17]). Most of researches centred around the investigation on existence and uniqueness of solution where this side of study for nonlinear FDEs have been extensively developed using especially the fixed point theory and other theoretical methods as iterative method, measures of non-compactness technic, Krasnoselskii-Krein and nagumo uniqueness theorems (see [4, 19]). However, the fixed point theorems staying the most used method to study the existence and

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*Corresponding author.

uniqueness of solutions of nonlinear FDEs and nonlinear fractional differential systems (see [2, 3, 5, 6, 10, 16, 21]). Beside, the mentioned published papers has been devoted to give the existence and uniqueness of solution of various classes of fractional differential and integral equations in the space of continuous functions $C([a,b])$ or $C(\mathbb{R}_+)$. But the discussion on measurable solutions of differential and integral equations remains relatively few compared to continuous solutions, we refer to some papers about this side as [9, 13, 14]. Where the L^p-solutions of fractional differential equations are discussed by Burton and Zhang in [9] using some techniques to show the belonging of solutions to $L^p(\mathbb{R}_+)$. In [13], Schauder's and Darbo's fixed point theorems are employed to study the existence of $L^p(\mathbb{R}_+)$-solutions of nonlinear quadratic integral equations. Also in [14], the authors give different existence results for $L^p[a,b]$ and $C([a,b])$-solutions of some nonlinear integral equations of the Hammerstein and Volterra types using some fixed point theorems combined with a general version of Gronwall's inequality.

In this paper, motivated by those valuable contributions mentioned above, we mainly discuss the existence of solutions for nonlinear FDEs of higher order q $(n-1<q<n)$ in a measurable weighted fractional Sobolev space. To achieve our mentioned purpose, we first transform the fractional differential equation (1.1) with conditions (1.2) into a equivalent integral equation with Green continuous function using Laplace transform technic of the Riemann-Liouville fractional derivative and some analytical skills, then we present the our study space which is based essentially on the classical concepts of weighted L^p-spaces and Sobolev spaces. Furthermore, we investigate the existence of solution of the system (1.1)-(1.2) using Schauder's fixed point theorem. The rest of this paper is organized as follows: in section 2 we present some auxiliary definitions and lemmas about fractional calculus theory and measurable functions theory that will be used to prove our main results, also we show the completeness of fractional Sobolev space. Section 3 devoted to the main result. Lastly, we present an example to show the effectiveness of our main result.

2. Preliminaries

We start by presenting some necessary definitions and lemmas that we will used for investigate our main results. For more details see [1, 7, 8, 12, 15, 17, 18, 20].

Definition 2.1 ([12, 15, 17]). The Riemann-Liouville fractional integral of the function u of order $q \geq 0$ is defined by

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-q}} d\tau,$$

where $\Gamma(q)$ is the Euler gamma function defined by $\Gamma(q) = \int_0^\infty e^{-t} t^{q-1} dt$.

Definition 2.2 ([12, 15, 17]). The Riemann-Liouville fractional derivative of the function u of order $q \in (n-1, n]$ is defined by

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{u(\tau)}{(t-\tau)^{q-n+1}} d\tau.$$

Definition 2.3 ([18]). $\Phi : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is called a Carathéodory function if
(i) $t \longmapsto \Phi(t, u)$ is measurable for every $u \in \mathbb{R}$,
(ii) $u \longmapsto \Phi(t, u)$ is continuous for almost all $t \in I$.

Definition 2.4. Let I be a measurable subset of $\mathbb{R}$, $g\colon I \times \mathbb{R}^d \to \mathbb{R}^{d'}$ satisfies the condition of Carathéodory. By a generalized Nemytskii operator we mean the mapping N_g taking a (measurable) vector functions $u = (u_1, \ldots, u_d)$ to the function $N_g u(t) = g(t, u(t))$, $t \in I$.

The continuity of the operator N_g is concerned in the following lemma.

Lemma 2.5. *Consider the same data of above definition. Let $p_j \in [0, \infty)$ for $j = 1, \ldots, d$ and $r \in [0, \infty)$ with $u_j \in L^{p_j}(I)$, $j = 1, \ldots, d$ and $b \in L^r(I)$, a constant $c > 0$, assume that*

$$|g(t, u)| \leq b(t) + c \sum_{j=1}^{d} |u_j|^{\frac{p_j}{r}}, \ \text{a.e. } t \in I, \ u \in \mathbb{R}^d,$$

then generalized Nemytskii operator

$$N_g u(t) = g(t, u(t)), \ \text{a.e. } t \in I, \ u = (u_1, \ldots, u_d) \in \prod_{j=1}^{d} L^{p_j}(I),$$

is bounded and continuous from $\prod_{j=1}^{d} L^{p_j}(I)$ to $L^r(I)$.

Proof. Lemma 2.5 is a generalization of Theorem 5.1 in [18] to a vector function case $u \in \prod_{j=1}^{d} L^{p_j}(I)$ instead of $u \in L^p(I)$. We use the Vitali's theorem [18] for show the continuity and boundedness of the generalized Nemytskii operator. Therefore, we omit the proof. □

Lemma 2.6 ([1, 8]). *Let $\mathcal{F}$ be a bounded set in $L^p([0, T])$ with $1 \leq p < \infty$. Assume that*
(i) $\lim_{|h| \to 0} \|\tau_h f - f\|_p = 0$ *uniformly on $\mathcal{F}$, where $\tau_h f(t) = f(t+h)$, and*
(ii) $\lim_{\epsilon \to 0} \int_{T-\epsilon}^{T} |f(t)|^p dt = 0$, *uniformly on $\mathcal{F}$.*
Then $\mathcal{F}$ is relatively compact in $L^p([0, T])$.

Remark 2.7. A first possible definition of solutions of problem (1.1)-(1.2) in the Lebesgue spaces of measurable function $L^p(I)$, is a function $u \in L^p(I)$ whose fractional derivative $D^s u$, $s \in (0,1)$ belongs to $L^p(I)$. On the other hand, from Definition 2.2, for some $s \in (0,1)$, it is obvious that the Riemann-Liouville fractional derivative of a function u is written in the form $D^s u = \left(I^{1-s}u\right)'$. That is, if $D^s u$ exists then the Riemann-Liouville fractional integral $I^{1-s}u$ is differentiable almost everywhere. Therefore, we use a more convenient definition of the solutions of (1.1)-(1.2) as the functions $u \in L^p(I)$, $I^{1-s}u \in L^p(I)$ and $\left(I^{1-s}u\right)' \in L^p(I)$, which form the structure of a Sobolev space that we denote him by $W_{RL}^{s,p}(I)$, defined as follows

$$W_{RL}^{s,p}(I) = \left\{u \in L^p(I) \text{ and } I^{1-s}u \in W^{1,p}(I)\right\}.$$

Before passing to show the completeness of $W_{RL}^{s,p}(I)$, we define the spaces:
$\mathfrak{D}'(I)$: space of distributions,
$C_c^1(I)$: space of $C^1(I)$-functions with compact support.

Lemma 2.8 ([11]). *$\left(W_{RL}^{s,p}(I), \|u\|_{W_{RL}^{s,p}(I)}\right)$ is a Banach space endowed with the norm*

$$\|u\|_{W_{RL}^{s,p}(I)} = \left(\|u\|_p^p + \left\|I^{1-s}u\right\|_{W^{1,p}(I)}^p\right)^{\frac{1}{p}}.$$

Remark 2.9. In [7], the authors discussed more broadly about fractional Sobolev space $W_{RL}^{s,p}(I)$ in the case where $p = 1$ to make the relation between this spaces and the classical spaces of functions of bounded variation BV. The authors shown also the completeness of the fractional Sobolev spaces $W_{RL}^{s,1}(I)$.

The following weighted fractional Sobolev space plays a fundamental role in our discussion. We define the weighted L^p-space

$$L^{p,\sigma}(I) = \left\{u \in L^p(I),\ \|u\|_{p,\sigma} < +\infty\right\},$$

where, $\|u\|_{p,\sigma}$ is the positive real valued function defined on $L^p(I)$ by

$$\|u\|_{p,\sigma} = \left(\int_I \sigma(t)\,|u(t)|^p\,dt\right)^{\frac{1}{p}} \text{ for all } u \in L^p(I).$$

Also, we define the weighted fractional Sobolev space with Riemann-Liouville fractional derivative by

$$E_\sigma = \left\{u \in L^{p,\sigma}(I) : I^{1-s}u \in W_1^{p,\sigma}(I)\right\},$$

equipped with the norm

$$\|u\|_\sigma = \left(\|u\|_{p,\sigma}^p + \left\|I^{1-s}u\right\|_{W_1^{p,\sigma}}^p\right)^{\frac{1}{p}},$$

where

$$W_1^{p,\sigma}(I) = \left\{v \in L^{p,\sigma}(I) : v' \in L^{p,\sigma}(I)\right\},$$

σ is a given function defined on I and such that there exists a real number $\sigma_* > 1$ satisfies $1 \le \sigma(t) \le \sigma_*$, for all $t \in I$, and

$$K' \in L^{p,\sigma}(I), \text{ for all } t \in I, \tag{2.1}$$

where

$$K(t) = \begin{cases} \int_0^t \frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^s} d\tau, & t \ge \tau, \\ 0, & t < \tau. \end{cases} \tag{2.2}$$

Clearly

$$\sigma(t-\tau) \ge 1, \text{ for all } t, \tau \in I \text{ with } t \ge \tau,$$

and $\|.\|_\sigma$ is a norm. Since $1 \le \sigma(t) \le \sigma_*$, then the two norms $\|.\|_{W^{s,p}_{RL}(I)}$ and $\|.\|_\sigma$ are equivalent. So, from Lemma 2.8, $(E_\sigma, \|.\|_\sigma)$ is a Banach space.

Definition 2.10. The solutions of the system (1.1)-(1.2) are functions $u \in E_\sigma(I)$ and u satisfies the system (1.1)-(1.2).

Lemma 2.11 ([17])**.** *Let $n-1 \le q < n$ and $p > 0$. The Laplace transform of the Riemann-Liouville fractional derivative $D^q u(t)$ and the power function $t \mapsto t^p$ are given respectively by*

(i) $L\{D^q u(t), z\} = z^q U(z) - \sum_{i=0}^{n-1} z^i \left[D^{q-i-1} u(t)\right]_{t=0}$,

(ii) $L\{t^p, z\} = \Gamma(p+1) z^{-(p+1)}$,

where $U(z) = L\{u(t), z\} = \int_0^\infty e^{-zt} u(t)\, dt$ denotes the Laplace transform of $u(t)$.

Lemma 2.12. *System (1.1)-(1.2) is equivalent to the following integro-differential equation*

$$u(t) = \int_0^T G(t,\tau)\, g(\tau, u(\tau), D^s u(\tau))\, d\tau, \ t \in I,$$

where $G(t,\tau)$ denotes the Green's function defined by

$$G(t,\tau) = \begin{cases} \frac{1}{\Gamma(q)} \left[(t-\tau)^{q-1} - \left(\frac{t}{T}\right)^{q-n+1} (T-\tau)^{q-1}\right], & 0 \le \tau \le t \le T, \\ \frac{1}{\Gamma(q)} \left[-\left(\frac{t}{T}\right)^{q-n+1} (T-\tau)^{q-1}\right], & 0 \le t \le \tau \le T. \end{cases} \tag{2.3}$$

Proof. We take $\left[D^{q-i} u(t)\right]_{t=0} = b_i$. Applying Laplace transform on both side of (1.1) and using Lemma 2.11, we get

$$z^q U(z) - \sum_{i=0}^{n-1} z^i \left[D^{q-i-1} u(t)\right]_{t=0} = G(z),$$

where $U(z)$ and $G(z)$ denote the Laplace transformers of $u(t)$ and $g(t)$ respectively.

In other words, we can write

$$U(z)=z^{-q}G(z)+\sum_{i=0}^{n-1}b_{i+1}z^{i-q}.$$

Inverse Laplace transform give us

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}g(\tau,u(\tau),D^su(\tau))\,d\tau+\sum_{i=0}^{n-1}\frac{b_{i+1}}{\Gamma(q-i)}t^{q-i-1}\\&=\frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}g(\tau,u(\tau),D^su(\tau))\,d\tau+\sum_{i=1}^{n}\frac{b_i}{\Gamma(q-i+1)}t^{q-i},\end{aligned}$$

we have $b_i=0$, $i=1,\dots,n$ for $i\neq n-1$ then

$$u(t)=\frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}g(\tau,u(\tau),D^su(\tau))\,d\tau+\frac{b_{n-1}}{\Gamma(q-n+2)}t^{q-n+1}.\quad(2.4)$$

By condition $u(T)=0$ we obtain

$$\frac{b_{n-1}}{\Gamma(q-n+2)}=\frac{-T^{n-q-1}}{\Gamma(q)}\int_0^T(T-\tau)^{q-1}g(\tau,u(\tau),D^su(\tau))\,d\tau,$$

substuting in (2.4), we get

$$u(t)=\int_0^T G(t,\tau)\,g(\tau,u(\tau),D^su(\tau))\,d\tau,$$

where $G(.,.)$ is the Green's kernel defined by (2.3). The proof is complete. □

Define the operator $A\colon E_\sigma(I)\to E_\sigma(I)$ by

$$Au(t)=\int_0^T G(t,\tau)\,g(\tau,u(\tau),D^su(\tau))\,d\tau.\quad(2.5)$$

We give in the following, Schauder's fixed point theorem which is the main ingredient in the proof of our existence result.

Theorem 2.13 (Schauder's fixed point theorem [20]). *Let M be a closed convex subset of a Banach space E. If $A:M\to M$ is continuous and the set $\overline{A(M)}$ is compact, then A has a fixed point in M.*

Obviously, all fixed points of A is a solution of system (1.1)-(1.2).

3. Main results

Theorem 3.1. *Assume the following hypothesis:*

(H_1) $g\colon I\times\mathbb{R}^2\to\mathbb{R}$ *satisfies the Carathéodory's condition.*

(H_2) *There exist a real constant* $c>0$ *and a function* $b\colon I\to\mathbb{R}_+$ *with* $b\in L^{1,\sigma}(I)$ *such that*

$$|g(\tau,u,v)|\le b(\tau)+c\left(|u|^p+|v|^p\right),$$

for any $\tau\in I$ *and any* $u,v\in\mathbb{R}$.

(H_3) *There exists a real number* $R>0$ *satisfies*

$$\|G_*\|_\infty\left[T\sigma_*+\frac{\sigma_*T^{1+p(1-s)}}{(\Gamma(2-s))^p}+\frac{\|K'\|_{p,\sigma}^p}{(\Gamma(1-s))^p}\right]^{\frac{1}{p}}\left[\|b\|_{1,\sigma}+cR^p\right]\le R,$$

where $G_*(t)=\sup_{\tau\in I}|G(t,\tau)|$.
Then the system (1.1)-(1.2) *has at least one solution in* E_σ.

Proof. Consider the operator A given by (2.5) and we define the set

$$B_R=\{u\in E_\sigma,\ \|u\|_\sigma\le R\},$$

where R is the same constant defined in (H_3). It is clear that B_R is convex, closed and bounded subset of E_σ.

Firstly, we show that $AB_R\subset B_R$. Let $u\in B_R$, $t\in I$, then by using (H_2) and (H_3), we get

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}}|Au(t)|\\
&\le\sigma(t)^{\frac{1}{p}}\int_0^T|G(t,\tau)|\,|g(\tau,u(\tau),D^su(\tau))|\,d\tau\\
&\le\sigma(t)^{\frac{1}{p}}\int_0^T\frac{|G(t,\tau)|}{\sigma(\tau)}\left[\sigma(\tau)\left(b(\tau)+c\left(|u(\tau)|^p+|D^su(\tau)|^p\right)\right)\right]d\tau\\
&\le\sigma_*^{\frac{1}{p}}G_*(t)\left[\|b\|_{1,\sigma}+c\left(\|u\|_{p,\sigma}^p+\left\|\left(I^{1-s}u\right)'\right\|_{p,\sigma}^p\right)\right]\\
&\le\sigma_*^{\frac{1}{p}}\|G_*\|_\infty\left[\|b\|_{1,\sigma}+c\|u\|_\sigma^p\right],
\end{aligned}$$

then

$$\|Au\|_{p,\sigma}^p\le T\sigma_*\|G_*\|_\infty^p\left[\|b\|_{1,\sigma}+cR^p\right]^p. \tag{3.1}$$

Similarly, we obtain the following estimates

$$\left\|I^{1-s}Au\right\|_{p,\sigma}^p\le\frac{\sigma_*T^{1+p(1-s)}}{(\Gamma(2-s))^p}\|G_*\|_\infty^p\left[\|b\|_{1,\sigma}+cR^p\right]^p, \tag{3.2}$$

and

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}} \left| \left(I^{1-s}Au\right)'(t) \right| \\
&= \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \left| \frac{d}{dt} \int_0^t \frac{1}{(t-\tau)^s} Au(\tau)\, d\tau \right| \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \left| \frac{d}{dt} \int_0^t \frac{1}{(t-\tau)^s} \left| Au(\tau) \right| d\tau \right| \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \left\| G_* \right\|_\infty \left[\|b\|_{1,\sigma} + c \left\| u \right\|_\sigma^p \right] \left| \frac{d}{dt} \int_0^t \frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^s} d\tau \right|,
\end{aligned}$$

thus,

$$\left\| \left(I^{1-s}Au\right)' \right\|_{p,\sigma}^p \leq \frac{\|K'\|_{p,\sigma}^p}{(\Gamma(1-s))^p} \left\| G_* \right\|_\infty^p \left[\|b\|_{1,\sigma} + cR^p \right]^p. \tag{3.3}$$

We combine (3.1)-(3.3), it yields

$$\|Au\|_\sigma \leq \|G_*\|_\infty \left[T\sigma_* + \frac{\sigma_* T^{1+p(1-s)}}{(\Gamma(2-s))^p} + \frac{\|K'\|_{p,\sigma}^p}{(\Gamma(1-s))^p} \right]^{\frac{1}{p}} \left[\|b\|_{1,\sigma} + cR^p \right] \leq R. \tag{3.4}$$

Hence $AB_R \subset B_R$.

Secondly, We prove that A is continuous operator. Let u_n, u in E_σ such that $\|u_n - u\|_\sigma \to 0$, using $(H_1) - (H_2)$, then for all $t \in I$ we have

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}} \left| Au_n(t) - Au(t) \right| \\
&\leq \sigma(t)^{\frac{1}{p}} \int_0^T \frac{|G(t,\tau)|}{\sigma(\tau)} \left[\sigma(\tau) \left| g(\tau, u_n(\tau), D^s u_n(\tau)) - g(\tau, u(\tau), D^s u(\tau)) \right| \right] d\tau \\
&\leq \sigma_*^{\frac{1}{p}} G_*(t) \left\| N_g u_n - N_g u \right\|_{1,\sigma} \\
&\leq \sigma_*^{\frac{1}{p}} \left\| G_* \right\|_\infty \left\| N_g u_n - N_g u \right\|_{1,\sigma},
\end{aligned}$$

applying L^p-norm, one gets

$$\|Au_n - Au\|_{p,\sigma} \leq (T\sigma_*)^{\frac{1}{p}} \|G_*\|_\infty \|N_g u_n - N_g u\|_{1,\sigma}. \tag{3.5}$$

Also

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}}\left|I^{1-s}Au_n(t)-I^{1-s}Au(t)\right| \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)}\int_0^t (t-\tau)^{-s}\int_0^T \frac{|G(\tau,\theta)|}{\sigma(\theta)}\left(\sigma(\theta)\left|g\left(\theta,u_n(\theta),D^s u_n(\theta)\right)\right.\right. \\
&\left.\left.-g\left(\theta,u(\theta),D^s u(\theta)\right)\right|\right) d\theta d\tau \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)}\int_0^t (t-\tau)^{-s} G_*(\tau)\left\|N_g u_n-N_g u\right\|_{1,\sigma} d\tau \\
&\leq \frac{\sigma_*^{\frac{1}{p}} T^{1-s}}{\Gamma(2-s)}\left\|G_*\right\|_\infty\left\|N_g u_n-N_g u\right\|_{1,\sigma},
\end{aligned}$$

then

$$\left\|I^{1-s}Au_n-I^{1-s}Au\right\|_{p,\sigma}\leq \frac{\sigma_*^{\frac{1}{p}} T^{\frac{1}{p}+1-s}}{\Gamma(2-s)}\left\|G_*\right\|_\infty\left\|N_g u_n-N_g u\right\|_{1,\sigma}. \tag{3.6}$$

Moreover

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}}\left|\left(I^{1-s}Au_n\right)'(t)-\left(I^{1-s}Au\right)'(t)\right| \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)}\left|\frac{d}{dt}\int_0^t (t-\tau)^{-s}\int_0^T \frac{|G(\tau,\theta)|}{\sigma(\theta)}\sigma(\theta)\left|\left[g\left(\theta,u_n(\theta),D^s u_n(\theta)\right)\right.\right.\right. \\
&\left.\left.\left.-\ g\left(\theta,u(\theta),D^s u(\theta)\right)\right]\right| d\theta d\tau\right| \\
&\leq \frac{\left\|G_*\right\|_\infty}{\Gamma(1-s)}\left[\sigma^{\frac{1}{p}}(t)\left|\frac{d}{dt}\int_0^t \frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^s}d\tau\right|\right]\left\|N_g u_n-N_g u\right\|_{1,\sigma},
\end{aligned}$$

so

$$\left\|\left(I^{1-s}Au_n\right)'-\left(I^{1-s}Au\right)'\right\|_{p,\sigma}\leq \frac{\left\|K'\right\|_{p,\sigma}\left\|G_*\right\|_\infty}{\Gamma(1-s)}\left\|N_g u_n-N_g u\right\|_{1,\sigma}. \tag{3.7}$$

Combining (3.5)-(3.7), one finds

$$\begin{aligned}
&\left\|Au_n-Au\right\|_\sigma \\
&\leq \left\|G_*\right\|_\infty\left[T\sigma_*+\frac{\sigma_* T^{1+p(1-s)}}{(\Gamma(2-s))^p}+\frac{\left\|K'\right\|_{p,\sigma}^p}{(\Gamma(1-s))^p}\right]^{\frac{1}{p}}\left\|N_g u_n-N_g u\right\|_{1,\sigma}.
\end{aligned} \tag{3.8}$$

Since the operator N_g is continuous on $L^{1,\sigma}$, then then the right side term of (3.8) tends to zero when n tends to infinity. This show that the operator A is continuous.

Thirdly, we prove that the set $AB_R = \{Au : u \in B_R\}$ is relatively compact in

E_σ using Lemma 2.6. For any $u \in B_R$ and any $\delta \geq 0$, we have

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}} |Au(t+\delta) - Au(t)| \\
&\leq \sigma(t)^{\frac{1}{p}} \int_0^T |G(t+\delta,\tau) - G(t,\tau)| \, |g(\tau, u(\tau), D^s u(\tau))| \, d\tau \\
&\leq \sigma(t)^{\frac{1}{p}} \int_0^T \frac{|G(t+\delta,\tau) - G(t,\tau)|}{\sigma(\tau)} \left[\sigma(\tau)\left(b(\tau) + c\left(|u(\tau)|^p + |D^s u(\tau)|^p\right)\right)\right] d\tau \\
&\leq \sigma(t)^{\frac{1}{p}} \sup_{\tau\in I} |G(t+\delta,\tau) - G(t,\tau)| \left[\|b\|_{1,\sigma} + c\left(\|u^p\|_{1,\sigma} + \|(D^s u)^p\|_{1,\sigma}\right)\right] \\
&= \sigma_*^{\frac{1}{p}} \sup_{t\in I}\left[\sup_{\tau\in I} |G(t+\delta,\tau) - G(t,\tau)|\right] \left[\|b\|_{1,\sigma} + c\left(\|u\|_{p,\sigma}^p + \left\|\left(I^{1-s}u\right)'\right\|_{p,\sigma}^p\right)\right] \\
&\leq \sigma_*^{\frac{1}{p}} \sup_{t\in I}\left[\sup_{\tau\in I} |G(t+\delta,\tau) - G(t,\tau)|\right] \left[\|b\|_{1,\sigma} + c\|u\|_\sigma^p\right],
\end{aligned}$$

therefore

$$\frac{\|Au_n(.+\delta) - Au(.)\|_{p,\sigma}}{(T\sigma_*)^{\frac{1}{p}}\left[\|b\|_{1,\sigma} + cR^p\right]} \leq \sup_{t\in I}\left[\sup_{\tau\in I} |G(t+\delta,\tau) - G(t,\tau)|\right]. \tag{3.9}$$

Similarly

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}} \left|I^{1-s}Au(t+\delta) - I^{1-s}Au(t)\right| \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \int_0^t (t-\tau)^{-s} \int_0^T |G(\tau+\delta,\theta) - G(\tau,\theta)| \, |g(\theta, u(\theta), D^s u(\theta))| \, d\theta d\tau \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \int_0^t (t-\tau)^{-s} \int_0^T \frac{|G(\tau+\delta,\theta) - G(\tau,\theta)|}{\sigma(\theta)} \sigma(\theta) \left[b(\theta)\right. \\
&\quad \left. + c\left(|u(\theta)|^p + |D^s u(\theta)|^p\right)\right] d\theta d\tau \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \int_0^t (t-\tau)^{-s} \sup_{\theta\in I} |G(\tau+\delta,\theta) - G(\tau,\theta)| \left[\|b\|_{1,\sigma}\right. \\
&\quad \left. + c\left(\|u^p\|_{1,\sigma} + \|(D^s u)^p\|_{1,\sigma}\right)\right] d\tau \\
&\leq \frac{T^{1-s}\sigma_*^{\frac{1}{p}}}{\Gamma(2-s)} \sup_{\tau\in I}\left[\sup_{\theta\in I} |G(\tau+\delta,\theta) - G(\tau,\theta)|\right] \left[\|b\|_{1,\sigma} + c\|u\|_\sigma^p\right],
\end{aligned}$$

it yields

$$\begin{aligned}
&\frac{\Gamma(2-s)\left\|\left(I^{1-s}Au_n\right)(.+\delta) - \left(I^{1-s}Au\right)(.)\right\|_{p,\sigma}}{T^{1-s}(T\sigma_*)^{\frac{1}{p}}\left[\|b\|_{1,\sigma} + cR^p\right]} \\
&\leq \sup_{\tau\in I}\left[\sup_{\theta\in I} |G(\tau+\delta,\theta) - G(\tau,\theta)|\right].
\end{aligned} \tag{3.10}$$

Using same method, one finds

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}} \left| \left(I^{1-s}Au\right)'(t+\delta) - \left(I^{1-s}Au\right)'(t) \right| \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \left| \frac{d}{dt} \int_0^t (t-\tau)^{-s} \int_0^T |G(\tau+\delta,\theta) - G(\tau,\theta)| \left| g\left(\theta, u(\theta), D^s u(\theta)\right) \right| d\theta d\tau \right| \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \left| \frac{d}{dt} \int_0^t (t-\tau)^{-s} \int_0^T \frac{|G(\tau+\delta,\theta) - G(\tau,\theta)|}{\sigma(\theta)} \right. \\
&\quad \times \left[\sigma(\theta) \left(b(\theta) + c\left(|u(\theta)|^p + |D^s u(\theta)|^p \right) \right) \right] d\theta d\tau \Big| \\
&\leq \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \left| \frac{d}{dt} \int_0^t (t-\tau)^{-s} \sup_{\tau \in I} |G(t+\delta,\tau) - G(t,\tau)| \, d\tau \right| \\
&\quad \times \left[\|b\|_{1,\sigma} + c\left(\|u^p\|_{1,\sigma} + \|(D^s u)^p\|_{1,\sigma} \right) \right] \\
&\leq \frac{\left[\sigma(t)^{\frac{1}{p}} \left| \frac{d}{dt} \int_0^t \frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^s} d\tau \right| \right]}{\Gamma(1-s)} \sup_{t \in I} \left[\sup_{\tau \in I} |G(t+\delta,\tau) - G(t,\tau)| \right] \\
&\quad \times \left[\|b\|_{1,\sigma} + c\left(\|u\|_{p,\sigma}^p + \left\| \left(I^{1-s}u\right)' \right\|_{p,\sigma}^p \right) \right],
\end{aligned}$$

then

$$\begin{aligned}
&\frac{\Gamma(1-s) \left\| \left(I^{1-s}Au\right)'(.+\delta) - \left(I^{1-s}Au\right)'(.) \right\|_{p,\sigma}}{\|K'\|_{p,\sigma} \left[\|b\|_{1,\sigma} + cR^p \right]} \\
&\qquad \leq \sup_{t \in I} \left[\sup_{\tau \in I} |G(t+\delta,\tau) - G(t,\tau)| \right]. \qquad (3.11)
\end{aligned}$$

From the continuity of the function $G(.,.)$ on I^2, we conclude that the second members of (3.9)-(3.11) tend to zero when δ tends to zero, these prove the condition (i) of Lemma 2.6.

It remains to prove condition (ii) of Lemma 2.6, before this end, for simplification we set the notations

$$c_1 = \sigma_*^{\frac{1}{p}} \|G_*\|_\infty \left[\|b\|_{1,\sigma} + c\|u\|_\sigma^p \right],$$

$$c_2 = \frac{\sigma_*^{\frac{1}{p}} T^{1-s}}{\Gamma(2-s)} \|G_*\|_\infty \left[\|b\|_{1,\sigma} + c\|u\|_\sigma^p \right],$$

and

$$\Phi(t) = \frac{\|G_*\|_\infty}{\Gamma(1-s)} \left[\sigma^{\frac{1}{p}}(t) |K'(t)| \right] \left[\|b\|_{1,\sigma} + c\|u\|_\sigma^p \right].$$

Obviously, from conditions (2.1), we deduce that Φ belongs to $L^{p,\sigma}(I)$. On the other hand, for all $v \in AB_R$, there exists $u \in B_R$ such that $v = Au$, taking into account assumption (H_2) and some calculation we get the following three estimates: for $t \in I$,

$$\sigma(t)^{\frac{1}{p}} |v(t)| \le c_1,$$

$$\sigma(t)^{\frac{1}{p}} \left|I^{1-s}v(t)\right| \le c_2,$$

and

$$\sigma(t)^{\frac{1}{p}} \left|\left(I^{1-s}v\right)'(t)\right| \le \Phi(t).$$

Then, for any $v \in AB_R$ and $t \in T$, we have

$$\begin{aligned} \lim_{\epsilon\to 0} \sup_{v\in AB_R} &\left[\int_{T-\epsilon}^{T} \sigma(t)^{\frac{1}{p}} |v(t)|^p dt + \int_{T-\epsilon}^{T} \sigma(t)^{\frac{1}{p}} \left|I^{1-s}v(t)\right|^p dt \right. \\ &\left. + \int_{T-\epsilon}^{T} \sigma(t)^{\frac{1}{p}} \left|I^{1-s}v(t)\right|^p dt\right] \\ &\le \lim_{\epsilon\to 0} \left[\int_{T-\epsilon}^{T} c_1^p dt + \int_{T-\epsilon}^{T} c_2^p dt + \int_{T-\epsilon}^{T} |\Phi(t)|^p dt\right] = 0. \end{aligned}$$

This proves the condition (ii) of Lemma 2.6.

Furthermore, from (3.4), we have $\|Au\|_\sigma \le R$ for all $u \in B_R$, this proves that AB_R is uniformly bounded. Consequently, AB_R is relatively compact in E_σ. Finally, using Schauder's fixed point theorem, we conclude that A has at least one fixed point in B_R and the proof of Theorem 3.1 is complete. □

4. Example

Consider the following boundary value problem of fractional differential equations in E_σ

$$\begin{cases} D^q u(t) = \frac{t^{\frac{-1}{2}} e^t + \arctan\left|(tu(t))^2 - (tD^s u(t))^2\right|}{(1+t)^3 e^{t+|u(t)|}}, \quad t \in I = [0,1], \\ D^{(q-i)} u|_{t=0} = 0, \ i = 1,2,3,5, \ u(1) = 0, \end{cases} \tag{4.1}$$

$q = \frac{9}{2}$, $s = \frac{1}{6}$, $g(t,u,v) = \frac{t^{\frac{-1}{2}} e^t + \arctan\left((tu)^2 - (tv)^2\right)^2}{(1+t)^3 e^{t+|u|}}$, then

$$\begin{aligned} |g(t,u,v)| &\le \frac{t^{\frac{-1}{2}} e^t + \left(u^4 + v^4\right)}{(1+t)^3 e^{t+|u(t)|}} \le \frac{t^{\frac{-1}{2}} e^t}{(1+t)^3 e^t} + \frac{\left(u^4+v^4\right)}{(1+t)^3 e^t} \\ &\le \frac{1}{t^{\frac{1}{2}}(1+t)^3} + \frac{\left(u^4+v^4\right)}{(1+t)^3 e^t} \le b(t) + c\left(u^4+v^4\right), \end{aligned}$$

where $b(t) = \frac{1}{t^{\frac{1}{2}}(1+t)^3}$ and $c = 1$. $p = 4$, $\sigma(t) = (1+t)^4$, it is clear that $1 \le \sigma(t-\tau)$ for $t \ge \tau$, and

$$K(t) = \int_0^t \frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^s} d\tau = \int_0^t \frac{1+z}{z^{\frac{1}{6}}} dz = \frac{6t^{\frac{5}{6}}(5t+11)}{55},$$

and

$$K'(t) = t^{\frac{5}{6}} + t^{-\frac{1}{6}},$$

some computations give us

$$\|K'\|_{4,\sigma} \simeq 3.187991075720807, \qquad \|b\|_{1,\sigma} = \frac{8}{3},$$

then condition (H_3) becomes

$$R^4 - 4.672639065946051R + 2.666666666666665 \leq 0,$$

so $R \in [0.598080985027521, 1.405251623483919]$.

Using theorem 3.1, we deduce that the nonlinear functional boundary value problem (4.1) has at least one solution in $B_R \subset E_\sigma$ for any

$$R \in [0.598080985027521, 1.405251623483919].$$

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Ahmed Hallaci
Departement of Mathematics, Faculty of Sciences, University of 08 Mai 1945 Guelma, P. Box 401, Guelma, 24000, Algeria.
hal.ahhmedguelm2@gmail.com

Hamid Boulares
Departement of Mathematics, Faculty of Sciences, University of 08 Mai 1945 Guelma, P. Box 401, Guelma, 24000, Algeria.
boulareshamid@gmail.com

Abdelouaheb Ardjouni
Department of Mathematics and Informatics, University of Souk-Ahras, P.O. Box 1553, Souk-Ahras, 41000, Algeria.
abd_ardjouni@yahoo.fr

Abderrazak Chaoui
Departement of Mathematics, Faculty of Sciences, University of 08 Mai 1945 Guelma, P. Box 401, Guelma, 24000, Algeria.
rchaoui@outlook.fr